TWISTED SUMS OF SEQUENCE SPACES
AND THE THREE SPACE PROBLEM[1]

BY

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ABSTRACT. In this paper we study the following problem: given a complete
locally bounded sequence space Y, construct a locally bounded space Z
with a subspace X such that both X and Z/X are isomorphic to Y, and such
that X is uncomplemented in Z. We give a method for constructing Z under
quite general conditions on Y, and we investigate some of the properties of
Z.

   In particular, when Y is l_p ($1 < p < \infty$), we identify the dual space of Z,
we study the structure of basic sequences in Z, and we study the endomor-
phisms of Z and the projections of Z on infinite-dimensional subspaces.

1. Introduction. Let X and Y be complete metric linear spaces. A *twisted
sum* of X and Y is a space Z which has a subspace X^1 isomorphic to X, with
Z/X^1 isomorphic to Y. The twisted sum Z is *trivial* if X^1 is complemented in
Z; otherwise, Z is *nontrivial*.

In this paper we give a general method of constructing nontrivial twisted
sums of sequence spaces, based on a modification of a construction first
described by Ribe. A special case of one of our main results is that for
$0 < p < \infty$ there is a nontrivial twisted sum of l_p with itself. For $0 < p < 1$,
this solves a problem raised in [4]. For $1 < p$ the twisted sum obtained is
actually a Banach space, by a result of Kalton. In particular, we have an
alternate solution to the 3-space problem for Hilbert spaces, which was solved
by Enflo, Lindenstrauss, and Pisier.

We summarize here the organization of the paper.

§§2 and 3 are technical in character; in them we collect the basic facts
needed for the construction. In §2 we show that nontrivial twisted sums arise
from–and give rise to–"quasi-linear" maps. §3 details two simplifications that
can be made in the construction of quasi-linear maps. One of these reduces
constructing quasi-linear maps to the problem of constructing "quasi-addi-
tive" maps; the section ends by studying quasi-additive maps on the real line.

In §4 we apply the results of §§2 and 3 to a quite general class of sequence

spaces. The main result of the section is that under very general conditions on the sequence space X, there is a nontrivial twisted sum of X with itself.

In §5 we restrict our attention to a class of twisted sums of l_p $(1 < p < \infty)$ with itself, which we denote by $l_p(\phi)$. We calculate the dual space of $l_p(\phi)$ and then study the structure of basic sequences in $l_p(\phi)$. Finally, in §6 we study the spaces Z_p $(1 < p < \infty)$ which are a subclass of the spaces $l_p(\phi)$. In a sense, the spaces Z_p are "extremal" solutions to the three space problem. They yield sharp estimates for the rate of growth of projections on l_p^n; the inclusion map and the quotient map defining the twisted sum are, respectively, strictly co-singular and strictly singular; and the endomorphisms of Z_p obey rather strong conditions.

We close this section by collecting some definitions and results we will need. A *quasi-norm* on a real vector space X is a function $\| \cdot \|: X \to \mathbf{R}$ satisfying

(1) $\|x\| > 0$ for $x \in X$, $x \neq 0$,

(2) $\|tx\| = |t| \, \|x\|$, for $x \in X$, $t \in \mathbf{R}$,

(3) $\|x + y\| \leqslant C(\|x\| + \|y\|)$, for $x,y \in X$.

The constant C in (3) will be called a *modulus of concavity* of $\| \cdot \|$, and $(X, \| \, \|)$ will be called a *quasi-normed space*.

If we take $U = \{x \in X: \|x\| \leqslant 1\}$, then the scalar multiples of U form a neighborhood base at 0 for a Hausdorff vector topology on X. With this topology, X is *locally bounded*, i.e. has a bounded neighborhood of 0. Conversely, every locally bounded topology on a vector space is given by a quasi-norm.

The locally bounded space X is *locally p-convex*, $0 < p \leqslant 1$, if it has a bounded neighborhood V of 0 such that $\alpha x + \beta y \in V$ whenever x,y are in V and $|\alpha|^p + |\beta|^p \leqslant 1$. X is a *p-Banach space* if it is p-convex and complete.

Let V be as above in the definition of p-convex. Then the gauge functional $\| \cdot \|$ of V is a quasi-norm with the additional property that $x \to \|x\|^p$ is a subadditive function of x.

A fundamental result of Aoki and Rolewicz [1], [9] is

THEOREM 1.1. *Every locally bounded space is p-convex for some $p > 0$.*

It follows that, if $(X, \| \, \|)$ is locally bounded, there are positive numbers p and L such that

$$\left\| \sum_{i=1}^{n} x_i \right\| \leqslant L \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}$$

for all n and $x_1, \ldots, x_n \in X$.

Throughout, $\mathbf{R}$, $\mathbf{Z}$, and $\mathbf{N}$ denote, respectively, the real numbers, the integers, and the positive integers. $\mathbf{R}^\infty$ is the space of all finitely-supported

real sequences. An *F-space* is a complete metrizeable linear space. The linear span of a set A will be denoted by lin A. And the symbol i_X denotes the identity map on a space X.

2. **Twisted sums.** Suppose that X, Y, and Z are F-spaces. Suppose further that Z contains a subspace X^1 which is isomorphic to X, with Z/X^1 isomorphic to Y. Loosely we may say that Z is a "twisted sum" of X and Y (in that order!). In other language, a *twisted sum* of the F-spaces X and Y is a short exact sequence of F-spaces $0 \to X \xrightarrow{j} Z \xrightarrow{q} Y \to 0$. If X, Y, and Z are quasi-normed F-spaces with

$$\|jx\| = \|x\|, \qquad x \in X,$$

and

$$\|y\| = \inf\{\|z\| : qz = y\}, \qquad y \in Y,$$

then we say that $0 \to X \xrightarrow{j} Z \xrightarrow{q} Y \to 0$ is an *isometric twisted sum* of X and Y. In this case Z has a subspace $j(X)$ isometric to X and $Z/j(X)$ is isometric to Y. In this section we collect a few basic facts about twisted sums; in particular, we show that they arise from–and give rise to–"quasi-linear" maps from Y into X.

Let X and Y be fixed F-spaces and let Z_1 and Z_2 be two twisted sums of X and Y.

DEFINITION 2.1. (a) The twisted sums Z_1 and Z_2 are *equivalent* if there exists a continuous linear operator $T: Z_1 \to Z_2$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \to & X & \xrightarrow{j_1} & Z_1 & \xrightarrow{q_1} & Y & \to & 0 \\
 & & \downarrow i_X & & \downarrow T & & \downarrow i_Y & & \\
0 & \to & X & \xrightarrow{j_2} & Z_2 & \xrightarrow{q_2} & Y & \to & 0
\end{array}
$$

(b) The twisted sums Z_1 and Z_2 are *projectively equivalent* if there exist a continuous linear operator $T: Z_1 \to Z_2$ and nonzero scalars α and β so that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \to & X & \xrightarrow{j_1} & Z_1 & \xrightarrow{q_1} & Y & \to & 0 \\
 & & \downarrow \beta i_X & & \downarrow T & & \downarrow \alpha i_Y & & \\
0 & \to & X & \xrightarrow{j_2} & Z_2 & \xrightarrow{q_2} & Y & \to & 0
\end{array}
$$

In both (a) and (b), T is necessarily one-to-one and onto, and by the open mapping theorem, T is an isomorphicm.

From now on we shall restrict ourselves to locally bounded F-spaces. A result of Roelcke (see also Kalton [4, 1.1]) asserts that the twisted sum of locally bounded F-spaces is a locally bounded F-space.

PROPOSITION 2.2. *Suppose X and Y are p-Banach spaces and that Z is a p-convex twisted sum of X and Y. Then Z is projectively equivalent (and hence isomorphic) to an isometric twisted sum of X and Y.*

PROOF. We may suppose that Z is quasi-normed by a p-subadditive quasi-norm. Suppose $j: X \to Z$ and $q: Z \to Y$ are the maps inducing the twisted sum.

From the open mapping theorem there exist constants $\alpha, \beta > 0$ such that

$$\beta \|jx\| \geqslant \|x\|, \qquad x \in X,$$

and

$$\alpha \|qz\| \geqslant \inf_{x \in X} \|z - jx\|, \qquad z \in Z.$$

Now define

$$\|z\|_0 = \inf_{x \in X} (\|z - jx\|^p + \beta^{-p}\|x\|^p)^{1/p}$$

and

$$\|z\|_1 = \max(\alpha\|qz\|, \|z\|_0).$$

Then

$$\|z\|_0^p \leqslant \|z\|^p$$

and

$$\|z\|_1 \leqslant (\max\{\alpha\|q\|, 1\})\|z\|.$$

Clearly $\| \cdot \|_1$ is a p-subadditive quasi-norm on Z. Let Z_1 be the quasi-normed space $(Z, \| \|_1)$, and let $T: Z \to Z_1$ be the identity map. Then the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \to & X & \overset{j}{\to} & Z & \overset{q}{\to} & Y & \to & 0 \\
 & & \downarrow \beta^{-1}i_X & & \downarrow T & & \downarrow \alpha i_Y & & \\
0 & \to & X & \overset{\beta j}{\to} & Z_1 & \overset{\alpha q}{\to} & Y & \to & 0
\end{array}
$$

Now,

$$\|\beta j x\|_1^p = \|\beta j x\|_0^p = \inf_{x' \in X} \left\{ \|j(\beta x - x')\|^p + \beta^{-p}\|x'\|^p \right\}$$

$$\geqslant \inf_{x' \in X} \left\{ \beta^{-p}\|\beta x - x'\|^p + \beta^{-p}\|x'\|^p \right\}$$

$$= \|x\|^p = \beta^{-p}\|\beta x\|^p \geqslant \|\beta j x\|_0^p.$$

Hence βj is an isometry. Clearly $\|\alpha qz\| \leqslant \|z\|_1$. For $\varepsilon > 0$ and $z \in Z$ we may choose $x \in X$ so that

$$\|z - jx\| \leqslant (1 + \varepsilon)\alpha\|qz\| = (1 + \varepsilon)\|\alpha qz\|.$$

Hence $\|z - jx\|_1 \leqslant (1 + \varepsilon)\|\alpha qz\|.$

These conditions together imply that Z_1 is complete, that it is an isometric twisted sum of X and Y, and that it is projectively equivalent to Z. The proof is complete.

DEFINITION 2.3. Let F be a map from Y to X. The map F is *quasi-linear* if it satisfies

(a) $F(ty) = tF(y)$ for $y \in Y$, $t \in \mathbf{R}$, and

(b) $\|F(y_1 + y_2) - F(y_1) - F(y_2)\| \leqslant M(\|y_1\| + \|y_2\|)$

for $y_1, y_2 \in Y$, where M is a constant independent of y_1 and y_2.

