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Analytic Theory of Continued Fractions II

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Equimodular Limit Periodic Continued Fractions*

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1. Introduction.

Let

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots \tag{1.1}$$

be a continued fraction. Then the nth numerator A_n and denominator B_n satisfy $A_{-1} = 1$, $A_0 = b_0$, $B_{-1} = 0$, $B_0 = 1$ and for $n \geq 1$ the difference equation

$$x_n = b_n x_{n-1} + a_n x_{n-2}. \tag{1.2}$$

This equation can be written in the matrix form

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = Q_n \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}, \quad n = 1, 2, \dots, \tag{1.3}$$

where

$$Q_n = \begin{pmatrix} b_n & a_n \\ 1 & 0 \end{pmatrix}.$$

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If $\lim_{n\to\infty} a_n = a \in \mathbb{C}$ and $\lim_{n\to\infty} b_n = b \in \mathbb{C}$, then $Q_n \to Q$ as $n \to \infty$, where

$$Q = \begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix} .$$

The characteristic equation of Q is

$$\lambda^2 - b\lambda - a = 0. \tag{1.4}$$

In this simple limit periodic case, we say (1.1) is <u>equimodular</u> if the roots of (1.4) have equal absolute value.

More generally, suppose

$$\begin{aligned} &\lim a_{2n-1} = \alpha_1,\ \lim a_{2n} = \alpha_2 \\ &\lim b_{2n-1} = \beta_1,\ \lim b_{2n} = \beta_2, \end{aligned} \tag{1.5}$$

where $\alpha_i,\ \beta_i \in C,\ i = 1,2.$

Using (1.3) we obtain

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = Q_{n+1}Q_n \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} = R_n \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} ,\ n \geq 1,$$

where

$$R_n = \begin{pmatrix} b_{n+1}b_n + a_{n+1} & b_{n+1}a_n \\ b_n & a_n \end{pmatrix} .$$

Under our hypothesis,

$$R_{2n-1} \to R = \begin{pmatrix} \beta_1\beta_2+\alpha_2 & \beta_2\alpha_1 \\ \beta_1 & \alpha_1 \end{pmatrix}$$

and

$$R_{2n} \to S = \begin{pmatrix} \beta_1\beta_2+\alpha_1 & \beta_1\alpha_2 \\ \beta_2 & \alpha_2 \end{pmatrix}$$

as $n\to\infty$. The matrices R and S have the common characteristic equation

$$\lambda^2 - \lambda(\alpha_1 + \alpha_2 + \beta_1\beta_2) + \alpha_1\alpha_2 = 0. \tag{1.6}$$

Under the conditions (1.5), we shall say (1.1) is <u>equimodular</u> if the roots of (1.6) have equal modulus.

Here we wish to point out what connection this definition has with the even and odd parts of (1.1) when they exist. If $b_{2k} \neq 0$, $k = 1, 2, \ldots$, then by [6, Theorem 2.10] the partial numerators and denominators a_k^* and b_k^*, respectively, of the even part satisfy

$$\left.\begin{aligned} a_k^* &= - a_{2k-2}a_{2k-1}b_{2k-4}b_{2k} \\ b_k^* &= a_{2k-1}b_{2k} + b_{2k-2}(a_{2k} + b_{2k-1}b_{2k}), \end{aligned}\right\} \quad k = 3, 4, \ldots$$

so

$$a_k^* \to a^* = -\alpha_1\alpha_2\beta_2^2$$

and

$$b_k^* \to b^* = \alpha_1\beta_2 + \beta_2(\alpha_1 + \beta_1\beta_2)$$

as $k \to \infty$. Applying (1.3) and (1.4) we have that the associated characteristic equation is

$$\lambda^2 - b^*\lambda - a^* = 0. \tag{1.7}$$

It is easy to see that, if the roots of (1.6) have equal modulus, then the roots of (1.7) have equal modulus. Similarly, if $b_{2k+1} \neq 0$, $k = 0, 1, 2, \ldots$, then by [6, Theorem 2.11] the odd part of (1.1) exists and under conditions (1.5) its partial numerators and denominators tend to limits $\hat{a}$ and $\hat{b}$, respectively. The associated characteristic equation

$$\lambda^2 - \hat{b}\lambda - \hat{a} = 0$$

also has roots of equal modulus if this is the case for (1.6).

Our purpose in this paper is to make a rather extensive study of equimodular limit periodic continued fractions of the types $-K(-1/b_n)$ and $K(a_n/1)$. Most convergence theorems to date for limit periodic continued fractions have depended on the stipulation that the roots of (1.4) have unequal modulus. Under this hypothesis, a number of fruitful convergence results such as those given in paragraph 19 of Perron's book [10] were obtained. These results and Poincaré's theorem on finite differences (see [7, Theorem A]) served as useful tools in the work of Lange [7] on limit periodic δ-fractions. But, if the roots of (1.4) have the same modulus, we can no longer employ these results. In fact Louboutin [8] has recently shown that Poincaré's theorem cannot be extended when two characteristic roots have the same absolute value, even if some arithmetical constraints are imposed. This may help to explain why it appears from our investigations that the behavior of limit periodic continued fractions in equimodular cases is quite complicated.

Up to now, there appears to be only a modest number of scattered results in the literature relating directly to the work we are presenting here. One of the earliest important results for which the roots of (1.4) have equal absolute value is the well known Stern (1860) - Stolz (1886) Theorem (see [6, p. 79]). This theorem states that $K(1/b_n)$ diverges if $\Sigma|b_n|$ converges. In more recent times, Thron and Waadeland [11] have generated interest in this topic through their 1980 paper where, besides considering convergence acceleration questions, they studied various cases and examples concerned with continued fractions $K(a_n/1)$ where $\lim a_n = -1/4$. This paper and some work of Waadeland [12] led Jacobsen and Magnus [4] in 1983 to prove the result that $K(a_n/1)$ diverges if

$$a_n = -\frac{1}{4} - \frac{c}{16(n+\theta)(n+\theta+1)},$$

where $c > 1$, $\theta > -1$ are constants. In 1981 Heller and Roach [3] proved that $K(1/b_n)$ diverges if $b_{2n-1} \to u$, $b_{2n} \to v$, $-4 < uv < 0$, and

both series $\Sigma \mid b_{2n-1} - u \mid$, $\Sigma \mid b_{2n} - v \mid$ converge. In 1984 Baltus [1] investigated continued fractions $K(a_n/1)$ with $a_n \to -1/4$ in section 6 of his doctoral thesis. Recently Jacobsen and Waadeland [5] have studied cases where $a_n \to -1/4$ or $a_n \to \infty$ and their connections.

In section 2 we study the convergence behavior of continued fractions $-K(-1/b_n)$, where $b_n \to b$, $-2 < b < 2$. The discriminant b^2-4 of (1.4) is negative in this case, so the roots of (1.4) not only have the same absolute value but they are unequal. Lemma 2.1 deals with the boundedness of solutions of certain second order difference equations. Besides possibly having applications in the study of difference equations, it has proved to be quite useful in our study of equimodular continued fractions. In Theorem 2.1 we show that $-K(-1/b_n)$ cannot converge finitely under the above conditions if the series $\Sigma \mid b_n - b_{n+1} \mid$ converges. Theorem 2.2 shows how to construct continued fractions $-K(-1/b_n)$ with $b_n \to b$, $0 \le b < 2$, that diverge by oscillation.

In section 3 the main emphasis is on the study of continued fractions $-K(-1/b_n)$, where $b_n \to 2$ from the left at $n \to \infty$. In this case the roots of (1.4) are equal. The convergence behavior is considerably different under these conditions than for the case studied in section 2. Through a series of five theorems assuming various growth patterns of the b_n and using a variety of techniques in their proofs, we feel we have shed much light on the behavior of these continued fractions. Theorem 3.1 is a general convergence theorem. Theorem 3.2 states that $-K(-1/b_n)$ converges finitely if $b_n \ge 2 - 1/(4n^2)$ for $n \ge 1$. Theorems 3.3 and 3.4 are designed to show what effect the speed of growth of the b_n has on the convergence of $-K(-1/b_n)$, and as a by-product they show that the coefficient $-1/4$ of n^{-2} in Theorem 3.2 is sharp.

Though we learned a fair amount about continued fractions $-K(-1/b_n)$ when $b_n \to 0$ in section 2, we have chosen in section 4 to study this case in greater detail. In Theorem 4.1 we settle a question of Wall showing that the convergence of the series Σb_n is not sufficient for the divergence of $K(1/b_n)$. In Theorem 4.2 we show how to construct finitely convergent continued fractions $-K(-1/b_n)$, where $\lim b_n = 0$. Theorem 4.3 deals with a case where the b_n vary with a parameter, and it is designed to give information about S- and H-fractions on the "cut". In example 4.1 we demonstrate how this study has application to the question of whether or not a continued fraction corresponding to a function can actually converge to this function.

In section 5 we again study continued fractions $-K(-1/b_n)$, but here we do not necessarily demand convergence of the sequence $\{b_n\}$. Instead we assume $b_{2n-1} \to a$ and $b_{2n} \to b$, where in some cases we allow a and b to have nonzero imaginary parts but always $0 \le ab < 4$. Under these conditions the discriminant $ab(ab - 4)$ of (1.6) is nonpositive. Therefore, the roots of (1.6) have equal absolute value. Lemma 5.1 is a statement about the boundedness of solutions of the difference equation

$$x_n = b_n x_{n-1} - x_{n-2}$$

that has proved to be quite useful in our study of equimodular continued fractions of this type. It may also prove to be useful to researchers in the field of difference equations. Theorem 5.1 essentially states that $-K(-1/b_n)$ cannot converge finitely if $a, b \in \mathbb{R}$, $0 < ab < 4$, $\Sigma |b_{2n+1} - b_{2n-1}| < \infty$, and $\Sigma |b_{2n+2} - b_{2n}| < \infty$. Theorem 5.2 deals with the construction of continued fractions with various types of convergence behavior for the two cases $a = b = 0$ and $a \neq 0$, $b = 0$. In our final result of this section, Theorem 5.3, we allow a and b to have nonzero imaginary parts. Theorem 5.3 states that, if $0 < ab < 4$ and a certain series converges, then $-K(-1/b_n)$ cannot converge finitely. If the case $a = b = 0$ is excluded, this theorem contains the main theorem in a paper by Heller and Roach [3] as a special case.

The following problem is considered in section 6: Given $0 < b < 2$, does there exist a finitely convergent continued fraction $-K(-1/b_n)$ such that $\lim b_n = b$? We, at this time, are able only to give a partial solution to this problem. In the proof of Theorem 6.1 we show how to construct a sequence $\{b_n\}$ such that $b_n \to b$ and $-K(-1/b_n)$ converges nonabsolutely to b, where $b = 2 \cos r\pi$ with $0 < r < 1/2$ a rational number. Theorem 6.2 says that for certain irrational numbers α, $0 < \alpha < 1/4$, there exists a sequence $\{b_n\}$ such that $b_n \to 2 \cos 2\alpha\pi$ and $-K(-1/b_n)$ converges absolutely to a finite limit.

After proving two needed lemmas in section 7, we give in Theorem 7.1 a matrix theoretic result whose conclusion involves the boundedness of products of matrices. This theorem has all the indications of being a powerful tool for investigating the convergence behavior of certain types of continued fractions. We point out through our Examples 7.1 how Theorem 7.1 might have been used to obtain some of our earlier results. It is our main force in the proofs of Theorems 7.2, 8.1, and 8.2. Theorem 7.2 is a divergence type theorem about continued fractions $-K(-1/b_n)$, where $b_n \to 0$ but $\Sigma | b_n - b_{n+1} |$ may diverge.

