ESSENTIAL NORM OF COMPOSITION OPERATORS
ON BANACH SPACES OF HÖLDER FUNCTIONS

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Dedicated to the memory of Nigel J. Kalton

FOREWORD

This paper deals with the computation of the essential norm of a composition operator induced on the Banach space of Hölder functions on a general metric space by a Lipschitz self map of the metric space. The structure of spaces of Lipschitz and Hölder functions and their preduals on general metric spaces was studied by Kalton in [8].

First of all, we give a lower bound on the essential norm of a composition operator using standard facts about weakly null sequences. Next, we make the assumption that the dual of the Banach space of Hölder functions has the approximation property to obtain an upper bound. The proof of this upper bound depends on some results involving shrinking compact approximating sequences that are derived from the work of Kalton [7].

It is natural to ask for some examples where the dual of the Banach space of Hölder functions has the approximation property. This happens, according to Kalton [8], if the metric space is uniformly discrete. Also, when the Banach space of Hölder functions is isomorphic to c_0, its dual is isomorphic to ℓ_1 and therefore it has the approximation property.

A classical result of Bonic, Frampton and Tromba [3] ensures that the Banach space of Hölder functions on a metric space is isomorphic to c_0 whenever the metric space is an infinite compact subset of a finite dimensional normed linear space. This result was corrected by Weaver [12], who asked whether such an isomorphism could be extended to any compact metric space. Kalton answered this question negatively by proving that a compact convex subset of a Hilbert space containing the origin has the property that the Banach space of Hölder functions on the convex set is isomorphic to c_0 if and only if the convex set is finite dimensional. Kalton conjectured that this holds in full generality for all Banach spaces.

Let us recall now that a metric space satisfies the doubling condition (or hasfinite Assouad dimension) if there is an integer n such that for any $\delta > 0$, every closed ball of radius δ can be covered by at most n closed balls of radius $\delta/2$. A theorem of Assouad [1] asserts that whenever a metric space satisfies the doubling condition, every snowflake of the metric space Lipschitz embeds in the euclidean space. Using this result, Kalton [8] observed that if a compact metric space satisfies the doubling condition, then the Banach space of Hölder functions on the metric space is isomorphic to c_0. Furthermore, he also showed that the converse is false by means of a counterexample.

ABSTRACT. Let (X, d) be a pointed compact metric space, let $0 < \alpha < 1$, and let $\varphi \colon X \to X$ be a base point preserving Lipschitz map. We prove that the essential norm of the composition operator C_φ induced by the symbol φ on the spaces $\mathrm{lip}_0(X, d^\alpha)$ and $\mathrm{Lip}_0(X, d^\alpha)$ is given by the formula

$$\|C_\varphi\|_e = \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}$$

whenever the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property. This happens in particular when X is an infinite compact subset of a finite-dimensional normed linear space.

1. INTRODUCTION

Let (X, d) be a compact metric space with a distinguished point $e \in X$ and $0 < \alpha < 1$. The formula $d^\alpha(x, y) = d(x, y)^\alpha$ defines a new metric on X, and the metric space (X, d^α) is said to be a *Hölder metric space of order* α. As usual, $\mathbb{K}$ denotes the field of real or complex numbers.

The *Lipschitz space* $\mathrm{Lip}_0(X, d^\alpha)$ is the Banach space of all Lipschitz functions $f \colon X \to \mathbb{K}$ on the Hölder metric space (X, d^α) for which $f(e) = 0$ under the standard Lipschitz norm

$$L_\alpha(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)^\alpha} : x, y \in X, \ x \neq y \right\}.$$

Notice that the Lipschitz functions on (X, d^α) are precisely the *Hölder functions of order* α on (X, d).

The *little Lipschitz space* $\mathrm{lip}_0(X, d^\alpha)$ is the closed subspace consisting of those functions $f \in \mathrm{Lip}_0(X, d^\alpha)$ that satisfy the following *local flatness condition*:

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} = 0.$$

The Lipschitz space $\mathrm{Lip}_0(X, d^\alpha)$ has a canonical predual $\mathcal{F}(X, d^\alpha)$, the *free Lipschitz space* on (X, d^α), also known as the *Arens-Eells space* in [12], that can be defined as the closed linear span of the point evaluations

$$\delta_x(f) = f(x) \qquad (x \in X, \ f \in \mathrm{Lip}_0(X, d^\alpha))$$

in the dual space $\mathrm{Lip}_0(X, d^\alpha)^*$. As it turns out, $\mathcal{F}(X, d^\alpha)$ is itself the dual space of $\mathrm{lip}_0(X, d^\alpha)$. The structure of spaces of Lipschitz and Hölder functions and their preduals on general metric spaces was studied by Kalton in [8]. We refer to the book [12] by Weaver for a complete study of the spaces of Lipschitz functions.

