A glimpse at Nigel Kalton's work

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Abstract. This article is a non exhaustive survey of Nigel Kalton's contribution to functional analysis. We focus on geometry and structure of Banach spaces and quasi-Banach spaces, non linear isomorphisms, isometric theory, interpolation theory, differentiation of interpolation lines and twisted sums, basis theory and applications.

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1 Introduction

Nigel Kalton's work concerns every domain of functional analysis, and goes in fact much further than that and even beyond the limits of analysis. His research spans over more than forty years, and the published part of his work represents thousands of dense, original and deep pages of concisely written mathematics.

Pretending to outline such an amount of mathematics in a few pages is simply pointless. The present survey therefore focuses on Nigel Kalton's work on the geometry and structure of Banach spaces and quasi-Banach spaces. But I should make it clear that even with this restriction, the present sketch is very far from comprehensive, and that many important articles (or Memoirs) will not even be mentioned.

Indeed a choice has been made, in order to provide the reader with some pieces of information that would not be too easily found elsewhere. This choice consisted into picking some works with which the author is relatively familiar, and on which he could provide, if not an original point of view, at least some facts which may not be commonly known, or recent references on related topics. I should make it clear that there is no claim on my side that this part of Nigel Kalton's contribution is the most important one. It is simply a part I understand, and which is clearly exciting enough to deserve a presentation, and hard enough to deserve an explanation.

This outline contains some sketches of proof and a few technical arguments, but it mainly consists into a glance at Nigel Kalton's work from a distance, in order to provide the reader with some intuition on what goes on. Obviously, this outline cannot be used as a substitute for an actual reading of the original articles. On the contrary, it is an invitation to dwell into Nigel Kalton's work, and to enjoy every line of it, and even all what can be found between the lines.

Let us now describe the contents of this note. Section 2 deals with non locally convex analysis, and the important concepts which Nigel Kalton discovered and used in this particular theory: extensions, quasi-linear maps, $\mathcal{K}$-spaces. Various objects from classical functional analysis, such as Maharam submeasures, entropy functions or distorted norms, turn out to be related with what happens when p is less than 1 and this shows in our presentation. Section 3 is devoted to some recent progress in non-linear geometry. This topic is relatively new and many basic questions are still unanswered. However asymptotic structures happen to provide usable invariants for Lipschitz and even uniform isomorphisms and this is displayed here. Section 4 is devoted to isometric theory, and to the wealth of information that can be deduced from the existence of special norms on a given space. Complex and real spaces behave quite differently there, and Hermitian operators, which are so useful but only in the complex case, partly explain why. We will also see that isometric theory sometimes has a quite algebraic flavor. Interpolation theory, differentiation of interpolation lines and the corresponding calculus on the "manifold" of Banach spaces are displayed in section 5, where unexpected connections with commutators and the trace class are shown. Finally, section 6 contains some applications of basis theory to the solution of important problems of classical analysis, showing that some of the work which has been done for its own sake some thirty years ago provides powerful tools and examples when geometry of Banach spaces joins forces with harmonic analysis or semigroup theory.

These sections have been created for the reader's convenience, but tight relations exist between them and the versatility of some of the concepts which are displayed below underlines the profound unity of mathematical analysis, when seen from Nigel Kalton's point of view. Quite exotic tools, such as non-linear liftings, discontinuous linear functionals, conditional bases of the Hilbert space, non locally convex twisted sums, and so on, are used below in such a way that they provide useful information on classical and main-stream analysis. And why dispensing with such tools when they can be so powerful?

2 The Kalton zone: $0 < p < 1$.

Hahn–Banach theorems are cornerstones of functional analysis, to such a point that analysts may get nervous when they have to dispense with them. But it turns out that non-locally convex spaces show up very naturally in many cases when there is no reason to "stop at $p = 1$": for instance, if $f : \Omega \to \mathbb{C}$ is an holomorphic function on some open subset Ω of $\mathbb{C}$ then $|f|^p$ is subharmonic for all $p > 0$. This very elementary but important example is an invitation to visit what I suggest to call the Kalton zone: $0 < p < 1$.

We recall that metrizable complete topological vector spaces (on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are called F-spaces. Their topology is induced by an *F-norm*, that is, a map Λ from the space X to $\mathbb{R}^+$ such that:

(i) $\Lambda(x) > 0$ if $x \neq 0$.
(ii) $\Lambda(\alpha x) \leq \Lambda(x)$ if $|\alpha| \leq 1$.
(iii) $\lim_{\alpha \to 0} \Lambda(\alpha x) = \Lambda(0) = 0$.
(iv) $\Lambda(x + y) \leq \Lambda(x) + \Lambda(y)$ for all $(x, y) \in X^2$.

The space X is locally bounded if and only if its topology can be generated by a *quasi-norm* $\|\cdot\|$, namely a map $\|\cdot\| : X \to \mathbb{R}^+$ such that:

(i) $\|x\| > 0$ if $x \neq 0$
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$
(iii) $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $(x, y) \in X^2$

where $C \geq 1$ is the "modulus of concavity" of the quasi-norm. An F-space is called a *quasi-Banach space* when its topology is generated by a quasi-norm, or equivalently by the Aoki–Rolewicz theorem, by a p-subadditive quasi-norm $|||\cdot|||$, which satisfies the condition

(iv) $|||x + y|||^p \leq |||x|||^p + |||y|||^p$

for all $(x, y) \in X^2$ and $p > 0$ given by $p = (1 + \log_2(C))^{-1}$.

We refer to [83] for an authoritative book on F-spaces. It is clear that the classical applications of Baire's lemma are not sensitive to local convexity assumptions, while Hahn–Banach theorem is, and in fact to the point where it leads to a characterization:

Theorem 2.1 ([56]). *A quasi-Banach space X is locally convex (i.e. is a Banach space) if and only if every continuous linear functional defined on a closed subspace E of X has an extension to a continuous linear functional on X.*

In other words, a quasi-Banach space is a Banach space if and only if the weak and quasi-norm topologies have the same closed subspaces.

The proof of Theorem 2.1 relies on the construction of Markushevich basic sequences (see [71, Proposition 3.4]), obtained by refining Mazur's classical argument. For being able to do so, one needs however a weaker topology, even if it is not "weak" in the classical sense. A quasi-Banach space is *minimal* if it does not have any weaker Hausdorff vector topology. A separable quasi-Banach space is minimal exactly when it contains no basic sequence. It turns out that quite general assumptions force the existence of basic sequences. Let us say that a p-subadditive quasi-norm $\|\cdot\|$ is *plurisubharmonic* (in short, $p.s.$) if

$$\|x\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\| d\theta$$

for all $(x, y) \in X^2$. It is shown in [131] that a complex quasi-Banach space X which has an equivalent $p.s.$ norm is *not* minimal, and this applies to subspaces of $L^p(0 < p < 1)$ and more generally to all *natural* quasi Banach spaces, where "natural" means "subspace of a lattice-convex quasi-Banach lattice".

However, minimal quasi-Banach spaces do exist and this is shown in [69]:

Theorem 2.2. *There is a quasi-Banach space X which contains a one-dimensional subspace E such that every infinite-dimensional closed subspace Y of X contains E. In particular, X contains no basic sequence and it is minimal.*

The reader may find it amusing to think of that space as a book: it may have many pages but they all meet on the one-dimensional binding.

The proof of Theorem 2.2 is the culmination of several works, which we now outline.

If X and Y are quasi-Banach spaces, Y is a *minimal extension* of X if there is a one-dimensional subspace E of Y such that $Y/E \simeq X$. The reader should be warned that the word "minimal" is used here with two different meanings. A minimal extension is usually *not* a minimal quasi-Banach space, but the above theorem asserts that this may happen. A minimal extension is said to be *trivial* if it splits, i.e. if $Y \simeq X \oplus E$. The case when X is actually a Banach space is important, and indeed Kalton [59], Ribe [119], Roberts [123] independently constructed non-trivial minimal extensions of $X = \ell_1$, thus solving negatively the three-space problem for local convexity.

All minimal extensions can be obtained in the following way [58]: let X be a quasi-Banach space over the field $\mathbb{K}$ and X_0 a dense linear subspace of X. A map $F : X_0 \to \mathbb{K}$ is *quasilinear* if:

(i) $F(\alpha x) = \alpha F(x)$ for $x \in X_0$ and $\alpha \in \mathbb{K}$.

(ii) There is $C \in \mathbb{R}$ such that $|F(x+y) - F(x) - F(y)| \leq C(\|x\| + \|y\|)$ for all $(x,y) \in X^2$.

Then one can define a quasi-norm on $\mathbb{K} \oplus X_0$ by

$$\|(\alpha, x)\|_F = |\alpha - F(x)| + \|x\|$$

and the completion of $\mathbb{K} \oplus X_0$ for this quasi-norm if a minimal extension of X denoted $\mathbb{K} \oplus_F X$. Conversely any minimal extension of X is obtained in this way, and $\mathbb{K} \oplus_F X$ splits if and only if there is a linear map $G = X_0 \to \mathbb{K}$ such that

$$|F(x) - G(x)| \leq C'\|x\| \tag{2.1}$$

for all $x \in X_0$. This approximation is related with Hyers–Ulam functional stability, for which we refer to [6, Chapter 15]. The Ribe space is then obtained by considering the quasilinear functional

$$F(x) = \sum_{k=1}^{\infty} x_k \log|x_k| - \Big(\sum_{k=1}^{\infty} x_k\Big) \log\Big|\sum_{k=1}^{\infty} x_k\Big|$$

on the dense subspace c_{00} of finitely supported sequences in ℓ_1. For showing that (2.1) fails for this F, it suffices to note that when X is a Banach space, (2.1) is equivalent to an estimate

$$\Big|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big| \leq C'' \sum_{i=1}^{n} \|x_i\|$$

for all $x_1, \ldots, x_n \in X_0$, and to compute

$$F\Big(\sum_{i=1}^{n} e_i\Big) - \sum_{i=1}^{n} F(e_i) = -n\log(n).$$

Ribe's function F is closely related to Shannon's entropy function from information theory. This motivates the following terminology. Let X be a Banach sequence space.

The quasilinear map Φ_X defined on c_{00}^+ by

$$\Phi_X(x) = \sup_{\|t\|_X \leq 1} \sum_{k=1}^{\infty} x_k \log |t_k| \tag{2.2}$$

and extended to c_{00} by

$$\Phi_X(x) = \Phi_X(x^+) - \Phi_X(x^-)$$

is called the *entropy function* of X [107]. For instance

$$\Phi_{\ell_1}(x) = \sum_{k=1}^{\infty} x_k \log \frac{|x_k|}{\|x\|_1},$$

while $\Phi_{\ell_p} = \frac{1}{p}\Phi_{\ell_1}$.