Given any quasi-linear map $F: Y \to X$ we can construct a twisted sum $X \oplus_F Y$ as follows: $X \oplus_F Y$ is the vector space $X \oplus Y$ with the quasi-norm $\|(x, y)\| = \|y\| + \|x - F(y)\|$. (We are using the same symbol $\| \cdot \|$ to denote the quasi-norm on $X \oplus Y$, Y, and X.) It is easy to see that the maps $jx = (x, 0)$ and $q(x, y) = y$ define, respectively, an embedding of X into $X \oplus_F Y$ and a quotient map of $X \oplus_F Y$ onto Y. It follows that $X \oplus_F Y$ is complete and hence a twisted sum. This construction was first described by Ribe [7]; a very similar construction is also given by Kalton [4, 4.6]. The following converse result is implicit in Proposition 3.3 of [4]:

THEOREM 2.4. *If Z is a twisted sum of X and Y, then there is a quasi-linear map $F: Y \to X$ such that Z is equivalent to $X \oplus_F Y$.*

PROOF. Let C be a modulus of concavity for the quasi-norms on X, Y, and Z. Let $0 \to X \xrightarrow{j} Z \xrightarrow{q} Y \to 0$ define the twisted sum. Then there is a linear, possibly discontinuous, map $\theta: Y \to Z$ such that $q\theta = i_Y$. Since q is open there are a constant K and a map $\phi: Y \to Z$ such that $\|\phi(y)\| \leqslant K\|y\|$ and $q\phi(y) = y$ for all $y \in Y$. We may suppose that $\phi(ty) = t\phi(y)$ for $y \in Y$.

Now let $F(y) = j^{-1}(\phi(y) - \theta(y))$. Let L be such that $\|j^{-1}z\| \leqslant L\|z\|$ for $z \in j(X)$. Then for $y_1, y_2 \in Y$,

$$\|F(y_1 + y_2) - F(y_1) - F(y_2)\| = \|j^{-1}(\phi(y_1 + y_2) - \phi(y_1) - \phi(y_2))\|$$
$$\leqslant L\|\phi(y_1 + y_2) - \phi(y_1) - \phi(y_2)\|$$
$$\leqslant C^2 L(\|\phi(y_1 + y_2)\| + \|\phi(y_1)\| + \|\phi(y_2)\|)$$
$$\leqslant 2C^3 KL(\|y_1\| + \|y_2\|).$$

Hence F is quasi-linear.

Now define $T: Z \to X \oplus_F Y$ by $Tz = (j^{-1}(z - \theta qz), qz)$. Then

$$\|Tz\| = \|j^{-1}(z - \phi(qz)), qz\| \leqslant L\|z - \phi(qz)\| + \|qz\|$$
$$\leqslant CL(\|z\| + \|\phi(qz)\|) + \|qz\| \leqslant (CL + K\|q\| + \|q\|)\|z\|.$$

Hence T is continuous. It is easy to see that T induces an equivalence between Z and $X \oplus_F Y$; the proof is complete.

We shall say that two quasi-linear maps F, $G: Y \to X$ are (*projectively*)

equivalent if $X \oplus_F Y$ and $X \oplus_G Y$ are (projectively) equivalent twisted sums.
Note that if $F \equiv 0$ on Y then F induces the ordinary topological direct sum
$X \oplus Y$. We shall say that F is *trivial* if it is equivalent to the identically zero
map. In this case the short exact sequence $0 \to X \to X \oplus_F Y \to Y \to 0$ splits.

THEOREM 2.5. (a) *The maps F and G are equivalent if and only if there exist
a constant M and a linear map $A \colon Y \to X$ such that*

$$\|F(y) - G(y) - A(y)\| \leqslant M\|y\|, \qquad y \in Y;$$

(b) *F and G are projectively equivalent if and only if there exist constants M
and α, $\alpha \neq 0$, such that*

$$\|F(y) - G(\alpha y) - A(y)\| \leqslant M\|y\|, \qquad y \in Y;$$

(c) *F is trivial if and only if it is projectively equivalent to the zero map.*

PROOF. (a) If F and G are equivalent there is a continuous linear map $T \colon$
$X \oplus_F Y \to X \oplus_G Y$ of the form $T(x, y) = (x + Ay, y)$, where $A \colon Y \to X$ is
linear. Then $T(F(y), y) = (F(y) + Ay, y)$ and hence

$$\|F(y) - G(y) + Ay\| \leqslant \|T\| \|y\|.$$

Conversely, if F, G and A satisfy the hypothesis, define $T \colon X \oplus_F Y \to$
$X \oplus_G Y$ by $T(x, y) = (x + Ay, y)$. It is easy to check that T is continuous
and induces an equivalence between $X \oplus_F Y$ and $X \oplus_G Y$.
   The proof of (b) is similar, and (c) is obvious.

3. Simplification of the construction of twisted sums. There are two im-
portant simplifications of the construction of quasi-linear maps and hence of
twisted sums. The first allows us to consider quasi-linear maps defined only
on a dense subspace of Y; the second allows us to drop the requirement of
homogeneity in the definition of quasi-linear map.

THEOREM 3.1. *Suppose that X and Y are quasi-normed F-spaces and that Y_0
is a dense subspace of Y. Suppose $F_0 \colon Y_0 \to X$ is a quasi-linear map. Then*
   (i) *there is a quasi-linear map $F \colon Y \to X$ such that $F(y) = F_0(y), y \in Y_0$;*
   (ii) *if $F^* \colon Y \to X$ is any quasi-linear map such that $F^*(y) = F_0(y), y \in Y_0$,
then F^* and F are equivalent.*

PROOF. Consider the space $X \oplus_{F_0} Y_0$ quasi-normed by

$$\|(x, y)\| = \|x - F_0(y)\| + \|y\|.$$

Let Z be the completion of this space; then Z is a twisted sum of X and Y.
To see this, define $j \colon X \to X \oplus_{F_0} Y_0$ by $j(x) = (x, 0)$ and $\hat{q} \colon X \oplus_{F_0} Y_0 \to Y$
by $\hat{q}(x, y) = y$. Then $\hat{q}$ extends to an open mapping q of Z onto Y whose
kernel is precisely $j(X)$. For this last point, note that (as in [4, 3.3]) if (x_n, y_n)
is a sequence in $X \oplus_{F_0} Y_0$ converging to z in Z and $\hat{q}(x_n, y_n) = y_n \to 0$, then

$(F_0(y_n), y_n) \to 0$ and hence $(x_n - F_0(y_n), 0) \to z$. Thus $(x_n - F_0(y_n))$ converges to some x_0 in X and $z = j(x_0)$.

Now by Theorem 2.4, Z is equivalent to $X \oplus_H Y$ for some quasi-linear H, i.e. there exists $T: Z \to X \oplus_H Y$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \to & X & \overset{j}{\to} & Z & \overset{q}{\to} & Y & \to & 0 \\
 & & \downarrow i_X & & \downarrow T & & \downarrow i_Y & & \\
0 & \to & X & \to & X \oplus_H Y & \to & Y & \to & 0
\end{array}
$$

Then $T|X \oplus_{F_0} Y_0$ has the form $T(x, y) = (x + Ay, y)$ where $A: Y_0 \to X$ is linear. From this,

$$\|F_0(y) + A(y) - H(y)\| \leq \|T\| \, \|y\|, \qquad y \in Y_0.$$

If we define

$$
\begin{aligned}
F(y) &= H(y) - A(y), & y \notin Y_0, \\
&= F_0(y), & y \in Y_0,
\end{aligned}
$$

then F is a quasi-linear extension of F. Uniqueness up to equivalence is guaranteed by the fact that any extension of F_0 defines a completion of $X \oplus_{F_0} Y_0$. The proof of Theorem 3.1 is complete.

Let Y_0 be a dense subspace of Y.

DEFINITION 3.2. A map $f: Y_0 \to X$ is *quasi-additive* if it satisfies

(i) $\|f(y_1 + y_2) - f(y_1) - f(y_2)\| \leq K(\|y_1\| + \|y_2\|), y_1, y_2 \in Y_0$;
(ii) $\lim_{t \to 0} f(ty) = 0, y \in Y_0$;
(iii) $f(-y) = -f(y), y \in Y_0$.

Thus we weaken the homogeneity assumption of quasi-linearity, replacing it by antisymmetry (iii) and continuity at zero along lines (ii). With K the constant in (i), we say that f is quasi-additive of *order K*.

The following lemma is a simple modification of Lemma 3.2 of [4]:

LEMMA 3.3. *Suppose X and Y are quasi-normed F-spaces. Then there exist positive constants r and L such that whenever Y_0 is a dense subspace of Y and f: $Y_0 \to X$ is quasi-additive of order K,*

$$\left\| f\left(\sum_{i=1}^n y_i \right) - \sum_{i=1}^n f(y_i) \right\| \leq KL \left(\sum_{i=1}^n \|y_i\|^r \right)^{1/r}$$

for any $y_1, \ldots, y_n \in Y_0$. If $X = Y = \mathbf{R}$, then L can be taken to equal 1 and r can be taken to equal $\frac{1}{2}$.

LEMMA 3.4. *Suppose X is a quasi-normed F-space. Then there is a constant B_X depending on X such that whenever $f: \mathbf{R} \to X$ is quasi-additive of order K, then*

$$\|f(t) - tf(1)\| \leq B_X K, \qquad 0 \leq t \leq 1.$$

PROOF. By replacing f by $g(t) = f(t) - tf(1)$, we see that it suffices to consider the case when $f(1) = 0$. In this case,

$$\|f(2^{-n}) - 2f(2^{-(n+1)})\| \leq K2^{-n}$$

and hence

$$\|2^n f(2^{-n}) - 2f(2^{-(n+1)})\| \leq K.$$

Now L and r can be chosen satisfying Lemma 3.3 with $Y = \mathbf{R}$ and also such that

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq L \left(\sum_{i=1}^{n} \|x_i\|^r \right)^{1/r}$$

for $x_1, \ldots, x_n \in X$ (by Theorem 1.1). We then have

$$\|2^n f(2^{-n})\| \leq LKn^{1/r}, \qquad n \in \mathbf{N}.$$

Let t be in $(0, 1)$; we suppose that $2^{-n} \leq t < 2^{1-n}$ for some integer n. Then

$$t = 2^{-n} \sum_{k=0}^{\infty} \varepsilon_k 2^{-k}$$

where $\varepsilon_0 = 1$ and each ε_k is either 0 or 1. Let

$$t_m = 2^{-n} \sum_{k=0}^{m} \varepsilon_k 2^{-k}.$$

Then

$$\left\| f(t_m) - \sum_{k=0}^{m} \varepsilon_k f(2^{-(n+k)}) \right\| \leq KL \left(\sum_{k=0}^{m} 2^{-(n+k)r} \right)^{1/r} \leq KL \left(\frac{1}{1 - 2^{-r}} \right)^{1/r}.$$

From this,

$$\|f(t_m)\| \leq KL^2 \left[(1 - 2^{-r})^{-1} + \sum_{k=0}^{m} \varepsilon_k (n + k) 2^{-(n+k)r} \right]^{1/r}$$

$$\leq KL^2 \left[(1 - 2^{-r})^{-1} + \sum_{k=0}^{\infty} k 2^{-kr} \right]^{1/r} = L^* K, \text{ say.}$$

Finally, $\| f(t) - f(t_m) - f(t - t_m)\| \leq K$ and hence

$$\|f(t)\| \leq L \left(\|f(t_m)\|^r + K^r \|f(t - t_m)\|^r + K^r \right)^{1/r}$$

$$\leq LK \left((L^*)^r + 1 + \|f(t - t_m)\|^r \right)^{1/r}.$$

Since $f(t - t_m) \to 0$, we have, on letting $m \to \infty$, that

$$\|f(t)\| \leq L((L^*)^r + 1)^{1/r} K$$

and the result is proved.