In our final section, section 8.1, we prove three theorems dealing with continued fractions $K(a_n/1)$ under various convergence conditions imposed on $\{a_n\}$. Theorem 8.1 says that, if $a_n < -1/4$, $a_n \to a < -1/4$, and $\Sigma | a_{n+1} - a_n | < \infty$, then $K(a_n/1)$ cannot converge finitely. Theorem 8.2 says that the same conclusion can be drawn about $K(a_n/1)$ if $a_{2n-1} \to a$, $a_{2n} \to b$, $K(a_n/1)$ is equimodular, and both series $\Sigma | a_{2n+1} - a_{2n-1}|$, $\Sigma | a_{2n+2} - a_{2n} |$ converge. Our concluding Theorem 8.3 deals with continued fractions $K(a_n/1)$, where $a_n \to -\infty$. It shows how to construct continued fractions of this type that diverge by oscillation and how to construct ones that converge finitely.

2. $-K(-1/b_n)$, where $\lim b_n = b(|b| < 2)$.

We begin this section with a fundamental lemma concerned with second order linear homogenous difference equations. The lemma plays an important role in the proofs of Theorems 2.1, 4.3 and 5.3.

Lemma 2.1: *If the sequence* $\{b_n\}$ *satisfies* $b_n \in \mathbb{C}$, $|b_n| < 2$, $b_n \to b \in \mathbb{R}$, $|b| < 2$, *and*

$$\sum_{n=1}^{\infty} |\bar{b}_n - b_{n+1}| < \infty, \tag{2.1}$$

then all solutions of the difference equation

$$x_n = b_n x_{n-1} - x_{n-2}, \quad n = 1, 2, \ldots \tag{2.2}$$

are bounded.

Proof: For n = 1, 2, ..., let

$$u_n = |x_n|^2 + |x_{n-1}|^2 - \mathrm{Re}(b_{n+1} x_n \bar{x}_{n-1})$$

and

$$u_n^* = |x_{n+1}|^2 + |x_n|^2 - \mathrm{Re}(\bar{b}_{n+1} x_{n+1} \bar{x}_n).$$

then, using (2.2), it is easily verified that

$$u_n^* = u_n, \quad n \geq 1.$$

Hence,

$$\begin{aligned} u_{n+1} - u_n &= u_{n+1} - u_n^* \\ &= \mathrm{Re}[(\bar{b}_{n+1} - b_{n+2})\, x_{n+1}\bar{x}_n] \\ &\leq |\bar{b}_{n+1} - b_{n+2}|\, |x_{n+1}|\, |x_n|. \end{aligned} \tag{2.3}$$

But,

$$\begin{aligned} u_n &\geq |x_n|^2 + |x_{n-1}|^2 - |b_{n+1}|\, |x_n|\, |x_{n-1}| \\ &\geq (2 - |b_{n+1}|)\, \frac{|x_n|^2 + |x_{n-1}|^2}{2} \\ &\geq (2 - |b_{n+1}|)\, |x_n|\, |x_{n-1}|. \end{aligned} \tag{2.4}$$

Similarly, using the fact that $u_n = u_n^*$, we obtain

$$\begin{aligned} u_n &\geq (2 - |b_{n+1}|)\, \frac{|x_{n+1}|^2 + |x_n|^2}{2} \\ &\geq (2 - |b_{n+1}|)\, |x_{n+1}|\, |x_n| . \end{aligned} \tag{2.5}$$

It follows from inequalities (2.3) and (2.5) that

$$u_{n+1} \leq u_n \left(1 + \frac{|\bar{b}_{n+1} - b_{n+2}|}{2 - |b_{n+1}|}\right),$$

from which we obtain

$$u_{n+1} \leq u_1 \prod_{k=1}^{\infty} \left(1 + \frac{|\bar{b}_{k+1} - b_{k+2}|}{2 - |b_{n+1}|}\right).$$

The last inequality shows that $\{u_n\}$ is bounded since $\sum_{n=1}^{\infty} |\bar{b}_n - b_{n+1}| < \infty$ and $|b_n| \to |b| < 2$ as $n \to \infty$. But from inequalities (2.4) we have

$$|x_n|^2 + |x_{n-1}|^2 \leq \frac{2\, u_n}{2 - |b_{n+1}|}$$

so that the x_n are bounded, and our proof is complete.

The following theorem tells us that, if a continued fraction $-K(-1/b_n)$ is to converge finitely when $\{b_n\}$ converges to a real limit $b(|b| < 2)$, restrictions must be put on the speed of convergence of $\{b_n\}$ and the sequence cannot be monotone when the b_n are real.

<u>Theorem</u> 2.1: <u>If the elements of the sequence</u> $\{b_n\}$ <u>satisfy</u> $b_n \in \mathbb{C}$, $|b_n| < 2$, $\lim b_n = b \in \mathbb{R}$, $|b| < 2$, <u>and</u>

(2.6) $$\sum_{n=1}^{\infty} |\bar{b}_n - b_{n+1}| < \infty ,$$

<u>then the continued fraction</u>

(2.7) $$\frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \cdots$$

<u>cannot converge to a finite limit.</u> <u>If the further restriction</u> $b_n \in \mathbb{R}$, $n \geq 1$, <u>is imposed, then condition</u> (2.6) <u>becomes</u>

(2.8) $$\sum_{n=1}^{\infty} |b_n - b_{n+1}| < \infty ,$$

<u>which is satisfied, in particular, if</u> $b_n \to b$ <u>monotonely as</u> $n \to \infty$.

<u>Proof:</u> The nth denominator B_n of (2.7) satisfies

$$B_{-1} = 0,\ B_1 = b_1;\ B_n = b_n B_{n-1} - B_{n-2},\ n \geq 1.$$

By Lemma 2.1, the sequence $\{B_n\}$ is bounded. In order for (2.7) to converge, at most a finite number of the B_n can be zero. So assume there exists a positive integer n_0 such that $B_n \neq 0$ for all $n \geq n_0$. But then the nth approximant f_n of (2.7) satisfies

$$f_n = f_{n_0} + \sum_{k=n_0+1}^{\infty} \frac{1}{B_k B_{k-1}} ,\ n > n_0 .$$

Since the B_k are bounded, the series

$$\sum_{k=n_0+1}^{\infty} \frac{1}{B_k B_{k-1}}$$

and therefore the sequence $\{f_n\}$ cannot converge to a finite limit. With this our proof is complete.

Our next theorem tells us how to create continued fractions $-K(-1/b_n)$ that diverge by oscillation when $\{b_n\}$ converges to a given limit b with the restriction $0 \leq b < 1$.

<u>Theorem</u> 2.2: <u>For</u> $n = 0, 1, 2, \ldots$ <u>let</u> $g_n = x_n + i\, y_n$, <u>where</u> $0 < y_n \leq 1$ <u>and</u> $x_n = (y_n(-y_n + 1/y_{n+1}))^{1/2}$. <u>Let</u> $b_n = g_n + 1/g_{n-1} = x_n + x_{n-1} y_n / y_{n-1}$. <u>If</u> $\lim y_n = L$, $0 < L \leq 1$, <u>then</u> $\lim b_n = 2(1-L^2)^{1/2}$ <u>and the continued fraction</u> (2.7) <u>diverges by oscillation.</u>

<u>Proof:</u> In the first part of our proof we let $b_n = g_n + 1/g_{n-1}$, where the only restriction on the g_n is that they are nonzero complex numbers. Let

$$f_n = \frac{1}{b_1} - \frac{1}{b_2} - \cdots - \frac{1}{b_n}$$

and set

$$F_n = \frac{1}{g_1} - \frac{1}{b_2} - \cdots - \frac{1}{b_n} .$$

Then

(2.9) $$f_n = \frac{1}{(1/g_0) + (1/F_n)} .$$

Hence, if $F_n \to \infty$ as $n \to \infty$, then $f_n \to g_0$ as $n \to \infty$ and the tails of (2.7) are right tails. If we set $F_n = C_n/D_n$, where $C_0 = 0$, $C_1 = 1$, $D_0 = 1$, $D_1 = g_1$, and C_n, D_n each satisfy the difference equation

$$z_n = b_n z_{n-1} - z_{n-2},\ n \geq 2,$$

then it can be established by induction that

$$D_n = g_1 g_2 \cdots g_n .$$

Now, since $F_n - F_{n-1} = 1/(D_n D_{n-1})$, we have that

$$F_n = F_0 + \sum_{k=1}^{n} 1/(D_k D_{k-1})$$

$$= \sum_{k=1}^{n} 1/(D_k D_{k-1}) \text{ (since } C_0 = 0).$$

$$(2.10) \qquad = g_0 \sum_{k=1}^{n} \prod_{j=1}^{k} 1/(g_j g_{j-1}) .$$

Finally, from (2.9) and (2.10) we have that

$$(2.11) \qquad f_n = \frac{g_0}{1 + 1/\sum_{k=1}^{n} \prod_{j=1}^{k} 1/(g_j g_{j-1})} .$$

Now at this stage let $g_n = x_n + iy_n$, where the x_n and y_n satisfy the hypotheses of our theorem. Then

$$|g_n| = (x_n^2 + y_n^2)^{1/2} = (y_n / y_{n+1})^{1/2}$$

so

$$\sum_{k=1}^{n} \prod_{j=1}^{k} \frac{1}{|g_j g_{j-1}|} = \sum_{k=1}^{n} \prod_{j=1}^{k} \left(\frac{y_{j+1}}{y_j}\frac{y_j}{y_{j-1}}\right)^{1/2}$$

$$= \sum_{k=1}^{n} \left(\frac{y_k}{y_0}\frac{y_{k+1}}{y_1}\right)^{1/2}.$$

Since $\lim_{k\to\infty} y_k = L$, $0 < L \le 1$, it follows that the sequence $(\sum_{k=1}^{n} (y_k y_{k+1}/(y_0 y_1))^{1/2})$ cannot converge to a finite limit. This same sequence cannot converge to ∞ either, for if so, $\{f_n\}$ and hence the continued fraction (2.7) would converge to g_0, a number whose imaginary part is not zero. This impossible because the b_n are real. Thus the conclusion must be that (2.7) diverges by oscillation.

3. $-K(-1/b_n)$, <u>where</u> $\lim b_n = 2$.

Our first theorem in this section is a general convergence type theorem for continued fractions $-K(-1/b_n)$ that does not demand that $\{b_n\}$ converge. It says that the continued fraction can converge

finitely if the b_n are real or complex provided that the modulus of b_n satisfies a certain growth condition. The theorem leads us to a method for constructing finitely convergent continued fractions of this type, where $b_n \to 2$ from the left.

<u>Theorem</u> 3.1: <u>Let</u> $\{g_n\}_{n\ge0}$ <u>be a sequence of positive real numbers, and let</u> $\{b_n\}$ <u>be a sequence of complex numbers satisfying</u>

$$(3.1) \qquad |b_n| \ge g_n + 1/g_{n-1}, \quad n = 1, 2, \ldots .$$

<u>Then the continued fraction</u>

$$(3.2) \qquad \frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \cdots$$

<u>converges to a finite value in the disk</u> $|z| \le g_0$. <u>If, in particular,</u>

$$(3.3) \qquad b_n \ge 2 - \frac{1}{4n(n+1/2)}, \quad n = 1, 2, \ldots,$$

<u>then</u> (3.2) <u>converges to a number in the interval</u> $[-3/2, 3/2]$.

<u>Proof:</u> If $\{\gamma_n\}_{n\ge0}$ is an arbitrary sequence of positive real numbers, then the continued fraction

$$K = \frac{1}{\gamma_0}\left(\frac{\gamma_0\gamma_1}{\gamma_1 b_1} - \frac{\gamma_1\gamma_2}{\gamma_2 b_2} - \frac{\gamma_2\gamma_3}{\gamma_3 b_3} - \cdots\right)$$

is equivalent to the continued fraction (3.2). By the Pringsheim Convergence Theorem (see [10, Satz 2.19]) K converges to a finite limit contained in the disk $|z| \le 1/\gamma_0$ if

$$(3.4) \qquad \gamma_n |b_n| \ge 1 + \gamma_n \gamma_{n-1}, \quad n = 1, 2, \ldots .$$

Thus to complete our proof it is sufficient to set $\gamma_n = 1/g_n$ in (3.4). Finally, in order to see that (3.2) converges if (3.3) holds, we have only to set

$$g_n = 1 + \frac{1}{2(n+1)}, \quad n = 0, 1, 2, \ldots .$$

We now show that condition (3.3) of Theorem 3.1 can be sharpened. The method of proof is somewhat novel in that it couples the theory of extensions for continued fractions with a 1905 convergence result of Pringsheim.

