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Quotients of $L_p(0,1)$ for $0 \leqslant p < 1$

by

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Abstract. One of the main results of this paper is a lifting theorem for operators from L_p, $0 \leqslant p < 1$, into a quotient space L_p/N. (The theorem is developed separately for L_0 and for L_p, $0 < p < 1$; the hypotheses on N are different in the two cases.) A corollary is that if N is a non-trivial finite dimensional subspace of L_p, $0 \leqslant p < 1$, then L_p/N is not isomorphic to L_p. Several similar results are obtained; at the end of the paper, the idea of a *K-space* (*K_p-space*) is introduced and studied in connection with the lifting theorems.

1. Introduction. Let $L_0 = L_0[0,1]$ be the space of all real (or complex) measurable functions on $[0,1]$ with the topology of convergence in measure. A. Pełczyński has asked whether the quotient of L_0 by a non-trivial finite-dimensional subspace is isomorphic to L_0. In this paper we prove a lifting theorem for operators on L_0; using this theorem, we can show that if B is a non-trivial closed subspace of L_0 which is either locally bounded or which admits a continuous linear functional, then $L_0/B \not\cong L_0$. Parallel results are developed for the spaces L_p $(0 < p < 1)$, where again we have that the quotient of L_p by a non-trivial finite-dimensional subspace cannot be isomorphic to L_p (contrasting of course with the case $p = 1$).

In Section 2, we show from certain general considerations that for $0 \leqslant p < 1$, $L_p/V \cong L_p/W$ whenever $\dim V = \dim W < \infty$. This enables us to define (L_p/n) to be the (unique) space obtained by forming the quotient of L_p by a subspace of dimension n. In Section 3 we prove our main lifting theorems and in Section 4 we apply them to show that $(L_p/n) \cong (L_p/m)$ if and only if $m = n$. We conclude Section 4 by giving an example of two isomorphic locally bounded subspaces of L_0, B_1 and B_2, such that $L_0/B_1 \not\cong L_0/B_2$.

In Section 5 we develop the idea of a *K-space*; this is an F-space X such that every short exact sequence of F-spaces $0 \to \mathbf{R} \to Y \to X \to 0$ splits. Using this notion we show that $L_0/N \cong L_0$ implies that N has no non-zero continuous linear functionals. Similar ideas for p-Banach spaces are also developed.

Throughout this paper an F-space will mean a complete metric topological vector space. An F-norm $x \to \|x\|$ on a space X is a mapping from X to $\mathbf{R}_+$ such that

(a) $\|x+y\| \leqslant \|x\|+\|y\|$ if $x, y \in X$,

(b) $\|\lambda x\| \leqslant \|x\|$ if $|\lambda| \leqslant 1$ and if $x \in X$,

(c) $\|\lambda x\| \to 0$, as $\lambda \to 0$ for each $x \in X$,

(d) $\|x\| = 0$ if and only if $x = 0$.

For $0 < p \leqslant 1$, a p-Banach space is an F-space with an F-norm $\| \ \|$ such that

(e) $\|\lambda x\| = |\lambda|^p \|x\|$ for all λ and $x \in X$.

If X and Y are p-Banach spaces and if $S: X \to Y$ is a continuous linear operator, we define

$$\|S\| = \sup(\|Sx\|: \|x\| \leqslant 1).$$

We denote by $\mathscr{L}(X)$ the space of all linear operators on X. If X is a p-Banach space, then so is $\mathscr{L}(X)$; if X is an F-space, then $\mathscr{L}(X)$ has, in general, no convenient F-norm topology. Unless otherwise stated, "linear map" and "linear operator" always refer to *continuous* maps.

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2. Transitive F-spaces. In this section we show that if V and W are two subspaces of L_p $(0 \leqslant p < 1)$ of the same finite dimension, then $L_p/V \cong L_p/W$. We approach this result through some general results on transitive F-spaces. An F-space X is said to be *transitive* if given $x, y \in X$ with $x \neq 0$, there exists $T \in \mathscr{L}(X)$ with $Tx = y$. We shall say that X is *strictly transitive*, if for any $k \in \mathbf{N}$, $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in X$ such that $\{x_1 \ldots x_k\}$ is linearly independent, there exists $T \in \mathscr{L}(X)$ with $Tx_i = y_i$.