In order to construct a minimal extension Y of ℓ_1 with no basic sequence and such that $Y/E \simeq \ell_1$, we need that every infinite-dimensional closed subspace of Y contains E. This is reminiscent of Gowers–Maurey's construction of a Banach space X_{GM} without unconditional basic sequence [45], which is such that for any infinite-dimensional subspaces U and V of X_{GM}

$$\inf \{ \|u - v\|; u \in U, v \in V, \|u\| = \|v\| = 1 \} = 0.$$

And indeed, Gowers' modification [43] of the original construction, used in his solution of the hyperplane problem, provides a space X whose entropy function Φ_X yields to a minimal extension $\mathbb{K} \oplus_{\Phi_X} \ell_1$ with no basic sequence [69], and which is therefore a minimal quasi-Banach space. Note that this function $\Phi_X = F$ fails to satisfy (2.1) when restricted to any infinite dimensional subspace J of c_{00} or equivalently

$$\sup \{ |F(x)| : x \in J, \|x\| \leq 1 \} = \infty,$$

hence $\Phi_X = F$ is distorted in the sense of [108].

Minimal quasi-Banach spaces M are pretty strange objects: every one-to-one continuous linear map from M into a Hausdorff topological vector space is actually an isomorphism on its range! However existing examples are "non-isotropic" in the sense where they contain a distinguished line, namely the orthogonal of the dual space. It in not known whether an even stranger example exists which would exhibit this behaviour everywhere.

Open problem. *Does there exist an "atomic" quasi-Banach space, that is, a quasi-Banach space which contains no infinite-dimensional proper closed subspace?*

The Ribe space is a non-trivial minimal extension of ℓ_1. However there exist infinite-dimensional quasi-Banach spaces X which are such that every minimal extension of X is trivial. Such spaces are called $\mathcal{K}$- *spaces* in [81] (but in my opinion Kalton spaces would sound perfect) and it is shown in [59] that if $0 < p < 1$, the spaces ℓ_p and L_p are $\mathcal{K}$-spaces, from which it follows in particular that if E is a one-dimensional subspace of $L_p(0 < p < 1)$ then L_p/E is not isomorphic to L_p [81].

Some Banach spaces are $\mathcal{K}$-spaces: it is shown in [86] that every quotient space of a $\mathcal{L}_\infty$-space is a $\mathcal{K}$-space, and in [59] that a Banach space with non-trivial type is a $\mathcal{K}$-space. In fact, Kalton conjectures that a Banach space is a $\mathcal{K}$-space exactly when its dual space has non-trivial cotype.

Minimal extensions of $\mathcal{L}_\infty$-spaces are trivial, in other words it is so that quasi-linear maps on "cubes" are close to linear ones. This creates a link between this field and the "Maharam problem", which we now outline.

The Maharam problem originates in von Neumann's question (Problem 163 in the Scottish book) of characterizing measure algebras. D. Maharam's conditional negative answer [104] to von Neumann's question (provided the Souslin hypothesis fails) lead her to consider what is now called a *Maharam submeasure*: if Σ is a σ-algebra of subsets of some set Ω, a Maharam submeasure is a map $\phi = \Sigma \to \mathbb{R}^+$ such that:

(M1) $\phi(\emptyset) = 0$.
(M2) $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ for $A, B \in \Sigma$.
(M3) If $(A_n) \in \Sigma$ is a disjoint sequence then $\lim \phi(A_n) = 0$.

The *Maharam problem* asks if for every Maharam submeasure ϕ, there is a (countably additive, positive, finite) measure λ *equivalent* to ϕ, in the sense where $\phi(A) = 0$ if and only if $\lambda(A) = 0$.

This problem has an equivalent form which turns it into a functional analysis problem: let X be an F-space and $G : \Sigma \to X$ be a countably additive vector measure. Does there exist a measure λ on Σ which *controls* G, that is $\lambda(A) = 0$ implies $G(A) = 0$? The answer to this "control measure problem" is positive when X is a Banach space by a classical result of Bartle–Dunford–Schwartz (see [65]), when $X = L_p$ $(0 < p < 1)$ through a change of density argument due to Maurey [105], and when $X = L_0$ as a consequence of the fact that L_0-valued vector measures are bounded [129, 82].

The equivalence between the Maharam problem and the control measure problem relies in part on the simple fact that is $G : \Sigma \to X$ is a vector measure, then its semi-variation $\|G\|$ defined by

$$\|G\|(A) = \sup\left\{\|G(B)\| : B \subseteq A,\ B \in \Sigma\right\}$$

is a Maharam submeasure.

The Maharam problem was investigated in [49] where a negative answer was shown to be equivalent to the existence of a *pathological* nonzero Maharam submeasure, where ϕ "pathological" means: for all $\varepsilon > 0$, there exists $B_1, B_2, \ldots, B_m$ in Σ such that

(P1) $\displaystyle \frac{1}{m}\sum_{i=1}^{m} \mathbf{1}_{B_i} > (1-\varepsilon)\mathbf{1}_\Omega$
(P2) $\phi(B_i) < \varepsilon$ for $i = 1, 2, \ldots, m$.

At this point, we should relate this line of thought with Roberts' construction of a compact convex set without extreme points [122]. This construction relies on the notion of a *needle point*: a point x in a quasi-Banach space X is a needle point if for all $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in X$ such that $\|x_i\| < \varepsilon$ for all $i = 1, 2, \ldots, n$, and if $y \in \mathrm{conv}\{x_1, x_2, \ldots, x_n\}$ then $\|y - tx\| < \varepsilon$ for some $t \in [0, 1]$.

We already met an example of needlepoint: if $(\mathbb{K} \oplus_F Y) = X$ is a non-trivial minimal extension of a Banach space Y, then $(1,0)$ is a needle point in X. It is also relevant to compare the definition of a needle point with the conditions (P1) and (P2) which characterize pathological submeasures.

A *needle space* is a quasi-Banach space such that every point is a needle point, and Roberts showed in [122] that every needle space contains a compact convex set without extreme points, and in [121] that L_p is a needle space if $0 < p < 1$.

Let us observe in passing a feature of quasi-Banach spaces which is quite unusual for those who usually work in Banach spaces where "all non-zero vectors are equivalent": given $x \neq 0$ and y in a quasi-Banach space X, there is in general *no* continuous linear operator T such that $Tx = y$. It may even be so that the *only* continuous linear map are scalar multiplications [85], and such "rigid spaces" are highly non-isotropic.

We now translate the Maharam problem into geometric terms. When a Maharam submeasure ϕ is equivalent to some measure λ, this submeasure satisfies a stronger uniform version of (M3), namely:

(M4) for all $\varepsilon > 0$, there is $N = N(\varepsilon)$ so that if $A_1, A_2, \ldots, A_N \in \Sigma$ are N disjoint sets then

$$\min_{1 \le i \le N} \phi(A_i) < \varepsilon.$$

Such submeasures are called *uniformly exhaustive*. When $\phi = \|G\|$ for a vecor measure $G : \Sigma \to X$, the relative compactness of $G(\Sigma)$ (or equivalently of $\mathrm{conv}(G(\Sigma)) \subseteq \overline{G(\Sigma)}$ by Liapunoff's theorem) provides, when available, the requested uniformity and ϕ is uniformly exhaustive.

The link between Roberts' construction and the Maharam problem is provided by [57] where it is shown that if $\overline{\mathrm{conv}}(G(\Sigma))$ is compact, then $\|G\|$ is equivalent to a measure (alternatively, G has a control measure) if and only if $\overline{\mathrm{conv}}(G(\Sigma))$ is locally convex.

This rises hope to obtain a negative solution to Maharam's problem (even with a uniformly exhaustive submeasure) by modifying Robert's approach to construct a non-locally convex compact *zonoid* $\overline{\mathrm{conv}}(G(\Sigma))$.

This however, cannot be done, and this is the content of the following Kalton–Roberts' theorem:

Theorem 2.3 ([86]). *A Maharam submeasure is equivalent to a measure if and only if it is uniformly exhaustive.*

The proof relies on the existence of remarkable bipartite graphs called concentrators (see [33]).

Therefore every compact zonoid is locally convex, every vector measure valued in an F-space with relatively compact range has a control measure, and every minimal extension of $Y = c_0$ is trivial (in other words, c_0 is a $\mathcal{K}$-space). Back to the set function side, Theorem 2.3 provides:

Theorem 2.4 ([86]). *There is a universal constant K such that if Σ is an algebra of subsets of some set Ω and $\phi : \Sigma \to \mathbb{R}$ is a set function such that*

$$|\phi(A \cup B) - \phi(A) - \phi(B)| \leq 1$$

if $A \cap B = \emptyset$, then there is an additive set function λ such that

$$|\phi(A) - \lambda(A)| \leq K$$

for all $A \in \Sigma$.

The best value of K belongs to the interval $[3/2, 45]$ but its precise value seems to be unknown.

The Maharam problem has recently been solved by Talagrand [130] who constructed a pathological Maharam submeasure. Talagrand's theorem provides a final negative answer to von Neumann's original question, and answers also negatively the control measure problem. On the other hand it shows that Kalton–Roberts' Theorem 2.3 is essentially optimal.

Let us note that if $G : \Sigma \to X$ is a vector measure with no control measure (which exists by Talagrand's theorem), then there is no one-to-one continuous linear operator $T : X \to L_0(\mu)$ with μ a measure, since $(T \circ G)$ would then be an L_0-valued vector measure for which the existence of a control measure is known. It is however simpler to show the existence of such "pathological" spaces, and in fact Kalton shows in [60] that if $0 < p < 1$ and

$$T = L_p(\mathbb{T})/H^p(\mathbb{T}) \to L_0$$

is a continuous linear operator then $T = 0$. The above space $L_p(\mathbb{T})/H^p(\mathbb{T})$ is a simple example of a quasi-Banach space which is not "natural" as defined above.

To conclude on the Maharam problem, let us mention that it has been shown by Louveau [103] that if ϕ is a pathological submeasure of an algebra Σ of subsets of Ω and if we denote for $B \in \Sigma$

$$t^{\omega}_{\phi}(B) = \sup \{t \geq 0 : \text{for all } n, \text{ there is a partition } \mathcal{P}_n \text{ of } B \text{ such that } \mathcal{P}_n \subseteq \Sigma, \ \text{card}\, \mathcal{P}_n \geq n, \text{ and } \phi(P) \geq t \text{ for all } P \in \mathcal{P}_n\}$$

then $t^{\omega}_{\phi} \geq \phi/3$. In particular $t^{\omega}_{\phi}(\Omega) \geq \phi(\Omega)/3$ and Kalton–Roberts' theorem follows since it means that $\phi(\Omega) = 0$ when $t^{\omega}_{\phi}(\Omega) = 0$. Louveau's argument is an adaptation of Kalton–Roberts' proof.