THEOREM 3.5. *Suppose X and Y are quasi-normed F-spaces and Y_0 is a dense subspace of Y. Then if $f: Y_0 \to X$ is quasi-additive, the map $F: Y_0 \to X$ defined by*

$$F(x) = \|x\| f(x/\|x\|), \qquad x \neq 0,$$
$$= 0, \qquad x = 0,$$

is quasi-linear.

PROOF. If $\|x\| \leqslant 1$ and f is quasi-additive of order K, then

$$\|F(x) - f(x)\| \leqslant B_X K$$

by Lemma 3.4. Let C denote a modulus of concavity for X. Then if $\|x\| + \|y\| \leqslant 1/C$, we have $\|x + y\| \leqslant 1$ and so

$$\|F(x + y) - f(x + y)\| \leqslant B_X K.$$

Now,

$$\|F(x + y) - F(x) - F(y)\| \leqslant C(\|f(x + y) - f(x) - f(y)\| + 3C^2 B_X K).$$

Hence

$$\|F(x + y) - F(x) - F(y)\| \leqslant C(K + 3C^2 B_X K).$$

Thus F is quasi-linear, and the proof is complete.

We now turn to the form of quasi-additive maps from $\mathbf{R}$ into $\mathbf{R}$. Let $B = B_{\mathbf{R}}$ (as obtained in Lemma 3.4).

LEMMA 3.6. *If $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are quasi-additive of order K and*

$$|f(2^n) - g(2^n)| \leqslant M \cdot 2^n, \qquad n \in \mathbf{Z},$$

then

$$|f(t) - g(t)| \leqslant (M + 4BK)|t|, \qquad t \in \mathbf{R}.$$

PROOF. For t in $\mathbf{R}^+$, choose n in $\mathbf{Z}$ so that $2^{n-1} \leqslant t < 2^n$, and define $f^{(n)}(r) = f(2^n r)$, $g^{(n)}(r) = g(2^n r)$, $r \in \mathbf{R}$. The functions $f^{(n)}$ and $g^{(n)}$ are quasi-additive of order $2^n K$. Hence

$$|f^{(n)}(2^{-n} t) - 2^{-n} t f^{(n)}(1)| \leqslant 2^n BK,$$
$$|g^{(n)}(2^{-n} t) - 2^{-n} t g^{(n)}(1)| \leqslant 2^n BK.$$

Then

$$|f(t) - g(t)| \leqslant 2^{-n} t |f(2^n) - g(2^n)| + 2^{n+1} BK$$
$$\leqslant (M + 4BK) t.$$

This proves the lemma for nonnegative t; since f and g are antisymmetric, we have the result for all real t.

THEOREM 3.7. (a) *If $\theta: \mathbf{R} \to \mathbf{R}$ is a Lipschitz function, then the map*

$$f(t) = t\theta(\log(1/|t|)), \qquad t \neq 0,$$
$$= 0, \qquad t = 0,$$

is quasi-additive, of order $L \log 2$ where L is the Lipschitz constant of θ.

(b) *Conversely, if f is quasi-additive there exists a Lipschitz function θ: $\mathbf{R} \to \mathbf{R}$ such that*

$$\sup_{0 < t < \infty} \left| \frac{f(t)}{t} - \theta\left(\log \frac{1}{|t|}\right) \right| < \infty.$$

PROOF. (a) For $t_1, t_2 > 0$,

$$\left| \frac{f(t_1 + t_2) - f(t_1) - f(t_2)}{t_1 + t_2} \right|$$

$$= \left| \theta\left(\log \frac{1}{t_1 + t_2}\right) - \frac{t_1}{t_1 + t_2}\theta\left(\log \frac{1}{t_1}\right) - \frac{t_2}{t_1 + t_2}\theta\left(\log \frac{1}{t_2}\right) \right|$$

$$\leqslant L\left(\frac{t_1}{t_1 + t_2}\log \frac{t_1 + t_2}{t_1} + \frac{t_2}{t_1 + t_2}\log \frac{t_1 + t_2}{t_2} \right)$$

where L is the Lipschitz constant of θ. Since

$$t \log \frac{1}{t} + (1 - t) \log \frac{1}{1 - t} \leqslant \log 2, \qquad 0 < t < 1,$$

we conclude that

$$|f(t_1 + t_2) - f(t_1) - f(t_2)| \leqslant L \log 2(t_1 + t_2).$$

It now follows easily that f is quasi-additive.

(b) If f is quasi-additive of order K, then

$$|f(2^{n+1}) - 2f(2^n)| \leqslant K \cdot 2^{n+1}, \qquad n \in \mathbf{Z}.$$

Hence

$$\left| \frac{f(2^{n+1})}{2^{n+1}} - \frac{f(2^n)}{2^n} \right| \leqslant K.$$

Now choose θ a Lipschitz function of Lipschitz constant K such that $\theta(n) = 2^n f(2^{-n})$, $n \in \mathbf{Z}$, and define

$$g(t) = t\theta(-\log|t|/\log 2), \qquad t \neq 0,$$
$$= 0, \qquad t = 0.$$

Then g is quasi-additive of order K and $f(2^n) = g(2^n)$, $n \in \mathbf{Z}$. Thus

$$\left| \frac{f(t)}{t} - \frac{g(t)}{t} \right| \leqslant 2BK, \qquad t \neq 0,$$

by the preceding lemma, and the result follows.

4. Twisted sums of sequence spaces. In this section we apply the results of the preceding two sections to the construction of quasi-linear functions on sequence spaces. We obtain that under very general conditions on the sequence space E, there is a short exact sequence

$$0 \to E \to Z \to E \to 0$$

which does not split. In particular, we give an alternate proof of the solution of the 3-space problem for Hilbert spaces.

DEFINITION 4.1. A *solid quasi-normed FK-space* is a sequence space E satisfying:

(i) if e_n denotes the nth basis vector $(e_n(k) = \delta_{nk})$, then the linear span $\mathbf{R}^\infty$ of (e_n) is dense in E;

(ii) the quasi-norm $\|\cdot\|$ on E satisfies

(a) $\|sx\| \leqslant \|s\|_\infty \|x\|$, $s \in l_\infty$, $x \in E$;

(b) $\|e_n\| = 1$, $n \in \mathbf{N}$;

(c) $\|x\|_\infty \leqslant \|x\|$, $x \in E$.

Under these circumstances (e_n) is an unconditional basis for E.

Notation. Let $\mathfrak{L}$ denote the class of Lipschitz functions $\phi: \mathbf{R} \to \mathbf{R}$ such that $\phi(t) = 0$ for $t \leqslant 0$.

For ϕ in $\mathfrak{L}$, we define a quasi-additive map $f: \mathbf{R}^\infty \to \mathbf{R}^\infty$ by

$$f(x)[k] = x(k)\phi(-\log|x(k)|), \qquad x(k) \neq 0, \quad k \in \mathbf{N},$$
$$= 0, \qquad x(k) = 0, \quad k \in \mathbf{N}.$$

To see that f is quasi-additive (for the quasi-norm on E), observe that if L is the Lipschitz constant of ϕ, then for x_1, x_2 in E,

$$\left| f(x_1)[k] + f(x_2)[k] - f(x_1 + x_2)[k] \right| \leqslant L \log 2(|x_1(k) + x_2(k)|).$$

From this,

$$|f(x_1) + f(x_2) - f(x_1 + x_2)| \leqslant L \log 2(|x_1| + |x_2|)$$

(using the obvious lattice operations on E). Thus

$$\|f(x_1) + f(x_2) - f(x_1 + x_2)\| \leqslant L \log 2(\| |x_1| + |x_2| \|)$$
$$\leqslant CL \log 2(\|x_1\| + \|x_2\|)$$

where C is a modulus of concavity of $\|\cdot\|$.

Next, by Theorems 3.5 and 3.1, f induces a quasi-linear map $F_0: \mathbf{R}^\infty \to \mathbf{R}^\infty$:

$$F_0(x) = \|x\| f(x/\|x\|), \qquad x \neq 0,$$
$$= 0, \qquad x = 0,$$

and F_0 may be extended to a quasi-linear map $F: E \to E$. We now study the properties of the twisted sum $E \oplus_F E$.

Note first that there is no gain of generality in relaxing the restriction $\phi(t) = 0$, $t \leqslant 0$; each ϕ may be replaced by

$$\bar{\phi}(t) = \phi(t) - \phi(0), \qquad t \geqslant 0,$$
$$= 0, \qquad t \leqslant 0,$$

and it is easily seen that $\bar{\phi}$ induces a twisted sum equivalent to $E \oplus_F E$. Secondly, note that although F is not unique, it is unique up to equivalence.

Notation. For ϕ in $\mathcal{L}$, $E(\phi)$ denotes the (unambiguously defined) twisted sum $E \oplus_F E$.

THEOREM 4.2. *Let E be a solid quasi-normed FK-space such that no subsequence of (e_n) is equivalent to the unit vector basis of c_0. Let ϕ and ψ be in $\mathcal{L}$. Then*

(a) *the two twisted sums $E(\phi)$ and $E(\psi)$ are equivalent if and only if*

$$\sup_{0<t<\infty} |\phi(t) - \psi(t)| < \infty;$$

(b) *$E(\phi)$ and $E(\psi)$ are projectively equivalent if and only if for some $a \neq 0$,*

$$\sup_{0<t<\infty} |\phi(t) - a\psi(t)| < \infty;$$

(c) *$E(\phi)$ is trivial if and only if ϕ is bounded.*

PROOF. (a) Suppose ϕ and ψ induce the quasi-linear maps $F \colon E \to E$ and $G \colon E \to E$. Then by 2.5, $E(\phi)$ and $E(\psi)$ are equivalent if and only if there is a linear map $A \colon E \to E$ such that

$$\|F(y) - G(y) - Ay\| \leqslant M\|y\|, \qquad y \in E.$$

It is immediate that if

$$\sup_{0<t<\infty} |\phi(t) - \psi(t)| < \infty,$$

the above condition holds with $a = 0$.