We do not know whether a transitive F-space is strictly transitive; however, it is possible to generalize standard arguments in Banach algebra theory (cf. Rickart [6], pp. 60–62) to yield the following:

PROPOSITION 2.1. *Let X be a transitive F-space; suppose that*

(a) *X is separable,*

(b) *The centre of $\mathscr{L}(X)$ consists only of scalar multiples of the identity operator.*

Then X is strictly transitive.

If X is a p-Banach space, condition (a) *may be omitted; if X is a complex p-Banach space, then conditions* (a) *and* (b) *may be omitted.*

Proof. By [6], Lemma 2.4.3, it is enough to show that given two linearly independent elements $v, w \in X$, there exists $T \in \mathscr{L}(X)$ such that $Tv = 0$ and $Tw \neq 0$. If not, we may define a (not necessarily continuous) operator D on X by $Dx = Tw$ when $Tv = x$ (cf. [6], Theorem 2.4.6). Then D commutes with each $T \in \mathscr{L}(X)$.

It is necessary to show that $D \in \mathscr{L}(X)$; at this point we require condition (a) in general. Consider $\mathscr{L}(X)$ with the topology of pointwise convergence. Then $\mathscr{L}(X)$ is a Souslin space and by the Open Mapping Theorem, the map ε_v: $\mathscr{L}(X) \to X$ defined by $\varepsilon_v(T) = Tv$ is open. Since $D \circ \varepsilon_v = \varepsilon_w$, it follows that $D \in \mathscr{L}(X)$. If X is a p-Banach space, then so is $\mathscr{L}(X)$ with its usual topology and the Open Mapping Theorem may be applied to this topology.

Now by condition (b), D is a multiple of I and we have a contradiction. If X is a complex p-Banach space, then it may be shown that the centre of $\mathscr{L}(X)$ is a field and by Żelazko's extension of the Gelfand–Mazur theorem [11], condition (b) must hold.

It is easy to check that each of the spaces L_p $(p \geqslant 0)$ satisfies conditions (a) and (b) of the proposition and is transitive (use an argument similar to [7], pp. 253–254; see also [5]), and hence is strictly transitive.

PROPOSITION 2.2. *Suppose X is a strictly transitive F-space and $X \cong X \oplus X$; then if $\{x_1 \ldots x_n\}$ and $\{y_1 \ldots y_n\}$ are two linearly independent sets in X, there exists an invertible $T \in \mathscr{L}(X)$ such that $Tx_i = y_i$, $1 \leqslant i \leqslant n$.*

Proof. First we prove that there exists a projection $P \in \mathscr{L}(X)$ such that $P(X) \cong X$, $(I-P)(X) \cong X$ and $\{Px_1 \ldots Px_n\}$ is linearly independent. Let F be the linear span of $\{x_1 \ldots x_n\}$; then we may choose a projection P so that $P(X) \cong X$, $(I-P)(X) \cong X$ and $\dim P(F)$ is maximal. Since $(I-P)(X) \cong X \cong X \oplus X$, there exists a projection $Q \in \mathscr{L}(X)$ such that $PQ = QP = 0$ and $Q(X) \cong (I-P-Q)(X) \cong X$. Then since $(P+Q)(X) \cong (I-P-Q)(X) \cong X$, we have $\dim(P+Q)(F) = \dim P(F)$. Hence P is one-one on $(P+Q)(F)$ and so if we have $\sum a_i Px_i = 0$, then also $\sum a_i(P+Q)x_i = 0$. Similarly we have $\sum a_i(I-Q)x_i = 0$; combining, $\sum a_i x_i = 0$ and $a_1 = a_2 = \ldots = a_n = 0$, i.e. $\{Px_1, \ldots, Px_n\}$ is linearly independent.

Now pick projections P_1 and $P_2 \in \mathscr{L}(X)$ so that $P_1(X) \cong (I-P_1)(X) \cong P_2(X) \cong (I-P_2)(X) \cong X$ and $\{P_1 x_1 \ldots P_1 x_n\}$, $\{P_2 y_1 \ldots P_2 y_n\}$ are linearly independent. Then there exists an invertible $T \in \mathscr{L}(X)$ such that $TP_1 = (I-P_2)T$. Since X is strictly transitive, there exists S: $(I-P_2)(X) \to P_2(X)$ such that $S(I-P_2)Tx_i = P_2(y_i - Tx_i)$, for $1 \leqslant i \leqslant n$. Similarly, there exists R: $P_2(X) \to (I-P_2)(X)$ so that $RP_2 y_i = (I-P_2)(y_i - Tx_i)$. Then $(I+R \circ P_2)$ and $(I+S \circ (I-P_2))$ are invertible, since $(R \circ P_2)^2 = (S \circ (I-P_2))^2 = 0$ and $(I+R \circ P_2)(I+S \circ (I-P_2))Tx_i = y_i$ $(1 \leqslant i \leqslant n)$.