The "Ribe–Roberts" circle of ideas lead to a number of results at the right of the Kalton zone, in the locally convex case $p = 1$. Indeed, the Ribe space is isomorphic [61] to the subspace of L^p $(0 < p < 1)$ generated by the constant function $\mathbf{1}$ and the sequence $(|\xi_n|)$, where (ξ_n) is a sequence of independent random variables with the Cauchy distribution (in other words, 1-stable variables), such that

$$\int e^{is\xi_n(t)} dt = e^{-|s|}.$$

It can be shown [61] that

$$\|\alpha_0 + \sum_k \alpha_k |\xi_k| \|_p \sim |\alpha_0 - \phi_{\ell_1}((\alpha_t))| + \sum_k |\alpha_k|$$

where ϕ_{ℓ_1} is the entropy function of the space ℓ_1 defined by (2.2). Through this construction, the constant function in the Ribe space appears as the most natural example of needle point.

Of course $\xi_n \notin L_1$, but a similar construction yields an interesting subspace of L_1 [39, Example 4.1.1]. If $p \in (1, 2]$, a random variable Y is called *p-stable* if for some constant C

$$\int e^{isY(t)} dt = e^{-C|s|^p}.$$

Let (Z_j) be a sequence of independent symmetric p_j-stable random variables such that $\|Z_j\|_1 = 1$, let $U_j = |Z_j| - 1$ and let

$$X = \overline{\text{span}}\{\mathbf{1}, (U_j)_{j\geq 1}\}$$

be the closed subspace of L_1 generated by $\mathbf{1}$ and the sequence (U_j). Then, for a proper choice of the sequence (p_j) with $\lim_{j\to\infty} p_j = 1$, the unit ball B_X of the space X is compact for the topology τ_m of convergence in measure, although $x^*(\mathbf{1}) = 0$ for every $x^* \in X^*$ whose restriction to B_X is τ_m-continuous; in particular (B_X, τ_m) is not locally convex.

The space X is the first example of a separable Banach space whose unit ball is compact but not locally convex for a Hausdorff t.v.s. topology. However it falls short from being the satisfactory example one could hope for, and indeed the following question, which could be related with the atomic space problem, remains open.

Open problem. *Does there exist a separable Banach space X such that B_X is compact for some Hausdorff topological vector space topology, but* $\text{Ext}(B_X) = \emptyset$*?*

It is not even known whether there is a subspace Y of L_1 such that B_Y is τ_m-compact but such that $\text{Ext}(B_Y) = \emptyset$, or merely such that Y fails the Radon–Nikodym property. Along these lines, let us mention the important Bourgain–Rosenthal example [9] of $Z \subsetneq L_1$ failing RNP but such that every *dyadic* bounded tree converges in Z. The closed unit ball B_Z is τ_m-*relatively* compact and the proof relies in part on Robert's argument.

If instead of taking absolute values of p_j-stable random variables one takes the variables themselves and the space $Z = \overline{\text{span}}^{\|\cdot\|}(\mathbf{1}, (Z_j)_{j\geq 1})$, one obtains another interesting subspace of L_1 [40, Section V], which shows for instance that Dor's Theorem B [22] does not extend from bases to FDD's: more precisely, for every $\varepsilon > 0$ the space Z is $(1+\varepsilon)$-isomorphic to an ℓ_1-sum of finite dimensional spaces; however, if $J : Z \to L_1$ denotes the canonical injection, then

$$\|\mathbb{E}^S - J\|_{L(Z,L_1)} \geq 1$$

where $\mathbb{E}^S$ denotes the conditional expectation with respect to the σ-algebra generated by an arbitrary measurable partition. A similar construction provides in [74] an example of a Banach space which isometrically embeds into L^p but not into L^r with $p < r < 1$, namely the span in L^p of a sequence of 1-stable Cauchy independent random variables and of the constant function. Note that the isomorphic embedding into L^r follows from the Maurey–Nikishin factorization theorem.

It is fairly natural that what happens in the Kalton zone $0 \leq p \leq 1$ overspills to $p = 1$. But what could there be at the *left* of the Kalton zone, where $p < 0$? For an answer to this question, we refer to [96, Section 4], and we state that an n-dimensional space X "embeds into L_p" $(-n < p < 0)$ if $\|\cdot\|_X^{-p}$ is a positive definite distribution. With this terminology, every n-dimensional space imbeds into L_{-n+3}. In particular 4-dimensional spaces embed into L_{-1} and this is important since embedding into L_{-1} characterizes intersection bodies [95, 94]. It follows, with more work, that the Buseman–Petty problem: "if K and L are symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap \xi^{\perp}) \leq \mathrm{vol}_{n-1}(L \cap \xi^{\perp})$$

for all $\xi \in S^{n-1}$, does it follow that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$?", has a positive answer if $n \leq 4$ (and a negative answer if $n \geq 5$) [34], [135].

Of course, going left of the Kalton zone has a price. When $p \in [0, 1)$, we loose the dual space. But when $p < 0$, the space itself disappears.

3 Non-linear geometry

Banach spaces are in particular metric spaces, and it is natural to wonder if two spaces are linearly isomorphic when they are Lipschitz isomorphic. We refer to [6, Chapter 7] for an account of this part of non-linear functional analysis.

It turns out that local theory provides non-trivial Lipschitz invariants [118, 27, 48] which permit to show that the moduli of uniform convexity or uniform smoothness are invariant under Lipschitz (and even uniform) homeomorphisms. Of course, alternative Lipschitz invariants are needed for dealing with non super-reflexive spaces, and it is shown in [38] and [52] that asymptotic structures can help.

Up to the notation, the following *modulus of asymptotic smoothness* is defined in [106]: if $x \in S_X$ and $Y \subseteq X$ is a closed linear subspace of X, let

$$\overline{\rho}(x, Y, \tau) = \sup\left\{\|x + y\| - 1; y \in Y, \|y\| = \tau\right\}$$

and let

$$\overline{\rho}(\tau) = \sup_{x \in S_X} \Big[\inf_{\substack{Y \subseteq X \\ \dim(X/Y) < \infty}} (\overline{\rho}(x, Y, \tau))\Big].$$

The space X is *asymptotically uniformly smooth* (in short, a.u.s.) if

$$\lim_{\tau \to 0} \frac{\overline{\rho}(\tau)}{\tau} = 0.$$

It is clear that uniformly smooth spaces are a.u.s. but the converse completely fails: for instance, the natural norm on $c_0(\mathbb{N})$ is such that $\overline{\rho}(\tau) = 0$ for all $\tau \in [0,1]$. This example points towards an interesting feature of the asymptotic modulus: if ℓ_p $(1 \leq p < \infty)$ is equipped with its natural norm then

$$\overline{\rho}_{\ell_p}(\tau) \sim \tau^p$$

and thus it distinguishes between the ℓ_p's when $2 \leq p$, while the modulus of uniform smoothness fails to do that. What happens is that well-chosen finite-codimensional subspaces Y permit to avoid the spherical subspaces provided by Dvoretzky's theorem, which show that the optimally smooth Banach space is ℓ_2, while the optimally a.u.s. Banach space is c_0.

The dual notion reads as follows: the norm of X is a.u.s. if and only if for every $\tau > 0$, there exists $\theta_X(\tau) > 0$ such that

$$\liminf_{n\to\infty} \|x^* + x_n^*\| \geq 1 + \theta_X(\tau)$$

whenever $\|x^*\| = 1, x_n^* \xrightarrow{w^*} 0$ and $\|x_n^*\| \geq \tau$.

Using this duality, the following result follows from [38, Theorem 5.4]:

Theorem 3.1. *Let X be an asymptotically uniformly smooth separable Banach space. If a Banach space Y is Lipschitz-isomorphic to X and $M > d_L(X,Y)$, then there is an equivalent norm on Y such that for all $\tau \in (0,1]$*

$$\overline{\rho}_Y(\tau/4M) \leq 2\overline{\rho}_X(\tau).$$

Here $d_L(X,Y)$ denotes the Lipschitz distance between X and Y.

If $U : X \longrightarrow Y$ is a Lipschitz isomorphism, the norm we need on Y is predual to the equivalent norm defined on Y^* by

$$\|y^*\| = \sup\left\{\frac{y^*(U(x_1) - U(x_2))}{\|x_1 - x_2\|} : x_1 \neq x_2\right\}$$

The rate of change which appears in this definition somehow provides a substitute to the lack of points of differentiability for Lipschitz map between a.u.s. spaces. We note that Theorem 3.1 also has a quite similar uniform version [38, Theorem 5.3] involving a sequence of norms.

The difficult part of the proof of Theorem 3.1 consists into controlling $\theta_Y(\tau)$ from below, since the natural "adjoint" of the Lipschitz map $U : X \longrightarrow Y$ maps the space $\mathrm{Lip}(Y)$ of real-valued Lipschitz functions on Y to $\mathrm{Lip}(X)$, but does not map Y^* to X^*. A proper use of the Gorelik principle ([42]; see [38, Prop. 5.1]) overcomes this obstruction by showing that if X_0 is a finite-codimensional subspace of X, then $U(B_{X_0})$ asymptotically norms the weak*-null sequences in Y^*. This principle therefore says that Lipschitz isomorphisms retain some restricted form of weak continuity, although they are of course not weakly continuous in general.

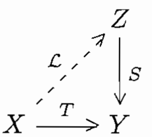
The quantitative Lipschitz invariance of a.u.s. norms leads (through [93] and [38]) to the Lipschitz invariance of Szlenk indices – at least when they are minimal, see [23] for more – and it also applies to the "optimal" case: indeed it is shown in [37] that a separable Banach space X is isomorphic to a subspace of $c_0(\mathbb{N})$ if and only if it has an equivalent norm such that $\rho_X(\tau_0) = 0$ for some $\tau_0 > 0$. Then Theorem 3.1 implies that the class of subspaces of $c_0(\mathbb{N})$ is stable under Lipschitz isomorphisms, from which it follows that if X is Lipschitz-isomorphic to $c_0(\mathbb{N})$ then it is actually linearly isomorphic to $c_0(\mathbb{N})$. [37, Theorem 2.2].

This looks like the right place to state an important question.

Open problem. *Let X and Y be two Lipschitz-isomorphic separable Banach spaces. Are X and Y linearly isomorphic?*

This problem is open even if $X = \ell_1(\mathbb{N})$, or if X and Y are assumed to be super-reflexive. Counterexamples are available in the non-separable case [1], and the relevance of separability is displayed in [36]. Let $\mathrm{Lip}_0(X)$ be the space of real-valued Lipschitz functions on a Banach space X which vanish at 0, and let $\mathfrak{F}(X)$ be the natural predual of $\mathrm{Lip}_0(X)$, whose weak*-topology coincide on the unit ball of $\mathrm{Lip}_0(X)$ with the pointwise convergence on X. The Dirac map $\delta : X \longrightarrow \mathfrak{F}(X)$ defined by $\langle g, \delta(x)\rangle = g(x)$ is an isometric (non-linear!) embedding from X to a subset of $\mathfrak{F}(X)$ which generates a dense linear subspace. This Dirac map has a linear left inverse $\beta : \mathfrak{F}(X) \longrightarrow X$ which is the quotient map such that $x^*(\beta(\mu)) = \langle x^*, \mu\rangle$ for all $x^* \in X^*$; in other words, β is the extension to $\mathfrak{F}(X)$ of the barycenter map. This setting provides canonical examples of Lipschitz-isomorphic spaces. Indeed, if we let $Z_X = Ker(\beta)$, it follows easily from $\beta\delta = Id_X$ that $Z_X \oplus X = \mathfrak{G}(X)$ is Lipschitz-isomorphic to $\mathfrak{F}(X)$.