<u>Theorem</u> 3.2: <u>The continued fraction</u> (3.2) <u>converges to a finite limit if</u>

$$b_n \geq 2 - \frac{1}{4n^2}, \quad n = 1, 2, \ldots .$$

<u>Proof</u>: The continued fraction (3.2) can be extended to

(3.5) $$\frac{1}{b_1+1} - \frac{1}{1} - \frac{1}{b_2+2} - \frac{1}{1} - \frac{1}{b_3+2} - \frac{1}{1} - \cdots,$$

and (3.5) can be extended to

(3.6) $$-1 + \frac{1}{1} - \frac{1}{b_1+2} - \frac{1}{1} - \frac{1}{b_2+2} - \frac{1}{1} - \frac{1}{b_3+2} - \frac{1}{1} - \cdots .$$

The 2n+1 st approximant of (3.6) is the nth approximant of (3.2). The continued fraction (3.6) (and therefore (3.2)) will converge finitely if

(3.7) $$\frac{1}{1} - \frac{1}{b_1+2} - \frac{1}{1} - \frac{1}{b_2+2} - \frac{1}{1} - \frac{1}{b_3+2} - \frac{1}{1} - \cdots$$

converges to a finite limit. Now let

$$p_1 = \frac{5}{4} > 1 \; ; \; p_n = \frac{2n-1}{n}, \quad n = 2, 3, \ldots,$$

and let

$$c_{2n-1} = 1, \; c_{2n} = b_n + 2, \quad n = 1, 2, \ldots .$$

Then, according to [10, Satz 2.21], (3.7) converges to a finite limit if

(3.8) $$\frac{1}{|c_n c_{n-1}|} \leq \frac{-1+p_n}{p_n p_{n-1}}, \quad n = 2, 3, \ldots .$$

Inequality (3.8) will be satisfied if

(3.9) $$|2 + b_n| \geq \frac{p_{2n}p_{2n-1}}{-1+p_{2n}}$$

and

(3.10) $$|2 + b_n| \geq \frac{p_{2n}p_{2n+1}}{-1+p_{2n+1}}$$

are both satisfied. Since

$$\frac{p_1 p_2}{-1+p_2} = \frac{p_2 p_3}{-1+p_3} = \frac{15}{4},$$

it follows that (3.9) and (3.10) are satisfied for $n = 1$ if $b_1 \geq 2 - 1/4$. Since

(3.11) $$\frac{p_n p_{n+1}}{-1+p_{n+1}} = 4 - \frac{1}{n^2}, \quad n = 2, 3, \ldots,$$

it follows that the sequence $\{p_n p_{n+1}/(1 + p_{n+1})\}$ is increasing. Hence, for $n \geq 2$, (3.9) will be satisfied if (3.10) is satisfied. But, in view of (3.11), (3.10) is equivalent to

$$|2 + b_n| \geq 4 - \frac{1}{4n^2}, \quad n = 2, 3, \ldots,$$

which is clearly satisfied if $b_n \geq 2 - 1/(4n^2)$. This completes our proof of Theorem 3.2.

<u>Theorem</u> 3.3: <u>Let</u> b <u>be a constant such that</u> $0 \leq b \leq 1$, <u>and let the sequence</u> $\{b_n\}_{n \geq 1}$ <u>be defined by</u>

$$b_n = \left(1 + \frac{1}{n+b}\right)^{1/2} + \left(1 - \frac{1}{n+b}\right)^{1/2}.$$

<u>Then</u> $0 < b_n < 2$, $\lim b_n = 2$, <u>and the continued fraction</u> (3.2) <u>converges to</u> $(1 + 1/b)^{1/2}$ <u>if</u> $0 < b \leq 1$ <u>and to</u> ∞ <u>if</u> $b = 0$.

<u>Remarks</u>. If we choose b_n as in Theorem 3.3 and let $b_n^* = 2 - 1/(4n^2)$, then after a few calculations it can be seen that $\lim(2-b_n)/(2-b_n^*) = 1$; so $2 - b_n \sim 2 - b_n^*$ as $n \to \infty$. This tells us that the coefficient -1/4 of n^{-2} in b_n^* is sharp with respect to having finite convergence of (3.2). For by Theorem 3.2, $-K(-1/b_n^*)$ converges to a finite limit, but as we shall show in the proof below of Theorem 3.3, $-K(-1/b_n)$ converges finitely or to ∞ according as $0 < b \leq 1$ or $b = 0$.

<u>Proof</u>: If we set $g_n = (1 + 1/(n + b))^{1/2}$ for $n \geq 0$ and $0 < b \leq 1$, then the fact that (3.2) converges finitely follows immediately from Theorem 3.1. But under these hypotheses we can give a more direct argument that tells us more. As in the proof of Theorem (2.2), the nth approximant f_n of (3.2) is given by

$$f_n = \frac{1}{\frac{1}{g_0} + \frac{1}{F_n}},$$

where

$$F_n = g_0 \sum_{k=1}^{n} \prod_{j=1}^{k} \frac{1}{g_j g_{j-1}}.$$

Simple calculations will show that

$$F_n = (1 + b) \sum_{k=1}^{n} \frac{1}{((k+1+b)(k+b))^{1/2}}, \quad 0 < b \leq 1,$$

so that $F_n \to \infty$ as $n \to \infty$. It follows that $\lim f_n = g_0 = (1 + 1/b)^{1/2}$.

If $b = 0$, it is easily verified by induction that the denominators B_n of (3.2) are given by $B_n = (n + 1)^{1/2}$, $n = 0, 1, 2, \ldots$. Hence,

$$f_n = \sum_{k=1}^{n} \frac{1}{B_k B_{k-1}} = \sum_{k=1}^{n} \frac{1}{((k+1)k)^{1/2}};$$

so $f_n \to \infty$ as $n \to \infty$, and our proof is complete.

<u>Theorem</u> 3.4: <u>For the continued fraction</u> (3.2) <u>let</u> B_n <u>be its nth denominator and let</u>

$$b_n \geq 2 - \frac{a(1-a)}{n(n-1+a)}, \quad n = 1, 2, \ldots,$$

<u>where</u> a <u>is a positive constant. Then</u>

$$h_n := B_n/B_{n-1} \geq 1 + a/n; \quad B_n \geq \prod_{k=1}^{n}(1+a/k) = O(n^a).$$

<u>Furthermore</u>, (3.2) <u>converges absolutely to a finite value if</u> $a > 1/2$; <u>it converges</u> (<u>possibly to</u> ∞) <u>if</u> $a > 0$; <u>and, in particular, it converges to</u> ∞ <u>if</u>

$$b_n = 2 - \frac{a(1-a)}{n(n-1+a)}$$

<u>with</u> $0 < a \leq 1/2$.

<u>Remarks</u>. If we set $c_n = 2 - a(1-a)/(n(n-1+a))$ for $a > 0$ and $d_n = 2 - 1/(4n^2)$, then

$$2 - c_n \sim 4a(1 - a)(2 - d_n)$$

and, in particular,

$$2 - c_n \sim 2 - d_n$$

if $a = 1/2$. By Theorem 3.2 $-K(-1/b_n)$ converges finitely if $b_n \geq d_n$, and by Theorem 3.4 $-K(-1/b_n)$ converges to ∞ if $b_n = c_n$, where $0 < a \leq 1/2$. Thus these examples again tell us that the estimate for the b_n in Theorem 3.2 is sharp if we wish to have finite convergence.

<u>Proof:</u> Since the h_n satisfy the recurrence relation

$$h_{n+1} = b_{n+1} - 1/h_n,$$

it is easily established by induction that

$$h_n \geq 1 + a/n.$$

Because

$$B_n = \prod_{k=1}^{n} h_k,$$

it follows that

$$B_n \geq \prod_{k=1}^{n} (1 + a/k),$$

where it can be verified by induction that equality holds if

$$b_n = 2 - \frac{a(1-a)}{n(n-1+a)}, \quad a > 0.$$

Since the nth approximant f_n of (3.2) is given by

$$f_n = \sum_{k=1}^{n} \frac{1}{B_k B_{k-1}},$$

we can use Raabe's test and $\sum_{k=1}^{\infty} 1/(C_k C_{k-1})$ as a comparison series, where $C_n = \prod_{k=1}^{n}(1+a/k)$, to verify that (3.2) has the convergence behavior indicated in the statement of our Theorem.

Stirling's formula for the Gamma function Γ comes in handy for determining the growth behavior of $\prod_{k=1}^{n}(1+a/k)$. Using it and the functional relation $\Gamma(x+1) = x\Gamma(x)$ we obtain

$$\prod_{k=1}^{n}(1+a/k) = \frac{1}{\Gamma(1+a)} \frac{\Gamma(n+1+a)}{\Gamma(n+1)}$$

$$\sim \frac{e^{-a}}{\Gamma(1+a)} (n+a)^a (1+a/n)^{1/2}, \quad n \to \infty.$$

Thus $\prod_{k=1}^{n}(1+a/k) = O(n^a)$ as asserted, and we are finished with the proof.

<u>Theorem</u> 3.5: <u>Let</u> $\{y_n\}_{n\geq 0}$ <u>be any sequence satisfying</u> $0 < y_n < 1$, $y_{n+1} \leq y_n$, $\lim y_n = 0$, $\lim y_n/y_{n+1} = 1$, <u>and</u> $\sum_{n=0}^{\infty} y_n = \infty$. <u>For</u> $n \geq 0$, <u>let</u> $x_n = (-y_n^2 + y_n/y_{n+1})^{1/2}$ <u>and</u> $g_n = x_n + iy_n$. <u>Finally, let</u>

$$b_n = g_n + 1/g_{n-1} = x_n + x_{n-1}y_n/y_{n-1}.$$

<u>Then</u> $0 < b_n < 2$, $\lim b_n = 2$, <u>and the continued fraction</u> (3.2)<u>diverges by oscillation</u>. <u>In particular</u>, (3.2) <u>diverges by oscillation if</u> $\{y_n\}$ <u>is defined by</u> $y_n = a/(n+1)$, <u>where</u> $0 < a < \sqrt{2}$.

<u>Remarks</u>. If $y_n = a/(n+1)$ in Theorem 3.5, where $0 < a < \sqrt{2}$, then b_n becomes

$$b_n = (1 + 1/(n+1) - a^2/(n+1)^2)^{1/2} + (1 - 1/(n+1) - a^2/(n+1)^2)^{1/2}.$$

Here we also use this formula to define b_n if $a = 0$. Setting $c_n = 2 - 1/(4n^2)$, it can be established that

$$2 - b_n \sim (1 + 4a^2)(2 - c_n), \quad n \to \infty.$$

By Theorem 3.1, taking $g_n = ((n+2)/(n+1))^{1/2}$, $-K(-1/b_n)$ converges finitely if $a = 0$. By Theorem 3.2, $-K(-1/c_n)$ converges finitely also. But, as we shall prove below, $-K(-1/b_n)$ diverges by oscillation if $0 < a < \sqrt{2}$. Thus our discussion here points out that the simple estimate in Theorem 3.2 for the elements of (3.2) provides a sharp dividing line between finite convergence and divergence by oscillation for these continued fractions whose elements approach 2 from the left.

<u>Proof</u>: We have that

$$|g_n| = (x_n^2 + y_n^2)^{1/2} = (y_n/y_{n+1})^{1/2}. \tag{3.12}$$

Let $\theta_n = \tan^{-1}(y_n/x_n)$, $n \geq 0$. Then

$$\theta_n = \tan^{-1}(y_n y_{n+1}/(1 - y_n y_{n+1}))^{1/2}. \tag{3.13}$$

It follows from the conditions on the y_n that $0 < \theta_n < \pi/2$, $\theta_{n+1} \leq \theta_n$, and $\theta_n \to 0$ as $n \to \infty$.