We now have:

THEOREM 2.3. *If $0 \leqslant p < 1$ and V and W are two subspaces of L_p with $\dim V = \dim W < \infty$, then $L_p/V \cong L_p/W$.*

Proof. This is immediate, since there is an invertible operator T on L_p such that $T(V) = W$.

Let us denote now by (L_p/n) the quotient of L_p by an n-dimensional subspace; of course $(L_p/0) = L_p$. Theorem 2.3 guarantees that (L_p/n) is well defined.

THEOREM 2.4. *For* $0 \leqslant p < 1$, $\big(L_p/(m+n)\big) \cong (L_p/m) \oplus (L_p/n)$ *when* $m, n > 0$.

Proof. Let V be a subspace of $L_p[0, \frac{1}{2}]$ of dimension m (embedded in L_p in the obvious way) and W a subspace of $L_p[\frac{1}{2}, 1]$ of dimension n. Then

$$L_p/(m+n) \cong L_p[0,1]/(V+W)$$
$$\cong (L_p[0, \tfrac{1}{2}]/V) \oplus (L_p[\tfrac{1}{2}, 1]/W) \cong (L_p/m) \oplus (L_p/n).$$

3. Lifting theorems. Let X be a p-Banach space $(0 < p \leqslant 1)$ and N a closed subspace of X. It is easy to see that any linear operator $S\colon l_p \to X/N$ may be lifted to a linear operator $\tilde{S}\colon l_p \to X$, so that $\pi\tilde{S} = S$ where $\pi\colon X \to X/N$ is the quotient map. In the case $p = 1$, a similar lifting property holds if l_p is replaced by any $\mathscr{L}_1$-space and N is isomorphic to a complemented subspace of a dual space (this is effectively proved by Lindenstrauss [4]). Not surprisingly there is a corresponding result for the case $p < 1$. We say that a p-Banach space Y is an *$\mathscr{L}_p$-space* $(0 < p < 1)$, if there is an increasing net $\{Y_a\colon a \in A\}$ of finite-dimensional subspaces of Y such that $\bigcup (Y_a\colon a \in A)$ is dense in Y and there exist linear maps $S_a\colon Y_a \to l_p^{(n_a)}$ and $T_a\colon l_p^{(n_a)} \to Y_a$ (where $n_a = \dim Y_a$), such that $\sup_a \|S_a\|\,\|T_a\| < \infty$ and $T_a S_a = I$ on Y_a. Clearly L_p is an $\mathscr{L}_p$-space.

We shall also call a p-Banach space Z *pseudo-dual* if there is a Hausdorff vector topology ϱ on Z such that the unit ball is relatively compact for ϱ. The space L_p is not pseudo-dual (see [1]), but the spaces l_p and H_p $(0 < p < 1)$ are pseudo-dual.

THEOREM 3.1. *Let Y be an $\mathscr{L}_p$-space $(0 < p \leqslant 1)$ and X a p-Banach space. Let N be a closed subspace of X and suppose N is isomorphic to a complemented subspace of a pseudo-dual p-Banach space Z. Then any operator $S\colon Y \to X/N$ may be lifted to an operator $\tilde{S}\colon Y \to X$.*

If $Y = L_p$, then the lifting is unique.

Proof. We observe that the unit ball of Z may be supposed to be ϱ-compact (by [1], Lemma 1). Then the argument is a straightforward imitation of the Lemma of [4]. We omit the details.

In the case $Y = L_p$, suppose T is any other lifting. Then $T - \tilde{S}$ maps L_p into N and there is a non-zero operator from L_p into Z. The induced map into (Z, ϱ) is compact, contradicting the results of [2].

We now give another result of a similar type for the space L_p.