For the proof of the converse, suppose there is a linear map $A \colon E \to E$ such that

$$\|F(y) - G(y) - Ay\| \leqslant M\|y\|, \qquad y \in E.$$

Let H be the quasi-linear map $F - G$. Observe that $H(e_n) = 0$ for all n, and hence $\|Ae_n\| \leqslant M$, $n \in \mathbf{N}$. In particular, $\|Ae_n\|_\infty \leqslant M$, $n \in \mathbf{N}$, and so we may select increasing sequences of integers $(n(k))$ and $(m(k))$ such that $A(e_{n(k)} - e_{m(k)}) \to 0$ coordinate-wise.

Now by a standard gliding hump argument we may select a subsequence $(f_n)_{n=1}^\infty$ of $(e_{n(k)} - e_{m(k)})_{k=1}^\infty$ and an increasing sequence (p_n) such that $p_0 = 0$ and

(1) $\|Af_n - \sum_{k=p_{n-1}+1}^{p_n} (Af_n(k))e_k\| \leqslant 2^{-n}$;
(2) $\operatorname{supp}\{f_n\} = \{q_n, r_n\} \subset \{p_{n-1} + 1, \ldots, p_n\}$.
Define a linear map $\hat{A} \colon \operatorname{lin}(f_n) \to E$ by

$$\hat{A}f_n = (Af_n)(q_n)e_{q_n} + (Af_n)(r_n)e_{r_n}.$$

Then $\|\hat{A}f_n\| \leqslant 2CM$, where C is a modulus of concavity for $\|\cdot\|$. Hence, $\|\hat{A}f_n\|_\infty \leqslant 2CM$, and so $|\hat{A}f_n| \leqslant 2CM|f_n|$, $n \in \mathbf{N}$.

For $x = \sum_{n=1}^N t_n f_n$ in $\mathrm{lin}(f_n)$, we have $\max |t_n| \leqslant \|x\|_\infty \leqslant \|x\|$. Hence

$$\left\| Ax - \sum_{n=1}^N t_n \sum_{k=p_{n-1}+1}^{p_n} (Af_n(k))e_k \right\| \leqslant M_1 \|x\|,$$

where M_1 depends only on the modulus of concavity of $\|\cdot\|$ (This follows from (1) above and Theorem 1.1: if r and L are chosen as in Theorem 1.1 and $\|y_i\| \leqslant 2^{-i}$, then

$$\left\| \sum_{i=1}^N y_i \right\| \leqslant L \left(\sum_{i=1}^N \|y_i\|^r \right)^{1/r} \leqslant L(1 - 2^{-r})^{-1/r}$$

for any N.) Thus

$$\left\| H(x) - \sum_{n=1}^N t_n \sum_{k=p_{n-1}+1}^{p_n} (Af_n(k))e_k \right\| \leqslant M_2^* \|x\|$$

and hence

$$\left\| H(x) - \sum_{n=1}^N t_n \hat{A}f_n \right\| \leqslant M_2 \|x\|,$$

with M_2^* and M_2 independent of x.

Now let $s_n = f_1 + \cdots + f_n$, $n \in \mathbf{N}$. Then

$$\left\| H(s_n) - \sum_{i=1}^n \hat{A}f_i \right\| \leqslant M_2 \|s_n\|.$$

However, $H(s_n) = (\phi(\log\|s_n\|) - \psi(\log\|s_n\|))s_n$ and so

$$\left\| |\phi(\log\|s_n\|) - \psi(\log\|s_n\|)| \, |s_n| - \sum_{i=1}^n |\hat{A}f_i| \right\| \leqslant M_2 \|s_n\|$$

for all n. Further,

$$\sum_{i=1}^n |\hat{A}f_i| \leqslant 2CM|s_n|, \qquad n \in \mathbf{N},$$

and hence if $|\phi(\log\|s_n\|) - \psi(\log\|s_n\|)| \geqslant 2CM$, then

$$|\phi(\log\|s_n\|) - \psi(\log\|s_n\|)| - 2CM \leqslant M_2.$$

Thus

$$|\phi(\log\|s_n\|) - \psi(\log\|s_n\|)| \leqslant 2CM + M_2, \qquad n \in N. \qquad (*)$$

Note that if $\sup_n \|s_n\| < \infty$, the sequence $\{e_{q_1}, e_{r_1}, e_{q_2}, e_{r_2}, \dots\}$ is equivalent to the usual basis of c_0. For assume $\sup_n \|s_n\| = S$ is finite; then if $x = \sum_i t_i e_{q_i} + \sum_j t_j e_{r_j}$,

$$\max_{i,j} \{|t_i|, |t_j|\} \leqslant \|x\| \leqslant S \cdot \max_{i,j} \{|t_i|, |t_j|\}.$$

Hence from our assumption it follows that $\|s_n\| \to \infty$. Also, $\|f_n\| \leqslant 2C$ and hence

$$\|s_n\| \leqslant C(\|s_{n-1}\| + 2C), \qquad n \in N.$$

Thus

$$\log\|s_n\| \leqslant \log C + \log(\|s_{n-1}\| + 2C)$$
$$\leqslant \log C(1 + 2C) + \log \|s_{n-1}\|.$$

It follows that for any $t \geqslant 1$ there exists an integer n such that

$$\log\|s_{n-1}\| \leqslant t \leqslant \log \|s_n\|$$

and

$$\log\|s_n\| - \log \|s_{n-1}\| \leqslant \log C(1 + 2C).$$

Since ϕ and ψ are Lipschitz, we conclude from inequality (*) and the above inequalities that $\sup_{0 < t < \infty} |\phi(t) - \psi(t)| < \infty$, which was to be proved.

(b) If $E(\phi)$ and $E(\psi)$ are projectively equivalent then $F(x)$ and $G(ax)$ are equivalent for some $a \neq 0$. (F and G are induced by φ and ψ, respectively.) Define $G_a(x) = G(ax)$; then $E \oplus_{G_a} E$ is equivalent to $E(\theta)$, where

$$\theta(t) = a\psi(t + \log 1/|a|) - a\psi(\log 1/|a|).$$

Hence

$$\sup_{0 < t < \infty} |\phi(t) - a\psi(t + \log 1/|a|) + a\psi(\log 1/|a|)| < \infty,$$

and since ϕ and ψ are Lipschitz,

$$\sup_{0 < t < \infty} |\phi(t) - a\psi(t)| < \infty.$$

This proves (b).

(c) is immediate from (a) and (b).

DEFINITION 4.3. A pair of F-spaces (X, Y) is said to *split* if every twisted sum of X and Y is trivial (i.e. every short exact sequence $0 \to X \to Z \to Y \to 0$ splits).

Since the class $\mathcal{L}$ contains (many!) unbounded functions, an immediate consequence of Theorem 4.2 is:

COROLLARY 4.4. *Let E be a solid quasi-normed FK-space such that no subsequence of (e_n) is equivalent to the usual basis of c_0. Then (E, E) fails to split.*

Suppose that E is a quasi-normed F-space with an unconditional basis, no subsequence of which is equivalent to a c_0-basis. Then E is isomorphic to a space satisfying the assumptions of Corollary 4.4, so (E, E) fails to split. In particular,

COROLLARY 4.5. *The pair (l_p, l_p) fails to split for $0 < p < \infty$.*

COROLLARY 4.6. *For* $0 < p \leqslant 1$, *there exists a quasi-normed F-space Z which is not p-convex, but which has a subspace Y isomorphic to* l_p *with* Z/Y *also isomorphic to* l_p.

Corollaries 4.5 and 4.6 solve a problem raised in [4] for $0 < p < 1$. For $p = 1$, these results are known; they are equivalent to results of Kalton [4], Ribe [7], and Roberts [8], which are constructions of a nonlocally convex *F*-space whose quotient by a one-dimensional subspace is a Banach space.

Theorem 2.6 of [4] implies that a twisted sum of two *B*-convex Banach spaces is (after renorming) a *B*-convex Banach space. This and Theorem 2.2 give

THEOREM 4.7. *If* $1 < p < \infty$ *and* $\phi \in \mathcal{L}$, *then* $l_p(\phi)$ *is isomorphic to a Banach space. In particular,* $l_p(\phi)$ *may be renormed so that it is projectively equivalent to an isometric twisted sum of* l_p *and* l_p.

Corollary 4.5 for $p = 2$ and the above theorem give

COROLLARY 4.8. *There is an isometric twisted sum of Banach spaces*

$$0 \to l_2 \to Z \to l_2 \to 0$$

which does not split.

This was originally established by Enflo, Lindenstrauss, and Pisier [2]. Their approach was "local" in nature; our construction seems more "global" and somewhat simpler. The corresponding result for $p \neq 2$ also follows simply enough.

It is possible to extend these results slightly by using the ideas of Turpin [12]. We modify Turpin's approach slightly but not essentially.

Suppose X is a (nontrivial) quasi-normed *F*-space. Then the *galb* $G(X)$ of X is defined to be the space of all real sequences $\{a_n\}$ such that $\sum a_n x_n$ converges whenever (x_n) is a bounded sequence in X. The space $G(X)$ is a quasi-normed *F*-space when equipped with the quasi-norm

$$\|\{a_n\}\| = \sup_m \sup_{\|x_i\| < 1} \left\| \sum_{i=1}^m a_i x_i \right\|.$$

It is easily verified that $G(G(X)) = G(X)$, and that $G(X)$ satisfies the hypotheses of Theorem 4.2 (see [12, 2.2 and 5.2]).

COROLLARY 4.9. *Suppose X is a nontrivial quasi-normed F-space. Then there is a quasi-normed F-space Z with a subspace Y isomorphic to* $G(X)$ *such that* Z/Y *is isomorphic to* $G(X)$ *but such that* $G(Z) \subsetneq G(X)$.

PROOF. From Corollary 4.4 there is a space Z with a subspace Y isomorphic to $G(X)$ with Z/Y isomorphic to $G(X)$ and Y not a direct summand of Z. Assume that $G(Z) = G(X)$. For each n define e_n in $G(X)$ by $e_n(k) = \delta_{nk}$.

Let q be the quotient map of Z onto $G(X)$; then by the openness of q we may choose z_n in Z with $qz_n = e_n$ and $\|z_n\| \leq M$, M independent of n. Then since $G(Z) = G(X)$,

$$\left\| \sum_{n=1}^{m} a_n z_n \right\| \leq M^* \|\{a_1, a_2, \ldots, a_m, 0, \ldots\}\|_{G(X)}$$

where M^* is independent of m and $a_1, \ldots, a_m$. Thus we may define a continuous linear operator $T: G(X) \to Z$ such that $qT = i_{G(X)}$. (T is defined by $Te_n = z_n$, $n \in \mathbf{N}$.) This implies that Y is a direct summand of Z; the proof is complete.

