THE KALTON CALCULUS

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Abstract. This article provides a glimpse at Nigel Kalton's contribution to interpolation of Banach spaces. Examples and concepts which look unrelated at first sight, such as quasi-linear maps, non-trivial twisted sums and interpolating operators are shown to be relevant to the same theory.

1. Introduction

Nigel Kalton's death occured on 31 August, 2010. He had suffered a devastating stroke two days earlier. When it happened, I was in charge of a short course in the summer school at Le Touquet, and the main topic of this course was Nigel's work. The last lecture began with the very sad announcement of his death. We were quite a few, in this school, who had been priviledged to collaborate with Nigel, and to benefit from his generosity, his tremendous mathematical power and his amazing insights. The feeling of loss was - and still is - dreadful. There is no such person as a substitute for Nigel. However, we can still try our best to attract attention towards his work and to the ideas which can be found in the gold mine of his articles - and sometimes, even, between his lines. The present work is such an attempt.

This short note focus on some of Nigel Kalton's contributions to interpolation theory of Banach spaces (surveyed in [27]) where he created a usable frame, in which it makes perfect sense to speak of the logarithm of a sequence space or of the derived space to an interpolation line at a given Banach space. Important examples, which might look strange when first met, figure in this frame as canonical objects: for instance, the Kalton-Peck space Z_2 is the derived space at l_2 to the interpolation line of the l_p spaces. We will therefore display differentiation of interpolation lines, extrapolation in given directions, and the corresponding calculus on the "manifold" of Banach spaces. This will lead us to unexpected connections with commutators and the trace class operators. Altogether, Nigel Kalton's global approach will reveal a network of links, which express the profound unity of analysis, when seen from his towering point of view. The reader may wish to consult [10] for further examples of his unifying power.

This note contains a few technical statements, but it is nothing more than a glance at Nigel Kalton's work from a distance, in order to provide the reader with some intuition on what goes on. This outline cannot be used as a substitute for an actual reading of the original articles. Hence any reader who wants to understand fully these results - and use them in his/her own research - is invited to dwell into Nigel's computations and proofs. It should be stressed that quite exotic tools, such as non-linear liftings, discontinuous linear functionals, quasi-linear maps and non locally

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convex twisted sums are used in this work, in such a way that they provide useful information on classical and main-stream analysis. There is no better way to underline the width and depth of Nigel Kalton's unique vision.

2. Distances between Banach spaces.

The Banach-Mazur functional d_{BM} is a classical tool for estimating the "distance" between two isomorphic Banach spaces - or equivalently, the distance between two equivalent norms and similar functionals such as the Lipschitz distance d_L can be defined when more general notions of isomorphisms are taken into consideration. However, it appears necessary to design notions which reflect similarities between Banach spaces which are not isomorphic (even in a weakened meaning of the word) but still share common features. Such notions show up in the modern theory of metric spaces developed by M. Gromov and his followers, where it is important to decide when two spaces have the same shape.

In [28], "distances" are defined between somewhat similar spaces, such as ℓ_p and ℓ_q when p and q are close to each other. Precisely, if X and Y are two subspaces of a Banach space Z, let $\Lambda(X,Y)$ be the Hausdorff distance between B_X and B_Y, that is

$$\Lambda(X,Y) = \max\{\sup_{x\in B_X}\inf_{y\in B_Y}\|x-y\|, \sup_{y\in B_Y}\inf_{x\in B_X}\|y-x\|\}.$$

The Kadets distance $d_K(X,Y)$ is the infimum of $\Lambda(\tilde{X},\tilde{Y})$ over all Banach spaces Z containing isometric copies $\tilde{X}$ and $\tilde{Y}$ of X and Y. The Kadets distance is a pseudo-metric which is controlled from above by d_{BM}, but there are non-isomorphic Banach spaces X and Y such that $d_K(X,Y)=0$.

The Gromov-Hausdorff distance d_{GH} is a non-linear analogue of the Kadets distance, defined along the same lines, except that the infimum is taken over all metric spaces containing isometric copies of X and Y. Of course, $d_{GH}\leq d_K$ and for instance $d_{GH}(\ell_p,\ell_1)\to 0$ as $p\to 1$ while $d_K(\ell_p,\ell_1)=1$ for all $p>1$.