We now investigate the convergence behavior of the series

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1}{g_k g_{k-1}}. \tag{3.14}$$

Using (3.12) and (3.13), (3.14) can be written in the form

$$\sum_{n=1}^{\infty} (y_{n+1}y_n/(y_1y_0))^{1/2}\exp(-i\sum_{j=1}^{n}(\theta_j + \theta_{j-1})). \tag{3.15}$$

Choose ε so that $0 < \varepsilon < \pi/2$. Then the hypotheses guarantee that there exists an integer N_0 such that for $N > N_0$ there exists an integer $m \geq 0$ for which

$$0 \leq \varepsilon/2 \leq \sum_{j=N+1}^{N+1+m} 2(y_j y_{j-1})^{1/2} \leq \sum_{j=N+1}^{N+1+m} 2\left(\frac{y_j y_{j-1}}{1-y_j y_{j-1}}\right)^{1/2} \leq \varepsilon.$$

Hence, for $N + 1 \leq k \leq N + 1 + m$, we have

$$0 \leq \sum_{j=N+1}^{k}(\theta_j + \theta_{j-1}) \leq 2\sum_{j=N+1}^{k}\theta_{j-1} \leq \sum_{j=N+1}^{N+1+m} 2\left(\frac{y_j y_{j-1}}{1-y_j y_{j-1}}\right)^{1/2} \leq \varepsilon,$$

so

$$\cos \sum_{j=N+1}^{k}(\theta_j\theta_{j-1}) \geq \cos \varepsilon > 0.$$

Let S_n denote the nth partial sum of (3.15). Then

$$|S_{N+m+1} - S_{N-1}| = |(y_{N+1}y_N/(y_1y_0))^{1/2} + S_{N+1+m} - S_N|$$

$$\geq (y_1y_0)^{-1/2}((y_{N+1}y_N)^{1/2}\sum_{k=N+1}^{N+1+m}(y_{k+1}y_k)^{1/2}\cos \varepsilon)$$

$$\geq (y_1y_0)^{-1/2}(\sum_{k=N+1}^{N+1+m}(y_{k+1}y_k)^{1/2}) \cos \varepsilon$$

$$\geq (y_1y_0)^{1/2}(\varepsilon/2) \cos \varepsilon > 0.$$

Hence the series (3.15) diverges and its nth partial sum S_n cannot converge to a finite limit. Suppose $S_n \to \infty$ as $n \to \infty$. Then the nth approximant f_n of (3.2) is given by

$$f_n = g_0/(1 + 1/S_n)$$

for n large enough. But now $f_n \to g_0$ as $n \to \infty$, which is impossible since $\mathrm{Im}(g_0) \neq 0$ and the elements of (3.2) are real. Hence the series (3.15), and therefore the continued fraction (3.2), must diverge by oscillation. This completes our proof of Theorem 3.5.

4. $-K(-1/b_n)$ where $b_n \to 0$ and a problem of Wall.

It is well known that the continued fraction $K(1/b_n)$ diverges if the series $\Sigma|b_n|$ converges. In 1948 Wall [13], on the bottom of page 33 of his book, raised the question of whether the simple convergence of the series Σb_n is sufficient for the divergence of $K(1/b_n)$. To our knowledge an answer to this question has not appeared in the literature to date. In our next theorem we give an example showing that convergence of Σb_n is not sufficient for the divergence of $K(1/b_n)$.

Theorem 4.1: Let the sequence $\{b_n\}$ be defined by $b_1 = 1$, $b_{2n+1} = 2(-1)^n/\sqrt{(n+1)}$, $b_{2n} = (-1)^{n-1}/(\sqrt{(n+1)} + \sqrt{n})$, $n \geq 1$. Then both the continued fraction $\overset{\infty}{\underset{n=1}{K}}(1/b_n)$ and the series $\sum_{n=1}^{\infty} b_n$ converge to finite limits.

Proof: We shall first establish that $\Sigma b_n < \infty$. If S_n denotes the nth partial sum of Σb_n, then

$$S_{2n+2} = 1 + \sum_{k=1}^{n+1}(-1)^{k-1}/(\sqrt{(k+1)} + \sqrt{k}) + \sum_{k=1}^{n} 2(-1)^k/\sqrt{(k+1)}.$$

Since the partial sums appearing on the right hand side of the last equation are partial sums of convergent alternating series, it follows that $\{S_{2n+2}\}$ converges. Because $S_{2n+1} = S_{2n+2} - b_{2n+2}$ and $b_{2n+2} \to 0$ as $n \to \infty$ it is therefore also true that $\{S_n\}$ converges. Thus Σb_n converges finitely as asserted.

We shall now establish the convergence of $K(1/b_n)$. It is easily verified that $K(1/b_n)$ is equivalent to $-K(-1/c_n)$, where

$$c_1 = b_1,\ c_{2n+1} = b_{2n+1},\ c_{2n} = -b_{2n},\ n \geq 1.$$

The denominators B_n of $-K(-1/c_n)$ satisfy

$$B_0 = B_1 = 1;\ B_n = c_nB_{n-1} - B_{n-2},\ n \geq 2.$$

Using this difference equation for the B_n it follows by induction that

$$B_{2n} = (-1)^n\sqrt{(n+1)},\ B_{2n-1} = 1,\ n \geq 1.$$

The nth approximant f_n of $-K(-1/c_n)$ is given by

$$f_n = \sum_{k=1}^{n} 1/(B_kB_{k-1}).$$

Hence,

$$f_{2n+1} = \sum_{k=1}^{2n+1} 1/(B_kB_{k-1}) = 1 + \sum_{k=1}^{n} 2/B_{2k} = 1 + \sum_{k=1}^{n} 2(-1)^k/\sqrt{(k+1)}.$$

Since $\sum_{k=1}^{\infty} 2(-1)^k/\sqrt{(k+1)}$ is a convergent alternating series, it follows that $\{f_{2n+1}\}$ converges. But then $\{f_{2n}\}$ converges to the same limit since $f_{2n} = f_{2n+1} - 1/B_{2n} = f_{2n+1} + (-1)^{n-1}/\sqrt{(n+1)}$. Thus $-K(-1/c_n)$ and $K(1/b_n)$ converge to finite limits, and our proof of Theorem 4.1 is complete.

Our next theorem shows that there is a wealth of finitely convergent continued fractions $-K(-1/b_n)$ with the property that $b_n \to 0$ through real values:

Theorem 4.2: Let $\{\alpha_k\}$ and $\{\beta_k\}$ be any sequences of real numbers satisfying

$$0 < \beta_k < \alpha_k,\ k \geq 1;\ \lim_{k\to\infty} \alpha_k = 0$$

with the additional properties

$$\sum_{k=1}^{\infty} \beta_k < \infty,\ \sum_{k=1}^{\infty} \alpha_k = \infty.$$

For $n \geq 1$, let

$$b_{2n-1} = \alpha_n / \prod_{k=1}^{n-1} \left(1 - \frac{\beta_k}{\beta_k + \sum_{j=1}^{k}(\alpha_j - \beta_j)}\right)^2$$

$$b_{2n} = \beta_n \alpha_1^{-2} / \prod_{k=1}^{n-1} \left(1 + \frac{\alpha_{k+1}}{\sum_{j=1}^{k}(\alpha_j - \beta_j)}\right)^2.$$

Then $\lim b_n = 0$ and the continued fraction

$$(4.1) \qquad \frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \cdots$$

converges to a finite limit.

Proof: After a modest amount of computation it can be verified that the sequence $\{B_n\}_{n\geq 0}$ of denominators of (4.1) is determined by the formulas

$$B_{2n} = (-1)^n \prod_{k=1}^{n} \left(1 - \frac{\beta_k}{\beta_k + \sum_{j=1}^{k}(\alpha_j - \beta_j)}\right)$$

$$(n \geq 0)$$

$$B_{2n+1} = (-1)^n \alpha_1 \prod_{k=1}^{n} \left(-1 + \frac{\alpha_{k+1}}{\sum_{j=1}^{k}(\alpha_j - \beta_j)}\right).$$

From these formulas it is readily seen that

$$b_{2n-1} = \alpha_n/(B_{2n-2})^2, \quad b_{2n} = \beta_n/(B_{2n-1})^2, \quad n \geq 1.$$

If we set $C_n = B_n B_{n-1}$, then

$$C_{2n} = \sum_{k=1}^{n}(\beta_k - \alpha_k) < 0, \quad C_{2n+1} = \alpha_{n+1} + \sum_{k=1}^{n}(\alpha_k - \beta_k) > 0.$$

Hence $C_{2n} \to -\infty$, $C_{2n+1} \to \infty$ as $n \to \infty$, and the C_n alternate in sign. Also

$$|C_{2n+1}| - |C_{2n}| = \alpha_{n+1} > 0, \quad |C_{2n}| - |C_{2n-1}| = \beta_n > 0,$$

so the sequence $\{|C_n|\}$ is increasing. Therefore, the series $\sum_{n=1}^{\infty} 1/C_n$ converges to a finite limit, so that (4.1) also converges finitely. This concludes our argument for Theorem 4.2.

Perron [10, p. 192] calls a continued fraction of the form

$$(4.2) \qquad \frac{1}{b_1 z} + \frac{1}{b_2} + \frac{1}{b_3 z} + \frac{1}{b_4} + \frac{1}{b_5 z} + \frac{1}{b_6} + \cdots$$

an H-fraction (after Hamburger) if $b_k \in \mathbb{R}$, $b_{2k-1} > 0$, and $b_{2k} \neq 0$, $k = 1, 2, \ldots$. If all the b_k are positive, he calls (4.2) an S-fraction (after Stieltjes). If we multiply (4.2) by -1, replace its parameter z by $-x^2$, and apply an equivalence transformation, we obtain the continued fraction

$$(4.3) \qquad \frac{1}{b_1 x^2} - \frac{1}{b_2} - \frac{1}{b_3 x^2} - \frac{1}{b_4} - \frac{1}{b_5 x^2} - \frac{1}{b_6} - \cdots.$$

The following result holds for these continued fractions:

Theorem 4.3: Let $x \in \mathbb{R}$ and let $\{b_n\}$ be a sequence of real numbers such that $\lim b_n = 0$ and $\Sigma |b_n - b_{n+1}| < 0$. Then the continued fraction (4.3) cannot converge to a finite limit.

Proof: Let B_n be the nth denominator of (4.3). Then, $B_0 = 1$, $B_1 = x^2 b_1$ and

$$B_{2n+1} = x^2 b_{2n+1} B_{2n} - B_{2n-1}; \quad B_{2n} = b_{2n} B_{2n-1} - B_{2n-2}, \quad n \geq 1.$$

Now define $\{C_n\}$ by

$$C_{2n+1} = B_{2n+1}, \quad C_{2n} = x B_{2n}, \quad n = 0, 1, 2, \ldots.$$

Then

$$C_n = x b_n C_{n-1} - C_{n-2}, \quad n \geq 2.$$

Let $N \geq 2$ be a fixed positive integer such that $|xb_n| < 2$ if $n \geq N$. Then

$$C_{n+N} = xb_{n+N}C_{n+N-1} - C_{n+N-2}, \quad n \geq 0.$$

That is, the C_{n+N} satisfy the difference equation

$$x_n = \hat{b}_n x_{n-1} - x_{n-2},$$

where $\hat{b}_n = xb_{n+N}$. Since

$$\Sigma\,|b_n - b_{n+1}| < \infty \Rightarrow \Sigma\,|x|\,|b_{n+N} - b_{n+1+N}| < \infty,$$

it follows from Lemma 2.1 that $\{C_{n+N}\}$ is bounded. Thus the sequence $\{C_n\}$ (hence also the sequence $\{b_n\}$) is bounded, so the continued fraction (4.3) cannot converge finitely.