THEOREM 3.2. *Let X be a p-Banach space $(p < 1)$ and let N be a closed subspace of X which is isomorphic to a q-Banach space, where $p < q \leqslant 1$. Then any linear operator $S\colon L_p \to X/N$ may be lifted uniquely to a linear operator $\tilde{S}\colon L_p \to X$.*

Proof. For $n \in \boldsymbol{N}$, let Y_n be the linear span of the functions χ_k^n $(1 \leqslant k \leqslant 2^n)$, where χ_k^n is the characteristic function of $\big((k-1)2^{-n}, k2^{-n}\big)$. Each Y_n is isometric to $l_p^{(2^n)}$. Let $S_n\colon Y_n \to X$ be a lift of $S\colon Y_n \to X/N$ with $\|S_n\| \leqslant 2\|S\|$. We shall show that for $y \in \bigcup Y_n$, $\lim_{n\to\infty} S_n y = \tilde{S}y$ exists. Then clearly $\|\tilde{S}\| \leqslant 2\|S\|$ and $\tilde{S}$ may be extended to L_p by continuity and is then a lift of S.

Since N is isomorphic to a q-Banach space, there exists a constant C such that the q-convex hull of the unit ball of N is bounded in norm by C. Hence, if $x_1, \ldots, x_n$ are in N, then

$$\Big\|\sum_{i=1}^{n} x_i\Big\| \leqslant C\Big(\sum_{i=1}^{n} \|x_i\|^{q/p}\Big)^{p/q}.$$

Now suppose $j \leqslant m \leqslant n$, and $1 \leqslant k \leqslant 2^j$; then

$$\|S_m\chi_k^j - S_n\chi_k^j\| = \Big\|(S_m - S_n)\Big(\sum_{i=1}^{2^{m-j}} \chi_{2^{m-j}k+i}^m\Big)\Big\|$$
$$\leqslant C\Big(\sum_{i=1}^{2^{m-j}} \|(S_m - S_n)\,\chi_{2^{m-j}k+i}^m\|^{q/p}\Big)^{p/q}$$
$$\leqslant 4C\|S\|\Big(\sum_{i=1}^{2^{m-j}} \|\chi_{2^{m-j}k+i}^m\|^{q/p}\Big)^{p/q}$$
$$= 4C\|S\|\,2^{-(jp/q+m(1-p/q))}$$

and so $\{S_m\chi_k^j;\ m \geqslant j\}$ is a Cauchy sequence. Hence we may find the lift $\tilde{S}$. As in Theorem 3.1, $\tilde{S}$ must be unique, since there are no non-zero operators from L_p into N (see Théorème 3.4.5 of [10] or Proposition 2 of [9], or use the argument above).

We now examine the case $p = 0$, which is rather different. Suppose X is an F-space with F-norm $\|\ \|$. For $x \in X$ we define $\sigma\colon X \to \boldsymbol{R} \cup \{\infty\}$, by

$$\sigma(x) = \sup_{t\in\boldsymbol{R}} \|tx\|.$$

In the case of L_0 with the F-norm

$$\|x\| = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\,d\mu(t),$$

we have that $\sigma(x) = \mu(\operatorname{supp} x)$.

In general, note that $\sigma(ax) = \sigma(x)$ if $a \neq 0$ and that $\sigma(x+y) \leqslant \sigma(x)+\sigma(y)$. If L is a linear subspace of X, we define $\sigma(L) = \sup(\sigma(x): x \in L)$.

We shall say that X *admits L_0-structure*, if for any $\varepsilon > 0$ there exist $n = n(\varepsilon)$ and subspaces $X_1, \ldots, X_n$ of X, such that $X = X_1 \oplus \ldots \oplus X_n$ and $\sigma(X_i) \leqslant \varepsilon$, $i = 1, 2, \ldots, n$. In addition to the obvious example of L_0 itself, any space of the type $L_0(Z)$ (all measurable functions from $[0,1]$ into an F-space Z) admits L_0-structure.

The following proposition is trivial.

PROPOSITION 3.3. *Suppose X admits L_0-structure and B is a locally bounded space. If $T: X \to B$ is continuous, then $T = 0$.*

We next prove two lemmas before giving the main lifting theorem.