Following [36], let us say that a Banach space X has the *lifting property* if when Y and Z are Banach spaces and $S : Z \to Y$ and $T : X \to Y$ are continuous linear maps, the existence of a Lipschitz map $\mathcal{L}$ such that $T = S\mathcal{L}$ implies the existence of a continuous linear operator L such that $T = SL$.



A diagram-chasing argument shows that $\mathfrak{G}(X)$ is linearly isomorphic to $\mathfrak{F}(X)$ if and only if X has lifting property.

It turns out that non-separable reflexive spaces, and also the spaces $\ell_\infty(\mathbb{N})$ and $c_0(\Gamma)$ when Γ is uncountable, fail the lifting property and this provides canonical examples of pairs of Lipschitz but not linearly isomorphic spaces. On the other hand, every *separable* space X has the lifting property: to prove it, one can pick a Gaussian measure γ whose support is dense in X and use the fact that $(\mathcal{L} * \gamma)$ is weakly Gâteaux-differentiable. Then $L = (\mathcal{L} * \gamma)'(0)$ satisfies $T = SL$.

The lifting property for separable spaces forbids the existence of canonical pairs of non-isomorphic separable spaces, but on the other hand it leads to

Theorem 3.2. *Let X be a separable Banach space. If there exists an isometric embedding from X into a Banach space Y, then Y contains a linear subspace which is isometric to X.*

Indeed a theorem due to Figiel [32] states that if $J : X \longrightarrow Y$ is an isometric embedding such that $J(0) = 0$ and $\overline{\text{span}}[J(X)] = Y$ then there is a linear quotient map with $\|Q\| = 1$ and $QJ = Id_X$, and then one applies the lifting property with $J = \mathcal{L}, S = Q$ and $T = Id_X$.

Theorem 3.2 does not extend to the non-separable case: if X is reflexive and non-separable then X embeds isometrically but not linearly into $\mathfrak{F}(X)$ [36, Proposition 4.1].

Non-linear theory is also an invitation to enter the Kalton zone $0 < p < 1$, following [73]. If $(X, \|\cdot\|)$ is a metric space and $\omega : [0, \infty) \longrightarrow [0, \infty)$ is a subadditive function such that $\lim_{t\to 0} \omega(t) = \omega(0) = 0$ and $\omega(t) = t$ if $t \geq 1$, then the space $\text{Lip}_\omega(X)$ of $(\omega \circ d)$-Lipschitz functions on X which vanish at 0 has a natural predual denoted $\mathfrak{F}_\omega(X)$, and the barycentric map $\beta_\omega : \mathfrak{F}_\omega(X) \longrightarrow X$ (whose adjoint is the canonical embedding from X^* to $\mathfrak{F}_\omega(X)$) is still a linear quotient map such that $\beta_\omega\delta = Id_X$. However, the Dirac map $\delta : X \longrightarrow \mathfrak{F}_\omega(X)$ is now uniformly continuous with modulus ω – e.g. α-Hölder when $\omega(t) = \max(t^\alpha, t)$ with $0 < \alpha < 1$.

Uniformly continuous functions usually fail the differentiability properties that Lipschitz functions may enjoy, and thus one can expect that this part of the theory is more "distant" from the linear theory than the Lipschitz one. It is indeed so, and [73, Theorem 4.6], reads as follows.

Theorem 3.3. *If ω satisfies $\lim_{t\to 0} \omega(t)/t = \infty$, then $\mathfrak{F}_\omega(X)$ is a Schur space – that is, weakly convergent sequences in X are norm convergent.*

It follows from Theorem 3.3 that the uniform analogue of the lifting property fails unless X has the (quite restrictive) Schur property. Moreover, $\mathfrak{F}_\omega(X)$ is (3ω)-uniformly homeomorphic to $[X \oplus \text{Ker}(\beta_\omega)]$ and as soon as $\lim_{t\to 0} \omega(t)/t = 0$ and X fails the Schur property we obtain canonical pairs of uniformly homeomorphic separable Banach spaces which are not linearly isomorphic. We refer to [120, 53] for other examples of such pairs.

Along with Hölder maps between Banach spaces, one may as well consider Lipschitz map between quasi-Banach spaces, and this is done in [2] where similar methods provide examples of separable quasi-Banach spaces which are Lipschitz but not linearly isomorphic.

It is also shown in [2, Theorem 3.2] that Theorem 3.2 drastically fails for quasi-Banach spaces: if $0 < p < 1$, there exists separable p-normed quasi-Banach spaces X and Y such that X embeds isometrically in Y, but if $T : X \longrightarrow Y$ is a continuous linear operator then $T = 0$. The proof relies in particular on the previously seen fact that if $0 < p < 1$, the space $X = L_p(\mathbb{T})/H^p(\mathbb{T})$ is not "natural" in Kalton's sense [62].

We conclude this section by mentioning a very recent embedding result which provides the optimal quantitative version of Aharoni's embedding theorem: if M is a separable metric space, there exists a 2-Lipschitz embedding from M into $c_0(\mathbb{N})$ and the constant 2 is best possible [75]. It is not known if a separable Banach space X which contain a Lipschitz copy of every separable metric space (or equivalently, a Lipschitz copy of $c_0(\mathbb{N})$) contains a closed subspace isomorphic to $c_0(\mathbb{N})$. However, differentiation shows that such an X fails the Radon–Nikodym property, and it is also known that $c_0(\mathbb{N})$ does not embed uniformly or coarsely into a reflexive space.

Embedding spaces, trees or groups in the Hilbert space, in a Lipschitz, uniform or coarse way, has recently become a major field of research, which happens to be related with deep geometric questions investigated in particular by M. Gromov and A. Naor. In order to remain within our present topic, let us simply mention the article [55] where it is shown that if $p > 2$, the space l_p does not coarsely embed into a Hilbert space.

4 Isometric theory

It is sometimes useful to work with special norms on Banach spaces: they might be canonical or easy to compute, or they can be tightly related with the structure of operators of the space, or they can provide isomorphic information on the space.

All this motivates the investigation of isometric theory, that is, the study of Banach spaces equipped with a given norm.

It should be pointed out that the real and complex isometric theory are quite different. On a complex Banach space X, one can define the notion of a *Hermitian* linear operator T by: $\|e^{ixT}\| = 1$ for all $x \in \mathbb{R}$. Let us say that $x \in X$ is Hermitian if there exists $x^* \in X^*$ such that $(x^* \otimes x)$ is a Hermitian operator. It is not difficult to check that a projection P is Hermitian if and only if $\|P + \lambda(I - P)\| = 1$ for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. In other words, Hermitian projections are "orthogonal". It follows that a complex Banach space with a 1-unconditional basis is the closed linear span of its Hermitian elements. A remarkable result of Kalton and Wood ([92]; see [126]) states the converse:

Theorem 4.1. *A complex Banach space X which is the closed linear span of its Hermitian elements has a* 1*-unconditional basis.*

An important corollary of Theorem 4.1 is that if X is a 1-complemented subspace of a complex Banach space Y with a 1-unconditional basis, then X has a 1-unconditional basis: indeed let us write

$$I_Y = \sum_{j=1}^{\infty} y_j^* \otimes y_j$$

where (y_j) is a 1-unconditional basis of Y, and let $P : Y \longrightarrow X$ be such that $P^2 = P$ and $\|P\| = 1$. If $T_j = y_j^* \otimes P(y_j)$ then $x = \sum_{j=1}^{\infty} T_j(x)$ for all $x \in X$ and

$$\|\sum_{j=1}^{\infty} \alpha_j T_j(x)\| \leq \|x\|$$

if $|\alpha_j| = 1$ for all j. But if $\|x^*\| = \|x\| = x^*(x) = 1$, we have

$$1 = \sum_{j=1}^{\infty} \langle x^*, T_j(x) \rangle$$

and

$$\left|\sum_{j=1}^{\infty} \alpha_j \langle x^*, T_j(x)\rangle\right| \leq 1$$

if $|\alpha_j| = 1$, and this implies that $\langle x^*, T_j(x)\rangle$ is real (and ≥ 0) for all j. It follows that $P(y_j) \in X$ is Hermitian for all j, and Theorem 4.1 concludes the proof.

It should be noted that in the *real* case, the existence of 1-unconditional bases does not pass to 1-complemented subspaces [101]. We refer to [117] for important results on 1-complemented subspaces of spaces with 1-unconditional bases. The Kalton–Wood theorem is one of the few available positive results, while the following question is still open:

Open problem. *Let X be a complemented subspace of a space Y with an unconditional basis. Does X have an unconditional basis?*

A negative answer looks plausible, since for instance it is observed in [14, §4] that a space with BAP but no FDD constructed by C. Read (unpublished) is complemented in a space with an unconditional FDD. On the other hand [14] contains a major positive result.

Theorem 4.2. *Let X be a separable Banach space. If X has the metric approximation property (MAP) then X has the commuting metric approximation property (CMAP).*

The proof goes as follows: if X has MAP then there exists a sequence of finite rank operators (T_n) such that $T_k T_n = T_n$ if $k > n$ and $\|T_n\| \leq 1 + \varepsilon_n$ with $\sum \varepsilon_n = \beta < \infty$. For $t > 0$ and $n \geq 1$, let

$$V_n(t) = e^{-nt} \exp(t \sum_{k=1}^{n} T_k).$$

Then $\|V_n(t)\| \leq e^{\beta t}$ and

$$\lim_{n\to\infty} V_n(t)x = S(t)x$$

exists for all $x \in X$. The operator $S(t)$ is compact for all $t > 0$, $\lim_{t\to 0} \|S(t)\| = 1$ and $\lim_{t\to 0} S(t)x = x$ for all $x \in X$. Now $S(t + s) = S(t)S(s)$ and thus of course $S(t)S(s) = S(s)S(t)$. Now the compact operators $S(1/n)$ can be approximated by finite rank operators which "nearly commute" and the conclusion follows through a perturbation argument. □

An alternative proof of Theorem 4.2 (with a slightly better control on the commuting sequence) is given in [35] for separable spaces which do not contain $\ell_1(\mathbb{N})$, and it is also shown there that the unconditional MAP is equivalent to its commutative version for all separable spaces (with a simpler proof in the complex spaces, using the Hermitian trick which is displayed above).

The following important problem is, however, still open.