Example 4.1. An immediate application of Theorem 4.3 and one of the type that helped motivate our study of equimodular limit periodic continued fractions is the following:

In 1982 Elbert [2] proved that the function

$$M(x) := \sum_{n=1}^{\infty} \frac{2n}{(n^2+x^2)^2}$$

is asymptotic to the series $\sum_{n=0}^{\infty} (-1)^n B_{2n} x^{-2n-2}$ as $x \to \infty$, where B_{2n} is the 2nth Bernoulli number. Thus it follows that

$$x^2 M(x) \sim \sum_{n=0}^{\infty} (-1)^n B_{2n} x^{-2n}.$$

We claim without supplying a proof here that the latter asymptotic series corresponds to the continued fraction

$$(4.4) \qquad 1 - \frac{x^{-2}/4}{b_1} - \frac{x^{-2}/4}{b_2} - \frac{x^{-2}/4}{b_3} - \cdots,$$

where $b_n = 1/n + 1/(n+1)$, $n = 1, 2, \ldots$.

For $x \neq 0$ the convergence behavior of (4.4) is the same as that of

$$(4.5) \qquad \frac{x^{-2}/4}{b_1} - \frac{x^{-2}/4}{b_2} - \frac{x^{-2}/4}{b_3} - \cdots.$$

But (4.5) is equivalent to

$$(4.6) \qquad \frac{1}{4x^2b_1} - \frac{1}{b_2} - \frac{1}{4x^2b_3} - \frac{1}{b_4} - \frac{1}{4x^2b_5} - \cdots.$$

Since

$$\sum_{n=1}^{\infty} |b_n - b_{n+1}| = 2\sum_{n=1}^{\infty}(n(n+2))^{-1} < \infty,$$

it follows from Theorem 4.3 that (4.6) cannot converge to a finite limit if x is real. Thus (4.4) cannot converge to $x^2M(x)$ for any real values of x, even though the continued fraction and the function are related through the indicated correspondence.

5. $-K(-1/b_n)$, where $b_{2n-1} \to a$, $b_{2n} \to b$, $0 \leq ab < 4$.

We begin this section with a few simple but interesting examples. The continued fraction

$$\frac{1}{0} - \frac{1}{b_2} - \frac{1}{0} - \frac{1}{b_4} - \frac{1}{0} - \frac{1}{b_6} - \cdots$$

is divergent because its odd denominators B_{2n-1} are zero. For the continued fraction

$$\frac{1}{1} - \frac{1}{0} - \frac{1}{1} - \frac{1}{0} - \frac{1}{1} - \frac{1}{0} - \cdots$$

the denominators are given by

$$B_{2n} = (-1)^n,\ n \geq 0;\ B_{2n-1} = (-1)^{n-1}n,\ n \geq 1.$$

Its nth approximant is $\sum_{k=1}^{n} 1/(B_k B_{k-1})$, so the continued fraction converges to 0 because

$$\sum_{k=1}^{\infty} 1/(B_k B_{k-1}) = 1 - 1 + 1/2 - 1/2 + 1/3 - 1/3 + 1/4 - \ldots = 0$$

More generally, the continued fraction

$$\frac{1}{b_1} - \frac{1}{0} - \frac{1}{b_3} - \frac{1}{0} - \frac{1}{b_5} - \frac{1}{0} - \ldots$$

has denominators B_n that satisfy

$$B_{2n} = (-1)^n,\ n \geq 0;\ B_{2n-1} = (-1)^{n-1}\sum_{k=1}^{n} b_{2k-1},\ n \geq 1.$$

It follows that this continued fraction cannot converge to a finite limit if $\Sigma b_{2k-1} < \infty$. Clearly, if the b_{2k-1} are nonnegative and $\sum_{k=1}^{\infty} b_{2k-1} = \infty$, then the continued does converge to a finite limit.

We are now ready to make a more comprehensive study of how continued fractions $-K(-1/b_n)$ behave when the sequences $\{b_{2n}\}$ and $\{b_{2n-1}\}$ have limits that are not necessarily the same. We first prove a lemma that is concerned with boundedness of solutions of difference equations. It is the heart of our proof of Theorem 5.1. Hopefully, the lemma will also be of interest to researchers in the field of difference equations.

Lemma 5.1: Let $\{b_n\}$ be a sequence of positive real numbers whose elements satisfy the conditions $0 < b_n b_{n+1} < 4$, $\lim b_{2n-1} = a$, $\lim b_{2n} = b$, $0 < ab < 4$, and

$$\sum_{n=1}^{\infty} |b_{2n+1} - b_{2n-1}| < \infty,\quad \sum_{n=1}^{\infty} |b_{2n+2} - b_{2n}| < \infty.$$

Then all solutions of the difference equation

(5.2) $$x_n = b_n x_{n-1} - x_{n-2},\ n = 1, 2, \ldots$$

are bounded.

Proof: Let

(5.3) $$u_n := b_{n+1}x_n^2 + b_{n+2}x_{n-1}^2 - b_{n+1}b_{n+2}x_n x_{n-1},\ n \geq 1,$$

where the x_n satisfy (5.2). Then it is easily verified that

(5.4) $$u_n = b_{n+1}x_{n+2}^2 + b_{n+2}x_{n+1}^2 - b_{n+1}b_{n+2}x_{n+2}x_{n+1}.$$

Using (5.3) and (5.4), we obtain

$$u_{n+2} = b_{n+3}x_{n+2}^2 + b_{n+4}x_{n+1}^2 - b_{n+3}b_{n+4}x_{n+2}x_{n+1}.$$

$$= u_n + b_{n+3}x_{n+2}^2(b_{n+3} - b_{n+1})/b_{n+3} + b_{n+4}x_{n+1}^2(b_{n+4} - b_{n+2})/b_{n+4}$$

$$+ (b_{n+1}b_{n+2} - b_{n+3}b_{n+4})x_{n+2}x_{n+1}$$

(5.5) $$\leq u_n + \gamma_n(b_{n+3}x_{n+2}^2 + b_{n+4}x_{n+1}^2) + (b_{n+1}b_{n+2} - b_{n+3}b_{n+4})x_{n+2}x_{n+1},$$

where

$$\gamma_n = \max\{|b_{n+3} - b_{n+1}|/b_{n+3},\ |b_{n+4} - b_{n+2}|/b_{n+4}\}.$$

With the aid of formula (5.3) we obtain from (5.5) that

$$u_{n+2} \leq u_n + \gamma_n u_{n+2} + (\gamma_n b_{n+3}b_{n+4} + b_{n+1}b_{n+2} - b_{n+3}b_{n+4})x_{n+2}x_{n+1}.$$

But,

$$u_n = b_{n+1}x_n^2 + b_{n+2}x_{n-1}^2 - b_{n+1}b_{n+2}x_n x_{n-1}$$

(5.6) $$\geq (\sqrt{(b_{n+1}b_{n+2})})(2 - \sqrt{(b_{n+1}b_{n+2})})|x_n x_{n-1}| \geq 0$$

Now let

$$\delta_n = \gamma_n + \frac{\gamma_n b_{n+3} b_{n+4} + b_{n+2}|b_{n+1} - b_{n+3}| + b_{n+3}|b_{n+2} - b_{n+4}|}{(\sqrt{(b_{n+3}b_{n+4})})(2 - \sqrt{(b_{n+3}b_{n+4})})}.$$

Then, using (5.6), it follows from (5.5) that

$$u_{n+2} \le u_n + \delta_n u_{n+2}.$$

Hence

(5.7) $$u_{n+2} \le u_n/(1 - \delta_n)$$

for n large enough, say $n \ge n_0$. The hypotheses of our Lemma guarantee that $\sum_{n=1}^{\infty} \delta_{2n-1}$ and $\sum_{n=1}^{\infty} \delta_{2n}$ converge. Therefore, the products $\pi(1-\delta_{2n})$ and $\pi(1 - \delta_{2n-1})$ converge, so it follows from (5.7) that for $n \ge n_1$

$$u_{2n+2} \le u_{2n_1}/\prod_{n=n_1}^{\infty}(1-\delta_{2n}),\ u_{2n+1} \le u_{2n_1-1}/\prod_{n=n_1}^{\infty}(1 - \delta_{2n-1}),$$

where n_1 is a fixed positive integer such that $2n_1 - 1 \ge n_0$. Hence the sequence $\{u_n\}$ is bounded. From (5.3) we obtain

(5.8) $$u_n \ge (b_{n+1}x_n^2 + b_{n+2}x_{n-1}^2)(2 - \sqrt{(b_{n+1}b_{n+2})})/2.$$

Since $\{u_n\}$ is bounded, it now follows from (5.8) that $\{x_n\}$ is bounded and our proof is complete.

Theorem 5.1: Let $\{b_n\}$ be a sequence of positive real numbers whose elements satisfy the conditions $0 < b_n b_{n+1} < 4$, $\lim b_{2n-1} = a$, $\lim b_{2n} = b$, $0 < ab < 4$, and

(5.9) $$\sum_{n=1}^{\infty}|b_{2n+1} - b_{2n-1}| < \infty,\ \sum_{n=1}^{\infty}|b_{2n+2} - b_{2n}| < \infty.$$

Then the continued fraction

(5.10) $$\frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \ldots$$

cannot converge to a finite limit. We note that condition (5.9) is satisfied, in particular, if $b_{2n-1} \to a$ and $b_{2n} \to b$ monotonely as $n\to\infty$.

Proof: By Lemma (5.1), the sequence of denominators $\{B_n\}$ of (5.10) is bounded. Hence, by an argument similar to the one given in the proof of Theorem 2.1, (5.10) cannot converge to a finite limit.

Our next theorem gives us some indication of what can happen with continued fractions $-K(-1/b_n)$ when at least one of the sequences $\{b_{2n-1}\}$, $\{b_{2n}\}$ converges to 0.

Theorem 5.2: Let $\{c_n\}_{n\ge0}$ be any sequence satisfying $0 < c_n < 1$. Let the sequence $\{b_n\}_{n\ge1}$ be defined by

$$b_{2n} = (c_{2n}/(1 - c_{2n+1}))\prod_{k=0}^{n}(1 - c_{2k+1})/(1 + c_{2k}),\ n \ge 1$$

$$b_{2n+1} = c_{2n+1}\prod_{k=0}^{n}(1 + c_{2k})/(1 - c_{2k+1}),\ n \ge 0.$$

(A) If $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim b_n = 0$ and the continued fraction (5.10) diverges by oscillation.

(B) If $\sum_{n=0}^{\infty} c_n = \infty$, $\sum_{n=0}^{\infty} c_n^2 < \infty$, and $c_{n+1} \le c_n$ for $n \ge 0$, then $\lim b_{2n} = 0$, $\lim b_{2n+1}$ may be finite or infinite, and (5.10) converges to $-1/(1 + c_0)$.

(C) In particular, if $a > 0$ is a constant and

$$c_{2n+1} = c_{2n} = 1/(2n+1 + 1/a),\ n \ge 0,$$

then

$$\lim b_{2n} = 0,\ \lim b_{2n+1} = a$$

<u>and</u> (5.10) <u>converges to</u> $-a/(2a+1)$.