In some cases, however, convergence in the Gromov-Hausdorff sense implies convergence for the Kadets distance. Let X be a Banach space, and let X_0 be a dense linear subspace of X. A map $F = X_0 \to \mathbb{K}$ is *quasilinear* if:

(i) $F(\alpha x) = \alpha F(x)$ for $x\in X_0$ and $\alpha\in\mathbb{K}$.

(ii) There is $C\in\mathbb{R}$ such that

$$|F(x+y)-F(x)-F(y)|\leq C(\|x\|+\|y\|)$$

for all $(x,y)\in X_0^2$.

A Banach space X is called a $\mathcal{K}-$space if any such F satisfies

$$|F(\sum_{i=1}^n x_i) - \sum_{i=1}^n F(x_i)| \leq C\sum_{i=1}^n \|x_i\| \tag{1}$$

for some constant C and all $x_1,\ldots,x_n\in X_0$.

This condition turns out to be equivalent with approximation of quasi-linear maps by linear ones. Along these lines, it turns out that if X is a $\mathcal{K}-$space then $d_{GH}(X_n,X)\to 0$ implies that $d_K(X_n,X)\to 0$. What makes this statement interesting is the existence of natural examples of $\mathcal{K}-$spaces: it is shown in [31] that every quotient space of a $\mathcal{L}_\infty$-space is a $\mathcal{K}$-space, and in [19] that a Banach space with non-trivial type is a $\mathcal{K}$-space. In fact, Nigel Kalton conjectured that a Banach space is a $\mathcal{K}$-space exactly when its dual space has non-trivial cotype. On the other hand, ℓ_1 is not a $\mathcal{K}$-space since the Kadets and Gromov-Hausdorff distances do not coincide at ℓ_1.

The space c_0 is particularly interesting in this respect: it follows from Sobczyk's theorem that if $d_{GH}(X_n, c_0) \to 0$ we have not only that $d_K(X_n, c_0) \to 0$ (since c_0 is a K-space), but actually $d_{BM}(X_n, c_0) \to 0$ [28]. It follows for instance that if the uniform distance between X and c_0 is small then X is linearly isomorphic to c_0 [11, Theorem 5.7]. We recall that it is not known whether a space which is uniformly homeomorphic to c_0 is linearly isomorphic to c_0 (see [11]).

These weaker distances naturally apply to interpolation theory, which provides families of Banach spaces which are not isomorphic but tightly related. For the convenience of the reader, we now outline the basics of complex interpolation. We restrict our discussion to an important special case. Let W be some complex Banach space and let X_0 and X_1 be two closed subspaces of W. We denote

$$S = \{z \in \mathbb{C}; 0 < Re(z) < 1\}$$

and $\mathfrak{F}$ is the space of analytic functions $F : S \longrightarrow W$ which extend continuously to $\bar{S}$ and such that $\{F(it); t \in \mathbb{R}\}$ is a bounded subset of X_0 and $\{F(1+it); t \in \mathbb{R}\}$ is a bounded subset of X_1. The space $\mathfrak{F}$ is normed by

$$\|F\|_{\mathfrak{F}} = \max_{j=0,1} \sup\{\|F(j+it)\|_{X_j}; t \in \mathbb{R}\}.$$

For $\theta \in (0,1)$ and $x \in W$, we define

$$\|x\|_\theta = \inf\{\|F\|_{\mathfrak{F}}; F(\theta) = x\}$$

and

$$X_\theta = \{x \in W; \|x\|_\theta < \infty\}$$

If $W_0 = \text{span}\{X_\theta;\ \theta \in (0,1)\}$, a linear map $T : W_0 \longrightarrow W_0$ is called *interpolating* if $F \longmapsto T \circ F$ is defined and bounded on $\mathfrak{F}$. If T is interpolating, then $T(X_\theta) \subseteq X_\theta$ for all $\theta \in (0,1)$.