We denote by $\mathcal{L}(E)$ the algebra of all bounded linear operators on a Banach space E, and by $\mathcal{K}(E)$, the closed ideal of all compact operators on E. The *essential norm* $\|T\|_e$ of an operator $T \in \mathcal{L}(E)$ is just the distance from T to $\mathcal{K}(E)$, that is,

$$\|T\|_e = \inf \left\{ \|T - K\| : K \in \mathcal{K}(E) \right\}.$$

It is clear that an operator $T \in \mathcal{L}(E)$ is compact if and only if $\|T\|_e = 0$.

Recall that a Banach space E is said to have the *approximation property* if the identity operator on E can be approximated uniformly on every compact subset of E by operators of finite rank.

Let (X, d) be a pointed compact metric space with base point $e \in X$, let $0 < \alpha < 1$, and let $\varphi \colon X \to X$ be a Lipschitz mapping that preserves the base point, that is, $\varphi(e) = e$ and

$$L(\varphi) := \sup \left\{ \frac{d(\varphi(x), \varphi(y))}{d(x,y)} : x, y \in X, \ x \neq y \right\} < \infty.$$

The *composition operator* $C_\varphi \colon \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ is defined by the expression

$$(C_\varphi f)(x) = f(\varphi(x)) \qquad (x \in X, \ f \in \mathrm{lip}_0(X, d^\alpha)).$$

The aim of this paper is to give lower and upper estimates for the essential norm of the composition operator C_φ on $\mathrm{lip}_0(X, d^\alpha)$ in terms of φ. Results along these lines were obtained by Montes-Rodríguez

[10, 11] and more recently by Galindo and Lindström [4], and also by Galindo, Lindström and Stević [5]. In Section 2, we compute the norm of C_φ. We show that

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}.$$

The prototype of a formula as above with $\alpha = 1$ was provided by Weaver in [12, Proposition 1.8.2] for the composition operator C_φ on the space $\mathrm{Lip}_0(X, d)$. In Section 3, we give a lower bound for the essential norm of the operator C_φ, namely

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \leq \|C_\varphi\|_e.$$

Section 4 contains our main result. When the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property, we show that

$$\|C_\varphi\|_e \leq \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}.$$

The proof of this inequality depends on some results involving shrinking compact approximating sequences on $\mathrm{lip}_0(X, d^\alpha)$. Using the fact that the space $\mathrm{Lip}_0(X, d^\alpha)$ is isometrically isomorphic to the second dual of $\mathrm{lip}_0(X, d^\alpha)$, and the relationship between the essential norm of an operator and its adjoint, we derive in Section 5 the same formula for the essential norm of the operator C_φ on $\mathrm{Lip}_0(X, d^\alpha)$.

It is natural to ask for some examples where the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property. For instance, this happens if X is uniformly discrete, that is, $\inf_{x \neq y} d(x, y) > 0$ (see [8, Proposition 4.4]). Also, $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property whenever the space $\mathrm{lip}_0(X, d^\alpha)$ is isomorphic to c_0 and hence $\mathrm{Lip}_0(X, d^\alpha)$ is isomorphic to ℓ_∞ and $\mathcal{F}(X, d^\alpha)$ is isomorphic to ℓ_1.

A classical result of Bonic, Frampton and Tromba [3] ensures that $\mathrm{lip}_0(X, d^\alpha)$ is isomorphic to c_0 whenever X is an infinite compact subset of a finite-dimensional normed linear space. This result was corrected by Weaver, who asked whether such an isomorphism could be extended to any compact metric space [12, p. 98]. Kalton answered this question negatively by proving that a compact convex subset X of a Hilbert space containing the origin has the property that $\mathrm{lip}_0(X, d^\alpha)$ is isomorphic to c_0 if and only if X is finite-dimensional [8, Theorem 8.3]. In fact this statement is true for every general Banach space in place of a Hilbert space if $0 < \alpha \leq 1/2$ [8, Theorem 8.5] and for any Banach space that has nontrivial Rademacher type if $0 < \alpha < 1$ [8, Theorem 8.4]. Kalton conjectured that this holds in full generality for all Banach spaces.