THEOREM 4.10. *Suppose E is a solid quasi-normed FK-space and $\phi \in \mathfrak{L}$. Then $E(\phi)$ has a basis.*

PROOF. Suppose ϕ induces the quasi-linear map $F: E \to E$. Then $E(\phi) = E \oplus_F E$. Let

$$u_{2n-1} = (e_n, 0),$$
$$u_{2n} = (0, e_n), \qquad n \in \mathbf{N}.$$

Then (u_n) spans a dense subspace of $E(\phi)$; we verify that it is a basic sequence. It is enough to show that there is a constant C^* such that if $p < q$ and $t_1, \ldots, t_q$ are in $\mathbf{R}$, then

$$\left\| \sum_{i=1}^{p} t_i u_i \right\| \leq C^* \left\| \sum_{i=1}^{q} t_i u_i \right\|.$$

Suppose first that p is even, $p = 2n$. Then

$$\left\| \sum_{i=1}^{p} t_i u_i \right\| = \left\| \sum_{i=1}^{n} t_{2i-1} e_i - F\left(\sum_{i=1}^{n} t_{2i} e_i \right) \right\| + \left\| \sum_{i=1}^{n} t_{2i} e_i \right\|$$

$$= \left\| \sum_{i=1}^{n} (t_{2i-1} - t_{2i}\phi(\log \alpha_n - \log|t_{2i}|)) e_i \right\| + \alpha_n,$$

where $\alpha_n = \|\sum_{i=1}^{n} t_{2i} e_i\|$. Now suppose $n \leq N$, and let $\alpha = \|\sum_{i=1}^{N} t_{2i} e_i\|$. Then

$$\left\| \sum_{i=1}^{p} t_i u_i \right\| \leq C \left[\left\| \sum_{i=1}^{n} (t_{2i-1} - t_{2i}\phi(\log \alpha - \log|t_{2i}|)) e_i \right\| \right.$$

$$\left. + L \left\| \sum_{i=1}^{n} t_{2i}(\log \alpha - \log \alpha_n) e_i \right\| \right] + \alpha_n,$$

where C is a modulus of concavity for $\| \cdot \|$ and L is the Lipschitz constant of ϕ. Hence

$$\left\| \sum_{i=1}^{2n} t_i u_i \right\| \leqslant C \left\| \sum_{i=1}^{N} (t_{2i-1} - t_{2i}\phi(\log \alpha - \log|t_{2i}|e_i)) \right\|$$

$$+ CL \log\left(\frac{\alpha}{\alpha_n}\right)\alpha_n + \alpha_n$$

$$\leqslant C \left\| \sum_{i=1}^{N} t_i u_i \right\| + \left(\frac{CL}{e} + 1\right)\alpha$$

$$\leqslant \left(C + \frac{CL}{e} + 1\right)\left\| \sum_{i=1}^{2N} t_i u_i \right\|.$$

(For the second inequality we have used that $|t \log t| \leqslant e^{-1}$ for $0 < t < 1$.)

Now if $p = 2n - 1$, $n \leqslant N$, then

$$\left\| \sum_{i=1}^{p} t_i u_i \right\| \leqslant C \left(\left\| \sum_{i=1}^{2n} t_i u_i \right\| + \|t_{2n} u_{2n}\| \right)$$

$$\leqslant C \left(C + \frac{CL}{e} + 1\right)\left\| \sum_{i=1}^{2N} t_i u_i \right\| + C|t_{2n}|,$$

where C is a modulus of concavity for $\| \cdot \|$. However,

$$|t_{2n}| \leqslant \left\| \sum_{i=1}^{N} t_{2i} e_i \right\| \leqslant \left\| \sum_{i=1}^{2N} t_i u_i \right\|,$$

so that

$$\left\| \sum_{i=1}^{p} t_i u_i \right\| \leqslant \left(C^2 + \frac{C^2 L}{e} + 2C\right)\left\| \sum_{i=1}^{2N} t_i u_i \right\|.$$

Finally, since

$$\left\| \sum_{i=1}^{2k} t_i u_i \right\| \leqslant \left\| \sum_{i=1}^{2k+1} t_i u_i \right\|$$

for any k, the result follows–with q equal either to $2N$ or $2N - 1$.

5. The Banach spaces $l_p(\phi)$, $1 < p < \infty$. For $1 < p < \infty$ and $\phi \in \mathcal{L}$, the space $l_p(\phi)$ is a reflexive Banach space, since l_p is reflexive. In this section we obtain some of its properties. We first identify the dual space of $l_p(\phi)$; then we study the structure of basic sequences in $l_p(\phi)$.

Toward our first objective: the adjoint of the short exact sequence

$$0 \to l_p \to l_p(\phi) \to l_p \to 0$$

is a short exact sequence

$$0 \to l_q \to l_p(\phi)^* \to l_q \to 0,$$

with $1/p + 1/q = 1$. It follows that $l_p(\phi)^*$ is a twisted sum $l_q \oplus_G l_q$, with the

duality given by

$$\langle (x, y), (w, z)\rangle = \langle y, w\rangle + \langle x, z\rangle$$

for x,y in l_p and w,z in l_q. The quasi-linear map G can readily be identified:

THEOREM 5.1. *The space $l_p(\phi)^*$ is equivalent to $l_q(\psi)$, where*

$$\psi(t) = -\phi((p-1)^{-1}t), \qquad t \geqslant 0.$$

PROOF. The crucial step in the proof is the following lemma:

LEMMA 5.2. *If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are in $\mathbf{R}$ then*

$$\left|\sum_{i=1}^{n} x_i y_i \left(\log \frac{|x_i|}{|y_i|^{p-1}}\right)\right| \leqslant \frac{1}{e}((p-1)\|x\|_q^q + \|y\|_p^p).$$

(Here, $1 < p < \infty$, $1/q + 1/p = 1$, and $0 \log 0 = 0 \log \infty = 0$.)

PROOF. Let J denote the set of indices in $\{1, \ldots, n\}$ such that $|x_i| \leqslant |y_i|^{p-1}$, $i \in J$; and let K denote the remaining indices. Then

$$\left|\sum_{i \in J} x_i y_i \left(\log \frac{|x_i|}{|y_i|^{p-1}}\right)\right| = \left|\sum_{i \in J} \frac{x_i}{|y_i|^{p-1}} \left(\log \frac{|x_i|}{|y_i|^{p-1}}\right) y_i |y_i|^{p-1}\right|$$

$$\leqslant \frac{1}{e}(\|y\|_p^p)$$

(using $|t \log|t|\,| \leqslant 1/e$, $|t| \leqslant 1$). Now,

$$\left(\frac{|y_i|}{|x_i|^{q-1}}\right)^{p-1} = \frac{|y_i|^{p-1}}{|x_i|};$$

so

$$\left|\sum_{i \in K} x_i y_i \left(\log \frac{|x_i|}{|y_i|^{p-1}}\right)\right| = \left|\sum_{i \in K} \frac{y_i}{|x_i|^{q-1}} \left(\log \frac{|x_i|}{|y_i|^{p-1}}\right) x_i |x_i|^{q-1}\right|$$

$$\leqslant \frac{(p-1)}{e}\|x\|_q^q.$$

The inequality follows from combining these two estimates.

PROOF OF 5.1. We suppose that ψ induces the quasi-linear map $H: l_q \to l_q$, so that for $x \in \mathbf{R}^\infty$,

$$(H(x))_n = x_n \psi\left(-\log \frac{|x_n|}{\|x_n\|}\right).$$

Let $\mathbf{R}_q^\infty$ denote the space $\mathbf{R}^\infty$ with the l_q norm, and define $T: \mathbf{R}_q^\infty \oplus_H \mathbf{R}_q^\infty \to l_p(\phi)^*$ by $T(w, z)(x, y) = \langle y, w\rangle + \langle x, z\rangle$ for x,y in $\mathbf{R}^\infty$. The inequalities which follow show that $T(w, z)$ defines a continuous linear functional and that T is a bounded operator.

Write

$$\langle y, w \rangle + \langle x, z \rangle = \langle x - F(y), z \rangle + \langle F(y), z \rangle$$
$$+ \langle y, H(z) \rangle + \langle y, w - H(z) \rangle.$$

By definition of F, H, and ψ,

$$|\langle F(y), z \rangle + \langle y, H(z) \rangle|$$

$$= \left| \sum_{i=1}^{n} y_i z_i \left[\phi\left(\log \frac{\|y\|}{|y_i|} \right) + \psi\left(\log \frac{\|z\|}{|z_i|} \right) \right] \right|$$

$$= \left| \sum_{i=1}^{n} y_i z_i \left[\phi\left(\log \frac{\|y\|}{|y_i|} \right) - \phi\left(\log \left(\frac{\|z\|}{|z_i|} \right)^{1/(p-1)} \right) \right] \right|$$

$$\leqslant L \sum_{i=1}^{n} \left| y_i z_i \log\left(\frac{\|y\|}{|y_i|} \cdot \left(\frac{|z_i|}{\|z\|} \right)^{1/(p-1)} \right) \right|$$

$$= \frac{L}{p-1} \sum_{i=1}^{n} \left| y_i z_i \log\left(\left(\frac{\|y\|}{|y_i|} \right)^{p-1} \cdot \frac{|z_i|}{\|z\|} \right) \right|.$$

Here, L is the Lipschitz constant of ϕ; and if any y_i or z_i is zero, the corresponding term in the sum is taken to be zero.

The last expression above equals

$$\frac{L}{p-1} \sum_{i=1}^{n} |y_i| z_i^* \log\left[\frac{\|y\|^{p-1} |z_i^*|}{|y_i|^{p-1} \|z^*\|} \right]$$

where

$$z_i^* = |z_i| \operatorname{sgn} \log\left[\frac{\|y\|^{p-1} |z_i|}{|y_i|^{p-1} \|z\|} \right].$$

Now let $u_i = |y_i|/\|y\|$ and let $v_i = z_i^*/\|z^*\|$. Then

$$|\langle F(y), z \rangle + \langle y, H(z) \rangle| \leqslant \frac{L}{p-1} \|y\| \|z\| \sum_{i=1}^{n} u_i v_i \log \frac{|v_i|}{|u_i|^{p-1}}$$

$$\leqslant \frac{Lp}{(p-1)e} \|y\| \|z\|$$

by Lemma 5.2.

Letting $K = \max\{ Lp/((p-1)e), 1 \}$, we see from the above and Hölder's inequality that

$$|\langle y, w \rangle + \langle x, z \rangle| \leqslant K(\|x - F(y)\| \|z\| + \|y\| \|z\| + \|y\| \|w - H(z)\|)$$
$$\leqslant K\|(x, y)\| \|(w, z)\|.$$

This shows that $T(w, z)$ is continuous on $\mathbf{R}_p^\infty \oplus_F \mathbf{R}_p^\infty$ and hence that $T(w, z)$ defines an element of $l_p(\phi)^*$; it also shows that T is a bounded operator from $\mathbf{R}_q^\infty \oplus_H \mathbf{R}_q^\infty$ into $l_p(\phi)^*$.

