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GREEDY ALGORITHMS AND BASES FROM THE POINT OF VIEW OF BANACH SPACE THEORY

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1. Introduction

Imagine that you are presented with the problem of transmitting a photograph over a telephone connection. The restrictions on you are the physical limitations of the telephone line, or time constraints, but you have perfect knowledge of the photograph. How do you decide what parameters to transmit? In mathematical terms the photograph may be represented by a function, f, or a family of functions representing color densities. So if we convert the problem to a more abstract level we are given a function and want to select a certain finite set of parameters to send to another person so that he or she can reconstruct the function as accurately as possible.

Of course one solution is simply to send as many points of the graph of f as is practical. If f were a completely random function, this might well be the best that can be done. But of course in practice our function is not random and contains many patterns. This suggests a series expansion of the function; one might think of Fourier series, but perhaps an expansion with respect to some wavelet basis is more appropriate.

Now the function is coded by its coefficients with respect to our given basic functions. We can only send some of these coefficients. Which ones should we choose to send? Suppose we can send a thousand coefficients. Mathematically, we are used to the idea that we pick the first one thousand as our approximation. But this may not be the most efficient for our given function; perhaps all these coefficients are small and some later coefficients are large. So, perhaps we should consider sending the thousand largest coefficients.

This is the basic idea behind the concept of a greedy basis. Of course we have been quite imprecise and we would need to specify some normalization

of our basis elements and some space functions within which to work. The idea has been around for some time; the author was first introduced to it at a conference on Littlewood-Paley theory in 1990 [9]. However, the formal development of a theory of greedy bases is more recent and was initiated in an important paper of Konyagin and Temlyakov [15], which we will discuss below. Subsequently the theory has been developed quite rapidly from the point of view of approximation theory by Temlyakov and others; we refer to [21] for a recent survey.

We are going to concentrate on the Banach space aspects of this theory, where rather unexpectedly the the theory of greedy bases has links to some old and classical results and also to some open problems. The idea of studying greedy bases and related greedy algorithms from a more abstract point of view seems to originate with the work of Dilworth, Kutzarova and Temlyakov [4].

Most of this article will focus on greedy bases, but in the final section we will try to draw the reader's attention to some other intriguing questions that concern convergence of greedy algorithms.

2. Greedy and quasi-greedy bases

In [15], Konyagin and Temlyakov introduced the formal notions of greedy and quasi-greedy bases in a Banach space. Suppose X is a (real) Banach space and $(e_n)_{n=1}^{\infty}$ is a Schauder basis of X. We will denote by $(e_n^*)_{n=1}^{\infty}$ the corresponding bi-orthogonal functions. Assume that $(e_n)_{n=1}^{\infty}$ is normalized, i.e. $\|e_n\| = 1$ for all n. Each $x \in X$ has a unique expansion

$$x = \sum_{n=1}^{\infty} e_n^*(x) e_n.$$

We will take these terms in decreasing order of magnitude; however, when two terms are of equal size we take them in the basis order. Thus for each n we let A_n be the unique subset of $\mathbb{N}$ of n elements with the properties that:

$$j \in A_n,\ k \notin A_n \implies |e_k^*(x)| < |e_j^*(x)| \quad \text{or} \quad |e_k^*(x)| = |e_j^*(x)| \text{ and } j \le k.$$

We define the *n-term greedy approximation* to x by

$$\mathcal{G}_n(x) = \sum_{j \in A_n} e_j^*(x) e_j.$$

Notice the operator $\mathcal{G}_n$ is highly nonlinear in x.

Now we define $(e_n)_{n=1}^\infty$ to be *quasi-greedy* if $\lim_{n\to\infty}\|\mathcal{G}_n(x)-x\| = 0$ for all $x \in X$. This means of course that the greedy approximations do converge, and is essentially the minimal requirement we might have to make this a reasonable method of approximation.