LEMMA 3.4. *Suppose X is an F-space and B is a closed locally bounded subspace of X; let $\pi: X \to X/B$ be the quotient map. Let δ be chosen so that the set $\{b \in B: \|b\| \leqslant \delta\}$ is bounded.*

Then if $\xi \in X/B$ and $\sigma(\xi) \leqslant \delta/3$, there is a unique $x \in X$ such that $\pi x = \xi$ and $\sigma(x) \leqslant \delta/3$. For this x, $\sigma(x) = \sigma(\xi)$.

Proof. We can find a sequence $\{x_n\} \in X$ such that $\pi x_n = \xi$ and

$$\|nx_n\| \leqslant \left(1+\frac{1}{n}\right)\|n\xi\|, \qquad n \in N.$$

Let $u_n = x_n - x_1$ $(n \in N)$; then $u_n \in B$ and if $m \geqslant n \geqslant 2$,

$$\begin{aligned}\|n(u_n-u_m)\| &= \|n(x_n-x_m)\| \\ &\leqslant \|nx_n\|+\|nx_m\| \\ &\leqslant \left(1+\frac{1}{n}\right)\|n\xi\|+\left(1+\frac{1}{m}\right)\|m\xi\| \\ &\leqslant \left(2+\frac{1}{n}+\frac{1}{m}\right)\delta/3 \leqslant \delta.\end{aligned}$$

By choice of δ, this implies that $\{u_n\}$ is a Cauchy sequence and hence so is $\{x_n\}$. If $x = \lim x_n$, then $\sigma(x) = \sigma(\xi) \leqslant \delta/3$. If y is any other point satisfying $\pi y = \xi$ and $\sigma(y) \leqslant \delta/3$, then $x - y \in B$ and $\sigma(x-y) \leqslant \frac{2}{3}\delta$; this implies $x - y = 0$.

LEMMA 3.5. *Under the assumptions of Lemma 3.4, let Y be a linear subspace of X/B with $\sigma(Y) \leqslant \frac{1}{6}\delta$. Then there is a linear operator $h: Y \to X$ such that $\pi \circ h(\xi) = \xi$ for $\xi \in Y$.*

Proof. For $\xi \in Y$, define $h(\xi)$ to be the unique $x \in X$ such that $\pi x = \xi$ and $\sigma(x) = \sigma(\xi)$. If $\alpha, \beta \in \boldsymbol{R}$ and $\xi, \eta \in Y$, then

$$\sigma\big(\alpha h(\xi)+\beta h(\eta)\big) \leqslant \sigma(\xi)+\sigma(\eta) \leqslant \delta/3.$$

Thus

$$h(\alpha\xi+\beta\eta) = \alpha h(\xi)+\beta h(\eta),$$

and h is linear.

Now suppose $\xi_n \to 0$ in Y; choose $x_n \in X$ such that $\pi x_n = \xi_n$ and $\|x_n\| \leqslant 2\|\xi_n\|$. Then $x_n - h(\xi_n) \in B$. If $x_n - h(\xi_n) \nrightarrow 0$, we may assume, by passing to a subsequence, that for some $a > 0$ we have

$$\big\|a\big(x_n - h(\xi_n)\big)\big\| \geqslant \delta$$

(since the set $\{b \in B: \|b\| \leqslant \delta\}$ is bounded).

Then

$$\|ax_n\| \geqslant \delta - \|h(a\xi_n)\| \geqslant \delta - \tfrac{1}{6}\delta = \tfrac{5}{6}\delta.$$

This is a contradiction since $\|x_n\| \to 0$. Hence we have $x_n - h(\xi_n) \to 0$ and so $h(\xi_n) \to 0$.

THEOREM 3.6. *Suppose X admits L_0-structure, Y is an F-space, and B is a closed locally bounded subspace of Y. Then if $S: X \to Y/B$ is a linear operator, there is a unique linear operator $\tilde{S}: X \to Y$ such that $\pi \circ \tilde{S} = S$ where $\pi: Y \to Y/B$ is the quotient map.*

Proof. Choose $\delta > 0$ so that $\{b \in B: \|b\| \leqslant \delta\}$ is bounded, and then $\varepsilon > 0$, so that if $\|x\| \leqslant \varepsilon$ $(x \in X)$, then $\|Sx\| \leqslant \delta/6$. Let $X_1, \ldots, X_n$ be closed subspaces of X such that $X = X_1 \oplus \ldots \oplus X_n$ and $\sigma(X_i) \leqslant \varepsilon$. Then $\sigma\big((S)X_i\big) \leqslant \delta/6$, and so there exist linear operators h_i: $S(X_i) \to Y$, such that $\pi h_i(\xi) = \xi$, $\xi \in S(X_i)$. If we define $\tilde{S}: X \to Y$ by

$$\tilde{S}(x_1+\ldots+x_n) = \sum_{i=1}^{n} h_i S x_i, \qquad x_i \in X_i,$$

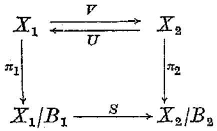
then $\tilde{S}$ is the required lifting of S.