The above space $X_\theta = [X_0, X_1]_\theta$ is said to be obtained from X_0 and X_1 by the complex interpolation method. The link with the Kadets distance is provided by the following result from [28]: for $0 < \theta < \phi < 1$

$$d_K(X_\theta, X_\phi) \leq 2\frac{sin[\pi\frac{(\phi-\theta)}{2}]}{sin[\pi\frac{(\phi+\theta)}{2}]}\ .$$

This continuity of the interpretation method with respect to the Kadets distance permits to apply connectedness arguments. Indeed, let us call a property (P) *stable* if there exists $\alpha > 0$ so that if X has (P), and $d_K(X,Y) < \alpha$, then Y has (P). For instance, each of the following properties (P) is stable: separability, reflexivity, $X \supseteq \ell_1$, super-reflexivity, type$(X) > 1$. Connectedness thus shows that if $0 < \theta < 1$ and $X_\theta = [X_0, X_1]_\theta$ has (P), then X_φ has (P) for every $\varphi \in (0,1)$. And this line of thought opens an exciting field of research. It can be shown that the connected component of any separable Banach space X contains all isomorphic copies of X. It follows from [35] that the connected component of ℓ_2 contains all super-reflexive Banach lattices, and it is not known whether it contains all super-reflexive spaces. It is conjectured that the component of c_0 consists of all spaces isomorphic to a subspace of c_0.

3. Twisted sums.

The Kadets and Gromov-Hausdorff distances clearly are metric notions, but interpolation points to some kind of differential structure, which we will outline. It turns out that this "tangent" structure leads to the consideration of twisted sums. And we will be bound to leave the locally convex world and to allow quasi-Banach spaces to enter the picture.

We recall that metrizable complete topological vector spaces (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are called F-spaces. Their topology is induced by an $F-$*norm*, that is, a map Λ from the space X to $\mathbb{R}^+$ such that

(i) $\Lambda(x) > 0$ if $x \neq 0$.
(ii) $\Lambda(\alpha x) \leq \Lambda(x)$ if $|\alpha| \leq 1$.
(iii) $\lim_{\alpha\to 0} \Lambda(\alpha x) = \Lambda(0) = 0$.
(iv) $\Lambda(x + y) \leq \Lambda(x) + \Lambda(y)$ for all $(x, y) \in X$.

The space X is locally bounded if and only if its topology can be generated by a *quasi-norm* $\|\cdot\|$, namely a map $\|\cdot\| : X \to \mathbb{R}^+$ such that:

(i) $\|x\| > 0$ if $x \neq 0$
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$and $\alpha \in \mathbb{K}$
(iii) $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $(x, y) \in X^2$

where $C \geq 1$ is the "modulus of concavity" of the quasi-norm. An F-space is called a *quasi-Banach space* when its topology is generated by a quasi-norm, or equivalently by the Aoki-Rolewicz theorem, by a p-subadditive quasi-norm $|||\cdot|||$, i.e. a quasi-norm which satisfies the condition

(iv) $|||x + y|||^p \leq |||x|||^p + |||y|||^p$

for all $(x, y) \in X^2$ and $p > 0$ given by $p = (1 + \log_2(C))^{-1}$.

We refer to [30] for an authoritative book on F-spaces.

Let X and Y be quasi-Banach spaces. We say that Z is an extension of X by Y if

$$Z/Y \simeq X$$

An extension Z is also called a *twisted sum* of X and Y (a non-trivial twisted sum if Y is not complemented in Z), and we refer to [4] for a comprehensive survey of this matter.

We extend the definition given above in the case of scalar-valued functions to call a map $\Omega : X \longrightarrow Y$ *quasi-linear* if $\Omega(\lambda x) = \lambda\Omega(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$, and if there is $C > 0$ such that

$$\|\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2)\| \leq C(\|x_1\| + \|x_2\|)$$

for all $x_1, x_2 \in X$. For any quasi-linear map Ω, we can define the extension $X \oplus_\Omega Y$ of X by Y to be the space $X \oplus Y$ equipped with the quasi-norm

$$\|(x, y)\| = \|x\| + \|y - \Omega(x)\|$$

Even when X and Y are Banach spaces $X \oplus_\Omega Y$ is not, unless Ω actually satisfies the analogue of condition (1): for all $n \geq 1$ and all (x_k)

$$\|\sum_{k=1}^{n} \Omega(x_k) - \Omega(\sum_{k=1}^{n} x_k)\| \leq C \sum_{k=1}^{n} \|x_k\|.$$