Let us recall now that a metric space (X, d) satisfies the *doubling condition* (or has *finite Assouad dimension*) if there is an integer n such that for any $\delta > 0$, every closed ball of radius δ can be covered by at most n closed balls of radius $\delta/2$. A theorem of Assouad [1] asserts that whenever a metric space (X, d) satisfies the doubling condition, every Hölder metric space (X, d^α) Lipschitz embeds in the euclidean space $\mathbb{R}^n$. Using this result, Kalton observed that if a compact metric space (X, d) satisfies the doubling condition, then the space $\mathrm{lip}_0(X, d^\alpha)$ is isomorphic to c_0 [8, Theorem 6.5]. Furthermore, he also showed that the converse is false by means of a counterexample [8, Proposition 6.8].

2. THE NORM OF C_φ ON $\mathrm{lip}_0(X, d^\alpha)$

The aim of this section is to derive a formula for the norm of the composition operator C_φ on $\mathrm{lip}_0(X, d^\alpha)$ in terms of the Lipschitz constant of φ. A similar expression was already provided by Weaver for the composition operator C_φ on the space $\mathrm{Lip}_0(X, d)$, obtaining in [12, Proposition 1.8.2] the following identity

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)}.$$

Theorem 2.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi\colon X \to X$ a base point preserving Lipschitz mapping. Then the norm of the composition operator $C_\varphi\colon \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ is given by the expression*

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}.$$

Proof. We follow the steps of the proof of Weaver's formula. One inequality is formally identical, while the other inequality needs an adjustment of the suitable attaining functions. For any $f \in \mathrm{lip}_0(X, d^\alpha)$ with $L_\alpha(f) \leq 1$, we have

$$\begin{aligned}
L_\alpha(C_\varphi f) &= \sup_{x \neq y} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)^\alpha} \\
&\leq \sup_{\varphi(x) \neq \varphi(y)} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x), \varphi(y))^\alpha} \cdot \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\
&\leq L_\alpha(f) \cdot \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\
&\leq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha},
\end{aligned}$$

and so

$$\|C_\varphi\| = \sup_{L_\alpha(f) \leq 1} L_\alpha(C_\varphi f) \leq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}.$$

For the converse inequality, fix two points $x, y \in X$ such that $\varphi(x) \neq \varphi(y)$ and choose β strictly between α and 1. Define $h\colon X \to \mathbb{R}$ by

$$h(z) = \frac{d(z, \varphi(y))^\beta - d(z, \varphi(x))^\beta}{2d(\varphi(x), \varphi(y))^{\beta - \alpha}},$$

and $f\colon X \to \mathbb{R}$ by

$$f(z) = h(z) - h(e).$$

It is not hard to show that $f \in \mathrm{lip}_0(X, d^\alpha)$ with $L_\alpha(f) = 1$ (see, for instance, [9]), so

$$\|C_\varphi\| \geq L_\alpha(C_\varphi f) \geq \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)^\alpha} = \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}.$$

Taking supremum over x and y, we conclude that

$$\|C_\varphi\| \geq \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}.$$

$\square$

3. THE LOWER ESTIMATE OF THE ESSENTIAL NORM OF C_φ ON $\mathrm{lip}_0(X, d^\alpha)$

Next we bound from below the essential norm of C_φ on $\mathrm{lip}_0(X, d^\alpha)$ by means of an assymptotic quantity that measures the local flatness of φ.

Theorem 3.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi\colon X \to X$ a base point preserving Lipschitz mapping. Then the essential norm of the operator $C_\varphi\colon \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ satisfies the lower estimate*

$$\lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \leq \|C_\varphi\|_e.$$

We will need the following description of the weak convergence in $\mathrm{lip}_0(X, d^\alpha)$. This result is part of the folklore and it is immediate from the Banach-Steinhaus theorem since $\mathrm{lip}_0(X, d^\alpha)^* = \overline{\mathrm{span}}\{\delta_x : x \in X\}$ (see Weaver [12, Theorem 3.3.3]).