If T is any other lifting, then $\tilde{S} - T$ maps X into B and hence $\tilde{S} = T$ by Proposition 3.3.

4. Quotient spaces of L_p $(0 \leqslant p < 1)$. In this section we treat the case $p = 0$ first and in more detail than the case $p > 0$; the arguments are analogous.

THEOREM 4.1. *Suppose X_1 and X_2 are two F-spaces with L_0-structure. Suppose B_1 and B_2 are closed locally bounded subspaces of X_1 and X_2, respectively. Then $X_1/B_1 \cong X_2/B_2$ if and only if there is an isomorphism $V: X_1 \to X_2$ mapping X_1 onto X_2 and such that $V(B_1) = B_2$.*

Proof. The "if" part is clear. For the "only if" part, let $S: X_1/B_1 \to X_2/B_2$ be an isomorphism, and let π_1, π_2 be the quotient maps. Then by Theorem 3.6, there exist lifts V, U of $S\pi_1\colon X_1 \to X_2/B_2$ and $S^{-1}\pi_2\colon X_2 \to X_1/B_1$.



Then $UV\colon X_1 \to X_1$ is a lift of $\pi_1\colon X_1 \to X_1/B_1$. By the uniqueness, $UV = I_{X_1}$; similarly $VU = I_{X_2}$, so V is an isomorphism of X_1 onto X_2. Clearly $V(B_1) \subset B_2$ and $U(B_2) = V^{-1}(B_2) \subset B_1$; hence $V(B_1) = B_2$.

COROLLARY 4.2. *If X admits L_0-structure and $B \subset X$ is a locally bounded subspace, then X/B admits L_0-structure if and only if $B = \{0\}$.*

Theorem 2.3 and Theorem 4.1 give

COROLLARY 4.3. *If B_1 and B_2 are locally bounded subspaces of L_0, then $L_0/B_1 \cong L_0/B_2$ if and only if there is an invertible operator $T\colon L_0 \to L_0$ such that $T(B_1) = B_2$.*

In particular, $(L_0/m) \cong (L_0/n)$ if and only if $m = n$, and $(L_0/1) \not\cong L_0$.

This solves the problem of Pełczyński (see Introduction). Also in this section we shall illustrate this corollary by showing that $B_1 \cong B_2$ does not imply $L_0/B_1 \cong L_0/B_2$. First however, we state the corresponding theorems for $p > 0$; the proofs are similar.

THEOREM 4.4. *Suppose B_1 and B_2 are two closed subspaces of L_p, each of which is either isomorphic to a complemented subspace of a pseudo-dual p-Banach space or to a q-Banach space where $p < q \leqslant 1$.*

Then $L_p/B_1 \cong L_p/B_2$ if and only if there is invertible operator $T\colon L_p \to L_p$ such that $T(B_1) = B_2$.

In particular, $(L_p/m) \cong (L_p/n)$ if and only if $m = n$, and $(L_p/1) \not\cong L_p$.

THEOREM 4.5. *If $B \subset L_p$ is isomorphic to a complemented subspace of a pseudo-dual p-Banach space and $B \neq \{0\}$, then L_p/B is not an $\mathscr{L}_p$-space.*

Proof. If L_p/B is an $\mathscr{L}_p$-space, then the identity map $I\colon L_p/B \to L_p/B$ may be lifted to a map $J\colon L_p/B \to L_p$. Then on L_p, $I - J\pi$ maps L_p into B. Hence, by applying the results of [2] as in Theorem 3.1, $I = J\pi$ and so $B = \{0\}$.