It will always be so when X is a $\mathcal{K}-$space [19]. It turns out that every extension can actually be obtained through such an Ω: if $q : Z \longrightarrow X$ is the quotient map, one takes $\Omega = S - R$ where $qS = qR = Id_X$, S is homogenous (but not necessarily linear) such that $\|S(x)\| \leq 2\|x\|$, and R is linear (but not necessarily continuous). The existence of a bounded linear projection from $X \oplus_\Omega Y$ onto Y (in other words, the triviality of the extension) is equivalent to the existence of a linear map $L : X \longrightarrow Y$ such that

$$\|\Omega(x) - L(x)\| \leq C\|x\|$$

for all $x \in X$.

When $X = Y$, the space $x \oplus_\Omega X$ is called a *self-extension* of X and it is denoted

$$X \oplus_\Omega X = d_\Omega X.$$

Let us illustrate this approach with three important examples. The Ribe space $E = \ell_1 \oplus_F \mathbb{R}$ from [36] is obtained by considering the quasilinear functional

$$F(x) = \sum_{k=1}^{\infty} x_k \log|x_k| - \Big(\sum_{k=1}^{\infty} x_k\Big) \log\Big|\sum_{k=1}^{\infty} x_k\Big|$$

on the dense subspace c_{00} of finitely supported sequences in ℓ_1. For showing that (1) fails for this F, it suffices to compute

$$F(\sum_{i=1}^{n} e_i) - \sum_{i=1}^{n} F(e_i) = -n\log(n).$$

The Ribe space is therefore a non-trivial twisted sum of ℓ_1 with a one-dimensional space and its existence solves negatively the three-space problem for local convexity.

We will see now that the Ribe space is closely related with another famous twisted sum. When $X = \ell_2$, a non-trivial self-extension of ℓ_2 is called a twisted Hilbert space, and it was shown in [9] that such spaces exist, and thus being isomorphic to a Hilbert space fails the three-space property. An alternative example, the Kalton-Peck space Z_2, is constructed in [29] with the help of the Ribe functional: let $\Omega = \ell_2 \longrightarrow \mathbb{R}^{\mathbb{N}}$ be defined by

$$\Omega((\xi_n)) = (\xi_n \log(\frac{|\xi_n|}{\|\xi\|_2}))_{n\geq 1}$$

(and $\Omega(0) = 0$). The space $Z_2 = d_\Omega \ell_2$ is then the space of pairs of sequences $((\xi_n), (\eta_n))$ such that

$$\|(\xi,\eta)\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \left|\eta_n - \xi_n \log\frac{|\xi_n|}{\|\xi\|_2}\right|^2\right)^{\frac{1}{2}} < \infty.$$

The space Z_2 is a Banach space since ℓ_2 is a $\mathcal{K}-$space. This space Z_2 exhibits remarkable features, which are not yet fully understood although that space was constructed more than 30 years ago. It is plain that Z_2 has an unconditional F.D.D. consisting of 2-dimensional spaces; however it has no unconditional basis and no local unconditional structure [15]. Actually, an unconditional F.D.D. with spaces of bounded dimension provides an unconditional basis which can be chosen from the subspaces *if* the space has local unconditional structure [3]. It is unknown, however, if a twisted Hilbert space can have local unconditional structure; the best result so far is that it has no unconditional basis in full generality [21]. The space Z_2 is also an example of a symplectic Banach space which is not the direct sum of two isotropic subspaces [32]. In fact, intuition suggests that the space Z_2 is "even-dimensional" and thus that it should not be isomorphic to its hyperplanes: this 30-years old conjecture is still open. We note along these lines that spaces with 2-dimensional unconditional F.D.D. but no unconditional basis (such as Z_2) show up in the classification results established in [33], which play a crucial role in Gowers' homogeneous space theorem [13].