Of course, any unconditional basis is quasi-greedy. Let us give a simple example of a non-quasi-greedy basis. Let $(s_n)_{n=1}^\infty$ be the summing basis in c_0. Then $\sum_{n=1}^\infty \xi_n s_n$ converges if and only if the (real) series $\sum_{n=1}^\infty \xi_n$ converges. Let

$$\xi_n = \left(1, -1, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{9}, -\frac{1}{9}, -\frac{1}{9}, \dots\right)$$

so that each $1/n$ is followed by n terms $-1/n^2$. Then

$$\left\|\mathcal{G}_{3n^2-n/2}\left(\sum_{k=1}^{\infty}\xi_k s_k\right)\right\| \ge \sum_{k=1}^{n^2}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k} \to \infty.$$

At the opposite extreme we may ask if it is true that the greedy algorithm in this instance gives the *best* method of approximating each $x \in X$. Konyagin and Temlyakov formalized this notion as follows. For each $x \in X$ and $n \in \mathbb{N}$ we define

$$\sigma_n(x) = \inf\left\{\|x - \sum_{j\in A}\alpha_j e_j\| : \qquad |A| = n,\ \alpha_j \in \mathbb{R},\ j \in A\right\} \tag{2.1}$$

to measure the minimal possible error in an n-term approximation. Then we say that $(e_n)_{n=1}^\infty$ is *greedy* if there is a constant C so that

$$\|x - \mathcal{G}_n(x)\| \le C\sigma_n(x) \qquad x \in X.$$

Thus, up to a constant C, one cannot improve on the greedy method of approximation.

If we have a basis which is not normalized, then we use the terms quasi-greedy and greedy if the corresponding normalized basis has these properties. Before proceeding, let us note that these definitions *seem* isometric in nature: they depend on the normalization of $(e_n)_{n=1}^\infty$ and so, passing to an equivalent norm (and then renormalizing $(e_n)_{n=1}^\infty$, should disrupt the behavior of the algorithm). This is not the case, however, and one can show that both definitions are invariant for equivalent norms.

We define a basis $(e_n)_{n=1}^\infty$ to be *democratic* if there is a constant Δ so that if A and B are two finite subsets of $\mathbb{N}$ with $|A| \le |B|$ then

$$\|\sum_{j\in A} e_j\| \le \Delta\|\sum_{j\in B} e_j\|.$$

Democracy means essentially that any n basis vectors have the same weight as any other n basis vectors.

The main theorem here is

Theorem 2.1. *(Konyagin-Temlyakov [15]) A basis $(e_n)_{n=1}^\infty$ of a Banach space X is greedy if and only if it is unconditional and democratic.*

This theorem is a very satisfactory characterization of a greedy bases. Of course symmetric bases are greedy, but, more importantly there are practical and useful examples. The Haar basis of $L_p[0,1]$ is greedy if $1 < p < \infty$ for example. See also [22].

In the case of quasi-greedy bases, Wojtaszczyk [22] showed that a basis is quasi-greedy if and only if the greedy operators $\mathcal{G}_n$ are uniformly bounded i.e. for some constant C

$$\|\mathcal{G}_n(x)\| \le C\|x\| \qquad x \in X,\ n = 1, 2, \dots.$$

Recall that these operators are nonlinear so this is not just the Uniform Boundedness Principle! He was also able to show that a Hilbert space has a quasi-greedy basis which is not greedy (note that a basis of a Hilbert space is greedy if and only if it is unconditional). We also note that Dilworth and Mitra [5] constructed a conditional quasi-greedy basis of ℓ_1. A recent very interesting example of a quasi-greedy basis in a space of functions of bounded variation is given in [1].

3. Almost greedy bases and duality

Let us look again at the definition of a greedy basis. It is important to notice that the definition of the optimal error $\sigma_n(x)$, (2.1), involves arbitrary coefficients. It would not be the same to consider

$$\tilde{\sigma}_n(x) = \inf\left\{\|x - \sum_{j\in A} e_j^*(x)e_j\| : \qquad |A| = n,\ \alpha_j \in \mathbb{R},\ j \in A\right\} \tag{3.2}$$

where one puts the expected coefficients $e_j^*(x)$ in place of the α_j. The author found this out quite by accident attempting to prove the Konyagin-Temlyakov theorem in a seminar with the wrong definition!