It is easy to check that T induces an equivalence between the diagrams

$$0 \to \mathbf{R}_q^\infty \to \mathbf{R}_q^\infty \oplus_H \mathbf{R}_q^\infty \to \mathbf{R}_q^\infty \to 0$$

and

$$0 \to \mathbf{R}_q^\infty \to l_p(\phi)^* \to \mathbf{R}_q^\infty \to 0.$$

The equivalence continues to hold when we pass to the completions of the spaces in the diagrams, and the proof of the theorem is complete.

We thank Haskell Rosenthal for simplifying the original proof of Lemma 5.2; the proof presented here is essentially due to him. We also thank Heinrich Lotz for pointing out an error in an earlier version of the deduction of Theorem 5.1 from Lemma 5.2.

We now turn to the structure of basic sequences in $l_p(\phi)$. For convenience, we suppose that ϕ is differentiable with bounded derivative. As before, the quasi-additive map f induced by ϕ is $f(t) = t\phi(-\log t)$. Then

$$f'(t) = \phi(-\log t) - \phi'(-\log t),$$

so that $|f|$ is increasing near 0 provided $|\phi(t)| \to \infty$ as $t \to \infty$. Indeed, for $p > 1$, $t^{-1}|f(t)|^p$ is increasing near 0.

Notation. $l_{p,f}$ is the space of all real sequences (x_n) such that $\sum_{n=1}^\infty |f(\tau x_n)|^p < \infty$ for all $\tau > 0$.

This is the usual Orlicz sequence space l_M, where $M(t) = |f(t)|^p$ in a neighborhood of 0 and M is extended so as to be an increasing function. The space l_M is a Banach space, since $t^{-1}M(t)$ is increasing near 0 and M is therefore equivalent to a convex Orlicz function.

LEMMA 5.3. *Let (v_n) be a normalized block basic sequence in l_p and let $w_n = (F(v_n), v_n)$ in $l_p(\phi)$. Then*

(a) *the sequence (w_n) is a basic sequence;*

(b) *if $\lim_{t\to\infty} \phi'(t) = 0$ monotonically, then (w_n) has a subsequence equivalent either to the usual basis of l_p or the usual basis of $l_{f,p}$;*

(c) *if $\phi(t) = ct$, $c \neq 0$, then (w_n) is equivalent to the usual basis of $l_{f,p}$.*

PROOF. (a) The sequence (w_n) is block basic with respect to the basis constructed in Corollary 4.9, and hence is a basic sequence.

(b) Assume first that $\sum_{n=1}^\infty t_n w_n$ converges. Then $\sum_{n=1}^\infty t_n v_n$ converges, so $\sum_{n=1}^\infty |t_n|^p < \infty$.

Now suppose $\sum_{n=1}^\infty |t_n|^p < \infty$. Let

$$\sigma_N = \left(\sum_{n=1}^N |t_n|^p \right)^{1/p}, \qquad n \in \mathbf{N},$$

and

$$\sigma = \left(\sum_{n=1}^{\infty} |t_n|^p \right)^{1/p}.$$

Assume that $v_n = \sum_{k=l_{n-1}+1}^{l_n} v_n(k)e_k$, where (l_n) is an increasing sequence of integers. Then

$$\left\| \sum_{n=1}^{N} t_n w_n \right\| = \sigma_N + \left(\sum_{n=1}^{N} \sum_{k=l_{n-1}+1}^{l_n} |t_n|^p |v_n(k)|^p \left| \phi \left(\log \frac{\sigma_N}{|t_n v_n(k)|} \right) \right. \right.$$

$$\left. \left. - \phi \left(\log \frac{1}{|v_n(k)|} \right) \right|^p \right)^{1/p}.$$

Since $\sigma_N \to \sigma$ and ϕ is Lipschitz, it is easy to see that if

$$\lim_{N \to \infty} \left\| \sum_{n=1}^{N} t_n w_n \right\|$$

exists, the limit must be

$$\sigma + \left(\sum_{n=1}^{\infty} \sum_{k=l_{n-1}+1}^{l_n} |t_n|^p |v_n(k)|^p \left| \phi \left(\log \frac{\sigma}{|t_n v_n(k)|} \right) - \phi \left(\log \frac{1}{|v_n(k)|} \right) \right|^p \right)^{1/p}.$$

$$(**)$$

Hence, since (w_n) is boundedly complete ($l_p(\phi)$ is reflexive), $\sum_{n=1}^{\infty} t_n w_n$ converges if and only if the expression $(**)$ is finite.

By assumption, $\lim_{t \to \infty} \phi'(t) = 0$ monotonically, so that

$$\left| \phi \left(\log \frac{\sigma}{|t_n v_n(k)|} \right) - \phi \left(\log \frac{1}{|v_n(k)|} \right) \right|$$

$$\leqslant \left| \phi \left(\log \frac{\sigma}{|t_n|} \right) - \phi(0) \right| = \left| \phi \left(\log \frac{\sigma}{|t_n|} \right) \right|.$$

Thus $\sum_{n=1}^{\infty} t_n w_n$ converges provided

$$\sum_{n=1}^{\infty} |t_n|^p \left| \phi \left(\log \frac{\sigma}{|t_n|} \right) \right|^p < \infty;$$

or, since ϕ is Lipschitz, provided

$$\sum_{n=1}^{\infty} |f(t_n)|^p < \infty. \qquad (***)$$

We now have two cases.

(i) If $a_n = \max\{|v_n(k)|: l_{n-1} + 1 \leqslant k \leqslant l_n\} \geqslant \varepsilon > 0$ for all n, we show that the finiteness of $(**)$ implies the finiteness of $(***)$. (Then (w_n) is equivalent to the usual basis of $l_{f,p}$.) In this case, the expression $(**)$ is greater than

$$\varepsilon\left(\sum_{n=1}^{\infty}|t_n|^p\left|\phi\left(\log\frac{\sigma}{|t_n\varepsilon|}\right)-\phi\left(\log\frac{1}{\varepsilon}\right)\right|^p\right)^{1/p}$$

$$\geqslant \varepsilon\left(\sum_{n=1}^{\infty}|t_n|^p\left|\phi\left(\log\frac{\sigma}{|t_n|}\right)\right|^p\right)^{1/p}-\varepsilon(L+1)\phi(\log 1/\varepsilon)\sigma;$$

this implies (∗∗∗) is finite. (L is the Lipschitz constant of ϕ.)

(ii) If $\inf_n a_n = 0$, there is a subsequence (v_{k_n}) of (v_n) such that $|\phi'(t)| \leqslant 2^{-n}$ for $t \geqslant \log(1/a_{k_n})$. For ease of notation we assume $k_n = n$. Then

$$\left|\phi\left(\log\frac{\sigma}{|t_n v_n(k)|}\right)-\phi\left(\log\frac{1}{|v_n(k)|}\right)\right| \leqslant 2^{-n}\log\frac{\sigma}{|t_n|}.$$

Now, using that $|t \log \sigma/t| \leqslant \sigma e^{-1}$ for $0 \leqslant t \leqslant \sigma$, we have that the expression (∗∗) is bounded above by

$$\sigma + \left(\sum_{n=1}^{\infty}2^{-np}|t_n|^p\left(\log\frac{\sigma}{|t_n|}\right)^p\right)^{1/p} \leqslant \sigma + \sigma e^{-1}\left(\sum_{n=1}^{\infty}2^{-np}\right)^{1/p} < \infty.$$

Thus in this case, (v_{k_n}) is equivalent to the usual basis of l_p. The alternatives (i) and (ii) imply (b).

For the proof of (c), note that if $\phi(t) = ct$, $c \neq 0$, then the expression (∗∗) equals

$$\sigma + |c|\left(\sum_{n=1}^{\infty}|t_n|^p\left|\log\frac{\sigma}{|t_n|}\right|^p\right)^{1/p};$$

thus (w_n) is equivalent to the usual basis of $l_{f,p}$.

THEOREM 5.4. *Suppose that either* $\lim_{t\to\infty}\phi'(t) = 0$ *monotonically or* $\phi(t) = ct$, $c \neq 0$. *Then every normalized basic sequence in* $l_p(\phi)$ *has a subsequence equivalent either to the usual basis of* l_p *or to the usual basis of* $l_{f,p}$.

PROOF. Suppose that $w_n = (u_n, v_n)$ is a normalized basic sequence in $l_p(\phi)$. If $\|v_n\| \to 0$, then

$$\|(u_n, v_n) - (u_n - F(v_n), 0)\| = \|(F(v_n), v_n)\| \to 0.$$

Also, $(u_n - F(v_n), 0)$ is in $j(l_p)$. Hence, by a standard perturbation of bases argument, (w_n) has a subsequence equivalent to a normalized basic sequence in l_p. From this the result follows easily.

Now suppose $\|v_n\| \nrightarrow 0$. Then by passing to a subsequence we may assume that $\|v_n\| \geqslant \varepsilon > 0$ for all n. From the fact that (w_n) is shrinking and the characterization of $l_p(\phi)^*$, it follows that $v_n \to 0$ weakly. Again, by passing to a subsequence, we may suppose that there is a block basic sequence (y_n) in l_p with $\|v_n - y_n\| \leqslant 2^{-n}$.

By passing to a further subsequence and using Lemma 5.3, we may suppose

that $(F(y_n), y_n)$ is a basic sequence equivalent either to the usual basis of l_p or to the usual basis of $l_{f,p}$. Let $z_n = (F(y_n), y_n)$; then $z_n \to 0$ weakly.

If $\|w_n - z_n\| \to 0$, a subsequence of (w_n) is equivalent to a subsequence of z_n; an application of Lemma 5.3 completes the argument in this case.

If $\|w_n - z_n\| \nrightarrow 0$, by passing to a further subsequence we may assume that $(w_n - z_n)$ is a basic sequence. However, $w_n - z_n = (u_n - F(y_n), v_n - y_n)$, and $\|v_n - y_n\| \to 0$. Hence by the very first part of the argument we may assume that $(w_n - z_n)$ is equivalent to the usual basis of l_p.

Thus if $\sum_{n=1}^{\infty} t_n w_n$ converges, then $\sum_{n=1}^{\infty} t_n v_n$ converges, and hence $\sum_{n=1}^{\infty} |t_n|^p < \infty$. Then $\sum_{n=1}^{\infty} t_n (w_n - z_n)$ and hence $\sum_{n=1}^{\infty} t_n z_n$ converge. Conversely, if $\sum_{n=1}^{\infty} t_n z_n$ converges, then since $\sum_{n=1}^{\infty} t_n y_n$ converges, certainly $\sum_{n=1}^{\infty} |t_n|^p < \infty$; and we may reverse the reasoning.