Finally, as so often in N. Kalton's work, the conceptual frame in which the construction is completed provides flexibility and leads to more results. If $F = \mathbb{R} \longrightarrow \mathbb{C}$ is any Lipschitz map and E is a Banach sequence space, we may consider the following quasi-linear map on E:

$$\Omega_F(\xi) = (\xi_n F(\log \frac{|\xi_n|}{\|\xi\|_E}))_{n\geq 1}$$

and then define

$$d_{\Omega_F} E = E \oplus_{\Omega_F} E.$$

Taking $E = \ell_2$ and $F(t) = t^{1+i\alpha}$ ($\alpha \neq 0$) provides a complex Banach space $Z(\alpha)$ (actually, a twisted Hilbert space) which is not complex-isomorphic to its conjugate space $\overline{Z(\alpha)} = Z(-\alpha)$. The existence of such spaces had been shown in [1] and [39] by probabilistic methods.

4. A differential structure.

The notation $d_\Omega X$ is reminiscent of differential calculus, and this is not a chance. With the above notation of the complex interpolation method, and following R. Rochberg and G. Weiss [38], we define a derived space $dX_\theta \subseteq W \times W$ by $dX_\theta = \{(x_1, x_2) : \ \|(x_1, x_2)\|_{dX_\theta} < \infty\}$ where

$$\|(x_1, x_2)\|_{dX_\theta} = \inf\{\|F\|_{\mathfrak{F}} : \ F(z) = x_1, F'(z) = x_2\}.$$

The space $Y = \{(x_1, x_2) \in dX_\theta : \ x_1 = 0\}$ is isometric to X_θ and so is dX_θ/Y. Hence dX_θ is a self-extension of X_θ. By the above, one has

$$dX_\theta = d_\Omega X_\theta$$

for some quasi-linear map $\Omega : X_\theta \longrightarrow W$. It turns out that $\Omega(x) = F'(z)$, where $F \in \mathfrak{F}$ is such that $\|F\|_{\mathfrak{F}} \leq C\|x\|_\theta$ and $F(z) = x$, does the work. Now, if T is an interpolating operator then $(x_1, x_2) \longrightarrow (Tx_1, Tx_2)$ is bounded on dX_θ and this translates into "commutator estimates":

$$\|T(\Omega(x)) - \Omega(T(x))\|_\theta \leq C\|x\|_\theta$$

for all $x \in X_\theta$.

A first example of interpolation line is provided by the sequence spaces ℓ_p ($1 \leq p \leq \infty$). The above calculations applied to $X_0 = \ell_1$ and $X_1 = \ell_\infty$ provide the Kalton-Peck space $Z_2 = dX_{1/2}$, which thus appears to be the derived space at l_2 to the interpolation line of the l_p spaces.

Similar calculations are possible for the function spaces $L_p(\mathbb{T})$. For this interpolation scale, the Hilbert transform H is a very important example of interpolating operator and the Rochberg-Weiss commutator estimate becomes in this case

$$\|H(f \log |f|) - H(f) \log |H(f)|\|_p \leq C_p\|f\|$$

for $1 < p < \infty$ and some $C_p < \infty$.

Following [22] and [23], we now relate this differential calculus with entropy functions of function spaces. For sake of simplicity, we restrict ourselves to the special case of sequence spaces. If X is a sequence space, its entropy function Φ_X is defined for positive sequences u by

$$\Phi_X(u) = \sup_{\|x\|_X \leq 1} \sum_{k=1}^{\infty} u_k \log |x_k|.$$

If X_0 and X_1 are separable sequence spaces, the interpolation spaces X_θ are given by the Calderon formula $X_\theta = X_0^{1-\theta} X_1^\theta$, that is:

$$\|x\|_\theta = \inf\{\|x_0\|_0^{1-\theta} \|x_2\|_1^\theta \ ; \ |x| = |x_0|^{1-\theta} |x_1|^\theta\}.$$

The entropy function Φ_X may be thought of as the *logarithm* of the sequence space X. Indeed one has

$$\Phi_{X_\theta} = (1-\theta)\Phi_{X_0} + \theta \Phi_{X_1}$$

and by the Lozanovsky factorization theorem

$$\Phi_X + \Phi_{X^*} = \Phi_{\ell_1}$$

where Φ_{ℓ_1} is the Ribe functional, while

$$\Phi_{\ell_p} = \frac{1}{p} \Phi_{\ell_1}$$

and $\Phi_{\ell_\infty} = 0$. It now becomes natural to see the Hilbert space as the geometric mean between any sequence space X and its dual X^*.