<u>Proof</u>: After setting $c_{-1} = 0$, we define the sequence $\{g_n\}_{n\geq 0}$ by

(5.11) $$g_{2n} = -\prod_{k=0}^{n}(1-c_{2k-1})/(1+c_{2k}),\quad g_{2n+1} = \prod_{k=0}^{n}(1+c_{2k})/(1-c_{2k+1}).$$

Then it can be verified that the b_n in our theorem are given by

(5.12) $$b_n = g_n + 1/g_{n-1},\quad n \geq 1.$$

Now let

(5.13) $$t_n = \prod_{k=1}^{n} 1/(g_k g_{k-1}) = (-1)^n \prod_{k=1}^{n}(1 + (-1)^k c_k),$$

and set F_n equal to the nth partial sum of the series

(5.14) $$\sum_{k=1}^{\infty} t_k.$$

Then the nth approximant f_n of (5.10) is given by

(5.15) $$f_n = g_0(1 + 1/F_n).$$

If the series $\sum_{k=1}^{\infty} c_k$ converges, then the product $\prod_{k=1}^{\infty}(1 + (-1)^k c_k)$ in (5.13) converges, so that $t_n \not\to 0$ as $n \to \infty$. Hence, in this case (5.14) cannot converge to a finite limit. We have

(5.16) $$F_{2n} = \sum_{k=1}^{n} (t_{2k} + t_{2k-1}) = \sum_{k=1}^{n}(c_{2k}/(1+c_{2k}))\prod_{j=1}^{2k}(1+(-1)^j c_j).$$

Therefore, $\{T_{2n}\}$ converges if $\Sigma c_{2n} < \infty$, which is the case since $\Sigma c_n < \infty$. But then (5.14) must diverge by oscillation. Thus (5.15) implies that (5.10) diverges by oscillation. This completes the proof of part (A).

If the c_n satisfy the stated conditions in part (B), then the product

$$\prod_{k=1}^{\infty}(1 + (-1)^k c_k)$$

converges. Hence from (5.16) we have that $F_{2n} \to \infty$ as $n \to \infty$. But then $F_{2n+1} \to \infty$ also as $n \to \infty$, since

$$F_{2n+1} = F_{2n} - \prod_{k=1}^{2n+1}(1 + (-1)^k c_k).$$

So from (5.15) we see that $f_n \to g_0$ as $n \to \infty$, and therefore (5.10) converges to $g_0 = -1/(1 + c_0)$ in this case.

It remains to justify part (C). If

$$c_{2n+1} = c_{2n} = 1/(2n+1 + 1/a),\quad n \geq 0,\ a > 0,$$

then $0 < c_n < 1$, $c_{n+1} \leq c_n$, $\Sigma c_n = \infty$, and $\Sigma c_n^2 < \infty$; so that by part (B), (5.10) converges to $g_0 = -a/(2a + 1)$. Since the hypotheses guarantee that $c_n \to 0$ monotonely in this case, it follows that $b_{2n} \to 0$ as $n \to \infty$. But now

$$\begin{aligned} b_{2n+1} &= c_{2n+1} \prod_{k=0}^{n}(1+c_{2k})/(1-c_{2k+1}) \\ &= (2n+1+1/a)^{-1}\prod_{k=1}^{n}(2(k+1)+1/a)/(2k+1/a) \\ &= (2n+1+1/a)^{-1}(2n+2+1/a)a, \end{aligned}$$

so $b_{2n+1} \to a$ as $n \to \infty$. With this the proof of our theorem is complete.

The following theorem extends a result of Heller and Roach [3].

<u>Theorem</u> 5.3: <u>Let the elements of the sequence</u> $\{b_n\}$ <u>satisfy</u> $b_n \in \mathbb{C}$, $\lim b_{2n-1} = u \in \mathbb{C}$, $\lim b_{2n} = v \in \mathbb{C}$, <u>where</u> $uv \in \mathbb{R}$ <u>and</u> $0 < uv < 4$. <u>Let</u> $\{c_n\}$ <u>be defined by</u>

$$c_{2n-1} = (v/u)b_{2n-1},\quad c_{2n} = b_{2n},\quad n = 1, 2, \ldots .$$

Then, if

(5.17) $\sum_{n=1}^{\infty} |\bar{c}_n / \bar{v} - c_{n+1} / v|$

converges, the continued fraction (5.10) cannot converge to a finite limit. In particular, (5.17) converges if both series

(5.18) $\sum_{n=1}^{\infty} |b_{2n-1}-u|, \sum_{n=1}^{\infty} |b_{2n}-v|$

converge.

Proof: The denominators B_n of (5.10) satisfy

$$B_0 = 1,\ B_1 = b_1;\ B_n = b_nB_{n-1}-B_{n-2},\ n \geq 2.$$

For n = 0, 1, 2, ... set

$$D_{2n+1} = B_{2n+1},\ D_{2n} = B_{2n}\sqrt{(uv)}/v.$$

Then

$$D_n = D_{n-1}c_n\sqrt{(uv)}/v - D_{n-2},\ n \geq 2.$$

There exists an integer N such that

$$x_n = d_nx_{n-1} - x_{n-2}$$

where $d_n = c_{n+N}\sqrt{(uv)}/v$ satisfies $|d_n| < 2$. Hence by Lemma 2.1, the sequence $\{D_{n+N}\}_{n\geq0}$ is bounded. Therefore, $\{D_n\}$ and $\{B_n\}$ are also bounded. Thus $\{1/(B_nB_{n-1})\}$ cannot converge to 0 so (5.10) cannot converge finitely. Since it is easily seen that

$$|\bar{c}_{2n} / \bar{v} - c_{2n+1}/v| \leq |b_{2n}-v|/|v| + |b_{2n+1} - u|/|u|$$

and

$$|\bar{c}_{2n-1}/ \bar{v} - c_{2n}/v| \leq |b_{2n-1}-u|/|u| + |b_{2n} - v|/|v|,$$

it follows that the convergence of the series (5.18) imply the convergence of (5.17). We are now finished with the proof of Theorem 5.3.

6. Construction of $\{b_n\}$, where $\lim b_n = b(0<b<2)$ and $-K(-1/b_n) < \infty$.

In this section we consider the problem of finding for a given $b(0<b<2)$ a corresponding real sequence $\{b_n\}$ such that $b_n \to b$ and $-K(-1/b_n)$ converges finitely. We conjecture that there always exists such a sequence $\{b_n\}$, though in this section we are able only to verify this conjecture for certain b's in the interval (0,2). A stronger conjecture is that the demand of finite convergence of $-K(-1/b_n)$ can be replaced by the demand of absolute finite convergence.

Theorem 6.1: Let p and q be positive integers with no common factor such that $0 < p/q < 1/2$, and let $b = 2\cos(\pi p/q)$. Let $\{\varepsilon_n\}$ be any sequence satisfying $0 < \varepsilon_n < 1$, $\lim \varepsilon_n = 0$, and $\sum_{n=1}^{\infty}\varepsilon_n = \infty$. Let the sequence $\{b_n\}$ be defined by $b_n = (1 - \varepsilon_m)b$ if $n = mq - 1$, $m \geq 1$, and let $b_n = b$ otherwise. Then $\lim b_n = b$, $0 < b < 2$, and the continued fraction

(6.1) $\frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \ldots$

converges nonabsolutely to b.

Proof: Let

(6.2) $D_n := (\sin(n+1)\alpha)/(\sin\alpha),\ n = 0, 1, \ldots, q-1,$

where $\alpha := \pi p/q$. Then

$$\sum_{k=1}^{q-2}(D_kD_{k-1})^{-1} = \sin^2\alpha\sum_{k=1}^{q-2}(\sin k\alpha\,\sin(k+1)\alpha)^{-1}$$

$$= \sin\alpha\sum_{k=1}^{q-2}(\cot k\alpha - \cot(k+1)\alpha)$$

$$= (\sin\alpha)(\cot\alpha - \cot(q-1)\alpha)$$

(6.3) $$= 2\sin\alpha\cot\alpha = b.$$

If θ is any real number such that $\cos(\theta+k\alpha) \neq 0$, $k = 0, 1, \ldots, q$, then

$$\sum_{k=1}^{q}(\cos(\theta+(k-1)\alpha)\cos(\theta+k\alpha))^{-1} = (\sin\alpha)^{-1}\sum_{k=1}^{q}(\tan(\theta+k\alpha) - \tan(\theta+(k-1)\alpha))$$

(6.4) $$= (\sin\alpha)^{-1}(\tan(\theta+q\alpha) - \tan\theta) = 0.$$

Now that we have established (6.3) and (6.4) our goal is to determine the denominators B_n of (6.1). It is easily verified that

(6.5) $$B_n = D_n,\ n = 0, 1, \ldots, q-2.$$

We claim that the B_n for $n \geq q-1$ are given by the formulas

$$B_{(2m-1)q-1} = (-1)^p b\sum_{k=1}^{2m-1}\varepsilon_k$$

$$B_{2mq-1} = b\sum_{k=1}^{2m}\varepsilon_k$$

(6.6) $$B_{(2m-1)q-2} = (-1)^{p+1} \qquad (m \geq 1)$$

$$B_{2mq-2} = -1$$

$$B_{mq+r} = D_{r+1}B_{mq-1} - D_rB_{mq-2},$$

where $r = 0, 1, \ldots, q-2$. These formulas can be established by mathematical induction, using the fact that the B_n satisfy

(6.7) $$B_0 = 1,\ B_1 = b,\ B_n = b_nB_{n-1} - B_{n-2},\ n \geq 2.$$

It follows from formulas (6.2), (6.5), and (6.6) that $B_n \neq 0$ for all $n \geq 0$. Since the B_n are not zero, the nth approximant f_n of (6.1) is given by

(6.8) $$f_n = \sum_{k=1}^{n} 1/(B_kB_{k-1}).$$

We assert that

(6.9) $$f_{(n+1)q-2} = b,\ n \geq 0.$$

We shall now verify (6.9) by induction. Clearly (6.9) is true for $n = 0$ by formula (6.3). Assume (6.9) holds for some n. Then

$$f_{(n+2)q-2} = \sum_{k=1}^{(n+2)q-2}(B_kB_{k-1})^{-1}$$

$$= \sum_{k=1}^{(n+1)q-2}(B_kB_{k-1})^{-1} + \sum_{k=(n+1)q-1}^{(n+2)q-2}(B_kB_{k-1})^{-1}$$

$$= b + \sum_{k=(n+1)q-1}^{(n+2)q-2}(B_kB_{k-1})^{-1}.$$

But there exist real constants θ and λ, $\lambda \neq 0$, such that

$$B_{(n+1)q-2+k} = \lambda\cos(\theta+k\alpha),\ k = 0, 1, \ldots, q.$$

Hence, we have from above that

$$f_{(n+2)q-2} = b + \lambda^{-2}\sum_{k=1}^{q}(\cos(\theta+(k-1)\alpha)\cos(\theta+k\alpha))^{-1}.$$

By formula (6.4), the term on the right involving Σ is zero; hence $f_{(n+2)q-2} = b$ and (6.9) holds for all $n \geq 0$ by the induction principle.

Since $\sum_{k=1}^{\infty}\varepsilon_k = \infty$, it follows from formulas (6.2), (6.5), and (6.6) that $|B_nB_{n-1}| \to \infty$ as $n \to \infty$. Hence from (6.8) and (6.9) we have $f_n \to b$, so that (6.1) converges to b. However, the series $\sum_{k=1}^{\infty} 1/(B_kB_{k-1})$ cannot converge absolutely. This follows from the fact that, with the aid of formulas (6.6), we have

$$\sum_{m=1}^{\infty}(|B_{2mq-1}B_{2mq-2}|)^{-1} = b^{-1}\sum_{m=1}^{\infty}\left(\sum_{k=1}^{2m}\varepsilon_k\right)^{-1} = \infty.$$

The latter series diverges by a theorem of Abel and Dini which states that if S_n is the nth partial sum of a positive term series Σa_n and $\Sigma a_n = \infty$, then $\Sigma a_n/S_n = \infty$. This concludes our proof of Theorem 6.1.

<u>Theorem</u> 6.2: <u>Let</u> α <u>be an irrational number satisfying</u> $0 \ \alpha < 1/4$ <u>such that, for any rational approximation</u> p/q,

(6.10) $\qquad |\alpha - p/q| \geq 1/(16q^3).$

<u>Let</u> $b = 2\cos 2\alpha\pi$. <u>Then there exists a sequence</u> $\{b_n\}$ <u>such that</u> $0 < b_n < 2$, $\lim b_n = b$, <u>and the continued fraction</u> (6.1) <u>converges absolutely to a finite limit</u>.