So we may define a basis $(e_n)_{n=1}^\infty$ to be *almost greedy* if there is a constant C so that

$$\|x - \mathcal{G}_n(x)\| \le C\tilde{\sigma}_n(x) \qquad x \in X.$$

It turns out, rather remarkably, that this definition has some nice equivalent formulations [3]:

Theorem 3.1. *Let X be a Banach space with a basis $(e_n)_{n=1}^{\infty}$. The following conditions are equivalent:*
(i) $(e_n)_{n=1}^{\infty}$ is almost greedy.
(ii) $(e_n)_{n=1}^{\infty}$ is quasi-greedy and democratic.
(iii) For some (respectively, every) $\lambda >$ there is a constant $C = C_\lambda$ so that

$$\|x - \mathcal{G}_{[\lambda n]}(x)\| \le C\sigma_n(x).$$

Here (iii) says that the greedy algorithm is essentially the best if one allows a small percentage increase in n, so the terminology is justified. There is in fact very little difference between the theory of almost greedy basis and that of greedy bases, except that an almost greedy basis need not be unconditional. However in a Hilbert space it is not difficult to show that any quasi-greedy basis is already almost greedy.

We now consider duality. If $(e_n)_{n=1}^{\infty}$ is a basis of X then $(e_n^*)_{n=1}^{\infty}$ is a basic sequence in X^* and if X is reflexive it is a basis for X^*. How do the three properties we have introduced dualize? In fact the Dilworth-Mitra example [5] of a quasi-greedy basis of ℓ_1 has the property that the dual basic sequence is not quasi-greedy [3]; in fact this example is almost greedy as well. Similarly, Oswald [19] showed that the Haar basis of H_1 is greedy while the dual basic sequence in BMO fails to be greedy; of course the dual sequence is unconditional and hence quasi-greedy in this instance.

Nevertheless there are positive results available [3]:

Theorem 3.2. *Let X be a Banach space with non-trivial Rademacher type. Suppose $(e_n)_{n=1}^{\infty}$ is an (almost) greedy basis of X; then $(e_n^*)_{n=1}^{\infty}$ is an almost greedy basic sequence in X^*.*

The counterexamples to such a theorem in general depend on the *fundamental function*

$$\varphi(n) = \|\sum_{j=1}^{n} e_j\|.$$

If the fundamental function increases slowly enough then the properties of being greedy or almost greedy pass to the dual. For a precise formulation we refer to [3].

4. Existence of quasi-greedy and almost greedy bases

It is clear that a Banach space cannot have a greedy basis unless it has an unconditional basis and this rules out many natural examples such as $L_1[0,1]$ and $C[0,1]$. One can also gives other examples of spaces failing to have a greedy basis such as $\ell_1 \oplus \ell_2$; here it is a classical result of Edelstein and Wojtaszczyk [7] that any normalized unconditional basis is equivalent to the canonical basis, which is plainly not democratic.

However it much easier to have an almost greedy basis and a very general construction was given in [2]. We start from the assumption that X has a basis, and then suppose that it has a complemented subspace S with a symmetric basis. Then X is isomorphic to $X \oplus S$. We now construct a new basis of the direct sum $X \oplus S$ which behaves very like the symmetric basis of S, in the sense that it inherits many of the good properties of the symmetric basis. The following theorem is proved in [2].

Theorem 4.1. *Let X be a Banach space with a basis and suppose X has a complemented subspace S with a symmetric basis. Then:*
(i) If S is not isomorphic to c_0 then X has a quasi-greedy basis.
(ii) If S has finite cotype then X has an almost greedy basis.

From this theorem one gets immediately that L_1 has an almost greedy basis, for example. It also appears that there is an obstruction around c_0. Of course c_0 has a greedy basis in the canonical basis, but we saw earlier that the summing basis is not quasi-greedy.

At this point we should observe the connection between quasi-greediness and unconditionality. A quasi-greedy basis need not be unconditional of course, but it preserves some vestige of unconditionality. For example there is a constant C so that we have

$$\|\sum_{j=1}^{n} \epsilon_j a_j e_j\| \le C\|\sum_{j=1}^{n} a_j e_j\|$$

whenever $\epsilon_j = \pm 1$ *provided* we make the requirement that all the non-zero coefficients (a_j) are approximately of the same size, e.g. $1 \le |a_j| \le 2$ if $a_j \neq 0$. This is an important restriction and it turns out that classical arguments in Banach space theory can be used to show it is not always possible to find such a basis.