It turns out that entropy functions of peculiar sequence spaces provide special quasi-Banach spaces. Here is an important example from [18]. We recall that a separable quasi-Banach space Y is called *minimal* if it does not have any weaker Hausdorff vector topology. This condition happens to be equivalent to the non-existence of basic sequences in Y. In [18], Nigel Kalton constructs a twisted sum Y of ℓ_1 and a one-dimensional space E, with no basic sequence since every infinite-dimensional closed subspace of Y contains E. This is reminiscent of Gowers-Maurey's construction of a Banach space X_{GM} without unconditional basic sequence [14], which is such that for any infinite-dimensional subspaces U and V of X_{GM}

$$\inf\{\|u - v\|; u \in U, v \in V, \|u\| = \|v\| = 1\} = 0.$$

And indeed, Gowers' modification [12] of the original construction, used in his solution of the hyperplane problem, provides a space X with an unconditional basis whose entropy function Φ_X yields to a minimal extension $\mathbb{K} \oplus_{\Phi_X} \ell_1$ with no basic sequence [18], and which is therefore a minimal quasi-Banach space. Note that for any infinite dimensional subspace J of c_{00}, this function $\Phi_X = F$ satisfies

$$\sup\{|F(x)| : x \in J, \|x\| \leq 1\} = \infty,$$

hence $\Phi_X = F$ is distorted in the sense of [34].

Minimal quasi-Banach spaces M are pretty strange objects: every one-to-one continuous linear map from M into a Hausdorff topological vector space is actually an isomorphism on its range! However existing examples are "non-isotropic" in the sense where they contain a distinguished line, namely the orthogonal of the dual space. It in not known whether an even stranger example exists which would exhibit this behaviour everywhere, in other words a quasi-Banach space containing no infinite-dimensional proper closed subspace.

To close the circle of ideas relating the entropy functions with derived spaces, we note that if $X_\theta = X_0^{1-\theta}X_1^\theta$ then $dX_\theta = d_\Omega X_\theta = X_\theta \oplus_\Omega X_\theta$ where the quasi-linear map Ω satisfies

$$|\langle x^*, \Omega(x)\rangle - \Phi(xx^*)| \leq C\|x\|_{X_\theta}\|x^*\|_{X_\theta^*}$$

where $\Phi = \Phi_{X_1} - \Phi_{X_0}$ and (xx^*) denotes the pointwise product of the sequences x and x^*.

5. Extrapolation.

The map "$X \mapsto \Phi_X$" is logarithmic-like, but in order to complete the picture we conversely need an *exponential* functor which associates a sequence space (or more generally, a function space) to a quasi-linear map Φ. This can be done as follows: if $\Phi : c_{00}^+ \longrightarrow \mathbb{R}$ is any functional, there exists a Banach sequence space X such that $\Phi_X = \Phi$ if and only if Φ and $(\Phi_{\ell_1} - \Phi)$ are convex functions and Φ is positively homogeneous [23], and the space X has unit ball

$$B_X = \{(x_k);\ \sum_{k=1}^{\infty} u_k \log|x_k| \leq \Phi(u)\,\text{for all}\,u \geq 0\}.$$

This exponential map leads to what I suggest to call the *Kalton calculus*, which bears an uncanny resemblance with the exponentation from a Lie algebra to its Lie group, and creates "lines" from infintesimals; in other words, yields to *extrapolation*. Here is a bunch of examples.

If X is a p-convex $(1 < p < 2)$ and p^*-concave discrete lattice, then $X = Y^{1/p}$ for some sequence space Y and so $(\frac{1}{p}\Phi_{\ell_1} - \Phi_X)$ is convex. Similarly p^*-concavity means that $(\frac{1}{p}\Phi_{\ell_1} - \Phi_{X^*})$ is convex. Now the equation

$$\begin{aligned}\Phi_X &= (1-\theta)\Phi + \theta\Phi_{\ell_2}\\ &= (1-\theta)\Phi + \frac{\theta}{2}\Phi_{\ell_1}\end{aligned}$$

provides a convex function Φ such that $(\Phi_{\ell_1} - \Phi)$ is also convex and thus $\Phi = \Phi_Z$ for some Z. Exponentiating, we find $X = Z^{1-\theta}\ell_2^\theta$ (a result from [35]).

