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GEOMETRY OF FINITE DIMENSIONAL SUBSPACES AND QUOTIENTS OF L_p

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The purpose of this paper is to present a series of results which are valid in the situation when E is a subspace and F a quotient of a finite dimensional space X, and $\dim E + \dim F > \dim X$. The "overlap" between E and F allows to prove, under some assumptions, results which assert a certain transfer of nice properties from E to F or vice-versa. The most productive cases occur, of course, when E or F, or both, are ℓ_p-spaces. In order to illustrate better the nature of the theorems proved in the paper, consider, for example, the situation when X is an arbitrary n-dimensional Banach space and E and F are ℓ_p-spaces; $1 \leq p \leq \infty$, of dimensions αn, respectively, βn and $\alpha + \beta > 1$. Then the fact proved in [2] that every ℓ_p subspace G of F contains, in turn, a well complemented subspace of dimension proportional to that of G yields, in the spirit described above, that also a fixed proportion of the unit vectors in E span a subpace which is well complemented in X.

The Hilbertian case is even more interesting. For instance, by using a quite well known result of V.D. Milman [18], one can prove that, whenever an n-dimensional space X contains an euclidean subspace E of dimension proportional to n then, for each $\varepsilon > 0$, X has a subspace X_ε of dimension $\geq n(1-\varepsilon)$ such that $X_\varepsilon \cap E$ is well complemented in X_ε. Another result of a slightly different nature asserts that a finite dimensional space X which contains a "very large" euclidean subspace already contains a sizeable euclidean subspace which is well complemented in X (the statement is made precise in the sequel).

A large part of the paper is devoted to the study of quotients of ℓ_p^n; $1 \leq p < 2$. The question considered here is on what extent these quotients themselves contain copies of ℓ_p-spaces. We prove below that a d-dimensional quotient of ℓ_p^n; $1 \leq p < 2$ contains a copy of ℓ_p^k with $k \geq c_1 d/(1+\log n/d)$ when $p = 1$ and $k \geq c_p(d^p/n^{2(p-1)})^{1/(2-p)}$ when $1 < p < 2$. Here

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c_p stands for a positive constant depending only on p. The remarkable fact about the above estimates is that they are best possible, as suitable examples will show. The lower estimate for k in the case $1 < p < 2$ is a direct consequence of a theorem which estimates precisely the dimension of an $\ell_{q'}^k$-space that can be found in the dual Y^* of a subspace Y of L_q; $q > 2$, $1/q + 1/q' = 1$, in terms of the euclidean distance of Y. This last result is based on a useful localization of a result of G. Pisier (see e.g., [10]) asserting that, for any Banach space X and any integer k, the type 2 and cotype 2 constants with k vectors can be computed by using only k normalized vectors. While a subspace X of ℓ_q^n; $2 < q \leq \infty$, of proportional dimension need not contain a copy of ℓ_q^k, for k proportional to n, this fact turns out to be true if X has an unconditional basis. The method which proves this assertion can be also used to prove a slight generalization of a result from [12] that symmetric bases of finite length, which are 2-concave, are unique, up to equivalence.

1. A Volumetric Result and its Applications

The aim of this section is to prove, by simple volumetric considerations, a theorem of a general nature from which one can derive easily several applications ot the case of ℓ_p-spaces.

Theorem 1.1. *There exists a constant $c > 0$ such that, whenever n, k and m are integers satisfying $k + m > n$, X is an n-dimensional Banach space, Q a quotient map from X onto a k-dimensional space F and $\{u_i\}_{i=1}^m$ a sequence of vectors in X spanning a subspace E so that*

$$\max_{1\leq i\leq m} |a_i| \geq \Big\| \sum_{i=1}^m a_i u_i \Big\| \geq \tau \sum_{i=1}^m |a_i|/m ,$$

for all $\{a_i\}_{i=1}^m$ and some $\tau > 0$, then

$$\int \Big\| \sum_{i=1}^m \varepsilon_i Q u_i \Big\| d\varepsilon \geq (c\tau)^{m/(k+m-n)} .$$

Proof: Since any uniformly distributed random variable taking values in the interval $[-1, +1]$ is clearly a convex combination of Bernoulli independent random variables we have that

$$\int \Big\| \sum_{i=1}^m \varepsilon_i Q u_i \Big\| d\varepsilon \geq \int_\Omega \Big\| \sum_{i=1}^m \eta_i(\omega) Q u_i \Big\| d\mu \geq \rho\mu\{\omega \in \Omega \; ; \; \Big\| \sum_{i=1}^m \eta_i(\omega) Q u_i \Big\| \geq \rho\} ,$$

where $0 < \rho < 1$ and $\{\eta_i\}_{i=1}^m$ is a sequence of independent uniformly distributed random variables over some probability space (Ω, Σ, μ). In order to estimate the expression

$$\mu\{\omega \in \Omega \; ; \; \Big\| \sum_{i=1}^m \eta_i(\omega) Q u_i \Big\| < \rho\} ,$$

we consider the convex subset of $\mathbb{R}^m$ defined by

$$C = \{ \sum_{i=1}^m a_i u_i \ ; \ \max_{1\le i\le m} |a_i| \le 1\}$$

and, for each $A \subset C$ for which it makes sense, we set

$$\nu(A) = \mu\{\omega \in \Omega \ ; \ \sum_{i=1}^m \eta_i(\omega) u_i \in A\} \ .$$

It follows that ν is a translation invariant probability measure on C and thus, it must coincide with the normalized Lebesgue measure, i.e.,

$$\nu(A) = \operatorname{vol} A / \operatorname{vol} C \ .$$

In particular, if B_E and B_F denote the unit balls of E, respectively F, then

$$\mu\{\omega \in \Omega \ ; \ \|\sum_{i=1}^m \eta_i(\omega) Q u_i\| < \rho\} = \nu\{C \cap Q^{-1}(\rho B_F)\} =$$
$$= \operatorname{vol}\{C \cap Q^{-1}(\rho B_F)\} / \operatorname{vol} C \le \operatorname{vol}\{B_E \cap Q^{-1}(\rho B_F)\} / \operatorname{vol} C \ .$$

For the sake of simplicity, we fix $0 < \theta < 1/2$ and estimate the expression $\operatorname{vol}\{(1-\theta)B_E \cap Q^{-1}(\theta B_F)\}$ rather than the expression appearing above. To this end, denote by L the kernel of Q and find vectors $\{v_j\}_{j=1}^N$ in the unit ball B_L of L so that

$$B_L \subset \bigcup_{j=1}^N \{v_j + \theta B_L\}$$

and $N \le (5/\theta)^{n-k}$. Take now an arbitrary vector $x \in (1-\theta)B_E \cap Q^{-1}(\theta B_F)$ and find first a vector $v \in L$ such that $\|x - v\| \le \theta$. It follows that $v \in B_L$ and, therefore, one can determine an integer $1 \le j_0 \le N$ such that $\|v - v_{j_0}\| \le \theta$ which further yields that $\|x - v_{j_0}\| \le 2\theta$.

Consider now the set

$$\sigma = \{1 \le j \le N \ ; \ d(v_j, B_E) \le 2\theta\}$$

and, for $j \in \sigma$, choose $w_j \in B_E$ so that $\|v_j - w_j\| \le 2\theta$. Since the index j_0 appearing above clearly belongs to σ it follows that $\|x - w_{j_0}\| \le 4\theta$, i.e,

$$(1-\theta)B_E \cap Q^{-1}(\theta B_F) \subset \bigcup_{j\in\sigma} \{w_j + (4\theta)B_E\} \ .$$

Hence,

$$\operatorname{vol}\left\{B_E \cap Q^{-1}\left(\frac{\theta}{1-\theta} B_F\right)\right\} \le (1-\theta)^{-m}(4\theta)^m(5/\theta)^{n-k} \operatorname{vol} B_E \le$$
$$\le 8^m \cdot 5^{n-k}\theta^{k+m-n} \operatorname{vol} B_E \ .$$

On the other hand, the conditions imposed on the vectors $\{u_i\}_{i=1}^m$ imply that

$$B_E \subset \tau^{-1} m B_{\ell_1^m}$$

i.e.,

$$\operatorname{vol} B_E \le \tau^{-m} \frac{m^m}{m!} \operatorname{vol} C \le (\tau^{-1} e)^m \operatorname{vol} C \ .$$

Thus, by taking $\rho = \theta/(1-\theta)$, it follows that

$$\mu\{\omega \in \Omega \ ; \ \|\sum_{i=1}^m \eta_i(\omega) Q u_i\| < \rho\} \le 8^m \cdot 5^{n-k}\theta^{k+m-n}(\tau^{-1}e)^m \le (120\tau^{-1})^m(\theta/5)^{k+m-n}$$

which, by choosing θ such that,

$$(120\tau^{-1})^m(\theta/5)^{k+m-n} = 2^{-m}$$

yields further that

$$\mu\{\omega \in \Omega \ ; \ \|\sum_{i=1}^m \eta_i(\omega) Q u_i\| \ge \rho\} \ge 1/2$$

and

$$\rho \ge 10(\tau/240)^{m/(k+m-n)} \ .$$

This, of course, completes the proof. □

Remarks. 1. We would like to comment on the condition imposed on E and the vectors $\{u_i\}_{i=1}^m$. Its meaning is, as easily noted, that the formal identity map from L_∞^n into L_1^n can be factorized through E in such a manner that the unit vectors in L_∞^n are mapped into the vectors $\{u_i\}_{i=1}^m$. This type of factorization was studied extensively in [23] where, among other facts, it was shown that it does not hold for every Banach space E unless $\tau = \tau(m) \to 0$, as $m \to \infty$. If E has a 1-symmetric basis $\{e_i\}_{i=1}^m$ then clearly the vectors $u_i = e_i / \|\sum_{i=1}^m e_i\|$; $1 \le i \le m$, satisfy the above condition with $\tau = 1$. The case of a space E with a normalized 1-unconditional basis $\{e_i\}_{i=1}^m$ is only slightly more complicated. In this situation, as is well known (see, e.g., [17],[8] or [11]), one can find positive reals $\{\lambda_i\}_{i=1}^m$ so that $\|\sum_{i=1}^m \lambda_i e_i\| \cdot \|\sum_{i=1}^m \lambda_i^{-1} e_i^*\| = m$, where $\{e_i^*\}_{i=1}^m$ denotes the corresponding biorthogonal sequence. Then the vectors $u_i = \lambda_i e_i / \|\sum_{i=1}^m \lambda_i e_i\|$; $1 \le i \le m$, satisfy the factorization condition, again with $\tau = 1$.

2. As easily seen from the proof of Theorem 1.1, the factorization condition imposed on the vectors $\{u_i\}_{i=1}^m$ can be replaced by weaker conditions. However, since all the applications that we have considered so far concern only ℓ_p-spaces the present hypothesis is quite satisfactory.

We present now an application of Theorem 1.1 to the case when both E and F are euclidean spaces.

Theorem 1.2. *There exists a constant $C < \infty$ such that, whenever α and β are positive reals with $\alpha + \beta > 1$, n, $m = \alpha n$ and $k = \beta n$ are integers, X is an n-dimensional Banach space, E a subspace of X of dimension m and F a quotient of X of dimension k, then for each $0 < \varepsilon < \alpha + \beta - 1$, there exists a projection P from X onto a subspace G of E such that $\dim G \geq (\alpha + \beta - 1 - \varepsilon)n$ and*

$$\|P\| \leq C^{1+(1-\beta)/\varepsilon} d_E^2 \cdot d_F \ ,$$

where d_E and d_F denote, as usual, the euclidean distance of E, respectively F.

Proof: We suppose first that E is isometric to ℓ_2^m and choose $S : F \longrightarrow \ell_2^k$ so that $\|S\| = 1$ and $\|S^{-1}\| = d_F$. By using the polar decomposition of the operator $SQ_{|E} : \ell_2^m \longrightarrow \ell_2^k$, we conclude the existence of orthonormal systems $\{e_i\}_{i=1}^m$ in E and $\{w_i\}_{i=1}^k$ in ℓ_2^k, and of reals $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ such that $SQe_i = \lambda_i w_i$; $1 \leq i \leq m$.