Lemma 3.2. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\{f_n\}$ a sequence in $\mathrm{lip}_0(X, d^\alpha)$. Then $\{f_n\}$ converges to 0 weakly in $\mathrm{lip}_0(X, d^\alpha)$ if and only if $\{f_n\}$ is bounded in $\mathrm{lip}_0(X, d^\alpha)$ and converges to 0 pointwise on X.*

Proof of Theorem 3.1. Since the mapping $\varphi\colon X \to X$ is Lipschitz, the function

$$t \mapsto \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \qquad (t > 0)$$

is well defined. It is easy to check that

$$(1) \qquad \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} = \inf_{t > 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}.$$

Now, for every natural number n we can take a real number t_n such that $0 < t_n < [2/n(1 + L(\varphi)^\alpha)]^{1/\alpha}$ and two points $x_n, y_n \in X$ such that $0 < d(x_n, y_n) < t_n$, satisfying

$$\sup_{0 < d(x,y) < t_n} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} < \inf_{t > 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} + \frac{1}{n}$$

and

$$\sup_{0 < d(x,y) < t_n} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} - \frac{1}{n} < \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha}.$$

In this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$(2) \qquad 0 < d(x_n, y_n) < \left[\frac{2}{n\,(1 + L(\varphi)^\alpha)} \right]^{\frac{1}{\alpha}}$$

and

$$\left| \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha} - \inf_{t > 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \right| < \frac{1}{n}$$

for all $n \in \mathbb{N}$, and this last inequality implies that

$$(3) \qquad \inf_{t > 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} = \lim_{n \to \infty} \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha}.$$

Now take $x_n, y_n \in X$ for which $\varphi(x_n) \neq \varphi(y_n)$ and choose $\beta \in\,]\alpha, 1[$. Define $h_n, f_n\colon X \to \mathbb{R}$ by

$$h_n(x) = \frac{d(x, \varphi(y_n))^\beta - d(x, \varphi(x_n))^\beta}{2d(\varphi(x_n), \varphi(y_n))^{\beta - \alpha}}$$

and

$$f_n(x) = h_n(x) - h_n(e).$$

Using inequality (2), we have $\|h_n\|_\infty \leq 1/n$. Clearly, $f_n \in \mathrm{lip}_0(X, d^\alpha)$ with $L_\alpha(f_n) = L_\alpha(h_n) = 1$ and $\|f_n\|_\infty \leq 2/n$ for all $n \in \mathbb{N}$. Moreover, an easy calculation shows that $|f_n(\varphi(x_n)) - f_n(\varphi(y_n))| = d(\varphi(x_n), \varphi(y_n))^\alpha$. Since

$$\frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha} = \frac{|f_n(\varphi(x_n)) - f_n(\varphi(y_n))|}{d(x_n, y_n)^\alpha} \leq L_\alpha(C_\varphi f_n)$$

for all $n \in \mathbb{N}$, we have

$$(4) \qquad \lim_{n \to \infty} \frac{d(\varphi(x_n), \varphi(y_n))^\alpha}{d(x_n, y_n)^\alpha} \leq \limsup_{n \to \infty} L_\alpha(C_\varphi f_n).$$

By Lemma 3.2, $\{f_n\} \to 0$ weakly in $\mathrm{lip}_0(X, d^\alpha)$. Thus, if K is any compact operator from $\mathrm{lip}_0(X, d^\alpha)$ into $\mathrm{lip}_0(X, d^\alpha)$, then $\lim_{n\to\infty} L_\alpha(Kf_n) = 0$ because compact operators map weakly convergent sequences into norm convergent sequences. It follows that

$$
\begin{aligned}
\limsup_{n\to\infty} L_\alpha(C_\varphi f_n) &= \limsup_{n\to\infty} \left(L_\alpha(C_\varphi f_n) - L_\alpha(Kf_n) \right) \\
&\leq \limsup_{n\to\infty} L_\alpha((C_\varphi - K)f_n) \\
&\leq \|C_\varphi - K\|.
\end{aligned}
$$

(5)

Combining (1), (3), (4) and (5), we conclude that

$$
\lim_{t\to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \leq \|C_\varphi - K\|.
$$

By taking the infimum on both sides of this inequality over all compact operators K on $\mathrm{lip}_0(X, d^\alpha)$, we obtain the lower estimate

$$
\lim_{t\to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \leq \|C_\varphi\|_e. \qquad \square
$$

4. The upper estimate of the essential norm of C_φ on $\mathrm{lip}_0(X, d^\alpha)$

Now we prove that the lower bound of the essential norm of C_φ on $\mathrm{lip}_0(X, d^\alpha)$ obtained in Section 3 is also an upper bound whenever the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property.