<u>Proof:</u> First we establish that there exists an α as described in the Theorem. There must be an irrational number β in the interval (0,1) such that

$$|\beta - p/q| \geq 1/(4q^3)$$

for any rational approximation p/q. For if not, then every irrational number in (0,1) is in an open interval of radius $(4q^3)^{-1}$ at some p/q, where $q \geq 2$, $0 < p < q$. The Lebesgue measure of the union of these intervals is at most

$$(1/2)\sum_{q=1}^{\infty}q(q^{-3}) = \pi^2/12 < 1.$$

This says that the measure of the set of irrationals in (0,1) is less than 1, which is a contradiction. Hence the desired β exists, and we get the desired α by setting $\alpha = \beta/4$.

By the Dirichlet Principle, for any integer $N > 1$ there exists integers m and n such that $1 \leq n < N$ and

(6.11) $\qquad |\alpha - m/n| \leq 1/(nN).$

Thus, in view of (6.10), it follows that $1/(16n^3) \leq 1/(nN)$ and $n \geq \sqrt{N}/4$. Note that

(6.12) $\qquad 1/(16n^2) \leq |n\alpha - m| \leq 1/N.$

Now for any β, $0 < \beta < 1$, there exists k, $1 \leq k \leq 16n^2$, such that

$$|kn\alpha - km - \beta| \leq 1/N.$$

Setting $\ell = kn$ we see that $\ell \leq 16n^3 \leq 16N^3$ and hence for any β, $0 < \beta < 1$, there exists $\ell \in \mathbb{N}$, $1 \leq \ell \leq 16N^3$, and $m \in \mathbb{Z}$ so that

(6.13) $\qquad |\ell\alpha - m - \beta| \leq 1/N.$

Now let $\theta = 2\alpha\pi$. Since tan x is continuously differentiable in a neighborhood of $\pi/2 - \theta$, we can find $0 < \delta < \pi/2$ and a constant $M > 0$ such that, if

$$|\Phi - (\pi/2 - \theta)| \leq \delta,$$

then

$$|\tan\Phi - \cot\theta| \leq M|\Phi + \theta - \pi/2|.$$

Now suppose $0 < \gamma < 1$. We claim that there is a constant $C = C(\alpha)$ such that whenever $\{x_k\}$ is a sequence of the form

$$x_k = \cos(k\theta + \mu),$$

where $0 < \pi/2 - \mu < 2\pi$, then for some $k \le C\gamma^{-3}$ we have

(6.14) $$|x_k| \le \gamma |x_{k-1}|.$$

In fact we may suppose $0 < \gamma < M\delta \sin\theta$, since if we establish it in this case, then by altering C we get the general case. Then, since $\gamma \le 2\pi M \sin\theta$, there is an integer N with

$$2\pi M\gamma^{-1}\sin\theta \le N \le 4\pi M\gamma^{-1}\sin\theta.$$

By (6.13) there exists $\ell \in \mathbb{N}$, $1 \le \ell \le 16N^3$, such that if $\beta = 1/4 - \mu/(2\pi)$ we have for some $m \in \mathbb{N}$ that

$$|\ell\alpha - m - \beta| \le 1/N.$$

Thus

$$|\ell\theta - 2m\pi - \pi/2 + \mu| \le 2\pi/N \le \gamma(M\sin\theta)^{-1} \le \delta,$$

and hence

$$|\tan((\ell-1)\theta + \mu) - \cot\theta| \le \gamma/(\sin\theta)$$

or

$$|\cos(\ell\theta + \mu| \le \gamma\ |\cos((\ell-1)\theta \pm \mu)|.$$

That is, $|x_\ell| \le \gamma|x_{\ell-1}|$. Now we note

$$\ell \le 16N^3 \le (16(4\pi)^3 M^3 \sin^3\theta)\gamma^{-3} = C\gamma^{-3}$$

and our claim (6.14) is established.

Now fix $0 < a < 1/3$. Let $b = 2\cos\theta$. We will select the sequence b_n by induction. We suppose $b_1 = b$, $B_0 = 1$, $B_1 = b$. Then the

denominators B_n of (6.1) are given by the difference equation

$$B_n = b_n B_{n-1} - B_{n-2}.$$

In order to fix the sequence $\{b_n\}$ we shall fix a sequence of indices n_k with $0 = n_0 < n_1 < n_2 < \dots$, and put

$$b_n = b \text{ if } n \notin \{n_k\}$$

$$= b(1 - (1/2)k^{-a}) \text{ if } n = n_k (k = 1, 2, \dots).$$

The sequence $\{n_k\}$ is determined by the condition that n_k is the first integer $\ell > n_{k-1}$ such that

$$|b\, B_{\ell-1} - B_{\ell-2}| < (b/4)k^{-a}|B_{\ell-1}|.$$

For convenience let us put $n_k = \infty$ if this condition is not satisfied for any $\ell > n_{k-1}$. We shall use C in the proof to denote a constant depending only on α (or θ or b) which may vary from line to line. We prove by induction that $n_k < \infty$ and $B_{n_k} \ne 0$ for all k. In fact this is true for $k = 0$. Suppose $n_{k-1} < \infty$ and $B_{n_{k-1}} \ne 0$. Then the difference equation

$$x_p = b\, x_{p-1} - x_{p-2}$$

subject to the initial conditions

$$x_{-1} = B_{n_{k-1}-1},\ x_0 = B_{n_{k-1}}$$

has a solution of the form

$$x_p = \lambda \cos(p\theta + \mu),$$

where $0 < \pi/2 - \mu < 2\pi$ and $\lambda \neq 0$. Hence, if $(b/4)k^{-a} < M\delta \sin\theta$, there exists $p \le C(b/4)^3 k^{-3a}$ such that

$$|x_p| \le (b/4)k^{-a}\,|x_{p-1}|.$$

Let p_0 be the first such p. Then

$$n_k = n_{n-1} + p_0$$

and

$$B_{n_{k-1}+p} = x_p$$

for $0 \le p < p_0$. We deduce that

$$(6.15) \qquad n_k - n_{k-1} \le C\, k^{-3a},$$

where C depends only on α. In particular, $n_k < \infty$. Next note that, if $p < p_0$, then

$$(6.16) \qquad |x_p| \ge (b/4)k^{-a}|x_{p-1}|;$$

so that $B_n \neq 0$ for $n_{k-1} \le n < n_k$. Now

$$(6.17) \qquad |B_{n_k}| = |bB_{n_k-1} - B_{n_k-2} - (b/2)k^{-a}B_{n_k-1}|$$

$$\ge (b/4)k^{-a}\,|B_{n_k-1}| \neq 0.$$

Thus we see that $n_k < \infty$ for all k and that $n_k - n_{k-1} \le C\, k^{-3a}$. Combining (6.16) and (6.17) we further have

$$(6.18) \qquad |B_n| \ge (b/4)k^{-a}\,|B_{n-1}|$$

for $n_{k-1} < n \le n_k$. Since, for all n,

$$|B_n| \le b|B_{n-1}| + |B_{n-2}|$$

we conclude for $n_{k-1} < n \le n_k$ that

$$(6.19) \qquad |B_n| \le (b + 4k^a/b)\,|B_{n-1}|.$$

Hence, from (6.18) and (6.19) we have

$$|B_{n-1}|^2 \le (4k^a/b)\,|B_nB_{n-1}|$$

$$|B_n|^2 \le (b + 4k^a/b)\,|B_nB_{n-1}|;$$

so that

$$(6.20) \qquad |B_n|^2 + |B_{n-1}|^2 \le C\,k^a\,|B_nB_{n-1}|,$$

where $C = C(\alpha)$.

Now let

$$E_n = B_n^2 + B_{n-1}^2 - bB_nB_{n-1}.$$

Then $E_n \ge 0$ and $E_n \le C\,k^a\,|B_nB_{n-1}|$. In fact, if $n_{k-1} < n < n_k$, $E_n = E_{n-1}$. If $n = n_k$,

$$B_n^2 + B_{n-1}^2 - b_nB_nB_{n-1} = B_{n-1}^2 + B_{n-2}^2 - b_nB_{n-1}B_{n-2};$$

so

$$E_n = E_{n-1} + (b_n - b)(B_n - B_{n-2})B_{n-1}.$$

Now

$$B_n - B_{n-2} = b_nB_{n-1} - 2\,B_{n-2}$$

$$= (b_n - 2b)B_{n-1} + 2(bB_{n-1} - B_{n-2}).$$

Hence,

$$(b_n - b)B_{n-1}(B_n - B_{n-2}) = (b_n - b)(b_n - 2b)B_{n-1}^2$$

$$+ 2(bB_{n-1} - B_{n-2})(b_n - b)B_{n-1}.$$

Now

$$|bB_{n-1} - B_{n-2}| \le (b/4)k^{-a} \; |B_{n-1}|.$$

So that

$$|2(bB_{n-1} - B_{n-2})B_{n-1}| \le (b/2)k^{-a} \; |B_{n-1}|^2$$

while

$$|(b_n - 2b)B_{n-1}^2| \ge b|B_{n-1}|^2.$$

We conclude that

$$(b_n - b)B_{n-1}(B_n - B_{n-2}) \ge (1/2)(b - b_n)(2b - b_n)B_{n-1}^2$$

$$\ge C^{-1}k^{-a}B_{n-1}^2,$$

where $C = C(\alpha)$. Thus

$$E_n - E_{n-1} \ge C^{-1}k^{-2a} \; |B_{n-1}B_{n-2}|$$

$$\ge C^{-1}k^{-3a}E_{n-1}$$

or

$$E_n \ge (1 + C^{-1}k^{-3a})E_{n-1}$$

if $n = n_k$. Therefore, if $e_k = E_{n_k}$, we have

$$e_k \ge \exp(C^{-1}k^{-3a})e_{k-1}, \; k = 2, 3, \ldots,$$

where $C = C(\alpha)$. Hence

$$e_k \ge \exp(C^{-1}k^{1-3a}), \; k = 1, 2, \ldots$$

for some constant C. If $n_{k-1} < n \le n_k$,

$$|B_nB_{n-1}| \ge C^{-1}k^{-a}E_n$$

$$\ge C^{-1}k^{-a}\exp(C^{-1}k^{1-3a}).$$

Hence,

$$|B_nB_{n-1}|^{-1} \le C\,k^{a}\exp(-C^{-1}k^{1-3a}).$$

Thus

$$\sum_{n_{k-1}+1}^{n_k} |B_nB_{n-1}|^{-1} \le C\,k^{4a}\exp(-C^{-1}k^{1-3a}).$$

Since $0 < a < 1/3$, we have $\sum_{n=1}^{\infty} |B_nB_{n-1}|^{-1} < \infty$. The nth approximant f_n of (6.1) is given by $f_n = \sum_{k=1}^{n} (B_kB_{k-1})^{-1}$, and hence (6.1) converges absolutely for our choice of $\{b_n\}$. This completes our proof of Theorem 6.2.

7. <u>A Matrix Theoretic Result with Applications to Continued Fractions.</u>

In this section we first state and sketch the proofs of two lemmas that are needed to prove Theorem 7.1. This theorem is a powerful matrix theoretic result that gives sufficient conditions for boundedness of products of matrices. In the latter part of this section and through the first two theorems in Section 8 we show how Theorem 7.1 can be used to obtain convergence information about equimodular limit periodic continued fractions. We have not attempted to obtain an exhaustive list of applications of Theorem 7.1, though we suspect there are many in the fields of continued fractions and difference equations.