The converse is trivial by using the sequences $(e_n, 0)$ and $(0, e_n)$. This completes the proof of Theorem 5.4.

COROLLARY 5.5. *If we take*
$$\phi_r(t) = t, \qquad 0 \leqslant t \leqslant 1,$$
$$= t^r, \qquad 1 < t < \infty,$$
then the spaces $l_p(\phi_r)$ $(0 < r \leqslant 1)$ are mutually nonisomorphic.

This is immediate from Theorem 5.4, since $l_p(\phi_r)$ contains $l_{f,p}$, where f_r is induced by ϕ_r. Since f_r and f_s are not equivalent at 0 for $r \neq s$ (see [5, 4.a.5]), the result follows.

Strictly speaking, in order to apply Theorem 5.4 we should first smooth ϕ_r.

COROLLARY 5.6. *Let $\phi(t) = ct$ $(c \neq 0)$ and let $(w_n) = (u_n, v_n)$ be a normalized basic sequence in $l_p(\phi)$ with inf $\|v_n\| > 0$. Then (w_n) has a subsequence equivalent to the usual basis of $l_{f,p}$.*

PROOF. This follows from an examination of the proof of Theorem 5.4. By Lemma 5.3(c), the sequence (z_n) in the proof of Theorem 5.4 is equivalent to the usual basis of $l_{f,p}$.

6. The Banach spaces Z_p, $1 < p < \infty$. In this section we examine more carefully the extremal example of the class of spaces we have constructed. For fixed p, the spaces $l_p(\phi)$ obtained by taking $\phi(t) = ct$, $c \neq 0$, are projectively equivalent twisted sums and hence isomorphic as Banach spaces.

Notation. Z_p denotes the (isomorphism class of) the Banach space $l_p(\phi)$, $\phi(t) = ct, t \neq 0$.

The next theorem summarizes the properties of Z_p obtained so far.

THEOREM 6.1. *For $1 < p < \infty$, Z_p is a reflexive Banach space with a basis. Z_p may be normed in such a way that it has a closed subspace M isometric to l_p with Z_p/M also isometric to l_p. Further, Z_p^* is isometric to Z_q, where $1/p + 1/q = 1$. However, Z_p is not isomorphic to l_p.*

We claim that Z_p is, in certain senses, an extremal solution to the three-space problem for l_p. In this section we will make the meaning of this more precise.

Let (r_i) denote the sequence of Rademacher functions. For a Banach space X and for $0 < p \leq 2$, the number $a_{n,p}(X)$ is defined to be the least constant a such that

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^p dt \right)^{1/p} \leq a \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}$$

for all $x_1, \ldots, x_n$ in X. The space X is of *type p* if $\sup_n a_{n,p}(X) < \infty$. Enflo, Lindenstrauss, and Pisier [2, Theorem 3] show that if X is a twisted sum of two Banach spaces of type 2, then $a_{n,2}(X) = O((\log n)^\alpha)$ for some constant α. Our next result sharpens this somewhat.

THEOREM 6.2. (a) *Let X be a twisted sum of two Banach spaces which are isomorphic to subspaces of L_p $(1 < p \leq 2)$. Then $a_{n,p}(X) = O(\log n)$.*
(b) *There is a constant $c > 0$ such that $a_{n,p}(Z_p) \geq c \log n$ $(1 < p \leq 2)$.*

PROOF. The space X is isomorphic to an isometric twisted sum of subspaces Y and Z of L_p. Then $a_{n,p}(Y) = a_{n,p}(Z) = 1$ for all n. Now the proof of Theorem 3 of [2] shows that if $p = 2$, $a_{n^2,p}(X) \leq 2a_{n,p}(X) + 1$, $n = 1, 2, \ldots$. The same inequality for $1 < p < 2$ follows from identical reasoning.
Thus if $a_{2,p}(X) = k$, then

$$a_{2^{2^n},p}(X) \leq 2^n k + 2^n - 1 \leq (k+1)2^n, \qquad n = 1, 2, \ldots.$$

Since $a_{n,p}(X)$ is increasing in n we easily obtain (a).
(b) Take $\phi(t) - t$; then Z_p is isomorphic to $l_p(\phi)$. In $l_p(\phi)$ let $u_n = (0, e_n)$. Then $\|u_n\| = 1$ and for $0 \leq t \leq 1$,

$$\left\| \sum_{i=1}^n r_i(t) u_i \right\| = \left\| F\left(\sum_{i=1}^n r_i(t) e_i \right) \right\| + \left\| \sum_{i=1}^n r_i(t) e_i \right\|$$

$$= \left\| \sum_{i=1}^n r_i(t) \log(n^{1/p}) e_i \right\| + \left\| \sum_{i=1}^n r_i(t) e_i \right\|$$

$$= \left(\frac{1}{p} (\log n) + 1 \right) n^{1/p};$$

while $(\sum_{i=1}^n \|u_i\|^p)^{1/p} = n^{1/p}$. Of course this calculation used the given quasinorm on $l_p(\phi)$, and we only conclude that for the norm on Z_p, $a_{n,p}(Z_p) \geq c \log n$ for some $c > 0$. This completes the proof.
In a similar vein we have

THEOREM 6.3. *For $1 < p < \infty$ there is a sequence of $2n$-dimensional normed spaces E_n satisfying*

(a) *each E_n has an n-dimensional subspace F_n such that F_n and E_n/F_n are isometric to l_p^n;*

(b) *if γ_n denotes the least norm of a projection of E_n onto F_n, then $\gamma_n \geqslant c \log n$ for some $c = c(p) > 0$.*

PROOF. By duality it is enough to consider the case $1 < p \leqslant 2$. Let $\phi(t) = t$; in $l_p(\phi) = l_p \oplus_F l_p$, let E_n be the linear span of $(0, e_j)$, $(e_j, 0)$ $(1 \leqslant j \leqslant n)$ and let F_n be the linear span of $(e_j, 0)$ $(1 \leqslant j \leqslant n)$.

Then F_n is isometric to l_p^n and E_n/F_n is isometric to l_p^n, although E_n as a subspace of $l_p(\phi)$ is only a quasi-normed space. However, $l_p(\phi)$ is isomorphic to the Banach space Z_p, and so we may consider E_n and F_n as subspaces of Z_p with its (Banach space) norm. Thus for some $K = K(p) < \infty$, F_n and E_n/F_n are K-isomorphic to l_p^n. Now by using the technique of Proposition 2.2, the (normed) spaces E_n may be *uniformly* renormed so that F_n and E_n/F_n are isometric to l_p^n. It therefore suffices to prove the result using the original quasi-norm on $l_p(\phi)$.

Suppose that $P: E_n \to F_n$ is a projection; then, for x in l_p^n,

$$\|P(F(x), x)\| \leqslant \|P\| \cdot \|x\|$$

where $F: l_p \to l_p$ is the induced quasi-linear map. Since P is a projection, it has the form $P(x, y) = (x + Ay, 0)$, where $A: l_p^n \to l_p^n$ is a linear map. Hence

$$\|F(x) + Ax\| \leqslant \|P\| \cdot \|x\|$$

and so $\|Ae_j\| \leqslant \|P\|$ for $j \leqslant n$ (since $F(e_j) = 0$). Now

$$\left[\int_0^1 \left\| F\left(\sum_{j=1}^n r_j(t)e_j\right) + \sum_{j=1}^n r_j(t)Ae_j \right\|^p dt \right]^{1/p}$$

$$\leqslant \|P\| \left\| \sum_{j=1}^n r_j(t)e_j \right\| = \|P\| n^{1/p}.$$

However,

$$\left[\int_0^1 \left\| F\left(\sum_{j=1}^n r_j(t)e_j\right) + \sum_{j=1}^n r_j(t)Ae_j \right\|^p dt \right]^{1/p}$$

$$\geqslant \left[\int_0^1 \left\| F\left(\sum_{j=1}^n r_j(t)e_j\right) \right\|^p dt \right]^{1/p} - \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t)Ae_j \right\|^p dt \right)^{1/p}$$

$$\geqslant \frac{1}{p}(\log n)n^{1/p} - \left(\sum_{j=1}^n \|Ae_j\|^p \right)^{1/p}$$

$$\geqslant n^{1/p}((1/p)\log n - \|P\|).$$

Hence $(1/p)\log n - \|P\| \leqslant \|P\|$, i.e., $\|P\| \geqslant (1/2p)\log n$.

This completes the proof.

REMARKS. (1) The construction of a sequence of spaces E_n as above with $\gamma_n \to \infty$ is the essential part of the solution to the three-space problem given by Enflo, Lindenstrauss, and Pisier in [2]. In their construction, $\gamma_n \geqslant c(\log n)^{1/2}$; the result above improves this estimate slightly.

A theorem of Figiel, Lindenstrauss, and Milman [2a, Theorem 6.5] shows that γ_n must be $O(\log n)^2$. Our Theorem 6.2(a) obtains the estimate in their proof: $a_{n,2}(X) = O(\log n)$.

(2) We conjecture that the rate of growth $\gamma_n \sim \log n$ is the best possible here. That is, there exists $d > 0$ such that whenever dim $E_n = 2n$ and F_n is an n-dimensional subspace of E_n with $F_n \cong l_p^n \cong E_n/F_n$, then there is a projection of E_n onto F_n of norm at most $d(\log n + 1)$.

It is perhaps worth noting that the Radamacher averaging technique used in Theorem 6.2 can be used to show directly that the pair (l_p, l_p) does not split for $1 < p \leqslant 2$.

ALTERNATE PROOF OF 4.5 FOR $1 < p \leqslant 2$. Let $\phi(t) = t$, $t \in \mathbf{R}$, and let F be the induced quasi-linear map. If $l_p(\phi)$ is trivial, there are a linear map $A: \mathbf{R}^\infty \to l_p$ and a constant M such that $\|F(x) - Ax\| \leqslant M\|x\|$, $x \in \mathbf{R}^\infty$. Since $F(e_i) = 0$ for all i, we have that $\|Ae_i\| \leqslant M$ for all i.

Now for $0 \leqslant t \leqslant 1$ and $n \in \mathbf{N}$, let $x_n(t) = \sum_{i=1}^n r_i(t)e_i$. We have that

$$\|x_n(t)\| = n^{1/p} \quad \text{and} \quad \|F(x_n(t))\| = \frac{1}{p}(\log n)n^{1/p}.$$

Now,

$$\|A(x_n(t))\| \geqslant (1/p)(\log n)n^{1/p} - Mn^{1/p}. \tag{1}$$

On the other hand,

$$\left(\int_0^1 \|A(x_n(t))\|^p \, dt\right)^{1/p} = \left(\int_0^1 \left\|\sum_{i=1}^n r_i(t)Ae_i\right\|^p \, dt\right)^{1/p}$$

$$\leqslant M'Mn^{1/p} \tag{2}$$

(again, as in Theorem 6.2, using that l_p is of type p for $p \leqslant 2$). Inequalities (1) and (2) cannot both hold for large n, and the argument is complete. (Note that the argument is valid for $0 < p \leqslant 1$ as well!)