Theorem 4.1. *Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property. Let $\varphi \colon X \to X$ be a base point preserving Lipschitz mapping. Then the essential norm of the composition operator $C_\varphi \colon \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ satisfies*

$$
\|C_\varphi\|_e \leq \lim_{t\to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}.
$$

The strategy for the proof of Theorem 4.1 is to work with a sequence $\{K_n\}$ of compact operators on $\mathrm{lip}_0(X, d^\alpha)$ that satisfies some prescribed conditions that are stated in Lemma 4.3 below. We borrow this technique from the work of Montes-Rodríguez [10].

First we recall some notions and results. A sequence $\{K_n\}$ is called a *compact approximating sequence* for a separable Banach space E if each $K_n \colon E \to E$ is a compact operator and $\lim_{n\to\infty} \|(I - K_n)f\| = 0$ for every $f \in E$, where I denotes the identity operator on E. Also, we say that $\{K_n\}$ is *shrinking* if $\lim_{n\to\infty} \|(I - K_n)^* f^*\| = 0$ for every $f^* \in E^*$.

Johnson [6, Theorem 2] showed that both the Banach space $\mathrm{lip}_0(X, d^\alpha)$ and its dual space $\mathrm{lip}_0(X, d^\alpha)^*$ are separable.

On the other hand, the *Banach-Mazur distance* between isomorphic Banach spaces E, F is defined by

$$
d(E, F) = \inf \left\{ \|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism of } E \text{ onto } F \right\}.
$$

We say that E *embeds almost isometrically into* F provided that for every $\varepsilon > 0$, there is a subspace $F_\varepsilon \subset F$ such that $d(E, F_\varepsilon) < 1 + \varepsilon$.

The next proposition is immediate from a result of Kalton [7].

Proposition 4.2. [7, Corollary 3] *Let $\{K_n\}$ be a sequence of compact operators between Banach spaces E and F and let us suppose that $\lim_{n\to\infty} \langle K_n^* f^*, f^{**} \rangle = 0$ for all $f^* \in F^*$ and $f^{**} \in E^{**}$. Then there exists a sequence $\{K_n^c\}$ of compact operators such that $K_n^c \in \mathrm{conv}\,\{K_m : m \geq n\}$ and $\lim_{n\to\infty} \|K_n^c\| = 0$.*

Lemma 4.3. *Let X be a pointed compact metric space and $0 < \alpha < 1$. Suppose that the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property. Then there is a shrinking compact approximating sequence $\{K_n\}$ on $\mathrm{lip}_0(X, d^\alpha)$ such that $\limsup_{n\to\infty} \|I - K_n\| \leq 1$.*

Proof. Since the dual space $\mathrm{lip}_0(X, d^\alpha)^*$ is separable and has the approximation property, it has the metric approximation property and therefore there is a shrinking compact approximating sequence $\{S_n\}$ on $\mathrm{lip}_0(X, d^\alpha)$. We claim that for every $j \in \mathbb{N}$, there exist a natural $n_j \geq j$ and a compact operator K_j on $\mathrm{lip}_0(X, d^\alpha)$ in the convex hull of the set $\{S_m \colon m \geq n_j\}$ such that $\|I - K_j\| < (1 + 1/j)^2$.

Fix $j \in \mathbb{N}$. Now, for the proof of our claim, we use a result that ensures that $\mathrm{lip}_0(X, d^\alpha)$ embeds almost isometrically into c_0. We refer to Kalton [8, Theorem 6.6] for a simple proof of this result due to Yoav Benyamini. Thus, there is a closed subspace $F_j \subset c_0$ and an isomorphism $T_j \colon \mathrm{lip}_0(X, d^\alpha) \to F_j$ such that $\|T_j\| \cdot \|T_j^{-1}\| < 1 + 1/j$. Now consider $M_n := T_j S_n T_j^{-1}$, and notice that $\{M_n\}$ is a shrinking compact approximating sequence on F_j. Next, let $\{P_n\}$ be the sequence of projections on c_0 defined by