Special properties of the derived space $d_\Omega X_\theta$ can "spread out" by exponentation to a segment $\{X_\varphi;\ |\varphi - \theta| < \varepsilon\}$. Our examples are function spaces. If X_0 and X_1 are acceptable function spaces on $\mathbb{T}$ and R is the vector-valued Riesz transform, then there is $\delta > 0$ such that R is bounded on X_θ for $|\theta - \theta_0| < \delta$ if and only if $\|R\Omega - \Omega R\|_{X_{\theta_0}} < \infty$. It follows that there exist twisted Hilbert spaces which are not U.M.D. [23] although the Kalton-Peck space Z_2 is U.M.D. [24]. We note at this point that higher order derivatives can be considered, and this has been done e.g. in [2] and [37].

As seen before, differentiating interpolation lines yield quasi-linear maps Ω such that $dX_\theta = d_\Omega X_\theta$. Let us consider the special case where the space X_1 is obtained from X_0 through a change of weight. In this case, the quasi-linear map Ω enjoys a commutation property with multiplication operators, namely

$$\|\Omega(ax) - a\Omega(x)\|_{X_\theta} \leq C\|a\|_\infty\|x\|_{X_\theta}.$$

These special maps are called *centralizers* in [22] and the corresponding space $d_\Omega X_\theta$ is a *lattice twisted sum*. Centralizers yield to extrapolation results: if for instance X is a super-reflexive Banach sequence space and Ω is a real centralizer on X, then there exist super-reflexive Banach sequence spaces X_0 and X_1 such that $X = X_0^{1/2} X_1^{1/2} = X_{1/2}$ and moreover $dX_{1/2} \simeq d_\Omega X$.

We now recall the Rochberg-Weiss commutator estimates: if $X_\theta = X_0^{1-\theta} X_1^\theta$ and $dX_\theta = d_\Omega X_\theta$ then

$$\|T(\Omega(x)) - \Omega(T(x))\|_{X_\theta} \leq C\|x\|_{X_\theta} \tag{2}$$

for interpolating operators T. When, for instance, Ω is a centralizer, this estimate says that Ω nearly commutes not only with multiplication operators, but with all interpolating operators.

Now the extrapolation technique allows a change of perspective: starting from an operator T on X, we may consider all pairs (X_0, X_1) such that $X = X_0^{1-\theta} X_1^\theta$ and T is interpolating between X_0 and X_1 and get a whole family of estimates on T. Indeed any such line through X gives birth to a quasi-linear map Ω, and thus to the corresponding Rochberg-Weiss estimates (2). We have seen that given a quasi-linear map Φ which satisfy mild necessary conditions, one can construct a space Y such that $\Phi = \Phi_Y$. Hence, given a space X, when the function Φ_X can be written in various ways as a convex combination of proper quasi-linear maps, a bunch of interpolation lines passing through X can be constructed. And since each interpolation line carries estimates which apply to every interpolating operator, it follows that operators wich interpolate around X satisfy a collection of inequalities, yielding for instance to a new symmetric tangent space which is the range of interpolating bilinear forms. We illustrate this approach with two basic examples.

The case $X = \ell_p$ yields to the family of quasi-linear maps

$$\Phi_G(u) = \sum_{n=1}^{\infty} u_n G(\log |u_n|)$$

where G runs through the family $\mathcal{G}$ of 1-Lipschitz maps from $\mathbb{R}$ to $\mathbb{R}$ whose derivative is compactly supported. This leads to considering the quasi-Banach space h_1^{sym} defined by

$$\|\xi\|_{h_1^{sym}} = \sum_{k=1}^{\infty} |\xi_k| + \sup_{G \in \mathcal{G}} \Phi_G(\xi) < \infty.$$