Open problem. *Let X be a separable Banach space with the bounded approximation property. Does X have the commuting bounded approximation property?*

By Theorem 4.2 an equivalent formulation of the problem is to know if a Banach space with BAP has an equivalent norm with MAP. It follows from results of Casazza [11] and Johnson [51] that still another formulation of this problem is to know if every "π-space" (that is, every space which has BAP with a sequence of finite rank *projections*) has a FDD. We refer to [12] for an updated survey on approximation properties.

An isometric concept which turned out to be very useful was defined by Alfsen and Effros in 1972 [3]: a closed subspace X of a Banach space Y is called an *M-ideal* in Y if there is a subspace V of Y^* such that

$$Y^* = V \oplus_1 X^\perp$$

where $\oplus_1$ means that $\|y^* + z^*\| = \|y^*\| + \|z^*\|$ for all $y^* \in V$ and all $z^* \in X^\perp$. We refer to [46] for the theory as it was in 1993, immediately after Kalton's breakthrough [68].

Although the notion of M-ideal is independent of any algebraic structure, it turns out to be tightly related with the notion of "ideal" from operator theory, and in fact ideals $K(X)$ of compact operators in spaces $L(X)$ of bounded operators provide a wealth of examples of M-ideals (see [46, Chapters V and VI]). It is shown in [68] that if X is a separable Banach space then $K(X)$ is an M-ideal in $L(X)$ (in short, $K(X) \subset_M L(X)$) if and only if there is a sequence (K_n) of compact operators such that

$$\lim_{n\to\infty} \|x - K_n(x)\| = 0$$

$$\lim_{n\to\infty} \|x^* - K_n^*(x^*)\| = 0$$

for all $x \in X$ and $x^* \in X^*$, such that

$$\lim_{n\to\infty} \|I - 2K_n\| = 1$$

and moreover X has *Propery (M)*, which means that for any sequence $(x_n) \subseteq X$ with $w - \lim_{n\to\infty}(x_n) = 0$, one has

$$\overline{\lim}\|x + x_n\| = \overline{\lim}\|y + x_n\|$$

for all $(x, y) \in X^2$ with $\|x\| = \|y\| = 1$. In other words, the norm of X is "asymptotically isotropic" and all vectors of the sphere "look the same" when seen from infinity. It is shown in this same paper that property (M) forbids distortion: if X has (M), there is some $p \in [1, +\infty]$ such that for every $\varepsilon > 0$, X contains a subspace $(1+\varepsilon)$-isomorphic to ℓ_p if $p < \infty$ and to c_0 if $p = \infty$.

Note that it is obvious that ℓ_p spaces equipped with their natural norms have (M), while L_p spaces fail to have it: this is related to the fact that they contain L_2 – and incidentally to the fact that the "asymptotic modulus of smoothness" (see Section 3 above) is better for ℓ_p than for L_p $(p > 2)$. In fact, if X is a separable order-continuous nonatomic Banach lattice and X has an equivalent norm with (M) then X is lattice-isomorphic to L_2 [68], and this is in particular the case when X is isomorphic to a subspace of an Orlicz sequence space h_F. In the article [4], twisted sums meet property

(M) and it is shown in particular that the Kalton–Peck space Z_2 has an equivalent norm with (M). On the other hand, it is shown in the recent work [24] that norms with (M) are optimally asymptotically uniformly smooth among all equivalent norms on a given Banach space. Since Theorem 3.1 above asserts that the corresponding modulus is a Lipschitz invariant, it follows for instance that Lipschitz-isomorphic Orlicz spaces contain the same l_p spaces.

The main result of [68] was soon improved in [91] where it was shown that the unconditionality condition (3) actually follows from the isotropy condition (M). Thus [91, Theorem 2.13] reads:

Theorem 4.3. *Let X be a separable Banach space. Then $K(X)$ is an M-ideal in $L(X)$ if and only if X does not contain a copy of ℓ_1, X has (M) and X has the metric compact approximation property.*

In this same paper, property (M) which, as seen before, yields to "discrete" spaces when applied to subspaces of L_p, is used to prove [91, Theorem 4.4].

Theorem 4.4. *Let $1 < p < \infty, p \neq 2$, and X be a closed infinite-dimensional subspace of L_p. Then B_X is $\|\cdot\|_1$-compact if and only if for any $\varepsilon > 0$, there is a subspace X_ε of ℓ_p such that $d(X, X_\varepsilon) < 1 + \varepsilon$.*

Note that if $p > 2$ the assumption amounts to say that X contains no subspace isomorphic to ℓ_2, and Theorem 4.4 improves a result of [54]. We mention at this point that it has been recently shown in [47] that if $p > 2$, $X \subseteq L_p$ and $\ell_p(\ell_2)$ does not embed in X, then X embeds into $\ell_p \oplus \ell_2$.

Theorem 4.4 has been pushed to the case $p = 1$ in [39], to characterize subspaces of L_1 which ε-embed into ℓ_1, but the non-reflexivity of L_1 (and the non-convexity of L_p if $p < 1$) somehow complicates the matter, and [39, Theorem 3.3] states that if X is a closed subspace of L^1 with the approximation property, then B_X is L_p-compact locally convex $(0 \leq p < 1)$ if and only if for every $\varepsilon > 0$, there is a quotient space E_ε of c_0 such that $d(X, E_\varepsilon^*) < 1 + \varepsilon$. We already saw in Section 2 above that the "Ribe–Roberts" approach provides examples where B_X is L_p-compact with $p < 1$ (or equivalently, τ_m-compact) but not locally convex.

Following [14], we say that $X \subseteq Y$ is a *u-ideal* in Y if

$$Y^* = V \oplus X^\perp$$

in such a way that $\|y^* + z^*\| = \|y^* + \lambda z^*\|$ for all $y^* \in V$, $z^* \in X^\perp$ and $|\lambda| = 1$.

The article [41] is devoted to this notion which, thanks again to Hermitian operators, is quite nicer in the complex case. Let us denote $Ba(X)$ the subspace of X^{**} consisting of weak*-limits of weak*-convergent sequences of elements of X. With this notation, it follows from [41, Theorem 6.5] that we have:

Theorem 4.5. *Let X be a separable complex Banach space which is a u-ideal in its bidual. Then there exists an Hermitian projection from X^{**} onto $Ba(X)$. Moreover for every $x^{**} \in Ba(X)$ and every $\varepsilon > 0$, there is a sequence (x_n) in X such that*

$$x^{**} = w^* - \lim_{n\to\infty} \sum_{j=1}^{n} x_j \quad \textit{and} \quad \Big\|\sum_{j=1}^{n} \theta_j x_j\Big\| \leq (1+\varepsilon)\|x^{**}\|$$

for all $n \geq 1$ and all $|\theta_j| = 1$.

If one thinks of $Ba(X)$ as the "band" generated by X in X^{**}, one can express Theorem 4.5 by saying that the embedding of a u-ideal in its bidual looks very much like the embedding of an order-continuous Banach lattice. However, u-ideals (such as $K(X)$ spaces, with X reflexive with the unconditional compact approximation property) bear in general no usable order structure. We refer to [87] for (order) ideal properties of (algebraic) ideals of operators between Banach lattices.

5 Interpolation, twisted sums, and the Kalton calculus

The Banach–Mazur functional d_{BM} is a classical tool for estimating the "distance" between two isomorphic Banach spaces – or equivalently, the distance between two equivalent norms and similar functionals such as the Lipschitz distance d_L can be defined when more general notions of isomorphisms are taken into consideration. In [80], "distances" are defined between spaces which are not isomorphic but are somewhat similar, such as ℓ_p and ℓ_q when p and q are close to each other. If X and Y are two subspaces of a Banach space Z, let $\Lambda(X,Y)$ be the Hausdorff distance between B_X and B_Y, that is

$$\Lambda(X,Y) = \max\big\{ \sup_{x\in B_X} \inf_{y\in B_Y} \|x-y\|, \sup_{y\in B_Y} \inf_{x\in B_X} \|y-x\|\big\}.$$

The Kadets distance $d_K(X,Y)$ is the infimum of $\Lambda(\tilde{X},\tilde{Y})$ over all Banach spaces Z containing isometric copies $\tilde{X}$ and $\tilde{Y}$ of X and Y respectively. The Kadets distance is a pseudo-metric which is controlled from above by d_{BM}, but there are non-isomorphic Banach spaces X and Y such that $d_K(X,Y) = 0$.

The Gromov–Hausdorff distance d_{GH} is the non-linear analogue of the Kadets distance, defined along the same lines, except that the infimum is taken over all metric spaces containing isometric copies of X and Y. Of course, $d_{GH} \leq d_K$ and for instance $d_{GH}(\ell_p, \ell_1) \to 0$ as $p \to 1$ while $d_K(\ell_p, \ell_1) = 1$ for all $p > 1$. However, if X is a $\mathcal{K}$-space then $d_{GH}(X_n, X) \to 0$ implies that $d_K(X_n, X) \to 0$. This can again be understood as an "approximation by linear maps" on $\mathcal{K}$-spaces.

Interpolation theory provides families of Banach spaces which are not isomorphic but tightly related, and Kadets distance will make this remark precise – and usable. Moreover, interpolation leads to a "differential calculus" on the "manifold" of Banach spaces. We will outline how N. Kalton's vision created links between this calculus, twisted sums, and quasi-linear maps.

Complex interpolation studies analytic families of Banach spaces. Let us restrict our discussion to a very important special case: let W be some complex Banach space and let X_0 and X_1 be two closed subspaces of W. We denote

$$S = \{z \in \mathbb{C}; 0 < \mathrm{Re}(z) < 1\}$$

and $\mathfrak{F}$ is the space of analytic functions $F : S \longrightarrow W$ which extend continuously to $\bar{S}$ and such that $\{F(it); t \in \mathbb{R}\}$ is a bounded subset of X_0 and $\{F(1+it); t \in \mathbb{R}\}$ is a bounded subset of X_1. The space $\mathfrak{F}$ is normed by

$$\|F\|_{\mathfrak{F}} = \max_{j=0,1} \sup\{\|F(j+it)\|_{X_j}; t \in \mathbb{R}\}.$$

For $\theta \in (0,1)$ and $x \in W$, we define

$$\|x\|_\theta = \inf\{\|F\|_{\mathfrak{F}}; F(\theta) = x\}$$

and

$$X_\theta = \{x \in W; \|x\|_\theta < \infty\}$$

If $W_0 = \mathrm{span}\{X_\theta; \theta \in (0,1)\}$, a linear map $T : W_0 \longrightarrow W_0$ is called *interpolating* if $F \longmapsto T \circ F$ is defined and bounded on $\mathfrak{F}$. If T is interpolating, then $T(X_\theta) \subseteq X_\theta$ for all $\theta \in (0,1)$.