Fix $0 < \varepsilon < \alpha + \beta - 1$ so that $\ell = (\alpha + \beta - 1 - \varepsilon)n$ is an integer, set $H_\varepsilon = [e_i]_{i=\ell+1}^m$ and notice that the Hilbert-Schmidt norm of $SQ_{|H_\varepsilon}$ satisfies

$$\|SQ_{|H_\varepsilon}\|^2(m-\ell) \geq \|SQ_{|H_\varepsilon}\|_{HS}^2 = \sum_{i=\ell+1}^m \|SQe_i\|^2 =$$

$$= \int \Big\| \sum_{i=\ell+1}^m \varepsilon_i SQe_i \Big\|^2 d\varepsilon \geq d_F^{-2} \Big(\int \Big\| \sum_{i=\ell+1}^m \varepsilon_i Qe_i \Big\| d\varepsilon \Big)^2 \ .$$

On the other hand, by Theorem 1, there exists a $c > 0$ so that

$$\int \Big\| \sum_{i=\ell+1}^m \varepsilon_i Qe_i \Big\| d\varepsilon \geq c^{1+(1-\beta)/\varepsilon} \sqrt{m-\ell}$$

from which it follows that

$$\lambda_{\ell+1} \geq \|SQ_{|H_\varepsilon}\| \geq d_F^{-1} c^{1+(1-\beta)/\varepsilon} \ .$$

Consequently, the restriction of SQ to the subspace $G = [e_i]_{i=1}^\ell$ of E is invertible and $\|(SQ_{|G})^{-1}\| \leq d_F c^{-(1+(1-\beta)/\varepsilon)}$. This clearly implies that there exists a linear projection P from X onto G so that

$$\|P\| \leq d_F c^{-(1+(1-\beta)/\varepsilon)} \ .$$

The general case follows from the case $E = \ell_2^m$ and the following fact: If Y is a finite dimensional subspace of a Banach space X and Y is isomorphic to a space Y_1 then one can find a space X_1 containing an isometric copy of Y_1 such that $d(X, X_1) \leq d(Y, Y_1)$. The space X_1 can be constructed in the following way: Let $T : Y \longrightarrow Y_1$ be an isomorphism satisfying $\|T\| = 1$ and $\|T^{-1}\| = d(Y, Y_1)$ and put

$$X_1 = (X \oplus_1 Y_1)/Z \ ,$$

where

$$Z = \{(y, -Ty); y \in Y\} \ .$$

As is readily verified, the map $\widetilde{T} : X \longrightarrow X_1$, which is defined by $\widetilde{T}x = (x, 0); x \in X$, is an isomorphism onto X_1 with $\|\widetilde{T}\| \leq 1$ and $\|\widetilde{T}^{-1}\| \leq d(Y, Y_1)$. Moreover, the map $\widetilde{T}T^{-1}$ is an isometry from Y_1 into X_1 which maps a vector $y_1 \in Y_1$ into the pair $(0, y_1) \in X_1$. □

Corollary 1.3. *There exists a constant $C < \infty$ such that, whenever X is an n-dimensional Banach space and E a subspace of X of dimension proportional to n, then, for each $0 < \varepsilon < \alpha = \dim E/n$, one can find a subspace X_ε of X and a linear projection P_ε from X_ε onto $E \cap X_\varepsilon$ such that $\dim X_\varepsilon \geq n(1 - \varepsilon)$ and*

$$\|P_\varepsilon\| \leq C d_E^2 \varepsilon^{-1} \log \varepsilon^{-1} \ .$$

Proof: By a well know result of V.D. Milman [18] (see also [9]), for each $\varepsilon > 0$, one can determine a subspace X_1 of X which admits as a quotient a space F with $\dim F \geq n(1 - \varepsilon/3)$ and $d_F \leq C_1 \varepsilon^{-1} \log \varepsilon^{-1}$, for some constant C_1, independent of ε and n. Since $\dim E \cap X_1 \geq (\alpha - \varepsilon/3)n$ one can apply Theorem 1.2 with $\varepsilon/3$ instead of ε provided $\varepsilon < \alpha$. It follows that there exists a projection P_1 from X_1 onto a subspace G of $E \cap X_1$ such that $\dim G \geq (\alpha - \varepsilon)n$ and

$$\|P_1\| \leq C^{4/3} d_E^2 d_F \ .$$

The proof can be now completed by passing from X_1 to a smaller subspace X_ε so that $E \cap X_\varepsilon = G$. □

There are other applications of Theorem 1.2 to spaces of a more specific nature. One such result can be proved for spaces of weak cotype 2, introduced in [20]. Recall that, by definition, for each space E of weak cotype 2 there exists a function $f(\lambda)$; $0 < \lambda < 1$, such that, for any $0 < \lambda < 1$ and any subspace E_0 of E, one can find a further subspace E_1 with $\dim E_1 / \dim E_0 > \lambda$ and $d_{E_1} \leq f(\lambda)$.

Corollary 1.4. *There exists a constant $C < \infty$ such that, whenever α and β are positive reals with $\alpha + \beta > 1$, X an n-dimensional Banach space, E a subspace of X of dimension αn endowed with a weak cotype 2 function $f(\lambda)$, as above, and F a quotient of X of dimension βn, then, for each $0 < \varepsilon < \alpha + \beta + 1$, there exists a projection P from X onto a subspace G of E so that*

(i) $\dim G \geq (\alpha + \beta - 1 - \varepsilon)n$

(ii) $d_G \leq f(1 - \varepsilon/2\alpha)$

(iii) $\|P\| \leq C^{1+2(1-\beta)/\varepsilon} f(1 - \varepsilon/2\alpha)^2 d_F$.

Proof: For $0 < \varepsilon < \alpha + \beta - 1$, let E_ε be a subspace of E such that $\dim E_\varepsilon \geq (\alpha - \varepsilon/2)n$ and $d_{E_\varepsilon} \leq f(1 - \varepsilon/2\alpha)$. By Theorem 1.2, applied for E_ε and F with $\varepsilon/2$ instead of ε, one can find a projection P from X onto a subspace G of E_ε such that $\dim G \geq (\alpha + \beta - 1 - \varepsilon)n$ and

$$\|P\| \leq C^{1+2(1-\beta)/\varepsilon} d_{E_\varepsilon}^2 d_F .$$

This completes the proof. □

Another application of Theorem 1.2 concerns the finite dimensional version of the so-called twisted sum Z_2 of two Hilbert spaces that was introduced in [13] (see also [14]). In order to define the space Z_{2n}, we fix n and, for $x \in \mathbb{R}^{2n}$, we consider the expression

$$\|x\| = \rho(x) + \left(\sum_{i=1}^{n} \left(x_{2i-1} - x_{2i} \log \frac{\rho(x)}{|x_{2i}|} \right)^2 \right)^{1/2} ,$$

with $\rho(x) = \left(\sum_{i=1}^{n} |x_{2i}|^2 \right)^{1/2}$ and the convention that $\log \frac{0}{0} = 0$ and $\log \infty = 0$. The expression $\|\cdot\|$ is not a norm although it is homogeneous and satisfies the condition

$$\|x + y\| \leq 3(\|x\| + \|y\|) ; \qquad x, y \in \mathbb{R}^{2n} .$$

Consider now the alternating form

$$\Omega(x, y) = \sum_{i=1}^{n} (x_{2i-1} y_{2i} - x_{2i} y_{2i-1}) ; \qquad x, y \in \mathbb{R}^{2n} ,$$

and put

$$|||x||| = \sup \{ \Omega(x, y) \; ; \; \|y\| \leq 1 \} .$$

This expression defines indeed a norm which satisfies the condition

$$|||x||| \leq \|x\| \leq 3|||x||| ,$$

for all $x \in \mathbb{R}^{2n}$. The space $\mathbb{R}^{2n}$ endowed with $|||\cdot|||$-norm will be called Z_{2n}.

The form Ω induces a linear map $L_\Omega : Z_{2n} \to Z_{2n-1}^*$, by setting

$$L_\Omega(x)(y) = \Omega(x, y) ,$$

and this map satisfies the condition

$$\|L_\Omega(x)\| \leq \|x\| \leq 9\|L_\Omega(x)\| ,$$

for all $x \in Z_{2n}$. Finally, we point out that $[e_{2i-1}]_{i=1}^n$, as well as $Z_{2n}/[e_{2i-1}]_{i=1}^n$, are isometric to ℓ_2^n. It turns out that, by using Theorem 1.2, one can show that, asymptotically, this is the largest dimension of a hilbertian subspace of Z_{2n}. More precisely, we have the following result.

Theorem 1.5. *Fix $D < \infty$ and suppose that, for each n, Z_{2n} contains a subspace H_n such that $d_{H_n} \leq D$, for all n. Then*

$$\dim H_n \leq n + o(n) .$$

Since Z_{2n} admits ℓ_2^n as a quotient space it follows that if Theorem 1.5 is false for some $D < \infty$ then, by Theorem 1.2, there exists a $\gamma > 0$ so that Z_{2n} contains a well complemented subspace of dimension $\geq \gamma n$ which is D-isomorphic to Hilbert space. This fact is not true, however, as the following proposition shows.

Proposition 1.6. *Fix $D < \infty$ and suppose that, for each n, Z_{2n} contains a D-complemented subspace G_n such that $d_{G_n} \leq D$, for all n. Then*

$$\dim G_n = O(n / \log n) .$$

Proof: Fix $D < \infty$ and n, and let P be a linear projection from Z_{2n} onto a subspace G such that $\|P\| \leq D$ and $d_G \leq D$. Notice that P^*, the Ω-adjoint of P, which is defined by

$$\Omega(Px, y) = \Omega(x, P^* y) ; \qquad x, y \in Z_{2n} ,$$

is also a linear projection of norm $\leq \|L_\Omega^{-1}\| \|L_\Omega\| D \leq 9D$. For each vector y in the range $\mathcal{R}(P^*)$ of P^*, define the functional

$$\varphi_y(x) = \Omega(x, y) = -\Omega(y, x) = -L_\Omega(y)(x) ; \qquad x \in Z_{2n} .$$

In view of the properties of L_Ω mentioned above, the map $S : \mathcal{R}(P^*) \longrightarrow G^*$, defined by

$$Sy = -L_\Omega(y)_{|G} \; ; \qquad y \in \mathcal{R}(P^*) \,,$$

is actually an isomorphism with $\|S\| \le 1$ and $\|S^{-1}\| \le 9D$. Then, $d_{\mathcal{R}(P^*)} \le 9D^2$.

Consider now the Ω-self-adjoint operator $T = P + P^*$ and observe that

$$\left(\int \Big\|\sum_{j=1}^{m} \varepsilon_j T e_{2j}\Big\|^2 d\varepsilon\right)^{1/2} \le 18D^3\sqrt{n} \,.$$

Next, for each value of $1 \le k \le n$, let Q_k denote the natural projection in Z_{2n} onto the span of the two vectors e_{2k-1} and e_{2k}. Since, as readily verified,

$$\Big\|\sum_{k=1}^{n} \eta_k Q_k\Big\| = 1 \,,$$

for any choice of $\eta_k = \pm 1$, $1 \le k \le n$, we easily get that

$$\left(\int \Big\|\sum_{h,k=1}^{n} \eta_h \eta_k Q_k T Q_h \Big(\sum_{j=1}^{n} \varepsilon_j e_{2j}\Big)\Big\|^2 d\varepsilon\right)^{1/2} \le 18D^3\sqrt{n} \,,$$

for all $\eta_k = \pm 1$, $1 \le k \le n$. Hence, by averaging over all choices of signs $\eta_k = \pm 1$; $1 \le k \le n$, it follows that the operator $A = \sum_{k=1}^{n} Q_k T Q_k$ satisfies

$$\left(\int \Big\|A\Big(\sum_{j=1}^{n} \varepsilon_j e_{2j}\Big)\Big\|^2 d\varepsilon\right)^{1/2} \le 18D^3\sqrt{n} \,.$$

On the other hand, A is almost diagonal in 2×2 blocks and

$$\operatorname{tr} A = \operatorname{tr} T = 2 \dim G \,.$$

Moreover, since we have that

$$e^*_{2k-1}(x) = \Omega(x, e_{2k})$$

and

$$e^*_{2k}(x) = -\Omega(x, e_{2k-1}) \,,$$

for all $x \in Z_{2n}$ and $1 \le k \le n$, we easily conclude that

$$e^*_{2k-1}(Te_{2k-1}) = \Omega(Te_{2k-1}, e_{2k}) = \Omega(e_{2k-1}, Te_{2k}) = -\Omega(Te_{2k}, e_{2k-1}) = e^*_{2k}(Te_{2k})$$

and, similarly, that

$$e^*_{2k-1}(Te_{2k}) = 0 \,,$$

for all $1 \le k \le n$. Hence, A is actually a diagonal operator and its diagonal entries are equal in pairs. Furthermore, for each permutation π of the intergers $\{1, 2, \ldots, n\}$, define the operator $S_\pi : Z_{2n} \longrightarrow Z_{2n}$, by putting,

$$\left.\begin{aligned} (S_\pi x)_{2k} &= x_{2\pi(k)} \\ (S_\pi x)_{2k-1} &= x_{2\pi(k)-1} \end{aligned}\right. \quad ; \qquad 1 \le k \le n \,,$$

and notice that S_π is an isometry on Z_{2n}. Since we clearly have that

$$B = \operatorname*{Average}_{\pi} S_\pi A S_\pi = \left(\frac{\operatorname{tr} A}{2n}\right) I$$

it follows that

$$\begin{aligned} 18D^3\sqrt{n} \ge \left(\int \Big\|B\Big(\sum_{j=1}^{n} \varepsilon_j e_{2j}\Big)\Big\|^2 d\varepsilon\right)^{1/2} &= \frac{\operatorname{tr} A}{2n}\left(\int \Big\|\sum_{j=1}^{n} \varepsilon_j e_{2j}\Big\|^2 d\varepsilon\right)^{1/2} = \\ &= \frac{\dim G}{n}\sqrt{n}(1 + \log\sqrt{n}) \end{aligned}$$

i.e.,

$$\dim G \le 18D^3 n/(1 + \log\sqrt{n}) \,.$$

This completes the proof. □

We present now a result which, in some sense, extends Theorem 1.2 to the case of ℓ_p-spaces; $1 \le p \le \infty$. However, even in the case $p = 2$ the result below has a more general range of applicability. On the other hand, its conclusion is slightly less precise.

Theorem 1.7. *For every $1 \le p \le 2$, $M < \infty$ and $D < \infty$, there exists a constant $\gamma = \gamma(p, M, D) > 0$ such that, whenever k, m and n are integers satisfying $k + m > n$ and $m/(k + m - n) \le M$, X is a Banach space of dimension n, E is an m-dimensional subspace of X which is D-isomorphic to ℓ_p^m and F a k-dimensional quotient of X which is D-isomorphic to ℓ_p^k, then one can find a projection P from X onto a subspace G of E such that*

(i) $\dim G = \gamma m$

(ii) $d(G, \ell_p^{\gamma m}) \le D$

(iii) $\|P\| \le \gamma^{-1}$.