This "tangent space" h_1^{sym} is conveniently described as the space of sequence (ξ_k) in ℓ_1 such that

$$\sum_{n=1}^{\infty} \frac{1}{n} |\xi_1 + \xi_2 + \cdots + \xi_n| < \infty.$$

The same steps applied to function spaces $L_p(\mu)$ lead to the symmetric Hardy function space $H^1_{sym}(\mu)$ of all functions $f \in L^1(\mu)$ such that

$$\|f\|_{H^1_{sym}} = \int |f| d\mu + \sup_{G \in \mathcal{G}} \int |f| G(\log |f|) d\mu < \infty.$$

Commutator estimates on interpolating operators then show the following theorem [22]:

Theorem 5.1. *Suppose* $1 < p_0 < p < p_1 < \infty$ *and* $p^{-1} + q^{-1} = 1$. *Suppose that* $T : L_{p_j} \longrightarrow L_{p_j}$ *is linear bounded for* $j = 0, 1$. *Then the bilinear form*

$$B_T(f, g) = f.T^* g - g.Tf$$

is bounded from $L_p \times L_q$ *to* H^1_{sym}.

The above dot denotes of course the pointivise product of functions.

This theorem can be applied to a variety of interpolating operators. The Riesz projection gives applications to harmonic analysis. Indeed, when T is the Riesz projection on $L^2(\mathbb{T})$, the theorem shows that if f and g belongs to $H^1(\mathbb{T})$ and $g(0) = 0$ the following inequality holds:

$$\|fg\|_{H^1_{sym}} \leq C\|f\|_2\|g\|_2$$

and thus since $H^1_0 = H^2.H^2_0$, one gets for every function $h \in H^1$ with $h(0) = 0$

$$\|h\|_{H^1_{sym}} \leq C\|h\|_1$$

This result, first shown in [5] and [6], somehow means that functions in $H^1(\mathbb{T})$ have a quite symmetric behavior around their singularities. Conversely, real-valued functions in $H^1_{sym}(\mathbb{T})$ are real parts of functions in $H^1(\mathbb{T})$.

Finally, the ideas developed above have non-commutative applications, and the bridge which brings to the non-commutative world is the concept of trace. If X is a symmetric Banach sequence space, we denote $\mathcal{C}_X$ the space of all operators T on ℓ_2 whose sequence $(s_n(T))_{n\geq 1}$ of singular numbers belongs to X. When X_0 and X_1 are reflexive then

$$[\mathcal{C}_{X_0}, \mathcal{C}_{X_1}]_\theta = \mathcal{C}_{X_0^{1-\theta}X_1^\theta} = \mathcal{C}_{X_\theta}$$

and interpolation tools apply to the spaces $\mathcal{C}_X$.

Let $\mathcal{C}_{\ell_1} = \mathcal{C}_1$ be the ideal of trace-class (or nuclear) operators on ℓ_2. A *trace* on $\mathcal{C}_1$ is a linear map τ such that $\tau(AB) = \tau(BA)$ for all $A \in \mathcal{C}_1$ and all bounded operators B. We denote $Comm(\mathcal{C}_1)$ the linear span of all commutators

$$[A, B] = AB - BA$$

with $A \in \mathcal{C}_1$ and B bounded. Clearly, if $S \in \mathcal{C}_1$ then $S \in Comm(\mathcal{C}_1)$ if and only if $\tau(S) = 0$ for every trace τ. It was shown in [41] that $Comm(\mathcal{C}_1)$ is strictly contained in $\{T \in \mathcal{C}_1; tr(T) = 0\}$, or equivalently that there exist *discontinuous* traces on $\mathcal{C}_1$. The precise description of $Comm(\mathcal{C}_1)$ was obtained in [25] by interpolation arguments and it reads as follows:

Theorem 5.2. *Let $T \in \mathcal{C}_1$ be a trace-class operator. Then $T \in Comm(\mathcal{C}_1)$ if and only if its eigenvalue sequence $\lambda_n(T))_{n\geq 1}$ belongs to h^1_{sym}.*

It was shown in [25] that every $T \in Comm(\mathcal{C}_1)$ is the sum of 6 commutators, but this number has now been put down to 3 and the case of general ideas of operators is also treated in [7, 8, 26].

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