$$(P_n x)(k) = x(k) \quad (1 \leq k \leq n), \qquad (P_n x)(k) = 0 \quad (k > n),$$

for all $x \in c_0$, and let $J \colon F_j \hookrightarrow c_0$ be the inclusion map. Then consider the sequence of compact operators $D_n := P_n J - J M_n$ defined from F_j into c_0. Notice that $\lim_{n \to \infty} \langle D_n^* f^*, f^{**} \rangle = 0$ for all $f^* \in c_0^*$ and $f^{**} \in F_j^{**}$. It follows from Proposition 4.2 that there is a sequence of operators $\{D_n^c\}$ such that $D_n^c \in \mathrm{conv}\{D_m \colon m \geq n\}$ and $\lim_{n \to \infty} \|D_n^c\| = 0$. This gives rise to a pair of shrinking compact approximating sequences $\{P_n^c\}$ and $\{M_n^c\}$ such that, for each $n \in \mathbb{N}$, $P_n^c \in \mathrm{conv}\{P_m \colon m \geq n\}$, $M_n^c \in \mathrm{conv}\{M_m \colon m \geq n\}$, and $\lim_{n \to \infty} \|P_n^c J - J M_n^c\| = 0$. Now, consider $L_n := T_j^{-1} M_n^c T_j$. We have

$$\|I - L_n\| = \left\|I - T_j^{-1} M_n^c T_j\right\| \leq \left\|T_j^{-1}\right\| \cdot \|T_j\| \cdot \|I - M_n^c\|$$

$$\leq \left(1 + \frac{1}{j}\right) \cdot \|I - M_n^c\| = \left(1 + \frac{1}{j}\right) \cdot \|J(I - M_n^c)\|$$

$$\leq \left(1 + \frac{1}{j}\right) \cdot (\|(I - P_n^c)J\| + \|P_n^c J - J M_n^c\|)$$

$$\leq \left(1 + \frac{1}{j}\right) \cdot (1 + \|P_n^c J - J M_n^c\|)$$

for all $n \in \mathbb{N}$. Finally, choose $n_j \geq j$ large enough so that $\left\|P_{n_j}^c J - J M_{n_j}^c\right\| < 1/j$ and conclude that $\left\|I - L_{n_j}\right\| < (1 + 1/j)^2$. The claim is proved if we take $K_j := L_{n_j}$.

The proof of the lemma will be finished if we show that $\{K_j\}$ is a shrinking compact approximating sequence on $\mathrm{lip}_0(X, d^\alpha)$. Let $f \in \mathrm{lip}_0(X, d^\alpha)$ and $\varepsilon > 0$. Since $\lim_{n \to \infty} L_\alpha(f - S_n f) = 0$, there exists $m_0 \in \mathbb{N}$ such that $L_\alpha(f - S_n f) < \varepsilon$ for $n \geq m_0$. If $j \geq m_0$, using that $K_j \in \mathrm{conv}\{S_m \colon m \geq n_j\}$ and $n_j \geq j$, we conclude that $L_\alpha(f - K_j f) < \varepsilon$. Hence $\lim_{j \to \infty} L_\alpha(f - K_j f) = 0$. This shows that $\{K_j\}$ is approximating on $\mathrm{lip}_0(X, d^\alpha)$ and similarly it is seen that $\{K_j\}$ is shrinking. □

There is another preliminary result that is needed for the proof of Theorem 4.1 and that can be stated as follows:

Lemma 4.4. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi \colon X \to X$ a base point preserving Lipschitz mapping. Let $\{K_n\}$ be a shrinking compact approximating sequence on $\mathrm{lip}_0(X, d^\alpha)$.*
Then, for each $t > 0$,

$$\lim_{n \to \infty} \sup_{L_\alpha(f) \leq 1} \sup_{d(x,y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x,y)^\alpha} = 0.$$

Proof. Fix $t > 0$. Since the inequality $\|f\|_\infty \leq \mathrm{diam}(X)^\alpha \cdot L_\alpha(f)$ is satisfied for all $f \in \mathrm{lip}_0(X, d^\alpha)$, there is a continuous injection $J \colon (\mathrm{lip}_0(X, d^\alpha), L_\alpha(\cdot)) \hookrightarrow (\mathrm{lip}_0(X, d^\alpha), \|\cdot\|_\infty)$. Moreover, it follows from the Arzelà–Ascoli Theorem that J is a compact operator, and by Schauder's theorem, its adjoint J^* is a compact operator, too.