Proof: Let Q be the quotient map from X onto F, T an isomorphism from ℓ_p^m onto E such that $\|T\| = 1$ and $\|T^{-1}\| = d(E, \ell_p^m)$ and S an isomorphism from F onto L_p^k so that $\|S\| = 1$ and $\|S^{-1}\| = d(F, \ell_p^k)$.

Since, as is readily verified, the vectors $u_i = Te_i/m^{1/p}$; $1 \le i \le m$, satisfy the assumption of Theorem 1.1 with $\tau = D^{-1}$ it follows immediately that

$$\int \|\sum_{i=1}^m \varepsilon_i QTe_i\| d\varepsilon \ge (cD^{-1})^{m/(k+m-n)} m^{1/p} ,$$

from which one deduces that

$$\int \|\sum_{i=1}^m \varepsilon_i SQTe_i\| d\varepsilon \ge c^M D^{-M-1} m^{1/p} .$$

We consider first the case $p = 2$. Then, with the notation $\delta = c^M D^{-M-1}$, one can find a subset σ_1 of $\{1, 2, \dots, m\}$ of cardinality $|\sigma_1| \ge \delta^2 m/2$ such that $\|SQTe_i\| \ge \delta_1/\sqrt{2}$, for all $i \in \sigma_1$. This puts us in a position to use the invertibility result [2] Theorem 1.2 and conclude the existence of a $\gamma > 0$, depending only on M and D, and of a subset σ of σ_1 of cardinality $|\sigma| \ge \gamma m$ so that

$$\|\sum_{i\in\sigma} a_i Te_i\| \ge \|\sum_{i\in\sigma} a_i SQTe_i\| \ge \gamma\big(\sum_{i\in\sigma} |a_i|^2\big)^{1/2} \ge \gamma \|\sum_{i\in\sigma} a_i Te_i\| ,$$

for any choice of $\{a_i\}_{i\in\sigma}$. Clearly, the subspace $G = [Te_i]_{i\in\sigma}$ of E has all the required properties since $[SQTe_i]_{i\in\sigma}$ is 1-complemented in L_2^k.

We pass now to the case $1 \le p < 2$. The conclusion obtained above asserts that

$$\|\big(\sum_{i=1}^m |SQTe_i|^2\big)^{1/2}\| \ge c_1 m^{1/p} ,$$

for some $c_1 > 0$, depending only on p, M and D. Moreover, we also have that

$$\|\big(\sum_{i=1}^m |SQTe_i|^p\big)^{1/p}\| \ge m^{1/p} ,$$

from which, by a simple interpolation argument, we deduce the existence of $c_2 = c_2(p, M, D) > 0$ so that

$$\|\max_{1\le i\le m} |SQTe_i|\| \ge c_2 m^{1/p} .$$

Therefore, there exists a partition $\{A_i\}_{i=1}^m$ of $\{1, 2, \dots, m\}$ into mutually disjoint subsets so that

$$\|\sum_{i=1}^m \chi_{A_i} SQTe_i\| \ge c_2 m^{1/p} .$$

This implies the existence of a subset σ_2 of $\{1, 2, \dots, m\}$ of cardinality $|\sigma_2| \ge c_2^p m/2$ such that

$$\|\chi_{A_i} SQTe_i\| \ge c_2/\sqrt{2} , \qquad i \in \sigma_2 .$$

Consequently, by using the invertibility result [2] Proposition 4.4 or Theorem 1.7, we get that there exist a constant $c_3 = c_3(p, M, D) > 0$ and a subset σ_3 of σ_2 of cardinality $|\sigma_3| \ge c_3 m$ for which

$$\|\sum_{i\in\sigma_3} a_i Te_i\| \ge \|\sum_{i\in\sigma_3} a_i SQTe_i\| \ge c_3\big(\sum_{i\in\sigma_3} |a_i|^p\big)^{1/p} \ge c_3 \|\sum_{i\in\sigma_3} a_i Te_i\| ,$$

for all $\{a_i\}_{i\in\sigma_3}$. In view of [2] Theorem 4.5, we can assume without loss of generality that $[SQTe_i]_{i\in\sigma_3}$ is c_3^{-1}-complemented in L_p^k. This, of course, yields that the subspace $G = [Te_i]_{i\in\sigma_3}$ of E has all the desired properties. □

Corollary 1.8. *For every $1 \le p \le \infty$, $D < \infty$ and $1 > \alpha$, $\beta > 0$ so that $\alpha + \beta > 1$, there exists a constant $\tau = \tau(p, D, \alpha, \beta) > 0$ with the property that, whenever X is an n-dimensional Banach space, E a subspace of X of dimension αn which is D-isomorphic to $\ell_p^{\alpha n}$ and F a quotient of X of dimension βn which is isomorphic to $\ell_p^{\beta n}$, then there exists a linear projection P from X onto a subspace G of E so that:*

(i) $\dim G = \tau n$

(ii) $d(G, \ell_p^{\tau n}) \le D$

(iii) $\|P\| \le \tau^{-1}$

Proof: For $1 \le p \le 2$, Corollary 1.8 is an immediate consequence of Theorem 1.7 since $\alpha n/(\alpha n + \beta n - n) = \alpha/(\alpha + \beta - 1)$ is a constant. Since also $\beta n/(\alpha n + \beta n - n) = \beta/(\alpha + \beta - 1)$ is a constant, the proof in the case $p > 2$ is deduced from the previous one by a simple duality argument. □

The final result presented in this section is an application of Theorem 1.7 to finite dimensional spaces that contain "very large" subspaces which are euclidean. The precise sense of this statement is given below.

Theorem 1.9. *For every $D < \infty$, there exists a constant $C = C(D) < \infty$ so that, whenever X is an n-dimensional Banach space and E a subspace of X of dimension $n - \ell$, for some $\ell \le \sqrt{n}/2D$, with $d_E \le D$, then there is a projection P from X onto a subspace G of E such that*

(i) $\dim G \ge n/C\ell$

(ii) $\|P\| \le C$.

Proof: Since E is clearly $\sqrt{\ell}$-complemented in X it follows that X^*, the dual of X, contains a subspace Y of dimension $n-\ell$ such that $d_Y \le D\sqrt{\ell}$. By a well-known result from [19], Y contains a further subspace Z of dimension $\ge (n-\ell)/2D^2\ell$ which is A-isomorphic to Hilbert space, for some constant A independent of n, ℓ or D. It is also obvious that X^* admits E^* as a quotient space (of dimension $n-\ell$).

We are now in a position to apply Theorem 1.7 to the situation $Z \subset X^*$ and E^* as a quotient space since the condition imposed above on ℓ yields that $\dim Z/(\dim Z + \dim E^* - n) \le 2$. It follows that there exist a $\gamma = \gamma(D) > 0$ and a projection Q from X^* onto a subspace H of Z so that $\|Q\| \le \gamma^{-1}$ and $\dim H \ge \gamma n/\ell$. Then the projection $P = Q^*$ has clearly all the required properties. □

Remark. It is quite possible that Theorem 1.9 remains valid with $\ell = o(n)$ but we have not checked this matter.

2. Quotients and Subspaces of ℓ_p^n of Proportional Dimension

Subspaces and quotients of ℓ_p^n-spaces can have quite a complicated structure and, in many cases, their properties differ radically from those of the underlying space ℓ_p^n. There are, however, some situations where a great deal of information can be obtained. The remainder of this paper is aimed to study such cases; in this section we consider the easier case of quotients of ℓ_p^n whose dimension is a fixed proportion of n.

Theorem 2.1. *For every $1 < p \le 2$ and $\alpha > 0$, there exists a $\gamma = \gamma(p,\alpha) > 0$ so that, whenever F is a quotient of ℓ_p^n of dimension $\ge \alpha n$, then F contains a subspace G of dimension γn for which $d(F, \ell_p^{\gamma n}) \le \gamma^{-1}$.*

Proof: We begin with the case $p = 1$ which is an immediate consequence of a result of J. Elton [5]. Indeed, if Q denotes the quotient map from ℓ_1^n onto a space F_1 with $\dim F_1 \ge \alpha n$ then, by Theorem 1.1, we have that

$$\int \Big\| \sum_{i=1}^{n} \varepsilon_i Q e_i \Big\| d\varepsilon \ge c^{1/\alpha} n .$$

Hence, by the afore mentioned result of J. Elton [5] in the real case or by A. Pajor [22] in the complex one, one can find a constant $\gamma_1 = \gamma_1(\alpha) > 0$ and a subset σ_1 of $\{1,2,,\ldots,n\}$ so that

$|\sigma_1| \ge \gamma_1 n$ and

$$\Big\| \sum_{i\in\sigma_1} a_i Q e_i \Big\| \ge \gamma_1 \sum_{i\in\sigma_1} |a_i| ,$$

for all $|a_i|_{i\in\sigma_1}$. This clearly completes the proof of the case $p = 1$.

In order to prove the theorem for $1 < p < 2$, we consider a quotient F of ℓ_p^n of dimension $\ge \alpha n$ and identify it with the dual Z^* of a subspace Z of $\ell_{p'}^n$, where $1/p + 1/p' = 1$. We shall consider Z also as a subspace of ℓ_∞^n in which case it will be denoted by Z_∞. Clearly, $F_1 = Z_\infty^*$ is a quotient of ℓ_1^n and the same map Q acts as a quotient map from ℓ_p^n onto F as well as from ℓ_1^n onto F_1.

Let $\gamma_1 = \gamma_1(\alpha) > 0$ and $\sigma_1 \subset \{1,2,\ldots,n\}$ be given by the first part of the proof for F_1 so that

$$\Big\| \sum_{i\in\sigma_1} a_i Q e_i \Big\|_{F_1} \ge \gamma_1 \sum_{i\in\sigma_1} |a_i| ,$$

for any choice of $\{a_i\}_{i\in\sigma_1}$.

The proof for F will be completed by using a standard exhaustion argument in order to show that there exists a subset $\sigma \subset \sigma_1$ of cardinality $|\sigma| \ge |\sigma_1|/2$ so that

$$\Big\| \sum_{i\in\sigma} a_i Q e_i \Big\|_F \ge \gamma \Big(\sum_{i\in\sigma} |a_i|^p \Big)^{1/p} ,$$

for all $\{a_i\}_{i\in\sigma}$ and $\gamma = \gamma_1^{1+1/p'}/8^{1/p'}$. Indeed, if this assertion is false then we can construct vectors $y_i = \sum_{j\in\sigma_1} a_{ij} Q e_j$ in F such that $\sum_{j\in\sigma_1} |a_{i,j}|^p = 1$ and $\|y_i\|_F < \gamma$, for $1 \le i \le m$, and the set

$$\tau_{\ell+1} = \{ j \in \sigma_1 \; ; \; \sum_{i=1}^{\ell} |a_{i,j}|^p < 1 \}$$

has cardinality $\ge |\sigma_1|/2$, for $1 \le \ell < m$, and $< |\sigma_1|/2$, for $\ell = m$. Notice that $m \ge |\sigma_1|/2$.

Choose now vectors $w_i \in \ell_p^n$ such that $Qw_i = y_i$ and $\|y_i\|_F = \|w_i\|_p$; $1 \le i \le m$, and let $\{\varphi_i\}_{i=1}^m$ be a sequence of p-stable independent random variables over a probability space (Ω, Σ, μ) which are normalized in $L_1(\mu)$. Then

$$\int_\Omega \Big\| \sum_{i=1}^{m} \varphi_i(\omega) y_i \Big\|_{F_1} d\mu \int_\Omega \Big\| \sum_{j\in\sigma_1} \Big(\sum_{i=1}^{m} a_{i,j} \varphi_i(\omega) \Big) Q e_j \Big\|_{F_1} \ge$$

$$\ge \gamma_1 \int_\Omega \sum_{j\in\sigma_1} \Big| \sum_{i=1}^{m} a_{i,j} \varphi_i(\omega) \Big| d\mu \ge \gamma_1 \sum_{j\in\sigma_1} \Big(\sum_{i=1}^{m} |a_{i,j}|^p \Big)^{1/p} .$$

Since the construction of the vectors $\{y_i\}_{i=1}^m$ yields that $\sum_{i=1}^{m} |a_{i,j}|^p < 2$, $j \in \sigma_1$, it follows that

$$\int_\Omega \Big\| \sum_{i=1}^{m} \varphi_i(\omega) y_i \Big\|_{F_1} d\mu \ge \gamma_1 2^{-1/p'} m .$$

On the other hand,

$$\int_\Omega \Big\| \sum_{i=1}^m \varphi(\omega) y_i \Big\|_{F_1} \le \int_\Omega \Big\| \sum_{i=1}^m \varphi_i(\omega) w_1 \Big\|_1 d\mu = \Big\| \int_\Omega \Big| \sum_{i=1}^m \varphi_i(\omega) w_i \Big| d\mu \Big\|_1 =$$

$$= \Big\| \Big(\sum_{i=1}^m |w_i|^p \Big)^{1/p} \Big\|_1 \le n^{1/p'} \Big\| \Big(\sum_{i=1}^m |w_i|^p \Big)^{1/p} \Big\|_p = n^{1/p'} \Big(\sum_{i=1}^m \|y_i\|_F^p \Big)^{1/p} \le n^{1/p'} m^{1/p} \gamma ,$$

i.e.,

$$\gamma_1 2^{-1/p'} m^{1/p'} \le \gamma n^{1/p'} .$$

This is, however, a contradiction in view of the choice of γ made above. □

Remark. For $p = 1$ the result is known and appears implicitly, e.g., in [1] or [4]. In this case, the same conclusion can be derived for a quotient F of a space X which contains a "large" copy of ℓ_1. More precisely, if X is n-dimensional, contains a copy of $\ell_1^{\alpha n}$ and admits as a quotient a space F for dimension $\ge \beta n$ with $\alpha + \beta > 1$ then F contains a copy of $\ell^{\gamma m}$, for some $\gamma = \gamma(\alpha, \beta) > 0$.