To conclude the paper we describe another sense in which Z_p is an extremal twisted sum of l_p's.

We need two definitions. If X and Y are Banach spaces, an operator $S: X \to Y$ is *strictly singular* if for any infinite-dimensional Banach space M and any operator $A: M \to X$, the composition SA fails to be an isomorphism. Dually, S is *strictly co-singular* if for any infinite-dimensional Banach space N and any operator $B: Y \to N$, the composition BS fails to be an open map. It

is easy to construct a quotient map which is strictly singular (e.g., an operator from l_1 onto l_2), or dually, an isomorphism which is strictly co-singular.

However, $0 \to l_p \xrightarrow{j} Z_p \xrightarrow{q} l_p \to 0$ provides an example of a short exact sequence such that j is strictly co-singular and q is strictly singular. We know of no other example of this.

THEOREM 6.4. *For $1 < p < \infty$, the map j: $l_p \to Z_p$ is strictly co-singular and the map q: $Z_p \to l_p$ is strictly singular.*

PROOF. By Theorem 5.1 the adjoint of the short exact sequence

$$0 \to l_p \to Z_p \to l_p \to 0$$

is (projectively equivalent to) the short exact sequence

$$0 \to l_q \to Z_q \to l_q \to 0,$$

$1/p + 1/q = 1$. So by duality we only need show that q: $Z_p \to l_p$ is strictly singular. Assume not; then (since l_p is hereditarily l_p), there is a normalized basic sequence (w_n) in Z_p equivalent to the usual l_p basis such that $\|qw_n\| \geqslant \varepsilon > 0$ for all n. By Corollary 5.6 this is impossible, and the proof is complete.

We now examine further the strictly singular operators on Z_p. For each of the known prime Banach spaces $X = l_p$ ($1 \leqslant p \leqslant \infty$) or c_0, the following property holds: if T: $X \to Y$ is not strictly singular, there is a subspace X_0 of X isomorphic to X such that T is an isomorphism on X_0. For l_p ($1 \leqslant p < \infty$) and c_0 this is well known. For l_∞, this follows from a theorem of Rosenthal [10, Theorem 3.7]: if T: $l_\infty \to Y$ is not an isomorphism on any subspace isomorphic to l_∞, then T is weakly compact. Since T has the Dunford-Pettis property, T is not an isomorphism on any infinite-dimensional subspace of l_∞. It turns out that Z_p has a similar but weaker property.

THEOREM 6.5. *If T: $Z_p \to Z_p$ is not strictly singular, there is a subspace W of Z_p which is isomorphic to Z_p such that T is an isomorphism on W.*

PROOF. Since Z_p is a twisted sum of l_p's it is hereditarily l_p; hence there is a subspace M of Z_p, M isomorphic to l_p, such that $T|M$ is an isomorphism. Let (u_n) be a basis of M equivalent to the usual l_p basis. Then by Corollary 5.6, $\|qu_n\| \to 0$ and $\|qTu_n\| \to 0$. By the first part of the proof of Theorem 5.4, we may assume (by passing to subsequences) that there are basic sequences (v'_n) and (w'_n) in $j(l_p)$ such that

$$\|u_n - v'_n\| \to 0 \quad \text{and} \quad \|Tu_n - w'_n\| \to 0.$$

Passing to subsequences again, we may assume that there are block basic sequences (v_n) and (w_n) in l_p such that

$$\|u_n - jv_n\| \leqslant 2^{-n}, \quad \|Tu_n - jw_n\| \leqslant 2^{-n}, \quad n \in \mathbf{N}.$$

Now for large enough N we may construct bounded operators $A: Z_p \to Z_p$ and $B: Z_p \to Z_p$ with $\|A\| \leq \frac{1}{2}$, $\|B\| \leq \frac{1}{2}$, and such that

$$A(jv_n) = u_n - jv_n, \qquad n \geq N,$$
$$B(Tu_n) = jw_n - Tu_n, \qquad n \geq N.$$

(The operators A and B are easily defined in terms of Hahn-Banach extensions of the coefficient functionals for the bases (jv_n) and (Tu_n), respectively.) Then

$$(I + B)T(I + A)jv_n = jw_n, \qquad n \geq N.$$

Let $S = (I + B)T(I + A)$. Since $I + B$ and $I + A$ are invertible, it suffices to prove the result for S.

Now define $V: \mathbf{R}^\infty \oplus \mathbf{R}^\infty \to \mathbf{R}^\infty \oplus \mathbf{R}^\infty$ by

$$V(e_n, 0) = (v_n, 0) = j(v_n), \qquad V(0, e_n) = (F(v_n), v_n)$$

and extending linearly. We will show that V is an isomorphism; then after extension V may be regarded as an isomorphism on Z_p. For x, y in $\mathbf{R}^\infty$,

$$V(x, y) = \left(\sum x_n v_n + \sum y_n F(v_n), \sum y_n v_n \right)$$

where the sums are finite. Hence (using the original quasi-norm on Z_p),

$$\|V(x, y)\| = \left\| \sum x_n v_n + \sum y_n F(v_n) - F\left(\sum y_n v_n \right) \right\| + \left\| \sum y_n v_n \right\|.$$

If $v_n = \sum_{l_{n-1}+1}^{l} v_n(k)e_k$, then

$$\sum_n y_n F(v_n) - F\left(\sum y_n v_n \right) = \sum_n \sum_{k=l_{n-1}+1}^{l_n} y_n v_n(k) \log\left(\frac{\|v_n\| \, |y_n|}{\left\| \sum y_n v_n \right\|} \right) e_k$$

$$= \sum_n y_n \log\left(\frac{\|v_n\| \, |y_n|}{\left\| \sum y_n v_n \right\|} \right) v_n.$$

Now there exist positive constants α, β so that $\alpha \leq \|v_n\| \leq \beta$, $n = 1, 2 \ldots$. Hence

$$\left\| \sum x_n v_n + \sum y_n F(v_n) - F\left(\sum y_n v_n \right) \right\| \leq \beta \left[\sum_n \left| x_n + y_n \log \frac{\|v_n\| \, |y_n|}{\left\| \sum y_n v_n \right\|} \right|^p \right]^{1/p}$$

$$\leq \beta \left(\sum_n \left| x_n + y_n \log \frac{|y_n|}{\|y\|} \right|^p \right)^{1/p} + \beta \left[\sum_n |y_n|^p \left| \log \frac{\|v_n\| \, \|y\|}{\left\| \sum y_n v_n \right\|} \right|^p \right]^{1/p}$$

$$\leq \beta \|x - F(y)\| + \beta \left(\log \frac{\beta}{\alpha} \right) \|y\|.$$

Hence $\|V\| \leqslant \beta + \beta \log(\beta/\alpha)$. For the other direction,

$$\left\|\sum x_n v_n + \sum y_n F(v_n) - F\left(\sum y_n v_n\right)\right\| \geqslant \alpha\left[\sum_n \left|x_n + y_n \log \frac{\|v_n\| \, |y_n|}{\left\|\sum y_n v_n\right\|}\right|^p\right]^{1/p}$$

$$\geqslant \alpha\|x - F(y)\| - \alpha\left(\log \frac{\beta}{\alpha}\right)\|y\|.$$

Thus,

$$\|V(x, y)\| \geqslant \max(\alpha\|x - F(y)\| - \alpha(\log(\beta/\alpha))\|y\|, \alpha\|y\|).$$

If $\|x - F(y)\| \geqslant 2(\log(\beta/\alpha))\|y\|$, then

$$\|V(x, y)\| \geqslant \frac{\alpha}{2}\|x - F(y)\| \geqslant \left(\frac{\alpha \log(\beta/\alpha)}{1 + 2\log(\beta/\alpha)}\right)\|(x, y)\|;$$

if $\|x - F(y)\| \leqslant 2(\log(\beta/\alpha))\|y\|$, then

$$\|V(x, y)\| \geqslant \|y\| \geqslant (1 + 2\log(\beta/\alpha))^{-1}\|(x, y)\|.$$

Thus, V is an isomorphism.

Similarly, if we let

$$W(e_n, 0) = (w_n, 0) = j(w_n), \qquad W(0, e_n) = (F(w_n), w_n),$$

then W defines an isomorphism of Z_p into Z_p.

Now consider the map $W - SV$. By construction, $(W - SV)(e_n, 0) = 0$, $n \geqslant N$. Hence there is a subspace Y of finite codimension such that $(W - SV)|Y$ factors through $q|Y$. It follows that $W - SV$ is strictly singular. As W is an isomorphism, it follows that there exists an integer n_0 such that the restriction of SV to $\overline{\lin}\{(e_j, 0), (0, e_j) : j \geqslant n_0\}$ is an isomorphism. Hence S is an isomorphism of $V(\overline{\lin}\{(e_j, 0), (0, e_j) : j \geqslant n_0\})$, which is isomorphic to Z_p. As remarked earlier, T is then an isomorphism on a subspace isomorphic to Z_p, and the proof is complete.

COROLLARY 6.6. *Every infinite-dimensional complemented subspace of Z_p contains a subspace isomorphic to Z_p.*

COROLLARY 6.7. *Z_p has no complemented subspace isomorphic to l_p.*

COROLLARY 6.8. *Z_2 can not be embedded in any L_p ($1 \leqslant p < \infty$).*

PROOF. If $p > 2$, then every Hilbertian subspace of L_p is complemented in L_p by a result of Kadec and Pełczynski [3, 3. Corollary 1]; the result then follows from 6.7. Assume $1 \leqslant p \leqslant 2$ and Z_2 embeds in L_p; then by a theorem of Rosenthal [11, Theorem 8] Z_2 embeds in $L_{p'}$ for some p' in (1, 2]. Now by a theorem of Pełczynski and Rosenthal [6, Theorem 3.1], since Z_2 contains a Hilbertian subspace it contains a complemented Hilbertian subspace; again this violates Corollary 6.7. This finishes the proof.

COROLLARY 6.9. Z_p *has no complemented infinite-dimensional subspace with an unconditional basis.*

PROOF. By 5.4, any basic sequence in Z_p has a subsequence which spans either l_p or $l_{f,p}$; if the original sequence is unconditional, the span of the subsequence is complemented in the span of the original sequence. Also, the space $l_{f,p}$ contains a complemented subspace isomorphic to l_p. (This follows from the fact that the function x^p is in the set $E_{M,1}$; see, for example, [5, p. 157].)

Corollary 6.7 and the above now give the desired conclusion.

REMARK. Z_p has an unconditional decomposition into two-dimensional subspaces. To see this, simply take E_n to be the span of the two vectors $(e_n, 0)$ and $(0, e_n)$, $n = 1, 2, \ldots$; the spaces E_n form the desired decomposition.

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