Recall the classical result of Lindenstrauss and Pełczyński [16]:

Theorem 4.2. *Any normalized unconditional basis of c_0 (or any $\mathcal{L}_\infty$-space) is equivalent to the canonical basis of c_0.*

Based on the same ideas, one can prove [2]:

Theorem 4.3. *Let X be a $\mathcal{L}_\infty$-space; if X has a quasi-greedy basis $(e_n)_{n=1}^{\infty}$*

then X is isomorphic to c_0 and $(e_n)_{n=1}^\infty$ is equivalent to the canonical basis of c_0.

In particular, the space $C[0,1]$ has no quasi-greedy basis.

5. Basic sequences: an open question

From the point of view of Banach space theory, it becomes natural to ask whether every Banach space at least contains a quasi-greedy basic sequence. In view of the remarks above, there is some connection here with the famous unconditional basic sequence problem, which was settled in the celebrated paper of Gowers and Maurey [11]. But it should be a lot easier to contain a quasi-greedy basic sequence. A natural related question is whether every weakly null sequence has a subsequence which is quasi-greedy; of course if this is true, then every Banach space has a quasi-greedy basic sequence by Rosenthal's theorem [20].

This problem was first examined in [2] and the *easy* case was settled:

Theorem 5.1. *Let X be a Banach space which does not have c_0 as a spreading model (e.g. suppose X has nontrivial cotype). Then every normalized weakly null sequence has a quasi-greedy subsequence.*

However the problem in general seems rather hard, and curiously intersects with ideas that have already been considered in Banach space theory. In 1978, Elton had considered a very similar problem [8]. Elton's theorem is the following:

Theorem 5.2. *For each $0 < \delta < 1$ there is a constant $C(\delta)$ with the following property. Let $(x_n)_{n=1}^\infty$ be a weakly null sequence in a Banach space: then $(x_n)_{n=1}^\infty$ has a subsequence $(y_n)_{n=1}^\infty$ such that*

$$\|\sum_{j\in A} a_j y_j\| \le C\|\sum_{j=1}^{\infty} a_j y_j\|$$

for all finitely nonzero sequences $(a_j)_{j=1}^\infty$ such that $\max_j |a_j| \le 1$ and all $A \subset \{j:\ |a_j| \ge \delta\}$.

Thus Elton's theorem gives a restricted form of unconditionality for subsequences of $(x_n)_{n=1}^\infty$. It is worth noting that at the time of Elton's work examples were already known of weakly null sequences which contain no unconditional basic sequence [18]; we also remark that there is a recent result of Johnson, Maurey and Schechtman [13] giving such an example in $L_1[0,1]$.

As pointed out in [2] Elton's argument gave a function $C(\delta) \sim \log(1/\delta)$. In order to find a quasi-greedy subsequence one needs that $C(\delta)$ is uniformly bounded. In a recent preprint [6] the authors give an in-depth examination of this problem and prove many results on *partial unconditionality* but the main question remains open.

Of course we have now come a long way from practical applications!

6. More general greedy algorithms

In the last section we consider a rather more general situation. Suppose X is a Banach space and we are given a subset X called a *dictionary* D. It is assumed that $d \in D$ implies $-d \in D$ and the closed linear span of D is X. We would like to approximate every $x \in X$ by a linear combination of the elements of D.

Let us suppose X is a Hilbert space. There is a natural greedy way to do this to achieve our approximation. Let us assume that the dictionary D is a *compact* set. If $x = x_1 \in X$ pick $d_1 \in D$ and $a_1 \ge 0$ so that $\|x_1 - a_1 d_1\|$ is minimized. Now let $x_2 = x_1 - a_1 d_1$ and repeat the procedure. This will develop a series $a_1d_1 + a_2d_2 + \cdots +$ and the natural question is whether this series converges to x.