While quotients of ℓ_p^n; $1 \le p \le 2$, of dimension proportional to n contain, by Theorem 2.1, copies of ℓ_p^k, for k also proportional to n, this fact is no longer true for their duals, i.e., for subspaces of ℓ_q^n; $q > 2$. The so-called random spaces (cf. [6]) form examples of subspaces of ℓ_q^n; $q > 2$, of dimension, e.g., $\ge n/2$, which contain ℓ_q^k only for $k \le n^{q'/2}$. The situation is, however, completely different if we consider subspaces of ℓ_q^n; $q > 2$, which have an unconditional basis. In order to present the result, we need first a proposition which turns out to have other applications, too. The notions of 2-convexity and 2-concavity, appearing below, are discussed in detail in [16] and we refer the reader to this book.

Proposition 2.2. *Let $\{y_i\}_{i=1}^n$ be a normalized 1-symmetric basis of a space Y which is 2-convex with 2-convexity constant equal to one. Fix $0 < \alpha \le 1$ so that $m = \alpha n$ is an integer and let $\{x_i\}_{i=1}^m$ be a normalized 1-unconditional basis of a space X which, again, is 2-convex with 2-convexity constant equal to one. If T is an isomorphism from X into Y then there exists a subset σ of $\{1, 2, \dots, m\}$ of cardinality $|\sigma| \ge n/2$ such that*

$$\Big\| \Big(\sum_{i\in\sigma} |c_i T x_i|^2 \Big)^{1/2} \Big\| \le \|T\| \sqrt{2/\alpha} \Big\| \sum_{i\in\sigma} c_i y_i \Big\| ,$$

for any choice of $\{c_i\}_{i\in\sigma}$. Moreover, if $Tx_i = \sum_{j=1}^m a_{i,j} y_j$; $1 \le i \le m$, then

$$\sum_{j=1}^n |a_{i,j}|^2 \le 2\|T\|^2/\alpha ,$$

for all $i \in \sigma$.

Proof: Let $\{x_i^*\}_{i=1}^m$ and $\{y_i^*\}_{i=1}^n$ be the biorthogonal functionals associated with $\{x_i\}_{i=1}^m$, respectively $\{y_i\}_{i=1}^n$.

Since $\{x_i^*\}_{i=1}^m$ is 2-concave with 2-concavity constant equal to one it follows that

$$\|T\| \ge \|T^* y_h^*\| = \Big\| \sum_{i=1}^m a_{i,h} x_i^* \Big\| \ge \Big(\sum_{i=1}^m |a_{i,h}|^2 \Big)^{1/2} ,$$

for all $1 \le h \le n$. Put

$$\sigma = \{1 \le i \le m \; ; \; \sum_{j=1}^n |a_{i,j}|^2 \le 2\|T\|^2/\alpha\}$$

and notice that

$$2\|T\|^2 |\sigma^c|/\alpha \le \sum_{j=1}^n \sum_{i=1}^m |a_{i,j}|^2 \le n\|T\|^2$$

from which we deduce that $|\sigma| \ge m/2$. The inequalities satisfied by the matrix $(a_{i,j})_{i=1,}^m{}_{j=1}^n$ imply that the matrix $(\alpha |a_{i,j}|^2 / 2\|T\|^2)_{i\in\sigma,}{}_{j=1}^n$ is doubly sub-stochastic. Hence, the map S, defined by

$$S y_i = \frac{\alpha}{2\|T\|^2} \sum_{j=1}^n |a_{i,j}|^2 y_j \; ; \qquad i \in \sigma ,$$

extends to a norm one linear operator on the 2-concavification $Y_{(2)}$ of Y. Since the norms in Y and $Y_{(2)}$ are related by the equality

$$\|y\|_{Y_{(2)}} = \big\| |y|^{1/2} \big\|_Y^2 , \qquad y \in Y_{(2)} ,$$

we can conclude that

$$\Big\| \sum_{j=1}^n \Big(\sum_{i\in\sigma} |c_i|^2 |a_{i,j}|^2 \Big)^{1/2} y_j \Big\|_Y \le \|T\| \sqrt{2/\alpha} \Big\| \sum_{i\in\sigma} c_i y_i \Big\|_Y ,$$

for all $\{a\}_{i\in\sigma}$. This, however, is exactly the assertion we wanted to prove. □

Theorem 2.3. *For every $0 < \varepsilon < 1$, $0 < \alpha < 1$, $K \ge 1$, $C \ge 1$ and $2 < r < q < \infty$, there exists a constant $M = M(\varepsilon, \alpha, K, C, r, q) < \infty$ such that if m and n are integers with $m \ge \alpha n$, X is an n-dimensional Banach space with a normalized 1-symmetric basis $\{u_i\}_{i=1}^n$ whose r-convexity and q-concavity constants are bounded by C and $\{x_i\}_{i=1}^m$ is a normalized K-unconditional basic sequence in X, then there is a subset σ of $\{1, 2, \dots, m\}$ with $|\sigma| \ge (1-\varepsilon)m$ such that*

$$M^{-1} \Big\| \sum_{i\in\sigma} c_i x_i \Big\| \le \Big\| \sum_{i\in\sigma} c_i u_i \Big\| \le M \Big\| \sum_{i\in\sigma} c_i x_i \Big\| ,$$

for all $\{c_i\}_{i\in\sigma}$.

Proof: This will follow clearly by iteration if we first prove that weaker statement that there exist constants $M_0 = M_0(\alpha, K, C, r, q) < \infty$ and $\beta = \beta(\alpha, K, C, r, q) > 0$ so that we can find a subset $\sigma_0 \subset \{1, 2, \ldots, m\}$ with $|\sigma_0| \geq \beta m$ and

$$M_0^{-1}\|\sum_{i\in\sigma} c_i x_i\| \leq \|\sum_{i\in\sigma} c_i u_i\| \leq M_0\|\sum_{i\in\sigma} c_i x_i\| ,$$

for all $\{c_i\}_{i\in\sigma}$.

To this end, notice that the space X has a type 2 constant bounded by a function of K, C, r and q alone. Hence, there exist a space Y with a 1-unconditional basis $\{y_i\}_{i=1}^m$, whose 2-convexity constant is equal to one, and a map $T : Y \longrightarrow X$ such that $Ty_i = x_i$, for all $1 \leq i \leq m$, and

$$\|\sum_{i=1}^m c_i y_i\| \leq \|\sum_{i=1}^m c_i x_i\| \leq M_1\|\sum_{i=1}^m c_i y_i\| ,$$

for all $\{c_i\}_{i=1}^m$, where M_1 depends only on K, C, r and Q.

We apply now Proposition 2.2 and conclude the existence of a subset σ_1 of $\{1, 2, \ldots, m\}$ with $|\sigma_1| \geq m/2$ so that

$$\left\|\left(\sum_{i\in\sigma_1} |c_i x_i|^2\right)^{1/2}\right\| \leq M_1\sqrt{2/\alpha}\|\sum_{i\in\sigma_1} c_i u_i\| ,$$

for each choice of $\{c_i\}_{i\in\sigma_1}$, where the lattice operations are taken with respect to the structure induced by the basis $\{u_i\}_{i=1}^n$ of X. Recall also that, by Proposition 2.2, we have that the matrix $(a_{i,j})_{i=1}^m{}_{j=1}^n$, defined by

$$Ty_i = x_i = \sum_{j=1}^n a_{i,j} u_i \quad ; \qquad 1 \leq i \leq m$$

satisfies

$$\sum_{j=1}^n |a_{i,j}|^2 \leq 2M_1^2/\alpha ,$$

whenever $i \in \sigma_1$.

However, for $i \in \sigma_1$,

$$1 = \|x_i\| = \|\sum_{j=1}^n a_{i,j} u_j\| \leq C\left(\sum_{j=1}^n |a_{i,j}|^r\right)^{1/r} ,$$

which yields that

$$\max_{1\leq j\leq n} |a_{i,j}| \geq \delta ,$$

for all $i \in \sigma$, where $\delta = \delta(\alpha, K, C, r, q) > 0$. Since

$$\left(\sum_{i=1}^m |a_{i,j}|^2\right)^{1/2} \leq M_1 ,$$

we conclude that, for each $1 \leq j \leq n$, $|a_{i,j}| \geq \delta$ for at most $M_1\delta^{-2}$ choices of i. Hence, there exist $\sigma_2 \subset \sigma_1$ with $|\sigma_2| \geq \delta^2|\sigma_1|/M_1$ and a one-to-one map $\pi : \sigma_2 \rightarrow \{1, 2, \ldots, n\}$ such that

$$|a_{i,\pi(i)}| \geq \delta ; \qquad i \in \sigma_2 .$$

Consequently,

$$\left\|\left(\sum_{i\in\sigma_2} |c_i x_i|^2\right)^{1/2}\right\| \geq \delta\|\sum_{i\in\sigma_2} c_i u_i\| ,$$

for all $\{c_i\}_{i\in\sigma_2}$. The assertion now follows since $\|\sum_{i\in\sigma_2} c_i x_i\|$ and $\left\|\left(\sum_{i\in\sigma_2} |c_i x_i|^2\right)^{1/2}\right\|$ are equivalent, up to a function of K, C, r and q. □

Theorem 2.3 implies in particular that, as long as $2 \leq p < \infty$, every normalized unconditional basic sequence in ℓ_p^n of proportional size is almost entirely equivalent to the usual basis of ℓ_p. The same result is true for $p = \infty$, but the proof is somewhat different because of lack of q-concavity. We need first the following result which is of independent interest.

Theorem 2.4. *For every* $0 < \varepsilon < 1$, $0 < \alpha < 1$ *and* $K \geq 1$, *there exists a constant* $M = M(\varepsilon, \alpha, K) < \infty$ *such that, whenever* m *and* n *are integers satisfying* $m \geq \alpha n$, X *is an* n*-dimensional Banach space with a normalized 1-unconditional basis* $\{u_i\}_{i=1}^n$ *containing a subspace* Y *which is* K*-isomorphic to* ℓ_1^m, *then there exists a subset* σ *of* $\{1, 2, \ldots, n\}$ *with* $|\sigma| \geq (1-\varepsilon)m$ *such that*

$$M\|\sum_{i\in\sigma} c_i u_i\| \geq \sum_{i\in\sigma} |c_i| ,$$

for any choice of $\{c_i\}_{i\in\sigma}$.

Proof: We first prove the following claim: for any $0 < \alpha < 1$ and $K \geq 1$, there exist constants $M = M(\alpha, K) < \infty$ and $\beta = \beta(\alpha, K) > 0$ such that if m and n are integers with $m \geq \alpha n$ and X is an n-dimensional Banach space with normalized 1-unconditional basis $\{u_i\}_{i=1}^n$ containing a subspace Y which is K-isomorphic to ℓ_1^m then there exists a subset $\sigma \subset \{1, 2, \ldots, n\}$ with $|\sigma| \geq \beta m$ so that

$$M\|\sum_{i\in\sigma} c_i u_i\| \geq \sum_{i\in\sigma} |c_i| ,$$

for all $\{c_i\}_{i\in\sigma}$.

Let us show first that the above claim actually implies Theorem 2.4. Indeed, under the hypotheses of Theorem 2.4, it follows from the remark to Theorem 2.1 that there is a constant $\gamma = \gamma(\varepsilon, \alpha, K) > 0$ such that, whenever $\tau \subset \{1, 2, \dots, n\}$ satisfies $|\tau| \geq n-(1-\varepsilon)m$, then there is a subspace Y_0 of $[u_i]_{i\in\tau}$ so that $k = \dim Y_0 \geq \gamma n$ and Y_0 is γ^{-1}-isomorphic to ℓ_1^k. Then, by the above claim, one can find a $\delta = \delta(\varepsilon, \alpha, K) > 0$ and a subset τ_0 of τ for which $|\tau_0| \geq \delta n$ and

$$\|\sum_{i\in\tau_0} c_i u_i\| \geq \delta \sum_{i\in\tau_0} |c_i| \ ,$$

for all $\{c_i\}_{i\in\tau_0}$. From this the proof of the theorem follows by an obvious iteration.

We shall prove now the claim made above. Consider X as a lattice induced by its 1-unconditional basis $\{u_i\}_{i=1}^n$ and let $\{y_i\}_{i=1}^m$ be a normalized basis for Y such that

$$K\|\sum_{i=1}^m c_i y_i\| \geq \sum_{i=1}^m |c_i| \ ,$$

for all $\{c_i\}_{i=1}^m$. Let $\{u_i^*\}_{I=1}^n$ be the sequence of the biorthogonal functionals associated with $\{u_i\}_{i=1}^n$ and, for each $A \subset \{1, 2, \dots, n\}$ with $|A| = k$, where k will be determined later, choose a 1/2-net in the unit ball of $[u_j^*]_{j\in A}$ whose cardinality is $\leq 5^k$.It follows that one can find vectors $\{\varphi_i\}_{i=1}^N$ in X^* such that $\|\varphi_i\| \leq 1$; $1 \leq i \leq N$, $5 \leq N \leq \binom{n}{k} 5^k \leq (5en/k)^k$ and

$$\max_{1\leq i\leq N} |\varphi_i(x)| \geq \tfrac{1}{2} \max_{\substack{A\subset\{1,2,\dots,n\}\\|A|=k}} \|\chi_A x\| \ ,$$

for all $x \in X$.