Let B be the unit ball of $(\mathrm{lip}_0(X, d^\alpha), \|\cdot\|_\infty)^*$. Since the bounded sequence of operators $\{(I - K_n)^*\}$ converges to zero pointwise on $\mathrm{lip}_0(X, d^\alpha)^*$ and $J^*(B)$ is a relatively compact set in $(\mathrm{lip}_0(X, d^\alpha), L_\alpha(\cdot))^*$,

it follows that $\lim_{n\to\infty} \|(I - K_n)^* J^*\| = 0$. Thus, $\lim_{n\to\infty} \|J(I - K_n)\| = 0$. Let $\varepsilon > 0$ be given and choose $m \in \mathbb{N}$ such that if $n \geq m$, then $\|(I - K_n)f\|_\infty < \varepsilon t^\alpha / 4$ for all $f \in \mathrm{lip}_0(X, d^\alpha)$ with $L_\alpha(f) \leq 1$. For $n \geq m$, we have

$$\frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \leq \frac{2 \|(I - K_n)f\|_\infty}{t^\alpha} < \frac{\varepsilon}{2}$$

whenever $f \in \mathrm{lip}_0(X, d^\alpha)$ with $L_\alpha(f) \leq 1$ and $x, y \in X$ such that $d(x, y) \geq t$. Finally, we get

$$\sup_{L_\alpha(f) \leq 1} \sup_{d(x,y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} < \varepsilon,$$

as required. $\square$

We now are ready to prove our main result.

Proof of Theorem 4.1. Let $\{K_n\}$ be the sequence of operators on $\mathrm{lip}_0(X, d^\alpha)$ provided by Lemma 4.3. Since each K_n is a compact operator, so is the product $C_\varphi K_n \colon \mathrm{lip}_0(X, d^\alpha) \to \mathrm{lip}_0(X, d^\alpha)$ and therefore

$$\|C_\varphi\|_e \leq \|C_\varphi(I - K_n)\|.$$

Next, fix $t > 0$ and notice that

$$\begin{aligned}
\|C_\varphi(I - K_n)\| &= \sup_{L_\alpha(f) \leq 1} L_\alpha\left(C_\varphi(I - K_n)f\right) \\
&= \sup_{L_\alpha(f) \leq 1} \sup_{x \neq y} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \\
&\leq \sup_{L_\alpha(f) \leq 1} \sup_{0 < d(x,y) < t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \\
&\quad + \sup_{L_\alpha(f) \leq 1} \sup_{d(x,y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha}.
\end{aligned}$$

Then, for every $f \in \mathrm{lip}_0(X, d^\alpha)$, we have

$$\begin{aligned}
\sup_{0 < d(x,y) < t} &\frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \\
&\leq \sup_{\varphi(x) \neq \varphi(y)} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(\varphi(x), \varphi(y))^\alpha} \cdot \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\
&\leq L_\alpha[(I - K_n)f] \cdot \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha},
\end{aligned}$$

so that

$$\sup_{L_\alpha(f) \leq 1} \sup_{0 < d(x,y) < t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha} \leq \|I - K_n\| \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha}.$$

Now, combining the above inequalities, we obtain

$$\begin{aligned}
\|C_\varphi\|_e &\leq \|I - K_n\| \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x, y)^\alpha} \\
&\quad + \sup_{L_\alpha(f) \leq 1} \sup_{d(x,y) \geq t} \frac{|[(I - K_n)f](\varphi(x)) - [(I - K_n)f](\varphi(y))|}{d(x, y)^\alpha}
\end{aligned}$$

Letting $n \to \infty$ and using Lemma 4.3 and Lemma 4.4, we conclude that

$$\|C_\varphi\|_e \leq \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}.$$

Finally, taking limits as $t \to 0$ yields the desired inequality.