Let us call this the *pure greedy algorithm* or (PGA). Note that in the (PGA) we inductively construct $(x_n)_{n=1}^\infty$, $(a_n)_{n=1}^\infty$ and $(d_n)_{n=1}^\infty$ by the rules:

$$a_n = (x, d_n) = \max_{d\in D}(x, d),$$

$$x_{n+1} = x_n - a_n d_n.$$

In general the dictionary is not a compact set and we must consider a variant of this algorithm involving a *weakness* parameter c. Let us formally describe the weak dual greedy algorithm (WDGA) in a Hilbert space. Fix $0 < c < 1$. Let $x = x_1$ and then choose $a_n \ge 0$, $d_n \in D$ and $x_n \in X$ so that

$$a_n = (x, d_n) > c \sup_{d\in D}(x, d),$$

$$x_{n+1} = x_n - a_n d_n.$$

The question is whether this procedure converges i.e. does $\lim_{n\to\infty} \|x_n\| = 0$ (regardless of the choices made along the way)? Then of course $x = \sum_{n=1}^\infty a_n d_n$. This procedure was first considered in the statistics literature [12] and convergence was proved by Jones [14].

Theorem 6.1. *If X is a Hilbert space then the weak dual greedy algorithm converges.*

Now let us consider what a similar algorithm would look like in an arbitrary Banach space. We suppose that X has a Gateaux smooth norm so that for each $x \neq 0$ there is a unique linear functional φ_x so that $\|\varphi_x\| = 1$ and $\varphi_x(x) = \|x\|$. Fix $0 < c < 1$ as before. Let $x = x_1$ and then choose $a_n \geq 0$, $d_n \in D$ and $x_n \in X$ so that

$$\varphi_{x_n}(d_n) > c \sup_{d \in D} \varphi_{x_n}(x),$$

$$\|x_n - a_n d_n\| = \min_{a \in \mathbb{R}} \|x_n - a d_n\|,$$

and

$$x_{n+1} = x_n - a_n d_n.$$

We call this the (WDGA) as before and the question is whether the (WDGA) converges, i.e. does $\lim_{n\to\infty} \|x_n\| = 0$.

This is an isometric question and should really depend on the choice of norm. It seems reasonable to restrict attention to spaces X which are uniformly convex or uniformly smooth or both. But even with these restrictions very little is known. However, Ganichev and the author [10] were able to extend the convergence from Hilbert spaces to the L_p–spaces where $1 < p < \infty$, although the result depends critically on rather delicate properties of the norm.

Theorem 6.2. *Let X be a subspace of a quotient of $L_p[0,1]$ where $1 < p < \infty$. Then the weak dual greedy algorithm converges in X.*

In a paper under preparation we will give some extensions of this result (including to the Schatten classes S_p where $1 < p < \infty$). However the general problem remains completely open. For example we do not know if the (WDGA) converges in every uniformly convex uniformly smooth space.

Of course the (PGA) makes sense in an arbitrary Banach space if the dictionary is compact. Let $x = x_1$ and then choose $a_n \geq 0$, $d_n \in D$ and $x_n \in X$ so that

$$\|x_n - a_n d_n\| = \min_{\substack{a \geq 0 \\ d \in D}} \|x_n - a d\|,$$

$$x_{n+1} = x_n - a_n d_n.$$

Unfortunately letting $c \to 0$ in the (WDGA) does not reduce to this algorithm then.

Little seems to be known about the (PGA). Livshits [17] gives an example of a space with a Gateaux smooth norm where this algorithm fails to converge.

A natural weak form of the (PGA) is as follows. Fix $0 < c < 1$ as before. Let $x = x_1$ and then choose $a_n \geq 0$, $d_n \in D$ and $x_n \in X$ so that

$$\|x_n - a_n d_n\|^2 \leq (1 - c^2)\|x_n\|^2 + c^2 \inf_{\substack{a \geq 0 \\ d \in D}} \|x_n - a d\|^2,$$

and

$$x_{n+1} = x_n - a_n d_n.$$

Let us call this the weak greedy algorithm (WGA). When does this algorithm converge?

A different variant of the (PGA) has been considered by Livshits [17] and he proves convergence under essentially the same hypotheses as Theorem 6.2 (taking into account results in [10]).

This whole subject appears wide open for exploration. These algorithms and many more are considered in the survey [21], which is a good place for the interested reader to start.

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