Put now $h = \log N$, $w = \left(\sum_{i=1}^m |y_i|^2\right)^{1/2} \in X$ and, for $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \{-1, +1\}^m$, set $v(\varepsilon) = \sum_{i=1}^m \varepsilon_i y_i$. Then, with λ being a positive integer which will be determined later and $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \{-1, +1\}^m$, consider the set

$$A_\varepsilon = \{1 \leq j \leq n \ ; \ |\sum_{i=1}^m \varepsilon_i u_j^*(y_i)| \geq \lambda u_j^*(w)\} \ .$$

Next, let Λ be the set of all $\varepsilon \in \{-1, +1\}^m$ such that $|A_\varepsilon| \geq k$ and notice that, with P being the usual probability measure on $\{-1, +1\}^m$, we have that

$$kP(\Lambda) \leq \int |A_\varepsilon| d\varepsilon = \sum_{j=1}^n P\{\varepsilon \in \{-1, +1\}^m \ , \ |\sum_{i=1}^m \varepsilon_i u_j^*(y_i)| \geq \lambda u_j^*(w)\} \leq$$

$$\leq \sum_{j=1}^n \lambda^{-2} \int \frac{|\sum_{i=1}^m \varepsilon_i u_j^*(y_i)|^2}{|u_j^*(w)|^2} d\varepsilon = \lambda^{-2} n$$

i.e.,

$$P(\Lambda) \leq k^{-1}\lambda^{-2} n \ .$$

On the other hand,

$$\int \max_{1\leq i\leq N} |\varphi_i(v(\varepsilon))| d\varepsilon \leq \left(\sum_{i=1}^N \int |\varphi_i(v(\varepsilon))|^h d\varepsilon\right)^{1/h}$$

and, for each $1 \leq i \leq N$,

$$\left(\int |\varphi_i(v(\varepsilon))|^h d\varepsilon\right)^{1/h} \leq \sqrt{h}\left(\sum_{j=1}^m |\varphi_i(y_j)|^2\right)^{1/2} \leq \sqrt{hm} \ ,$$

in view of Khinchine's inequality. Hence,

$$\int \max_{1\leq i\leq N} |\varphi_i(v(\varepsilon))| d\varepsilon \leq N^{1/h}\sqrt{hm} \leq e(mk \log 5en/k)^{1/2} \ .$$

Thus, there exists a universal constant C_1 such that

$$\int \max_{\substack{A\subset\{1,2,\dots,n\}\\|A|=k}} \|\chi_A v(\varepsilon)\| d\varepsilon \leq C_1\left(mk(1 + \log n/k)\right)^{1/2} \ .$$

Now, if $\varepsilon \notin \Lambda$ then there exists a set A containing A_ε so that $|A| = k$ and

$$|v(\varepsilon) - \chi_A v(\varepsilon)| \leq \lambda w \ .$$

Hence,

$$\|v(\varepsilon)\| \leq \max_{\substack{A\in\{1,2,\dots,n\}\\|A|=k}} \|\chi_A v(\varepsilon)\| + \lambda\|w\| \ ,$$

which yields that

$$\int \|v(\varepsilon)\| d\varepsilon \leq \int_\Lambda \|v(\varepsilon)\| d\varepsilon + C_1\left(mk(1 + \log n/k)\right)^{1/2} + \lambda\|w\| \leq$$

$$\leq mnk^{-1}\lambda^{-2} + C_1\left(mk(1 + \log n/k)\right)^{1/2} + \lambda\|w\| \ .$$

However, we also have that

$$\int \|v(\varepsilon)\| d\varepsilon \geq mK^{-1}$$

so we can conclude that

$$\lambda\|w\| \geq m\left(K^{-1} - nk^{-1}\lambda^{-2} - C_1(km^{-1})^{1/2}(1 + \log n/k)^{1/2}\right) \ .$$

This means that if n is sufficiently large then one can choose first a $k = k(\alpha, K, n)$ and then a $\lambda = \lambda(\alpha, K)$ as to deduce the estimate

$$\|w\| \geq \eta m ,$$

for some $\eta = \eta(\alpha, K) > 0$. Furthermore, since

$$\|w\|^2 \leq \Big\| \sum_{i=1}^{m} |y_i| \Big\| \Big\| \max_{1\leq i\leq m} |y_i| \Big\|$$

we conclude that

$$\Big\| \max_{1\leq i\leq m} |y_i| \Big\| \geq \eta^2 m .$$

Therefore, we also have that

$$\Big\| \sum_{i=1}^{n} u_i \Big\| \geq \eta^2 m$$

from which one can easily derive the desired assertion. □

We can now summarize our results for ℓ_p^n-spaces.

Theorem 2.5. *For any* $2 \leq p \leq \infty$, $0 < \alpha < 1$, $0 < \varepsilon < 1$ *and* $K \geq 1$, *there exists a constant* $M = M(p, \alpha, \varepsilon, K) < \infty$ *such that if* m *and* n *are integers with* $m \geq \alpha n$ *and* $\{x_i\}_{i=1}^m$ *is a normalized* K*-unconditional basic sequence in* ℓ_p^n *then there is a subset* σ *of* $\{1, 2, \ldots, m\}$ *such that* $|\sigma| \geq (1 - \varepsilon)m$ *and*

$$M^{-1}\Big(\sum_{i\in\sigma} |c_i|^p\Big)^{1/p} \leq \Big\| \sum_{i\in\sigma} c_i x_i \Big\| \leq M\Big(\sum_{i\in\sigma} |c_i|^p\Big)^{1/p} , \quad \text{if } 2 \leq p < \infty ,$$

or

$$M^{-1} \max_{i\in\sigma} |c_i| \leq \Big\| \sum_{i\in\sigma} c_i x_i \Big\| \leq M \max_{i\in\sigma} |c_i| , \quad \text{if } p = \infty ,$$

for all $\{c_i\}_{i\in\sigma}$.

Proof: For $p = 2$, this is entirely trivial while for $2 < p < \infty$ it follows from Theorem 2.3. We thus turn to the case $p = \infty$ which clearly will follow by iteration from the weaker claim that there are $M = M(\alpha, K) < \infty$ and $\beta = \beta(\alpha, K) > 0$ such that we can find a subset σ of $\{1, 2, \ldots, m\}$ for which $|\sigma| \geq \beta m$ and

$$\Big\| \sum_{i\in\sigma} c_i x_i \Big\| \leq M \max_{i\in\sigma} |c_i| ,$$

for any choice of $\{c_i\}_{i\in\sigma}$. In order to prove this assertion, put $X = [x_i]_{i=1}^m$ and observe that X^* is a quotient of ℓ_1^n. Hence, by Theorem 2.1, X^* contains a subspace Y with $\dim Y \geq \gamma n$

and $d(Y, \ell_1^{\dim Y}) \leq \gamma^{-1}$, where γ is a constant depending only on α and K. By Theorem 2.4, one can find a subset σ of $\{1, 2, \ldots, n\}$ such that $|\sigma| \geq \beta n$ and

$$\Big\| \sum_{i\in\sigma} c_i x_i^* \Big\| \geq \beta \sum_{i\in\sigma} |c_i| ,$$

for all $\{c_i\}_{i\in\sigma}$ and some $\beta = \beta(\alpha, K) > 0$, where $\{x_i^*\}_{i=1}^m$ is the biorthogonal sequence associated to $\{x_i\}_{i=1}^m$. Thus,

$$\Big\| \sum_{i\in\sigma} c_i x_i \Big\| \leq \beta^{-1} \max_{i\in\sigma} |c_i| ,$$

for all $\{c_i\}_{i\in\sigma}$, and this completes the argument. □

It turns out that Proposition 2.2 can be also used to give a short proof of a slight generalization of [12] Theorem 3.8.

Theorem 2.6. *Each member of the family* $\mathcal{F}$ *of all finite dimensional Banach spaces having a 1-symmetric normalized basis, which is 2-convex or 2-concave with corresponding constant equal to one, has a unique symmetric basis, up to equivalence.*

Proof: We consider here only the 2-convex case; the 2-concave one follows clearly by duality. Let X and Y be two spaces with normalized 1-symmetric basis $\{x_i\}_{i=1}^n$, respectively $\{y_i\}_{i=1}^n$, whose 2-convexity constants are both equal to one, and let $T = X \to Y$ be an isomorphism.

By Proposition 2.2, one can find a $\sigma \subset \{1, 2, \ldots, m\}$ so that $|\sigma| \geq n/2$ and

$$\Big\| \Big(\sum_{i\in\sigma} |c_i T x_i|^2\Big)^{1/2} \Big\| \leq \|T\| \sqrt{2} \Big\| \sum_{i\in\sigma} c_i y_i \Big\| ,$$

for all $\{c_i\}_{i\in\sigma}$. On the other hand, by Grothendieck's inequality (cf. [16] Theorem 1.f.14), it follows that

$$\Big\| \sum_{i\in\sigma} c_i x_i \Big\| \leq K_G \|T^{-1}\| \Big\| \Big(\sum_{i\in\sigma} |c_i T x_i|^2\Big)^{1/2} \Big\| ,$$

again, for all $\{c_i\}_{i\in\sigma}$. Hence,

$$\Big\| \sum_{i\in\sigma} c_i x_i \Big\| \leq K_G \|T\| \cdot \|T^{-1}\| \sqrt{2} \Big\| \sum_{i\in\sigma} c_i y_i \Big\| ,$$

which easily implies that

$$\Big\| \sum_{i=1}^{n} c_i x_i \Big\| \leq 2\sqrt{2} K_G \|T\| \|T^{-1}\| \Big\| \sum_{i=1}^{n} c_i y_i \Big\| ,$$

for all $\{c_i\}_{i=1}^n$, since $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are 1-symmetric and $|\sigma| \geq n/2$. This completes the proof for the roles of $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ can be interchanged. □

3 Quotients and Subspaces of ℓ_p^n of Lower Dimension

The aim of this section is to extend the assertion of Theorem 2.1 to the case of quotients of ℓ_p^n; $1 \le p < 2$, of dimension "much" lower than n. The weaker dimension constraint imposed in this case makes the results completely different. Moreover, the theorems presented in the sequel have a threshold dependent on p below which nothing can be proved. The reason lies in the fact that, for $p = 1$, any space of dimension $[\log n]$ is a quotient of ℓ_1^n while, for $1 < p < 2$, the space ℓ_p^n admits ℓ_2^m as a quotient provided $m \le n^{2(p-1)/p}$. Hence, nothing can be said about the size of a copy of ℓ_p^k that embeds in such a small quotient.

Before presenting the results on quotients and subspaces of ℓ_p^n, we prove a theorem of quite a general nature asserting that the type 2 and cotype 2 constants calculated with a fixed number of vectors can be computed, up to an equivalence constant, by using only equal norm vectors in the same number as above. This result localizes a theorem of G. Pisier (see [10]) which proves the same statement without, however, any restriction on the number of vectors used in the calculation.

We shall use the following notation for k being a positive integer and X an arbitrary Banach space: $T_2^{(k)}(X)$ and $C_2^{(k)}(X)$ will denote the (Rademacher) type 2, repectively cotype 2, constants calculated with exactly k vectors. For instance, $T_2^{(k)}(X)$ is the infimum of all $C < \infty$, for which

$$\int \Big\| \sum_{i=1}^{k} \varepsilon_i x_i \Big\| d\varepsilon \le C \Big(\sum_{i=1}^{k} \|x_i\|^2 \Big)^{1/2},$$

for any choice of $\{x_i\}_{i=1}^k$ in X. Similarly, $\alpha_2^{(k)}(X)$ and $\beta_2^{(k)}(X)$ will denote the Gaussian type 2, respectively cotype 2, constants calculated with k vectors. If we impose, in addition, the condition that all these k vectors have norm equal to one then we shall use instead the respective notation $\widehat{T}_2^{(k)}(X), \widehat{C}_2^{(k)}(X), \widehat{\alpha}_2^{(k)}(X)$ and $\widehat{\beta}_2^{(k)}(X)$. It follows from the definition that

$$\widehat{T}_2^{(k)}(X) \le T_2^{(k)}(X)\ ,\ \widehat{C}_2^{(k)}(X) \le C_2^{(k)}(X)\ ,\ \widehat{\alpha}_2^{(k)}(X) \le \alpha_2^{(k)}(X)\ ,\ \widehat{\beta}_2^{(k)}(X) \le \beta_2^{(k)}(X)\ .$$

The particular interest generated by the type 2 and cotype 2 constants defined above was stirred in part by the very useful result of N. Tomczak-Jaegermann [25] that, for any n-dimensional space X, the absolute type 2 and cotype 2 constants of X are respectively equivalent to $T_2^{(n)}(X)$ and $C_2^{(n)}(X)$.