The above space $X_\theta = [X_0, X_1]_\theta$ is said to be obtained from X_0 and X_1 by the *complex interpolation method* and one has [80] for $0 < \theta < \phi < 1$

$$d_K(X_\theta, X_\phi) \leq 2 \frac{\sin[\pi \frac{(\phi-\theta)}{2}]}{\sin[\pi \frac{(\phi+\theta)}{2}]}.$$

This continuity of the interpolation method with respect to the Kadets distance permits to apply connectedness arguments. Indeed, let us call a property (P) *stable* if there exists $\alpha > 0$ so that if X has (P), and $d_K(X,Y) < \alpha$, then Y has (P). For instance, each of the following properties (P) is stable: separability, reflexivity, $X \supseteq \ell_1$, super-reflexivity, type$(X) > 1$. Connectedness thus shows that if $0 < \theta < 1$ and $X_\theta = [X_0, X_1]_\theta$ has (P), then X_φ has (P) for every $\varphi \in (0,1)$.

This line of thought opens an exciting field of research. It can be shown that the connected component of any separable Banach space X contains all isomorphic copies of X. It follows from [111] that the connected component of ℓ_2 contains all super-reflexive Banach lattices, and it is not known whether it contains all super-reflexive spaces. It is conjectured that the component of c_0 consists of all spaces isomorphic to a subspace of c_0.

These concepts are also relevant to non-linear isomorphisms: it follows from instance from Sobczyk's theorem that if $d_{GH}(X_n, c_0) \to 0$ then we have not only that $d_K(X_n, c_0) \to 0$ (since c_0 is a $\mathcal{K}$-space), but actually $d_{BM}(X_n, c_0) \to 0$ [80]. This implies for instance that if the uniform distance between X and c_0 is small then X is linearly isomorphic to c_0 [38, Theorem 5.7]. It is not known whether a space which is uniformly homeomorphic to c_0 is linearly isomorphic to c_0.

The Kadets and Gromov–Hausdorff distances clearly are topological notions, but interpolation points to some kind of differential structure, which we now outline.

Minimal extensions have been displayed in Section 2. Let us say more generally that if X, Y and Z are quasi-Banach spaces, Z is an extension of X by Y if

$$Z/Y \simeq X.$$

An extension Z is also called a *twisted sum* of X and Y (a non-trivial twisted sum if Y is not complemented in Z), and we refer to [17] for a comprehensive survey of this matter.

A map $\Omega : X \longrightarrow Y$ is called *quasi-linear* (see Section 2) if $\Omega(\lambda x) = \lambda\Omega(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$, and if there is $C > 0$ such that

$$\|\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2)\| \leq C(\|x_1\| + \|x_2\|)$$

for all $x_1, x_2 \in X$. The extension $X \oplus_\Omega Y$ of X by Y is the space $X \oplus Y$ equipped with the quasi-norm

$$\|(x, y)\| = \|x\| + \|y - \Omega(x)\|$$

Even when X and Y are Banach spaces $X \oplus_\Omega Y$ is not, unless Ω actually satisfies for all $n \geq 1$ and all (x_k)

$$\|\sum_{k=1}^{n} \Omega(x_k) - \Omega(\sum_{k=1}^{n} x_k)\| \leq C \sum_{k=1}^{n} \|x_k\|.$$

It will always be so when X is a $\mathcal{K}$-space [59]. Every extension can actually be obtained with such an Ω: if $q : Z \longrightarrow X$ is the quotient map. Take $\Omega = S - R$ where $qS = qR = Id_X$, S is homogenous (but not necessarily linear) such that $\|S(x)\| \leq 2\|x\|$, and R is linear (but not necessarily continuous). As in the case of minimal extensions, the existence of a bounded linear projection from $X \oplus_\Omega Y$ onto Y is equivalent to the existence of a linear map $L : X \longrightarrow Y$ such that

$$\|\Omega(x) - L(x)\| \leq C\|x\|$$

for all $x \in X$. When $X = Y$, the space $X \oplus_\Omega X$ is called a *self-extension* of X and it is denoted by

$$X \oplus_\Omega X = d_\Omega X.$$

When $X = \ell_2$, a non-trivial self-extension of ℓ_2 is called a twisted Hilbert space, and it was shown in [28] that such spaces exist. An alternative example, the Kalton–Peck space Z_2, is constructed in [81] with the help of the Ribe functional (see Section 2): let $\Omega = \ell_2 \longrightarrow \mathbb{R}^{\mathbb{N}}$ be defined by

$$\Omega((\xi_n)) = \left(\xi_n \log(\frac{|\xi_n|}{\|\xi\|_2})\right)_{n\geq 1}$$

(and $\Omega(0) = 0$). The space $Z_2 = d_\Omega \ell_2$ is therefore the space of pairs of sequences $((\xi_n), (\eta_n))$ such that

$$\|(\xi, \eta)\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \left|\eta_n - \xi_n \log \frac{|\xi_n|}{\|\xi\|_2}\right|^2\right)^{\frac{1}{2}} < \infty.$$

Of course Z_2 is a Banach space since ℓ_2 is a $\mathcal{K}$-space. The Kalton–Peck space Z_2 exhibits remarkable features, which are not yet fully understood although that space was constructed 30 years ago. It is plain that Z_2 has an unconditional FDD consisting of 2-dimensional spaces; however it has no unconditional basis and no local unconditional structure [53]. Actually, an unconditional FDD with spaces of bounded dimension provides an unconditional basis which can be chosen from the subspaces *if* the space has local unconditional structure [15]. It is unknown, however, if a twisted Hilbert space can have local unconditional structure; the best result so far is that it has no unconditional basis in full generality [72]. The space Z_2 is also an example of a symplectic Banach space which is not the direct sum of two isotropic subspaces [88]. In fact, intuition suggests that the space Z_2 is "even-dimensional" and thus that it should not be isomorphic to its hyperplanes: this 30-years old conjecture is still open, although examples of infinite-dimensional Banach spaces which are not isomorphic to their hyperplanes are now known [43]. In fact, there are even (necessarily non-separable) $C(K)$ spaces which are indecomposable and not isomorphic to their hyperplanes [98, 114]. These spaces are obtained by techniques which are completely independent from the Gowers–Maurey approach. The complexity of their construction relies on the underlying compact space K, which is obtained through an elaborate transfinite induction. Coming back to Z_2, we note that spaces with 2-dimensional unconditional FDD but no unconditional basis show up in the classification results shown in [97], which play a crucial role in Gowers' homogeneous space theorem [44].

As so often in N. Kalton's work, the conceptual frame in which the construction is completed provides flexibility and leads to more results. If $F : \mathbb{R} \longrightarrow \mathbb{C}$ is any Lipschitz map and E is a Banach sequence space, let

$$\Omega_F(\xi) = \big(\xi_n F(\log \frac{|\xi_n|}{\|\xi\|_E})\big)_{n\geq 1}$$

and let

$$d_{\Omega_F} E = E \oplus_{\Omega_F} E.$$

Taking $E = \ell_2$ and $F(t) = t^{1+i\alpha}$ $(\alpha \neq 0)$ provides a complex Banach space $Z(\alpha)$ which is not complex-isomorphic to its conjugate space $\overline{Z(\alpha)} = Z(-\alpha)$. The existence of such spaces had been shown in [8] and [128] by probabilistic methods. We refer to [30] for recent work on this topic, and the existence of Banach spaces with exactly n complex structures, for any given integer n, and also to [31].

The notation $d_\Omega X$ is reminiscent of differential calculus, and this is not a chance. With the above notation of the complex interpolation method, and following [125], we define a derived space $dX_\theta \subseteq W \times W$ by $dX_\theta = \{(x_1, x_2) : \ \|(x_1, x_2)\|_{dX\theta} < \infty\}$ where

$$\|(x_1, x_2)\|_{dX_\theta} = \inf\big\{\|F\|_{\mathfrak{F}} : \ F(z) = x_1, F'(z) = x_2\big\}.$$

The space $Y = \{(x_1, x_2) \in dX_\theta : \ x_1 = 0\}$ is isometric to X_θ and so is dX_θ/Y. Hence dX_θ is a self-extension of X_θ. By the above, one has

$$dX_\theta = d_\Omega X_\theta$$

for some quasi-linear map $\Omega : X_\theta \longrightarrow W$ and $\Omega(x) = F'(z)$, where $F \in \mathfrak{F}$ is such that $\|F\|_{\mathfrak{F}} \leq C\|x\|_\theta$ and $F(z) = x$, does the work. Now, if T is an interpolating operator then $(x_1, x_2) \longrightarrow (Tx_1, Tx_2)$ is bounded on dX_θ and this translates into "commutator estimates":

$$\|T(\Omega(x)) - \Omega(T(x))\|_\theta \leq C\|x\|_\theta$$

for all $x \in X_\theta$.

The sequence spaces ℓ_p $(1 \leq p \leq \infty)$ provide a first example of interpolation line and the above calculations applied to $X_0 = \ell_1$ and $X_1 = \ell_\infty$ provide the Kalton–Peck space

$$Z_2 = dX_{1/2}.$$

Similar calculations are possible for the function spaces $L_p(\mathbb{T})$. For this interpolation scale, the Hilbert transform H is a very important example of interpolating operator and the Rochberg–Weiss commutator estimate becomes in this case

$$\|H(f \log |f|) - H(f) \log |H(f)|\|_p \leq C_p\|f\|$$

for $1 < p < \infty$ and some $C_p < \infty$.

Following [67] and [64], we now relate this differential calculus with entropy functions (see Section 2). If X is a sequence space, its entropy function Φ_X is defined for positive sequences u by

$$\Phi_X(u) = \sup_{\|x\|_X \leq 1} \sum_{k=1}^{\infty} u_k \log |x_k|.$$

If X_0 and X_1 are separable sequence spaces, the interpolation spaces X_θ are given by the Calderon formula $X_\theta = X_0^{1-\theta} X_1^\theta$, that is:

$$\|x\|_\theta = \inf\{\|x_0\|_0^{1-\theta}\|x_2\|_1^\theta \ ; \ |x| = |x_0|^{1-\theta}|x_1|^\theta\}$$

and the entropy function Φ_X can conveniently be described as the *logarithm* of the sequence space X. Indeed one has

$$\Phi_{X_\theta} = (1-\theta)\Phi_{X_0} + \theta\Phi_{X_1}$$

and by the Lozanovsky factorization theorem,

$$\Phi_X + \Phi_{X^*} = \Phi_{\ell_1}$$

where Φ_{ℓ_1} is the Ribe functional while (see Section 2)

$$\Phi_{\ell_p} = \frac{1}{p}\Phi_{\ell_1}$$

and $\Phi_{\ell_\infty} = 0$. It now becomes natural to see the Hilbert space as the geometric mean between any sequence space X and its dual X^*. The map "$X \mapsto \Phi_X$" is logarithmic-like, but in order to complete the picture we conversely need an *exponential* function which maps a quasi-linear map Φ to a Banach space. If $\Phi : c_{00}^{+} \longrightarrow \mathbb{R}$ is any functional, there exists a Banach sequence space X such that $\Phi_X = \Phi$ if and only if Φ and $(\Phi_{\ell_1} - \Phi)$ are convex functions and Φ is positively homogeneous [67], and the space X has unit ball

$$B_X = \big\{(x_k);\ \sum_{k=1}^{\infty} u_k \log|x_k| \leq \Phi(u) \text{ for all } u \geq 0\big\}.$$

This exponential map leads to what I suggest to call the *Kalton calculus*, which bears an uncanny resemblance with the exponentation from a Lie algebra to its Lie group, and creates "lines" from infinitesimals; in other words, yields to *extrapolation*. For instance, if X is p-convex $(1 < p < 2)$ and p^*-concave, then $X = Y^{1/p}$ for some sequence space Y and so $(\frac{1}{p}\Phi_{\ell_1} - \Phi_X)$ is convex. Similarly p^*-concavity means that $(\frac{1}{p}\Phi_{\ell_1} - \Phi_{X^*})$ is convex. Now the equation

$$\begin{aligned} \Phi_X &= (1-\theta)\Phi + \theta\Phi_{\ell_2} \\ &= (1-\theta)\Phi + \frac{\theta}{2}\Phi_{\ell_1} \end{aligned}$$

provides a convex function Φ such that $(\Phi_{\ell_1} - \Phi)$ is also convex and thus $\Phi = \Phi_Z$ for some Z. Exponentiating, we find $X = Z^{1-\theta}\ell_2^{\theta}$ (a result from [111]).