$\square$

5. THE ESSENTIAL NORM OF C_φ ON $\mathrm{Lip}_0(X, d^\alpha)$

Now we extend the estimates on the essential norm of a composition operator to the spaces $\mathrm{Lip}_0(X, d^\alpha)$. Recall that $\mathrm{lip}_0(X, d^\alpha)^{**}$ is isometrically isomorphic to $\mathrm{Lip}_0(X, d^\alpha)$ whenever (X, d) is a pointed compact metric space and $\alpha \in (0, 1)$ (see [12, Theorem 3.3.3 and Proposition 3.2.2]). As a matter of fact, the mapping $\Delta \colon \mathrm{lip}_0(X, d^\alpha)^{**} \to \mathrm{Lip}_0(X, d^\alpha)$ defined by

$$\Delta(F)(x) = F(\delta_x) \qquad (F \in \mathrm{lip}_0(X, d^\alpha)^{**}, \ x \in X),$$

is an isometric isomorphism.

If T is a bounded linear operator on a Banach space, then $\|T^*\| = \|T\|$. However, this identity is no longer true for the essential norm. Since the adjoint of a compact operator is again a compact operator, we always have $\|T^*\|_e \leq \|T\|_e$ and therefore $\|T^{**}\|_e \leq \|T^*\|_e \leq \|T\|_e$. Axler, Jewell and Shields [2] showed that in fact $\|T^{**}\|_e = \|T^*\|_e$, but they gave a counterexample where $\|T^*\|_e < \|T\|_e$.

Theorem 5.1. *Let X be a pointed compact metric space, $0 < \alpha < 1$ and $\varphi \colon X \to X$ a point preserving Lipschitz mapping. Then the essential norm of the operator $C_\varphi \colon \mathrm{Lip}_0(X, d^\alpha) \to \mathrm{Lip}_0(X, d^\alpha)$ satisfies the lower estimate*

$$\|C_\varphi\|_e \geq \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}.$$

If, in addition, the space $\mathrm{lip}_0(X, d^\alpha)^*$ has the approximation property, then we have the upper estimate

$$\|C_\varphi\|_e \leq \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha}.$$

Proof. Let us start with the lower estimate. Let $\{f_n\}$ be the weakly null sequence in $\mathrm{lip}_0(X, d^\alpha)$ that we constructed for the proof of Theorem 3.1. Then the sequence $\{f_n\}$ is weakly null in $\mathrm{Lip}_0(X, d^\alpha)$. Thus, if K is any compact operator on $\mathrm{Lip}_0(X, d^\alpha)$, we have $\lim_{n \to \infty} L_\alpha(Kf_n) = 0$. Hence, the same computation we performed in Theorem 3.1 yields the lower estimate

$$\lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha} \leq \|C_\varphi\|_e.$$

Now, for the upper estimate, given $F \in \mathrm{lip}_0(X, d^\alpha)^{**}$ and $x \in X$, notice that

$$(\Delta^{-1} C_\varphi \Delta)(F)(\delta_x) = ((C_\varphi \Delta)(F))(x) = C_\varphi(\Delta(F))(x) = \Delta(F)(\varphi(x))$$

$$= F\left(\delta_{\varphi(x)}\right) = F\left(\delta_x \circ C_\varphi|_{\mathrm{lip}_0(X, d^\alpha)}\right) = \left(F \circ \left(C_\varphi|_{\mathrm{lip}_0(X, d^\alpha)}\right)^*\right)(\delta_x) = \left(C_\varphi|_{\mathrm{lip}_0(X, d^\alpha)}\right)^{**}(F)(\delta_x).$$

Since $\mathrm{lip}_0(X, d^\alpha)^* = \overline{\mathrm{span}}\{\delta_x : x \in X\}$, we conclude that $\Delta^{-1} C_\varphi \Delta = \left(C_\varphi|_{\mathrm{lip}_0(X, d^\alpha)}\right)^{**}$. Finally, using the relationship between the essential norm of an operator and that of its second adjoint, and applying Theorem 4.1, we get

$$\|C_\varphi\|_e = \left\|\Delta^{-1} C_\varphi \Delta\right\|_e = \left\|\left(C_\varphi|_{\mathrm{lip}_0(X, d^\alpha)}\right)^{**}\right\|_e \leq \left\|C_\varphi|_{\mathrm{lip}_0(X, d^\alpha)}\right\|_e \leq \lim_{t \to 0} \sup_{0 < d(x,y) < t} \frac{d(\varphi(x), \varphi(y))^\alpha}{d(x,y)^\alpha},$$

as we wanted. $\square$

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