Theorem 3.1. *There exists a constant $A < \infty$ such that, for any integer k and every Banach space X, we have that*

$$\alpha_2^{(k)}(X) \le A\widehat{\alpha}_2^{(k)}(X)\ ,\ \beta_2^{(k)}(X) \le A\widehat{\beta}_2^{(k)}(X)\ ,\ \text{and}\ T_2^{(k)}(X) \le A\widehat{T}_2^{(k)}(X)\ .$$

If, in addition, the space X has cotype q with constant $B < \infty$, for some value of $q < \infty$, then there exists a constant $C < \infty$, depending only on q and B, so that

$$C_2^{(k)}(X) \le C\widehat{C}_2^{(k)}(X)\ .$$

Proof: We begin with the proof of the case of the Rademacher type 2 constant. Choose vectors $\{x_i\}_{i=1}^k$ in X so that $\sum_{i=1}^{k} \|x_i\|^2 = 1$ and

$$\int \Big\| \sum_{i=1}^{k} \theta_i x_i \Big\| d\theta = T_2^{(k)}(X)\ ,$$

where $\theta_i = \pm 1$, for all $1 \le i \le k$. Then, for $r \ge 1$, put

$$\sigma_r = \{1 \le i \le k\ ;\ 1/2^r < \|x_i\| \le 1/2^{r-1}\}$$

and notice that

$$\sum_{r=1}^{\infty} |\sigma_r| = k$$

and

$$\sum_{r=1}^{\infty} |\sigma_r|/4^r < \sum_{r=1}^{\infty} \sum_{i \in \sigma_r} \|x_i\|^2 = 1\ .$$

Fix now an integer r_0 such that $16k < 4^{r_0} \le 64k$ and, for $1 \le r \le r_0$ and $i \in \sigma_r$, consider a set of vectors $\{y_j\}_{j \in \tau_i}$ with $y_j = x_i/2^{r_0 - r}$; $j \in \tau_i$, where $\{\tau_i\}_{i \in \sigma_r, 1 \le r \le r_0}$ are mutually disjoint sets of integers for which $|\tau_i| = 4^{r_0 - r}$ whenever $i \in \sigma_r$. Set

$$\eta_r = \bigcup_{i \in \sigma_r} \tau_i$$

and observe that $|\eta_r| = 4^{r_0 - r}|\sigma_r|$; $1 \le r \le r_0$, from which it follows that

$$\sum_{r=1}^{r_0} |\eta_r| = 4^{r_0} \sum_{r=1}^{r_0} |\sigma_r|/4r < 4^{r_0} \le 64k\ .$$

Hence,

$$2^{r_0}\widehat{T}_2^{(64k)}(X) \ge \int \Big\| \sum_{r=1}^{r_0} \sum_{j \in \eta_r} \varepsilon_j y_j/\|y_j\| \Big\| d\varepsilon = \int \Big\| \sum_{r=1}^{r_0} \sum_{i \in \sigma_r} \Big(\sum_{j \in \tau_i} \varepsilon_j/\|y_j\| \Big) x_i/2^{r_0 - r} \Big\| d\varepsilon =$$

$$= 2^{r_0 - r} \int\!\!\int \Big\| \sum_{r=1}^{r_0} \sum_{i \in \sigma_r} \Big| \sum_{j \in \tau_i} \varepsilon_j/\|y_j\| \Big| \theta_i x_i \Big\| d\varepsilon\, d\theta \ge$$

$$\ge 2^{r_0 - r} \int \Big\| \sum_{r=1}^{r_0} \sum_{i \in \sigma_r} \Big(\int \Big| \sum_{j \in \tau_i} \varepsilon_j/\|y_j\| d\varepsilon \Big) \theta_i x_i \Big\| d\theta\ .$$

However, for $i \in \sigma_r$, we have that

$$\Big(\sum_{j\in\tau_i} 1/\|y_j\|^2\Big)^{1/2} \geq |\tau_i|^{1/2} 2^{r_0-1} = 2^{r_0-r}2^{r_0-1} .$$

Hence, by Khintchine's inequlity, it follows that

$$2\sqrt{2}\widehat{T}_2^{(64k)}(X) \geq \int \|\sum_{r=1}^{r_0}\sum_{i\in\sigma_r} \theta_i x_i\| d\theta \geq T_2^{(k)}(X) - \int \|\sum_{r=r_0+1}^{\infty}\sum_{i\in\sigma_r}\theta_i x_i\| d\theta \geq$$

$$\geq T_2^{(K)}(X)\Big(1 - \Big(\sum_{r=r_0+1}^{\infty}\sum_{i\in\sigma_r}\|x_i\|^2\Big)^{1/2}\Big) \geq$$

$$\geq T_2^{(k)}(X)\Big(1 - \sum_{r=r_0+1}^{\infty} |\sigma_r|/4^{r-1}\Big)^{1/2} \geq$$

$$\geq T_2^{(k)}(X)\Big(1 - \Big(\sum_{r=r_0+1}^{\infty}|\sigma_r|\Big)^{1/2}/2^{r_0-1}\Big) \geq T_2^{(k)}(X)(1 - k^{1/2}/2^{r_0-1}) .$$

On the other hand, we clearly have that $k^{1/2}/2^{r_0} < 1/4$ and

$$\widehat{T}_2^{(64k)}(X) \leq 8\widehat{T}_2^{(k)}(X) .$$

Therefore,

$$32\sqrt{2}\widehat{T}_2^{(k)}(X) \geq T_2^{(k)}(X) ,$$

thus completing the proof of this part.

The proof for $\alpha_2^{(k)}(X)$ is quite similar and we omit it. We proceed now with the case of $\beta_2^{(k)}(X)$ which is actually easier.

Choose first vectors $\{x_i\}_{i=1}^k$ so that $\sum_{i=1}^k \|x_i\|^2 = 1$ and

$$\beta_2^{(k)}(X)\int \|\sum_{i=1}^k g_i(\omega)x_i\| d\omega = 1 ,$$

where $\{g_i\}_{i=1}^k$ denotes a sequence of independent Gaussian random variables which are normalized in L_1. We shall maintain in the sequel the same notation for $\{\sigma_r\}_{r=1}^\infty$, r_0, $\{\tau_i\}_{i\in\sigma_r}$, η_r and $\{y_j\}_{j\in\tau_i}$, as in the first part of the proof. Since

$$\sum_{r=1}^{r_0}|\eta_r| = \sum_{r=1}^{r_0} 4^{r_0-r}|\sigma_r| \geq 4^{r_0-1}\sum_{r=1}^{r_0}\sum_{i\in\sigma_r}\|x_i\|^2 \geq 4^{r_0-1}(1-k/4^{r_0}) \geq 4^{r_0-2}$$

we conclude, by using the infinite divisibility of the Gaussian random variables, that

$$2^{r_0-2} \leq \widehat{\beta}_2^{(64k)}(X)\int \|\sum_{r=1}^{r_0}\sum_{i\in\sigma_r}\sum_{j\in\tau_i} g_j(\omega)y_j/\|y_j\|\| d\omega \leq$$

$$\leq 2^{r_0}\widehat{\beta}_2^{(64k)}(X)\int \|\sum_{r=1}^{r_0}\sum_{i\in\sigma_r}\sum_{j\in\tau_i} g_j(\omega)y_j\| d\omega \leq$$

$$\leq 8\cdot 2^{r_0}\widehat{\beta}_2^{(k)}(X)\int \|\sum_{r=1}^{r_0}\sum_{i\in\sigma_r} g_i(\omega)x_i\| d\omega \leq 8\cdot 2^{r_0}\widehat{\beta}_2^{(k)}(X)/\beta_2^{(k)}(X) ,$$

i.e.,

$$\beta_2^{(k)}(X) \leq 32\widehat{\beta}_2^{(k)}(X) .$$

Finally, the assertion made for $C_2^{(k)}(X)$ in the case of a Banach space of cotype q follows immediately from the well known fact, proved, e.g., in [20] Theorem II.1, that

$$\int \|\sum_{i=1}^k g_i(\omega)x_i\| d\omega \leq K\int \|\sum_{i=1}^k \varepsilon_i x_i\| d\varepsilon ,$$

for any $\{x_i\}_{i=1}^k$ in X and $k = 1, 2, \ldots$, and a constant K depending only on q and the cotype q constant of X. □

Remark. Theorem 3.1 is no longer valid for the type p and cotype q constants if $1 \leq p < 2$ or $q > 2$. This is a consequence of [26] and a simple compactness argument.

We are prepared now to begin the study of quotients of ℓ_p^n. Since the methods of proof differ considerably from the case $p = 1$ to that of $1 < p < 2$ we discuss separately the case $p = 1$.

Theorem 3.2. *There exists a constant $c > 0$ such that any quotient X of ℓ_1^n whose dimension is d contains a subspace of dimension $k \geq cd/(1+\log n/d)$ which is c^{-1}-isomorphic to ℓ_1^k.*

The proof of Theorem 3.2 requires two preliminary results the first of which is proved by so-called empirical method of B. Maurey (cf. [24]).

Lemma 3.3. *Let $\{y_i\}_{i=1}^m$ be a sequence of vectors in the unit ball of an arbitrary Banach space Y, put $A = \mathrm{conv}\{y_i\}_{i=1}^m$ and, for $1 \leq k \leq m$, denote by A_k the set of all convex combinations of exactly k elements from A. Then, for each $y \in A$, we have that*

$$d(y, A_k) = \inf_{a\in A_k}\|y - a\| \leq 2T_2^{(k)}(Y)/\sqrt{k} .$$

Proof: Fix a vector $y = \sum_{i=1}^{m} \lambda_i y_i$ in A with $0 \le \lambda_i \le 1$; $1 \le i \le m$, and $\sum_{i=1}^{m} \lambda_i y_i = y$. Then define on the space $\Omega = \{1, 2, \dots, m\}$ a probability measure λ, by setting, $\lambda(i) = \lambda_i$; $1 \le i \le m$. Furthermore, by putting $\xi(i) = y_i$; $1 \le i \le m$, we define a random variable $\xi : \Omega \to A$ so that

$$\int_\Omega \xi(i) d\lambda = y \ .$$

Let now $\{\xi_j\}_{j=1}^k$ be a sequence of independent random variables on a probability space which, for simplicity, will be denoted again by (Ω, λ) so that ξ_j and ξ have the same distribution, for any value of $1 \le j \le k$. Since

$$d(y, A_k) \le \|k^{-1} \sum_{j=1}^{k} \xi_j(\omega) - y\| \ ,$$

for all $\omega \in \Omega$, we get that

$$d(y, A_k) \le \int_\Omega \|k^{-1} \sum_{j=1}^{k} \xi_j(\omega) - y\| d\lambda = k^{-1} \int_\Omega \Big\| \sum_{j=1}^{k} \Big(\xi_j(\omega) - \int_\Omega \xi_j(\omega) d\lambda \Big) \Big\| d\lambda \ .$$

Let (Ω', λ') be an independent copy of (Ω, λ) and $\{\xi'_j\}_{j=1}^k$ a sequence of independent random variables on (Ω', λ') each of which having the same distribution as ξ. Then

$$\begin{aligned} d(y, A_k) &\le k^{-1} \int_\Omega \Big\| \int_{\Omega'} \sum_{j=1}^{k} (\xi_j(\omega) - \xi'(\omega')) d\lambda' \Big\| d\lambda \le \\ &\le k^{-1} \int_\Omega \int_{\Omega'} \int \Big\| \sum_{j=1}^{k} \varepsilon_j (\xi_j(\omega) - \xi'_j(\omega')) \Big\| d\varepsilon \, d\lambda' \, d\lambda \le \\ &\le 2 T_2^{(k)}(Y)/\sqrt{k} \ . \end{aligned}$$

□

Lemma 3.4. *Let T be a linear operator of norm ≤ 1 from ℓ_1^n into a Banach space X of dimension $d \le n$. Put $A = \operatorname{conv}\{\pm Te_i \ ; \ 1 \le i \le n\}$ and $k = [d/\log e^5 n/d]$. Then there exists a vector x in the unit ball B_X of X such that*

$$d(x, A_k) \ge 1/100 \ ,$$

where A_k has the same meaning as in the previous lemma.

Proof: We shall present a proof based on a simple entropy argument. Let σ be a subset of $\{1, 2, \dots, n\}$ of cardinality $|\sigma| = k$ and consider the symmetric convex set

$$C_\sigma = \operatorname{conv}\{\pm Te_i \ ; \ i \in \sigma\}$$

which, as easily seen, is the set of all combinations of the form $\sum_{i \in \sigma} \lambda_i Te_i$ with $0 \le |\lambda_i| \le 1$; $i \in \sigma$, and $\sum_{i \in \sigma} |\lambda_i| \le 1$. Then with $\theta = 1/2e^3$ choose a maximal set of vectors $\{v_j^\sigma\}_{j=1}^{N(\sigma)}$ in C_σ such that $\|v_j^\sigma - v_h^\sigma\| > \theta$, for $1 \le j \ne h \le N(\sigma)$. Since the balls $\{v_j^\sigma + \frac{\theta}{2} C_\sigma\}_{j=1}^{N(\sigma)}$ are mutually disjoint it follows immediatley that $N(\sigma) \le (2+\theta)/\theta)^k \le (3/\theta)^k$. On the other hand, we also have that

$$(1+\theta)C_\sigma \subset \bigcup_{j=1}^{N(\sigma)} \{v_j^\sigma + (2\theta)B_X\} \ .$$

Now consider all the subsets σ of $\{1, 2, \dots, n\}$ which have cardinality equal to k and observe that

$$\operatorname{vol}\{ \bigcup_{|\sigma|=k} (1+\theta)C_\sigma \} \le \binom{n}{k} (3/\theta)^k (2\theta)^d \operatorname{vol} B_X \le (6en/k)^k (2\theta)^{d-k} \operatorname{vol} B_X \ .$$

However, by using the definitions of k and θ, one easily verifies that

$$(6en/k)^k (2\theta)^{d-k} < 1$$

from which it follows that

$$\operatorname{vol}\{B_X \sim \bigcup_{|\sigma|=k} (1+\theta)C_\sigma\} > 0 \ .$$

Consequently, there exists a vector $x \in B_X$ such that

$$d(x, A_k) \ge d(x, \bigcup_{|\sigma|=k} C_\sigma) \ge \theta > 1/100 \ .$$

□

Proof of Theorem 3.2. Since the claim of the theorem is stated up to a constant c there is no loss of generality in assuming that $k = [d/\log e^5 n/d]$. Let Q be the quotent map from ℓ_1^n onto a d-dimensional space and put $A = \{\pm Qe_i \ ; \ 1 \le i \le n\}$. By Lemma 3.4, it follows that with k as above there exists a vector $x \in B_X$ such that $d(x, A_k) \ge 1/100$. However, since Q is a quotient map A coincides with the unit ball of X and thus, by Lemma 3.3, we get that

$$T_2^{(k)}(X) \ge \sqrt{k}/200 \ .$$

Hence, by Theorem 3.1 or directly, one can conclude the existence of a constant $a > 0$ and of norm one vectors $\{x_i\}_{i=1}^k$ in X such that

$$\int \Big\| \sum_{i=1}^{k} \varepsilon_i x_i \Big\| d\varepsilon \ge ak \ .$$

The proof is now completed by using [5] in the real case or [22] in the complex one. □

Remark. The estimate given in the statement of Theorem 3.2 for k is asymptotically sharp. In order to check this fact, we construct the following example (which was considered in [6]). For the sake of simplicity, we shall assume that all the fractions and logarithms of integers are themselves integers.