To close the circle of ideas relating the entropy functions with derived spaces, we note that if $X_\theta = X_0^{1-\theta}X_1^{\theta}$ then $dX_\theta = d_\Omega X_\theta = X_\theta \oplus_\Omega X_\theta$ where the quasi-linear map Ω satisfies

$$|\langle x^*, \Omega(x)\rangle - \Phi(xx^*)| \leq C\|x\|_{X_\theta}\|x^*\|_{X_\theta^*}$$

where $\Phi = \Phi_{X_1} - \Phi_{X_0}$ and (xx^*) denotes the pointwise product of the sequences x and x^*.

Special properties of the derived space $d_\Omega X_\theta$ can "spread out" by exponentation to a segment $\{X_\varphi;\ |\varphi - \theta| < \varepsilon\}$. Indeed, if X_0 and X_1 are acceptable function spaces on $\mathbb{T}$ and R is the vector-valued Riesz transform, then R is bounded on X_θ for $|\theta - \theta_0| < \delta$ if and only if $\|R\Omega - \Omega R\|_{X_{\theta_0}} < \infty$. It follows that there exist twisted Hilbert spaces which are not UMD [67] although Z_2 is UMD [63]. We note at this point that higher order derivatives can be considered, and this has been done e.g. in [10, 106, 124].

As seen before, differentiating interpolation lines yield quasi-linear maps Ω such that $dX_\theta = d_\Omega X_\theta$. If for instance X_1 is obtained from X_0 through a change of weight, the map Ω enjoys a commutation property, namely

$$\|\Omega(ax) - a\Omega(x)\|_{X_\theta} \leq C\|a\|_\infty\|x\|_{X_\theta}.$$

These special maps are called *centralizers* in [64] and the corresponding space $d_\Omega X_\theta$ is a *lattice twisted sum*.

Centralizers yield to extrapolation results: if for instance X is a super-reflexive sequence space and Ω is a real centralizer on X, then there exist super-reflexive Banach sequence spaces X_0 and X_1 such that $X = X_0^{1/2}X_1^{1/2} = X_{1/2}$ and moreover $dX_{1/2} \simeq d_\Omega X$. Note that the Kalton calculus, which we displayed here (following [79]) for sequences, is designed for function spaces and this is what is done in [67] and [64].

We now recall the Rochberg–Weiss commutator estimates: if $X_\theta = X_0^{1-\theta} X_1^\theta$ and $dX_\theta = d_\Omega X_\theta$ then

$$\|T(\Omega(x)) - \Omega(T(x))\|_{X_\theta} \leq C\|x\|_{X_\theta}$$

for interpolating operators T. When, for instance, Ω is a centralizer, this estimate says that Ω nearly commutes not only with multiplication operators, but with all interpolating operators.

Now the extrapolation technique allows a change of perspective: starting from an operator T on X, we may consider all pairs (X_0, X_1) such that $X = X_0^{1-\theta} X_1^\theta$ and T is interpolating between X_0 and X_1 and get a whole family of estimates on T. Doing this for $X = \ell_p$ yields to the family of quasi-linear maps

$$\Phi_G(u) = \sum_{n=1}^{\infty} u_n G(\log |u_n|)$$

where G runs through the family $\mathcal{G}$ of 1-Lipschitz maps from $\mathbb{R}$ to $\mathbb{R}$ whose derivative is compactly supported. This leads to considering the quasi-Banach space h_1^{sym} defined by

$$\|\xi\|_{h_1^{\text{sym}}} = \sum_{k=1}^{\infty} |\xi_k| + \sup_{G \in \mathcal{G}} \Phi_G(\xi) < \infty.$$

This "tangent space" h_1^{sym} is conveniently described as the space of sequence (ξ_k) in ℓ_1 such that

$$\sum_{n=1}^{\infty} \frac{1}{n} |\xi_1 + \xi_2 + \cdots + \xi_n| < \infty.$$

The same steps applied to function spaces lead to the symmetric Hardy function space $H^1_{\text{sym}}(\mu)$ of all functions $f \in L^1(\mu)$ such that

$$\|f\|_{H^1_{\text{sym}}} = \int |f| d\mu + \sup_{G \in \mathcal{G}} \int |f| G(\log |f|) d\mu < \infty.$$

Commutator estimates on interpolating operators then show the following theorem [64]:

Theorem 5.1. *Suppose $1 < p_0 < p < p_1 < \infty$ and $p^{-1} + q^{-1} = 1$. Suppose that $T : L_{p_j} \longrightarrow L_{p_j}$ is linear bounded for $j = 0, 1$. Then the bilinear form*

$$B_T(f, g) = f.T^* g - g.Tf$$

is bounded from $L_p \times L_q$ to H^1_{sym}.

The above dot denotes of course the pointwise product of functions. This theorem can be applied to a variety of interpolating operators. When applied to the Riesz projection on $L_2(\mathbb{T})$, it gives when f and g belongs to $H^1(\mathbb{T})$ and $g(0) = 0$ the inequality

$$\|fg\|_{H^1_{\text{sym}}} \leq C\|f\|_2 \|g\|_2$$

and thus since $H_0^1 = H^2.H_0^2$, one gets for every function $h \in H^1$ with $h(0) = 0$

$$\|h\|_{H^1_{\text{sym}}} \leq C\|h\|_1.$$

This result, first shown in [18] and [19], somehow means that functions in $H^1(\mathbb{T})$ have a quite symmetric behavior around their singularities. Conversely, real-valued functions in $H^1_{\text{sym}}(\mathbb{T})$ are real parts of functions in $H^1(\mathbb{T})$.

The ideas developed above have non-commutative analogues, and the bridge which brings to the non-commutative world is the concept of trace. If X is a symmetric Banach sequence space, we denote $\mathcal{C}_X$ the space of all operators T on ℓ_2 whose sequence $(s_n(T))_{n\geq 1}$ of singular numbers belongs to X. When X_0 and X_1 are reflexive then

$$[\mathcal{C}_{X_0}, \mathcal{C}_{X_1}]_\theta = \mathcal{C}_{X_0^{1-\theta}X_1^\theta} = \mathcal{C}_{X_\theta}$$

and interpolation tools apply to the spaces $\mathcal{C}_X$.

Let $\mathcal{C}_{\ell_1} = \mathcal{C}_1$ be the ideal of trace-class (or nuclear) operators on ℓ_2. A *trace* on $\mathcal{C}_1$ is a linear map τ such that $\tau(AB) = \tau(BA)$ for all $A \in \mathcal{C}_1$ and all bounded operators B. We denote $\text{Comm}(\mathcal{C}_1)$ the linear span of all commutators

$$[A, B] = AB - BA$$

with $A \in \mathcal{C}_1$ and B bounded. Clearly, if $S \in \mathcal{C}_1$ then $S \in \text{Comm}(\mathcal{C}_1)$ if and only if $\tau(S) = 0$ for every trace τ. It was shown in [133] that $\text{Comm}(\mathcal{C}_1)$ is strictly contained in $\{T \in \mathcal{C}_1; tr(T) = 0\}$, or equivalently that there exist *discontinuous* traces on $\mathcal{C}_1$. The precise description of $\text{Comm}(\mathcal{C}_1)$ was obtained in [66] by interpolation arguments and it reads as follows:

Theorem 5.2. *Let $T \in \mathcal{C}_1$ be a trace-class operator. Then $T \in$ Comm$(\mathcal{C}_1)$ if and only if its eigenvalue sequence $\lambda_n(T))_{n\geq 1}$ belongs to h^1_{sym}.*

It was shown in [66] that every $T \in \text{Comm}(\mathcal{C}_1)$ is the sum of 6 commutators, but this number has now been put down to 3 and the case of general ideas of operators is also treated in [25, 26, 70]. We refer to [29] for characteristic determinants of trace-class operators and their use in this context.

6 Multipliers and some of their uses

Bases and their various substitutes provide coordinate systems in which some computations can take place, and this is important for understanding the geometry of the Banach spaces these bases generate, and for applied mathematics as well: let us refer for instance to frame theory (see [16, 13] and Casazza's subsequent work), and to wavelets and greedy bases (see [134, 21]).

Bases allow in particular to define "diagonal" operators, shift operators and more generally Toeplitz-like $Q(S)$ operators, where

$$S(\sum_j x_j e_j) = \sum_j x_j e_{j+1}$$

is the shift and $Q(\xi) = \sum_{n\in\mathbb{Z}} a_n \xi^n$ is a formal bilateral series. We stress that such operators can be bounded or not, and their norm can be controlled or not by some norm of Q, and this is where the fun begins. Of course, more general "matrices" can be considered which would define operators, but it seems to be a fact of life that if something significant happens far away from the diagonal, then it is very hard to control.

Let us now see how diagonal operators, when properly used, actually suffice for obtaining important examples. We begin with Hilbertian theory. A linear operator T on a complex Hilbert space H is called *power bounded* if $\sup_{n\geq 0} \|T^n\| < \infty$ and *polynomially bounded* if there is $C < \infty$ such that

$$\|P(T)\| \leq C\|P\|_\infty = C \sup\{\|P(z)\|; |z| \leq 1\}$$

for every $P \in \mathbb{C}[\xi]$. J. von Neumann's classical inequality shows that every operator T which is similar to a contraction (that is, there exists A invertible such that $\|A^{-1}TA\| \leq 1$) is polynomially bounded, and an example due to Pisier (see [112]) shows that the converse is false. A weaker requirement would be to show that T is similar to an operator such that $\sup_n \|T^n\|$ is close to 1 if it is power-bounded, and this question is asked in [110]. Note that a classical theorem of Rota ([127]) shows that power-bounded operators are similar to operators of norm close to 1.