EXAMPLE. Let $B_1 \subset L_0$ be the closed linear span of the Rademacher functions r_k on $[0, 1]$

$$(r_k(t) = \operatorname{sgn}(\sin 2^k \pi t)),$$

and let B_2 be the closed linear span of a sequence of independent random variables normally distributed with mean zero and variance one. Then $B_1 \cong B_2 \cong L_2$; we shall show, however, that $L_0/B_1 \not\cong L_0/B_2$.

For suppose $L_0/B_1 \cong L_0/B_2$; then there is an invertible linear operator $T\colon L_0 \to L_0$ such that $T(B_1) = B_2$. By Kwapień's Representation Theorem [3], T takes the form

$$Tx(t) = \sum_{i=1}^{\infty} \varphi_i(t)\, x(\Phi_i t) \quad \text{a.e.}$$

where

(i) $\varphi_n \in L_0$, $n \geqslant 1$,

(ii) $m\{t\colon \varphi_n(t) \neq 0$ for infinitely many $n\} = 0$,

(iii) Φ_n maps $[0, 1]$ into $[0,1]$; if A is measurable, then $\Phi_n^{-1}(A)$ is measurable; if $m(A) = 0$, then $m(\Phi_n^{-1}(A) \cap \operatorname{Supp} \varphi_n) = 0$.

Thus for almost every $t \in [0, 1]$, the sequence $\{Tr_k(t)\}$ assumes only finitely many values. Hence for some j, k, with $j \neq k$, we must have $m\{t\colon Tr_j(t) = Tr_k(t)\} > 0$. However $Tr_j - Tr_k$ is normally distributed and hence $Tr_j = Tr_k$. Thus T is not injective, and we have a contradiction.

Remarks. For $p > 0$, let $x \in L_p$ be non-zero and let V be the linear span of x. Let $\mathscr{L}(L_p)$ and $\mathscr{L}(L_p/V)$ be the p-Banach algebras of all bounded linear operators on L_p and L_p/V, respectively. If $S \in \mathscr{L}(L_p/V)$, let $\hat{S}\colon L_p \to L_p$ be the unique lift of $S \circ \pi$. Then the map $S \to \hat{S}$ is an algebra homorphism, and in fact an embedding of $\mathscr{L}(L_p/V)$ into $\mathscr{L}(L_p)$. Thus $\mathscr{L}(L_p/V)$ is isomorphic to the closed subalgebra of $\mathscr{L}(L_p)$ consisting of all $T \in \mathscr{L}(L_p)$ such that $Tx \in V$. We may define a multiplicative linear functional φ on $\mathscr{L}(L_p/V)$ by

$$\varphi(S)x = \hat{S}x.$$

5. *K*-spaces. In this section, we abstract a particular property of the spaces L_p and consider it in more generality. We restrict to the real case, but the complex case is identical.

If X is an F-space, we shall say that X is a *K-space* if every short exact sequence $0 \to \mathbf{R} \to Y \to X \to 0$, with Y an F-space, splits. Alternatively, if $S\colon Y \to X$ is onto and $\dim S^{-1}(0) = 1$, then there exists an operator $T\colon X \to Y$ such that $ST = I_X$.

If X is a p-Banach space $(0 < p \leqslant 1)$, we shall say that X is a *K_p-space* if every short exact sequence $0 \to \mathbf{R} \to Y \to X \to 0$, with Y a p-Banach space, splits.

THEOREM 5.1. *An F-space [p-Banach space] X is a K-space [K_p-space] if and only if whenever Y and Z are F-spaces [p-Banach spaces] and $S\colon Y \to Z$ is a surjective operator with $\dim S^{-1}(0) = 1$, then each linear operator $T\colon X \to Z$ may be lifted to an operator $\tilde{T}\colon X \to Y$ such that $S\tilde{T} = T$.*

Proof. We prove the statement for K-spaces. Suppose X is a K-space. Let $V \subset X \oplus Y$ be the subspace of all (x, y) such that $Tx = Sy$, and define $P: V \to X$ by $P(x, y) = x$. Then $P: V \to X$ is surjective, and $\dim P^{-1}(0) = 1$. Hence there exists a linear operator $R: X \to V$ such that $PR = I_X$. Then $Rx = (x, \tilde{T}x)$; clearly $S\tilde{T} = T$.

For the converse take $Z = X$ and T to be the identity.

We remark now that if X admits L_0-structure, then X is a K-space (Theorem 3.6), and that an $\mathscr{L}_p$-space is a K_p-space.