Fix $n, d \le n/e$, $k = d/(1+\log n/d)$ and put $N = n/k$. Since the space ℓ_1^N admits $\ell_2^{\log N}$ as a quotient space it follows that also ℓ_1^n admits as a quotient space the direct sum in the sense of ℓ_1 of k copies of $\ell_2^{\log N}$. This space, which will be denoted by Y, has dimension

$$k \log N = \frac{d}{1+\log n/d}\big(\log n/d + \log(1+\log n/d)\big) .$$

Therefore, $d/2 \le \dim Y \le 2d$. On the other hand, it is easily checked that, up to a constant, Y contains ℓ_1^m for $m \le k$ only.

We pass now to the case $p > 1$. We shall prove below a result on quotients of ℓ_p^n; $1 < p < 2$, that is similar to Theorem 3.2 except, of course, for the numerical estimates which depend on p. In fact, this result will be deduced from a more general theorem which estimates sharply in terms of the euclidean distance the dimension k of a copy of ℓ_p^k that embeds in a quotient of L_p.

Theorem 3.5. *For every $q > 2$, there exists a constant $a = a(q) > 0$ such that, whenever X is a subspace of L_q with euclidean distance $d_X < \infty$, then its dual X^* contains a subspace a^{-1}-isomorphic to $\ell_{q'}^k$, where $q' = q/(q-1)$ and*

$$k \ge a d_X^{(1/2-1/q)^{-1}} = a d_X^{2q/(q-2)} .$$

The proof of Theorem 3.5 requires two preliminary lemmas.

Lemma 3.6. *For every $q > 2$, there exists a constant $C_q < \infty$ such that, whenever X is a subspace of L_q with euclidean distance $d_X < \infty$, then one can find a system $\{\varphi_i\}_{i=1}^m$ of norm one functions in X with $m = [d_X^{2q/(q-2)}]$, for which*

$$m^{1/q} \le \Big\| \Big(\sum_{i=1}^m |\varphi_i|^2 \Big)^{1/2} \Big\|_q \le C_q m^{1/q} .$$

Proof: By a well known result of S. Kwapien [15], there exists a constant $D_0 < \infty$ such that

$$d_X \le D_0 T_2(X) C_2(X)$$

which, in view of the fact that L_q is of type 2, yields the existence of a constant $c_1 > 0$ so that

$$C_2(X) \ge c_1 d_X .$$

It follows easily that there exists an integer k and a k-dimensional subspace X_0 of X for which

$$C_2^{(k)}(X_0) \ge c_1 d_X/2 .$$

Hence, by the last part of Theorem 3.1, we have that

$$\widehat{C}_2^{(k)}(X_0) \ge c_2 d_X ,$$

for some constant $c_2 > 0$, depending only on q. Therefore, one can find norm one functions $\{x_i\}_{i=1}^k$ in X_0 such that

$$\Big\| \Big(\sum_{i=1}^k |x_i|^2 \Big)^{1/2} \Big\|_q \le D_1 k^{1/2}/d_X ,$$

for some $D_1 < \infty$, depending only on q.

The main step of the proof, which comes now, consists of a probabilistic selection among the vectors $\{x_i\}_{i=1}^k$. To this end, fix

$$\tau = (d_X/D_1)^{2q/(q-2)}/k$$

and notice that $0 < \tau \le 1$ since $\big\|\big(\sum_{i=1}^k |x_i|^2\big)^{1/2}\big\|_q \ge k^{1/q}$. Then let $\{\xi_i\}_{i=1}^k$ be a sequence of $\{0,1\}$-valued independent random variables of mean τ over some probability space (Ω, Σ, μ). Set

$$J = \Big(\int_\Omega \Big\|\Big(\sum_{i=1}^k \xi_i(\omega)|x_i|^2\Big)^{1/2}\Big\|_q^q d\mu \Big)^{1/q}$$

and observe that

$$J^q = \int_\Omega \Big\| \sum_{i=1}^k \xi_i(\omega)|x_i|^2 \Big\|_{q/2}^{q/2} d\mu \le 2\Big[\int_\Omega \Big\|\Big(\sum_{i=1}^k (\xi_i(\omega) - \tau)|x_i|^2\Big)^{1/2}\Big\|_q^q d\mu + \big(D_1(\tau k)^{1/2}/d_X\big)^q \Big] .$$

Let now (Ω', Σ', μ') be an independent copy of (Ω, Σ, μ) and $\{\xi_i'\}_{i=1}^k$ a sequence of independent random variables over (Ω', Σ', μ') which have the same distribution as $\{\xi_i\}_{i=1}^k$. Then, with B_τ

standing for the constant in Khinchine's inequality in L_r, we have that

$$\int_\Omega \Big\|\big(\sum_{i=1}^k (\xi_i(\omega)-\tau)|x_i|^2\big)^{1/2}\Big\|_q^q d\mu = \int_\Omega \Big\| \int_{\Omega'} \sum_{i=1}^k (\xi_i(\omega)-\xi_i'(\omega'))|x_i|^2 d\mu'\Big\|_{q/2}^{q/2} d\mu \le$$

$$\le \int_\Omega\int_{\Omega'} \Big\|\sum_{i=1}^k (\xi_i(\omega)-\xi_i'(\omega'))|x_i|^2\Big\|_{q/2}^{q/2} d\mu' d\mu \le$$

$$\le B_{q/2}^{q/2}\int_\Omega\int_{\Omega'} \Big\|\big(\sum_{i=1}^k (\xi_i(\omega)-\xi_i'(\omega'))^2|x_i|^4\big)^{1/4}\Big\|_q^q d\mu' d\mu \le$$

$$\le 2^{q/4}B_{q/2}^{q/2}\int_\Omega \Big\|\big(\sum_{i=1}^k \xi_i(\omega)|x_i|^4\big)^{1/4}\Big\|_q^q d\mu .$$

It follows that

$$J \le D_2\left[\Big(\int_\Omega \|(\sum_{i=1}^k \xi_i(\omega)|x_i|^4)^{1/4}\|_q^q d\mu\Big)^{1/q} + (\tau k)^{1/2}/d_X\right] ,$$

for some constant $D_2 < \infty$, again depending on q only.

Suppose now that

$$J > 2D_2(\tau k)^{1/2}/d_X .$$

Then, by using the Cauchy-Schwartz inequality twice, we get that

$$J \le 2D_2\Big(\int_\Omega \|(\sum_{i=1}^k \xi_i(\omega)|x_i|^4)^{1/4}\|_q^q d\mu\Big)^{1/q} \le$$

$$\le 2D_2\Big(\int_\Omega \|\max_{1\le j\le k}\xi_j(\omega)|x_j|^{1/2}\cdot\Big(\sum_{i=1}^k \xi_i(\omega)|x_i|^2\Big)^{1/4}\|_q^q d\mu\Big)^{1/q} \le$$

$$\le 2D_2\Big(\int_\Omega \|\max_{i\le j\le k}\xi_j(\omega)|x_j|^{1/2}\|_{2q}^q\cdot\|(\sum_{i=1}^k \xi_i(\omega)|x_i|^2)^{1/4}\|_{2q}^q d\mu\Big)^{1/q} \le$$

$$\le 2D_2\Big(\int_\Omega \|\max_{1\le j\le k}\xi_j(\omega)|x_j|^{1/2}\|_{2q}^{2q} d\mu\Big)^{1/2q}\cdot\Big(\int_\Omega \|(\sum_{i=1}^k \xi_i(\omega|x_i|^2)^{1/4}\|_{2q}^{2q} d\mu\Big)^{1/2q} .$$

On the other hand,

$$\|\max_{1\le j\le k}\xi_j(\omega)|x_j|^{1/2}\|_{2q}^{2q} \le \Big\|\Big(\sum_{j=1}^k \xi_j(\omega)|x_j|^q\Big)^{1/q}\Big\|_q^q = \sum_{j=1}^k \xi_j(\omega) ,$$

for all $\omega\in\Omega$, which yields that

$$J \le 2D_2(\tau k)^{1/2q}\Big(\int_\Omega \Big\|\Big(\sum_{i=1}^k \xi_i(\omega)|x_i|^2\Big)^{1/2}\Big\|_q^q d\mu\Big)^{1/2q} ,$$

i.e.,

$$J \le (2D_2)^2(\tau k)^{1/q} .$$

Hence, by combining the two estimates obtained above for J, we conclude that

$$J \le \max\{2D_2(\tau k)^{1/2}/d_X\ ,\ (2D_2)^2(\tau k)^{1/q}\}$$

which, in view of our choice of τ, implies that

$$J \le D_3 d_X^{2/(q-2)} ,$$

for some constant D_3, depending only on q. Then, by a standard concentration of measure argument, we get that there exists a subset σ of $\{1,2,\dots,k\}$ such that

$$\tau k/2 \le |\sigma| \le 2\tau k$$

and

$$\|(\sum_{i\in\sigma}|x_i|^2)^{1/2}\|_q \le D_3 d_X^{2/(q-2)} .$$

□

Lemma 3.7. *For any $2 < q < r$ and $C < \infty$, there exists a constant $c = c(q,r,C) > 0$ such that, whenever ν is a probability measure and Y a finite dimensional subspace of $L_q(\nu)$ containing a sequence $\{\varphi_i\}_{i=1}^m$ of norm one functions which satisfies*

$$\|(\sum_{i=1}^m |\varphi_i|^2)^{1/2}\|_q \le Cm^{1/q} ,$$

then there exists a density ϕ, a subset $\sigma\subset\{1,2,\dots,m\}$ and a sequence $\{g_i\}_{i\in\sigma}$ in $L_{q'}(\lambda)$, where $d\lambda/d\nu=\phi$, so that

(i) $|\sigma| \ge cm$

(ii) $c\|\sum_{i\in\sigma} a_i g_i\|_{q'} \le (\sum_{i\in\sigma}|a_i|^{q'})^{1/q'}$,

and

(iii) $\|\sum_{i\in\sigma} b_i g_i\|_{(\widetilde{Y}_r)^*} \ge c\sum_{i\in\sigma}|b_i|/|\sigma|^{1/q}$,

for any choice of $\{a_i\}_{i\in\sigma}$ and $\{b_i\}_{i\in\sigma}$, where $\widetilde{Y}$ denotes the isometric image of Y in $L_q(\lambda)$ and $\widetilde{Y}_r$ stands for $\widetilde{Y}$ when it is considered as a subspace of $L_r(\lambda)$.

Proof: Let ϕ_0 be defined by

$$\phi_0 = 1 + (\sum_{i=1}^m |\varphi_i|^2)^{q/2}/m ,$$

put $\phi = \phi_0/\|\phi_0\|_1$, and denote by λ the probability measure that satisfies $d\lambda/d\nu = \phi$. The map $\varphi \to \widetilde{\varphi} = \varphi/\phi^{1/q}$ defines an isometry from $L_q(\nu)$ onto $L_q(\lambda)$ which takes Y into a subspace $\widetilde{Y}$ of $L_q(\lambda)$. The advantage of this change of density is that it transforms the original system $\{\varphi_i\}_{i=1}^m$ into a system $\{\widetilde{\varphi}_i\}_{i=1}^m$ whose square function is pointwise bounded. Indeed, it follows from the definition of ϕ that the functions $\widetilde{\varphi}_i = \varphi_i/\phi^{1/q}$; $1 \le i \le m$, satisfy

$$\Big(\sum_{i=1}^m |\widetilde{\varphi}_i|^2\Big)^{1/2} \le (\phi_0 m/\phi)^{1/q} \le \big(\|\phi_0\|_1 m\big)^{1/q} \le \big((1+C^q)m\big)^{1/q}$$

i.e.,

$$\Big(\sum_{i=1}^m |\widetilde{\varphi}_i|^2\Big)^{1/2} \le (1+C^q)^{1/q} m^{1/q} ,$$

pointwise.