Basis theory joins forces with harmonic analysis to provide a negative answer to Peller's question in [78]. A *weight* w on $\mathbb{T}$ is a non-zero function from $L_1(\mathbb{T})$ with $w \geq 0$. We denote by $L_2(w)$ the corresponding weighted L_2-space, and

$$H^2(w) = \overline{\text{span}}\{e^{in\theta}; n \geq 0\} \subseteq L_2(w).$$

The Riesz projection is formally defined on $L_2(w)$ by

$$R(\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta}) = \sum_{n\geq 0} \widehat{f}(n)e^{in\theta}$$

and w is called an *A_2-weight* if R is a bounded projection from $L_2(w)$ onto $H^2(w)$. If $\|R\|_w$ denotes the norm of the Riesz projection and $0 \leq \varphi < \pi/2$, then [78, Proposition 2.2] proves the following Helson–Szegö estimate: $\|R\|_w \leq (\cos(\varphi))^{-1}$ if and only if there exists $h \in H^1(\mathbb{T})$ such that $|w - h| \leq w \sin(\varphi)$.

If $\alpha \in (0, 1)$ and $w(e^{i\theta}) = |\theta|^\alpha$ for $\theta \in (-\pi, \pi]$, this estimate leads to

$$\inf\{\|R\|_v; v \sim w\} = (\cos(\frac{\pi\alpha}{2}))^{-1}$$

where $v \sim w$ means that (w/v) and (v/w) both belong to $L_\infty(\mathbb{T})$.

Following [78], let us consider a basis $(e_n)_{n\geq 0}$ of H and call $T : H \longrightarrow H$ a *fast monotone multiplier* (with respect to (e_n)) if

$$T(\sum_{k=0}^{\infty} a_k e_k) = \sum_{k=0}^{\infty} \lambda_k a_k e_k$$

where $(\lambda_k) \in (0, 1)$ is an increasing sequence such that

$$\lim_{k\to\infty} \frac{1-\lambda_k}{1-\lambda_{k-1}} = 0.$$

Easy computations then show that $\sup_{n\geq 0} \|T^n\|$ is at most equal to the basis constant b of (e_n).

If we now consider the (usually conditional!) basis $(e_k)_{k\geq 0}$ of $H^2(w)$, where $e_k(\theta) = e^{ik\theta}$, then its basis constant is $b = \|R\|_w$. The main result of [78] implies in particular the following:

Theorem 6.1. *Let $\alpha \in (0, 1)$ and $w_\alpha(e^{i\theta}) = |\theta|^\alpha$ for $\theta \in (-\pi, \pi]$. Let T be a fast monotone multiplier with respect to the basis of exponential functions $(e_n)_{n\geq 0}$ of $H^2(w_\alpha)$. Then T is power bounded, and*

$$\inf_A \{ \sup_n \|(A^{-1}TA)^n\|\} = (\cos(\frac{\pi\alpha}{2}))^{-1}$$

where the infimum is taken over all invertible operators A.

Showing Theorem 6.1 amounts to proving

$$\inf_A\{\sup_n\|(A^{-1}TA)^n\|\} = \inf\{\|R\|_v; v \sim w\}$$

and "$\geq$" is the hard part of the above equation. If $S(f) = e_1 f$ is the shift operator on $H^2(w)$ and if $\sup_n\|(A^{-1}TA)^n\| = \sigma$ then consider a Banach limit

$$\langle f, g\rangle = \lim_{n\to\mathcal{U}} (AS^n f, AS^n g)$$

where (.,.) is the inner product of $H^2(w)$ and $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. This defines an equivalent inner-product norm

$$|||f||| = \langle f, f\rangle^{\frac{1}{2}}$$

on $H^2(w)$. Since T is a fast monotone multiplier, its powers provide good approximations of the "tails" $Q_n = I - P_n$ of the basis (e_n), and it follows that the basis constant of (e_n) in the new norm $|||\cdot|||$ is at most σ.

Now the sequence $C_k = \langle e_0, e_k\rangle$ if $k \geq 0$ and $c_k = \overline{c_{-k}}$ if $k < 0$ is positive definite. Then Bochner's theorem gives a weight $v \sim w$ which defines $\|\cdot\|_v = |||\cdot|||$, and $\|R\|_v \leq \sigma$ since $\|R\|_v$ is equal to the basis constant of (e_n) in the norm $\|.\|_v$.

Hence a negative answer to Peller's question is obtained with fast monotone multipliers with respect to Babenko's conditional basis of the Hilbert space [5]. More general weights are also considered in [78], which show that the infimum is usually *not* attained in Theorem 6.1. On the other hand, polynomially bounded operators which are obtained by this method are actually similar to contractions. It is conjectured that there exist polynomially bounded operators which are not similar to operators T such that $\sup_n \|T^n\|$ is close to 1. Along these lines, we mention the existence of uniformly bounded representations of the free group with 2 generators which are not similar to representations with small norm ([113, Corollary 5.10]).

Multipliers can as well be unbounded, and such objects provide a negative answer to another important open question. Let us consider the following Cauchy problem:

$$u'(t) + B(u(t)) = f(t)$$

with the initial condition $u(0) = 0$, where $t \in [0, T)$, $-B$ is the closed, densely defined infinitesimal generator of a bounded analytic semi-group on a complex Banach space X, and u and f are X-valued functions on [0,T). One says that B satisfies *maximal regularity* if $u' \in L_2(X)$ as soon as $f \in L_2(X)$. The importance of this notion lies in the fact that it permits to solve quasi-linear parabolic P.D.E. through fixed point arguments.

When X is the Hilbert space, every such B has maximal regularity [20], and the question of the converse occurs, in particular when $X = L_p$ with $1 < p < \infty$. A quite general answer is obtained in [76]:

Theorem 6.2. *If a Banach space X has an unconditional basis, then every closed densely defined operator B such that $(-B)$ generates a bounded analytic semi-group on X has maximal regularity if and only if X is isomorphic to the Hilbert space.*

The proof goes as follows: if B satisfies maximal regularity, then solving the Cauchy problem for well-chosen functions $f \in L_2(X)$ amounts to showing that for every X-valued trigonometric polynomial $g(t) = \sum_{n=-N}^{N} \widehat{g}(n) e^{int}$ one has

$$\|\sum_{n\in\mathbb{Z}} in(in + B)^{-1}\widehat{g}(n)e^{int}\|_{L^2(X)} \leq C\|g\|_{L^2(X)}.$$

This is applied to closed density defined operators B of the form

$$B(\sum_{n=1}^{\infty} a_n e_n) = \sum_{n\geq 1} a_n b_n e_n$$

where (e_n) is an unconditional basis of X and (b_n) is an increasing sequence of real numbers such that $b_n > 0$. Any such multiplier $B = M((b_n))$ is *sectorial* of type ω for any $\omega \in (0, \pi)$ and in particular $(-B)$ generates an analytic bounded semigroup.

Note that when the sequence $(\widehat{g}(n))$ is unconditional, then

$$\|g\|_{L_2(X)} \simeq \|\sum_{-N}^{N} \widehat{g}(n)\|_X.$$

Now the maximal regularity inequality applied with proper choices of scalars (b_n) shows after some work that for any block basis (u_j) of any permutation $(e_{\pi(n)})$ of (e_n), the space $\overline{\text{span}}((u_j))$ is complemented in X. It then follows from [102] that (e_j) is equivalent to the canonical basis of c_0 or ℓ_p for some $p \in [1, \infty)$, and since $\ell_p \simeq (\sum_{n\geq 1} \oplus \ell_2^n)_p$ has non-equivalent unconditional bases if $1 < p < \infty, p \neq 2$ [109], it follows that X is c_0, ℓ_1 or ℓ_2. Finally, c_0 and ℓ_1 can be discarded with the help of a multiplier relative to the (conditional) summing basis of c_0.

Since the Haar system is an unconditional basis of L_p, Theorem 6.1 answers negatively the maximal regularity problem for L_p $(1 < p < \infty, p \neq 2)$. The case of L_1 had been done in [99]. The unconditionality assumption in Theorem 6.2 can be released, but it is not known if the conclusion holds for every space X with a basis (see [77]). We refer to [100] for examples of quasi Banach spaces which have unique unconditional bases up to permutation.

Theorem 6.2 started, together with [132], an important and very active field of research. The above discussion shows how basis theory, developed for its own sake in the 60's and 70's, turns out to be very useful in abstract P.D.E. problems, since (via multipliers), bases – or merely Schauder decompositions – provide sectorial operators. More elaborate concepts, such as R-bounded families of operators, allow an intensive application of geometric tools to maximal regularity and Kato's square root estimates, for which I refer to the joint works of N. Kalton and L. Weis [90, 89] and to [50]. Let us mention that the discrete time analogue of the maximal regularity problem, which presents some specific difficulties since there is no differentiation with respect to time in this case, is investigated in [7, 115, 116] and [84].

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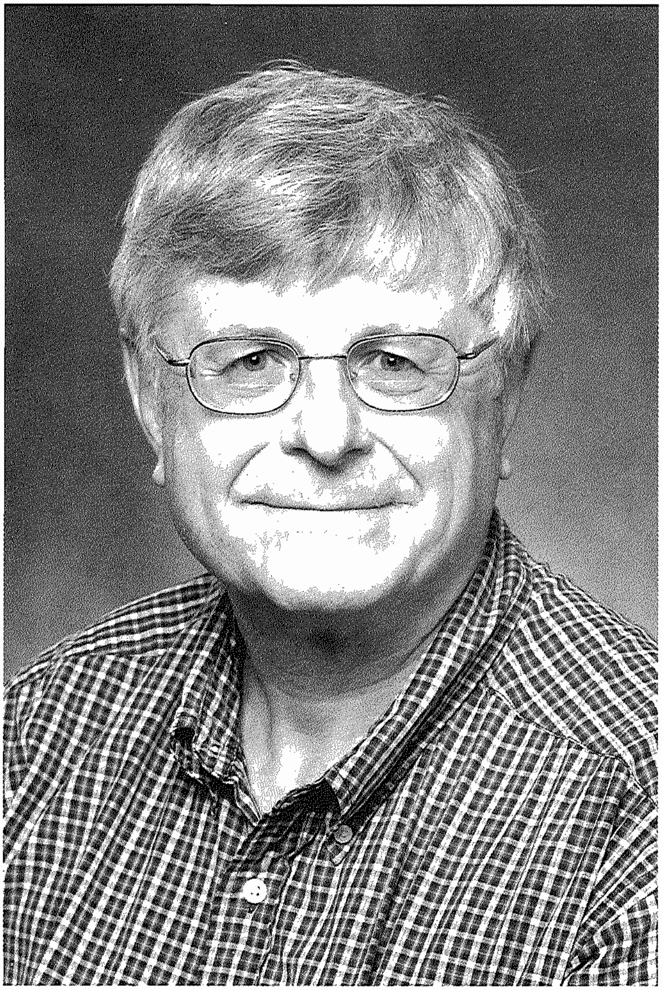

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