THEOREM 5.2. *If X is an F-space [p-Banach space] and N is a closed subspace of X such that X/N is a K-space [K_p-space], then N has the Hahn–Banach Extension Property in X.*

Proof. Again we restrict to the K-space case. Suppose $\varphi \in N'$ is non-zero; let $M = \varphi^{-1}(0) \subset N$. Consider the natural quotient map π: $X/M \to X/N$; then there is a map $S: X/N \to X/M$ such that $\pi S = I$ on X/N. Then $S(X/N)$ is a closed subspace of co-dimension one in X/M and so there exists $\psi \in (X/M)'$ such that $\psi \neq 0$ and $\psi \circ S = 0$. If $q: X \to X/M$ is the natural quotient map, then $\psi q \in X'$. If $x \in N$, then $\psi q(x) = 0$ if and only if $q(x) \in S(X/N)$; then $q(x) = S\pi q(x) = 0$. Thus $(\psi q)^{-1}(0) \cap N = M$ and so a suitable multiple of ψq extends φ.

There is also a converse to Theorem 5.2.

THEOREM 5.3. *If X is a K-space [K_p-space] and $N \subset X$ is a closed subspace with* HBEP, *then X/N is a K-space [K_p-space].*

Proof. Suppose we have a short exact sequence

$$0 \to \boldsymbol{R} \to Z \xrightarrow{\varrho} X/N \to 0,$$

and let $\pi: X \to X/N$ be the quotient map. Then there is a lifting of π, $S: X \to Z$, so that $\varrho S = \pi$ (by Theorem 5.1). Suppose first S is not surjective; then $S(X)$ has co-dimension one in Z and $\varrho|S(X)$ is one-one. Define $R: X/N \to Z$ by $R\xi = z$ where $z \in S(X)$ and $\varrho z = \xi$. If $\xi_n \to 0$ in X/N, then there exists a sequence $\{x_n\}$ in X such that $x_n \to 0$ and $\pi x_n = \xi_n$. Since $Sx_n \to 0$ and $Sx_n = R\xi_n$, $R\xi_n \to 0$, i.e. R is continuous.

Now suppose S is surjective; then $S^{-1}(0)$ has co-dimension one in N. Let $\varphi \in N'$ be a non-zero linear functional with kernel $S^{-1}(0)$. Then φ may be extended to $\psi \in X'$. Now define $\tilde{S}: X \to Z$ by $\tilde{S}x = S\big(x - \psi(x)u\big)$ where $u \in N$ is chosen so that $\psi(u) = \varphi(u) = 1$. Then $\varrho\,\tilde{S}x = \pi\big(x - \psi(x)u\big) = \pi x$; and $\varrho\tilde{S}x = 0$ implies $x \in N$. Hence $\tilde{S}x = S\big(x - \varphi(x)u\big) = 0$. Thus ϱ is one-one on $\tilde{S}(X)$ and $\tilde{S}(X)$ has co-dimension one in Z; we can apply the previous part of the proof.

COROLLARY 5.4. *X is a K_p-space if and only if $X \cong l_p(I)/N$ where I is some index set and $N \subset l_p(I)$ has the* HBEP.

We remark that if $l_p/N \cong L_p$, then N has HBEP and the extension is unique, since $L_p' = \{0\}$.

COROLLARY 5.5. (i) *If N is a closed subspace of L_0, then L_0/N is a K-space if and only if $N' = \{0\}$. In particular, if L_0/N has L_0-structure, then $N' = \{0\}$.*

(ii) *If N is a closed subspace of L_p, then L_p/N is a K_p-space if and only if $N' = \{0\}$. In particular, if L_p/N is an $\mathscr{L}_p$-space, then $N' = \{0\}$.*

Note here that if we take for N the closed linear span of a sequence of functions with disjoint supports in L_0, then $N \cong \omega$, and hence $L_0/N \ncong L_0$. However $L_0/N \cong \omega(L_0/1)$ (the countable product of copies of $L_0/1$); hence $\omega(L_0/1) \ncong L_0$.

PROBLEM. Is L_p or l_p a K_r-space for any $r < p$, or even a K-space? In particular, is l_1 (or any Banach space) a K_p-space for any $p < 1$? This latter question is essentially the same as a problem of Stiles [8]: if l_p/N is locally convex, must N have the HBEP?

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