Next, choose norm one functions $\{f_i\}_{i=1}^m$ in $L_{q'}(\lambda)$ so that $\langle f_i, \widetilde{\varphi}_i\rangle = 1$, for all $1 \le i \le m$. Since the sequence $\{f_i\}_{i=1}^m$ does not necessarily satisfy an upper q'-estimate we consider instead the functions

$$g_i = f_i \chi_{\{|f_i| > \varepsilon (\sum_{j=1}^m |f_j|^2)^{1/2}\}} ,$$

where $\varepsilon = \big[4(1+C^q)^{1/q}\big]^{-2/(2-q')}$. Notice that, for any t in the underlying measure space, the set

$$\eta_t = \Big\{1 \le i \le m \;;\; |f_i(t)| > \varepsilon\Big(\sum_{j=1}^m |f_j(t)|^2\Big)^{1/2}\Big\}$$

satisfies the condition

$$\sum_{i=1} |f_i(t)|^2 \ge \sum_{i\in\eta_t} |f_i(t)|^2 > \varepsilon^2 |\eta_t| \sum_{j=1}^m |f_j(t)|^2$$

i.e.,

$$|\eta_t| < 1/\varepsilon^2 ,$$

for all t. Hence,

$$\Big|\sum_{i=1}^m a_i g_i(t)\Big| = \Big|\sum_{i\in\eta_t} a_i f_i(t)\Big| \le |\eta_t| \max_{i\in\eta_t} |a_i f_i(t)| \le \varepsilon^{-2}\Big(\sum_{i\in\eta_t} |a_i f_i(t)|^{q'}\Big)^{1/q'} ,$$

which yields that

$$\Big\|\sum_{i=1}^m a_i g_i\Big\|_{q'} < \varepsilon^{-2}\Big(\sum_{i=1}^m |a_i|^{q'}\Big)^{1/q'} = [4(1+C^q)^{1/q}]^{4/(2-q')}\Big(\sum_{i=1}^m |a_i|^{q'}\Big)^{1/q'} ,$$

for any choice of $\{a_i\}_{i=1}^m$.

We want now to estimate from below the expression $\langle g_i, \widetilde{\varphi}_i\rangle$; $1 \le i \le m$. To this end, observe that

$$\begin{aligned}\sum_{i=1}^m |\langle f_i - g_i, \widetilde{\varphi}_i\rangle| &\le \Big\langle \Big(\sum_{i=1}^m |f_i - g_i|^2\Big)^{1/2}, \Big(\sum_{i=1}^m |\widetilde{\varphi}_i|^2\Big)^{1/2}\Big\rangle \le \\ &\le (1+C^q)^{1/q} m^{1/q} \Big\|\Big(\sum_{i=1}^m |f_i - g_i|^2\Big)^{1/2}\Big\|_{q'} .\end{aligned}$$

However,

$$\begin{aligned}\Big(\sum_{i=1}^m |f_i - g_i|^2\Big)^{1/2} &\le \Big(\sum_{i=1}^m |f_i|^{q'} \cdot |f_i - g_i|^{2-q'}\Big)^{1/2} \le \\ &\le \Big(\sum_{i=1}^m |f_i|^{q'}\Big)^{1/2} \max_{i\le j\le m} |f_j - g_j|^{1-q'/2} \le \\ &\le \Big(\sum_{i=1}^m |f_i|^{q'}\Big)^{1/2} \Big[\varepsilon\Big(\sum_{j=1}^m |f_j|^2\Big)^{1/2}\Big]^{1-q'/2} \le \\ &\le \varepsilon^{1-q'/2}\Big(\sum_{j=1}^m |f_j|^{q'}\Big)^{1/q'}\end{aligned}$$

from which, in view of our choice of ε made above, we get that

$$\begin{aligned}\Big(\sum_{i=1}^m |\langle f_i - g_i, \widetilde{\varphi}_i\rangle| &\le (1+C^q)^{1/q} \varepsilon^{1-q'/2} m^{1/q} \Big\|\Big(\sum_{j=1}^m |f_j|^{q'}\Big)^{1/q'}\Big\|_{q'} \le \\ &\le (1+C^q)^{1/q}\varepsilon^{1-q'/2} m \le m/4 .\end{aligned}$$

This, of course, ensures the existence of a subset η_0 of $\{1,2,\dots,m\}$ such that $|\eta_0| \ge m/2$ and

$$\langle g_i, \widetilde{\varphi}_i\rangle \ge 1/2 ,$$

for all $i \in \eta_0$.

Take now $r > q$ and notice that

$$\begin{aligned}\int \Big\|\sum_{i\in\eta_0} \varepsilon_i g_i\Big\|_{(\widetilde{Y}_r)^*} d\varepsilon &\ge \sum_{i\in\eta_0} \langle g_i, \widetilde{\varphi}_i\rangle \Big/ \int \Big\|\sum_{i\in\eta_0} \varepsilon_i \widetilde{\varphi}_i\Big\|_r d\varepsilon \ge \\ &\ge |\eta_0|/2B_r \Big\|\Big(\sum_{i\in\eta_0} |\widetilde{\varphi}_1|^2\Big)^{1/2}\Big\|_r \ge m^{1/q'}/4B_r(1+C^q)^{1/q} ,\end{aligned}$$

i.e., the functions $h_i = 4B_r(1+C^q)^{1/q} m^{1/q} g_i$; $i \in \eta_0$, satisfy the estimate

$$\int \Big\|\sum_{i\in\eta_0} \varepsilon_i h_i\Big\|_{(\widetilde{Y}_r)^*} d\varepsilon \ge m \ge |\eta_0| .$$

Moreover, the fact that the sequence $\{g_i\}_{i=1}^m$ has a good upper q'-estimate implies that, for any $\eta \subset \eta_0$, we have

$$\Big\|\sum_{i\in\eta} h_i\Big\|_{(\widetilde{Y}_r)^*} \le \Big\|\sum_{i\in\eta} h_i\Big\|_{r'} \le \Big\|\sum_{i\in\eta} h_i\Big\|_{q'} \le$$
$$\le 4B_r(1+C^q)^{1/q} m^{1/q}\Big\|\sum_{i\in\eta} g_i\Big\|_{q'} \le A m^{1/q}|\eta|^{1/q'} ,$$

where A is a constant depending on q, r and C. This means that the sequence $\{h_i\}_{i\in\eta_0}$ satisfies the conditions of [2] Theorem 5.2. Hence, there exist a constant $a > 0$ and a subset σ of η_0 so that $|\sigma| \ge am$ and

$$\Big\|\sum_{i\in\sigma} b_i h_i\Big\|_{(\widetilde{Y}_r)^*} \ge a\sum_{i\in\sigma}|b_i| ,$$

for all $\{b_i\}_{i\in\sigma}$. Then the proof can be completed by substituting back the functions $\{y_i\}_{i\in\sigma}$.□

Proof of Theorem 3.5. Suppose that $q > 2$, ν is a probability measure and X a subspace of $L_q(\nu)$ with $d_X < \infty$. By Lemma 3.6, there exist a constant $C_q < \infty$ and norm one functions $\{\varphi_i\}_{i=1}^m$ in X, where $m = [d_X^{2q/(q-2)}]$, such that

$$\Big\|\Big(\sum_{i=1}^m |\varphi_i|^2\Big)^{1/2}\Big\|_q \le C_q m^{1/q} .$$

Thus, if we put $Y = [\varphi_i]_{i=1}^m$, fix $r > q$ and use Lemma 3.7, there is no loss of generality in assuming the existence of a constant $c = c(q,r) > 0$, of a subset σ of $\{1,2,\ldots,m\}$ and of norm one functions $\{g_i\}_{i\in\sigma}$ in $L_{q'}(\nu)$ for which the conditions (i),(ii) and (iii) of that lemma are satisfied.

Since

$$\|g\|_{Y^*} \le \|g\|_{X^*} \le \|g\|_{q'} ,$$

for all $g \in L_{q'}(\nu)$, the proof of the theorem will be completed provided we show the existence of a subset σ_0 of σ of cardinality $|\sigma_0| \ge |\sigma|/2$ so that

$$\Big\|\sum_{i\in\sigma_0} a_i g_i\Big\|_{Y^*} \ge c'\Big(\sum_{i\in\sigma_0}|a_i|^{q'}\Big)^{1/q'} ,$$

for all $\{a_i\}_{i\in\sigma_0}$ and some choice of $c' = c'(q,r) > 0$ which will be determined later. This is done by a standard exhaustion arguemnt. Indeed, if the claim made above is false then one can easily construct vectors $y_i = \sum_{j\in\sigma} a_{i,j}g_j$; $1 \le i \le \ell$, such that $\|y_i\|_{Y^*} < c'$, $\sum_{j\in\sigma}|a_{i,j}|^{q'} = 1$, $\sum_{i=1}^{\ell}|a_{i,j}|^{q'} \le 2$, for all $1 \le i \le \ell$ and $j \in \sigma$, and if we set

$$\delta_{\ell+1} = \{j \in \sigma \; ; \; \sum_{i=1}^{\ell}|a_{i,j}|^{q'} < 1\}$$

then $|\delta_{\ell+1}| < |\sigma|/2$. A simple argument shows that, under these conditions, $\ell \ge |\sigma|/2$.

Now, for each $1 \le i \le \ell$, choose a vector z_i in the kernel of the quotient map from the space $L_{q'}(\nu)$ onto Y^* such that $u_i = y_i + z_i$ satisfies $\|y_i\|_{Y^*} = \|u_i\|_{q'}$, for all $1 \le i \le \ell$. Next, let $\{\psi_i\}_{i=1}^{\ell}$ be a sequence of q'-stable independent random variables over some probability space (Ω, Σ, μ) which are normalized in $L_1(\mu)$. Then, on one hand, we have that

$$\int_\Omega \Big\|\sum_{i=1}^{\ell}\psi_i(\omega)y_i\Big\|_{(Y_r)^*}d\mu \le \int_\Omega\Big\|\sum_{i=1}^{\ell}\psi_i(\omega)u_i\Big\|_{r'}d\mu \le \|\psi_1\|_{r'}\Big\|\Big(\sum_{i=1}^{\ell}|u_i|^{q'}\Big)^{1/q'}\Big\|_{r'} \le$$
$$\le \|\psi_1\|_{r'}\Big\|\Big(\sum_{i=1}^{\ell}|u_i|^{q'}\Big)^{1/q'}\Big\|_{q'} < c'\|\psi_1\|_{r'}\ell^{1/q'} .$$

On the other hand, we also have that,

$$\int_\Omega \Big\|\sum_{i=1}^{\ell}\psi_i(\omega)y_i\Big\|_{(Y_r)^*}d\mu = \int_\Omega\Big\|\sum_{j\in\sigma}\Big(\sum_{i=1}^{\ell}\psi(\omega)a_{i,j}\Big)g_j\Big\|_{(Y_r)^*}d\mu \ge$$
$$\ge c\sum_{j\in\sigma}\int_\Omega\Big|\sum_{i=1}^{\ell}\psi_i(\omega)a_{i,j}\Big|d\mu/|\sigma|^{1/q} =$$
$$= c\sum_{j\in\sigma}\Big(\sum_{i=1}^{\ell}|a_{i,j}|^{q'}\Big)^{1/q'}/|\sigma|^{1/q} .$$

However, since

$$2^{1/q} \ge \Big(\sum_{i=1}^{\ell}|a_{i,j}|^{q'}\Big)^{1/q'} ,$$

we get that

$$c'\|\psi_1\|_{r'}\ell^{1/q'} > c\sum_{j\in\sigma}\sum_{i=1}^{\ell}|a_{i,j}|^{q'}/(2|\sigma|)^{1/q} = c\ell/(2|\sigma|)^{1/q}$$

i.e.,

$$c'\|\psi_1\|_{r'} > c/4^{1/q} .$$

Therefore, if we choose

$$c' = c/4^{1/q}\|\psi_1\|_{r'}$$

then we reach a contradiction, thus proving the theorem. □

Corollary 3.8. *For every $1 < p < 2$, there exists a constant $C_p < \infty$ such that if, for some n and d, X is a d-dimensional quotient space of ℓ_p^n then X contains a subspace which is C_p-isomorphic to ℓ_p^k with*

$$k \ge C_p^{-1}(d^p/n^{2(p-1)})^{1/(2-p)} .$$

Proof: Since the dual X^* of X is a subspace of $\ell_{p'}^n$ it follows from Theorem 3.5 that X contains a subspace which is a^{-1}-isomorphic to ℓ_p^k, for some constant $a = a(p) > 0$ and $k \geq a d_X^{2p'/(p'-2)}$. On the other hand, by a well known result from [19], X^* contains a 10-hilbertian subspace of dimension $[d/2d_X^2]$. However, this dimension cannot exceed n^2/p' which yields that

$$d_X \geq d^{1/2}/2^{1/2}n^{1/p'} .$$

It follows that

$$k \geq a'(d^{p'}/n^2)^{1/(p'-2)} = a'(d^p/n^{2(p-1)})^{1/(2-p)} ,$$

for some a' depending only on p. □

Remark. As in the case of Theorem 3.2, the estimate given by Corollary 3.8 is precise. This fact is verified by the following example. Fix $1 < p < 2$, n and $d \leq n$, take $k = (d^p/n^{2(p-1)})^{1/(2-p)}$ and put $N = n/k$. Under the assumption that k and N are integers, it is known that the space ℓ_p^N admits ℓ_2^m as a quotient space, for $m = N^{2/p'}$. Hence, ℓ_p^n admits as a quotient space the direct sum X_p in the sense of ℓ_p of k copies of ℓ_2^m. Notice that

$$\dim X_p = kN^{2/p'} = n^{2/p'}k^{(2-p)/p} = d$$

and the argument is completed by observing that X_p contains ℓ_p^h for $h \leq k$ only.

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