Lemma 7.1: Let A_0 be an nxn-matrix. Suppose λ_0 is an eigenvalue of A_0 which is a simple root of the characteristic polynomial. Suppose $A_0\underline{x}_0 = \lambda_0\underline{x}_0$ where $\underline{x}_0 \neq \underline{0}$. Then there is an open neighborhood U of A_0 and C^1-maps $\sigma: U \to \mathbb{C}^n$, $\lambda: U \to \mathbb{C}$ such that

$$\sigma(A_0) = \underline{x}_0$$
$$\lambda(A_0) = \lambda_0$$
$$A\sigma(A) = \lambda(A)\sigma(A).$$

Proof: Consider the (n+1)-equations

$$A\underline{x} = \lambda\underline{x}$$

$$\underline{x}^*\underline{x}_0 = \|\underline{x}_0\|^2 \qquad (\underline{x}^* = \bar{\underline{x}}^T),$$

where the norm is the Euclidean norm. The result follows from the Implicit Function Theorem once we check that the following matrix is non-singular.

$$B_0 = \left(\begin{array}{c|c} A_0 - \lambda_0 I & -\underline{x}_0 \\ \hline \underline{x}_0^* & 0 \end{array}\right).$$

Suppose

$$B\begin{pmatrix}\underline{y}\\ \mu\end{pmatrix} = 0.$$

Then

$$(A_0 - \lambda_0 I)\underline{y} - \mu\underline{x}_0 = 0$$
$$\bar{\underline{x}}_0^*\,\underline{y} = 0$$

So

$$(A_0 - \lambda_0 I)^2\underline{y} = (A - \lambda_0 I)\mu\underline{x}_0 = 0.$$

Hence, (see Nering [9, Corollary 8.4]), since λ_0 is a simple root,

$$A_0\underline{y} = \lambda_0\underline{y}$$

and

$$\underline{y} = \alpha\underline{x}_0.$$

Now $\alpha\underline{x}_0^*\underline{x}_0 = 0 \Rightarrow \alpha = 0 \Rightarrow \underline{y} = 0$, and therefore $\mu = 0$. This completes our proof of Lemma 7.1.

Lemma 7.2: Let Q be an nxn-matrix with n distinct eigenvalues. Suppose $Q_m \to Q$ as $m \to \infty$ and

$$\Sigma\,\|Q_m - Q_{m+1}\| < \infty,$$

where the matrix norm is the operator norm. Then there exists an integer N and invertible matrices $P_m (m \geq N)$ such that

$$P^{-1}Q_mP_m = \Delta_m$$

is diagonal $(m \geq N)$ and

$$\Sigma\,\|P_m - P_{m+1}\| < \infty.$$
$$\Sigma\,\|P_m^{-1} - P_{m+1}^{-1}\| < \infty.$$

<u>Proof:</u> There is an open neighborhood U of Q and C^1-maps $\sigma_i : U \to \mathbb{C}^n (i \le n)$, $\lambda_i : U \to \mathbb{C}$ such that

$$\sigma_i(Q) = \underline{x}_i, \ \lambda_i(Q) = \lambda_i,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Q and $\underline{x}_1, \ldots, \underline{x}_n$ are the eigenvectors. Let

$$P_m = [\sigma_1(Q_m), \ldots, \sigma_n(Q_m)]$$

as long as $Q_m \in U$. Since the irreducible matrices are open and $P \to P^{-1}$ is C^1, we can determine N so that the P_m for $m \ge N$ satisfy the conditions. This finishes our sketch of a proof for Lemma 7.2.

<u>Theorem</u> 7.1: <u>Let</u> $\{Q_m\}$ <u>be a sequence of</u> nxn-<u>matrices such that</u>

(i) $Q_m \to Q$ <u>as</u> $m \to \infty$, <u>where</u> Q <u>has distinct eigenvalues</u>.
(ii) $\Sigma \|Q_m - Q_{m+1}\| < \infty$.

<u>Let</u> $R_m = \mu_m I + \tau_m Q_m$ for $m \ge 1$, <u>where</u> $\{\mu_m\}$ <u>and</u> $\{\tau_m\}$ <u>are arbitrary sequences of real numbers</u>. <u>Let</u> r_m <u>be the spectral radius of</u> R_m. <u>Suppose</u>

$$\sup_n \left(\prod_{k=1}^{n} r_k\right) < \infty.$$

<u>Then there exists a constant</u> $K < \infty$ <u>such that</u>

$$\|R_m R_{m-1} \cdots R_1\| \le K, \ m \ge 1.$$

<u>Proof:</u> If $X \in \mathbb{C}^n$, let

$$X_m = R_m \cdots R_1 X.$$

Then

$$X_m = R_m X_{m-1}, \ X_0 = X, \ m \ge 1.$$

So by Lemma 7.2, for each $m \ge N$ there exists P_m such that

$$P_m^{-1} X_m = P_m^{-1} R_m P_m R_m^{-1} X_{m-1}$$
$$= D_m P_m^{-1} X_{m-1},$$

where D_m is diagonal. In what follows the norm of a vector will be the Euclidean norm and the norm of a matrix will be the operator norm. From the above we obtain

$$\|P_m^{-1} X_m\| \le r_m \|P_m^{-1} X_{m-1}\|$$
$$\le r_m (\|P_{m-1}^{-1} X_{m-1}\| + \|(P_m^{-1} - P_{m-1}^{-1}) X_{m-1}\|)$$
$$\le r_m (\|P_{m-1}^{-1} X_{m-1}\| + \|(P_m^{-1} P_{m-1} - I) P_{m-1}^{-1} X_{m-1}\|)$$
$$\le r_m (1 + \|P_m^{-1}\| \|P_m - P_{m-1}\|) \|P_{m-1}^{-1} X_{m-1}\|$$

Now, if

$$\sup_k \prod_{m=N}^{k} r_m (1 + \|P_m^{-1}\| \|P_m - P_{m-1}\|) < \infty,$$

we are done. In fact $\{\|P_m^{-1}\|\}$ is bounded, $\Sigma \|P_m - P_{m-1}\| < \infty$, and hence

$$\|P_m^{-1} X_m\| \le K$$

for some constant K independent of m. Since $\{P_m\}$ is bounded,

$$\|X_m\| \le K',$$

and as this is true for all X, $\|R_m \cdots R_1\|$ is bounded.

Examples 7.1.

Given $\{b_n\}$, $b_n \in \mathbb{R}$, suppose

$$x_n = b_n x_{n-1} - x_{n-2}, \quad n = 1, 2, \ldots .$$

Let

$$\underline{x}_n = \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} .$$

Then $\underline{x}_n = Q_n \underline{x}_{n-1}$, where

$$Q_n = \begin{pmatrix} b_n & -1 \\ 1 & 0 \end{pmatrix} .$$

If $b_n \to b$, then $Q_n \to Q$, where

$$Q = \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix} .$$

Q has distinct eigenvalues if $|b| < 2$. Letting r_n denote the spectral radius of Q_n, we note that $r_n = 1$ if $|b_n| \le 2$. Hence, if $b_n \in \mathbb{R}$, $|b_n| \le 2$, $b_n \to b(|b| < 2)$ and $\Sigma|b_n - b_{n+1}| < \infty$, the requirements for Q_n in Theorem 7.1 are satisfied. By taking $\mu_n \equiv 0$ and $\tau_n \equiv 1$ so that $R_n = Q_n$, it follows from this theorem that $\{x_n\}$ is bounded. Thus Lemma 2.1 follows from Theorem 7.1 when the b_n are real.

If we change the definition of $\underline{x}_n$ above to

$$\underline{x}_n = \begin{pmatrix} x_{2n} \\ x_{2n-1} \end{pmatrix} ,$$

then $\underline{x}_n = Q_n \underline{x}_{n-1}$, where now

$$Q_n = \begin{pmatrix} b_{2n} b_{2n-1} & -b_{2n} \\ b_{2n-1} & -1 \end{pmatrix} .$$

It is not difficult to see that if the b_n meet the hypotheses of Lemma 5.1, then Q_n meets the requirements of the Q_n in Theorem 7.1. Hence again, if we set $\mu_n \equiv 0$ and $\tau_n \equiv 1$ in Theorem 7.1, it follows that Lemma 5.1 is implied by this theorem. We have now finished the discussion we intended under Examples 7.1.

We have seen earlier that if $b_n \in \mathbb{R}$, $|b_n| < 2$, $\lim b_n = 0$, and $\Sigma\ |b_n - b_{n+1}| < \infty$, then $-K(-1/b_n)$ cannot converge to a finite limit. In our next theorem we give some indication of what can be said if the condition $\Sigma|b_n - b_{n+1}| < \infty$ is not required, that is if $\Sigma|b_n - b_{n+1}| = \infty$ is allowed.

Theorem 7.2: *Let $\{b_n\}$ be a sequence whose elements satisfy $b_n \in \mathbb{R}$, $|b_n| < 2$, $0 < b_n b_{n+1} < 4$, $\lim b_n = 0$, and*

(i) $\lim b_{2n-1}/b_{2n} = L \neq 0, \infty$

(ii) $\Sigma|b_{2n-1} - b_{2n+1}| < \infty$ *or* $\Sigma|b_{2n} - b_{2n+2}| < \infty$

(iii) $\Sigma|b_{2n-1}/b_{2n} - b_{2n+1}/b_{2n+2}| < \infty$.

Then

$$\text{(7.1)} \qquad \frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \cdots$$

cannot converge to a finite limit. In particular, if a, b, c, d are constants satisfying

$$a > 0,\ c > 0,\ a+b > 1/2,\ c+d > 1/2,\ a \neq c$$

and

$$b_{2n-1} = 1/(an+b),\ b_{2n} = 1/(cn+d),$$

then (7.1) cannot converge finitely.

Proof: If we again let B_n denote the nth denominator of (7.1), then

$$\begin{pmatrix} B_{2n+2} \\ B_{2n+1} \end{pmatrix} = M_n \begin{pmatrix} B_{2n} \\ B_{2n-1} \end{pmatrix},$$

where

$$M_n = \begin{pmatrix} b_{2n+2}b_{2n+1} - 1 & -b_{2n+2} \\ b_{2n+1} & -1 \end{pmatrix}.$$

Now

$$M_n = b_{2n+2}Q_n - I$$

and

$$M_n = b_{2n+1}R_n - I,$$

where

$$Q_n = \begin{pmatrix} b_{2n+1} & -1 \\ b_{2n+1}/b_{2n+2} & 0 \end{pmatrix}$$

and

$$R_n = \begin{pmatrix} b_{2n+2} & -b_{2n+2}/b_{2n+1} \\ 1 & 0 \end{pmatrix}.$$

We have that

$$Q_n \to Q = \begin{pmatrix} 0 & -1 \\ L & 0 \end{pmatrix}$$

and

$$R_n \to R = \begin{pmatrix} 0 & -1/L \\ 1 & 0 \end{pmatrix}.$$

The characteristic roots of Q are $\pm\ i\sqrt{L}$ and those of R are $\pm\ i/\sqrt{L}$. The series $\Sigma\|Q_n - Q_{n+1}\|$ converges if (i) and (iii) holds and $\Sigma|b_{2n-1} - b_{2n+1}| < \infty$, or the series $\Sigma\|R_n - R_{n+1}\|$ converges if (i) and (iii) holds and $\Sigma|b_{2n} - b_{2n+2}| < \infty$. Since $0 < b_nb_{n+1} < 4$, the characteristic roots of M_n are distinct with common modulus 1. Thus it follows from Theorem 7.1 that

$$\{\|M_nM_{n-1} \cdots M_1\|\}_{n\geq 1}$$

is bounded. Hence the sequences $\{B_{2n}\}$ and $\{B_{2n-1}\}$ are bounded so that $\{B_n\}$ is bounded. It follows that the nth term of the series $\Sigma\ (B_nB_{n-1})^{-1}$ cannot converge to 0, which implies that the continued fraction (7.1) cannot converge to a finite limit.

8. Equimodular Limit Periodic $K(a_n/1)$.

In this section we study continued fractions $K(a_n/1)$; where the a_n lie on the negative real axis and the convergence structure of the sequence $\{a_n\}$ is varied. Theorem 8.1 deals with the case $a_n \to a < -1/4$. Theorem 8.2 is concerned with the case $a_{2n-1} \to a < 0$, $a_{2n} \to b < 0$, where a and b are chosen so that $K(a_n/1)$ is equimodular. Finally, Theorem 8.3 gives information about what happens when $a_n \to -\infty$.

Theorem 8.1: If $a_n < -1/4$, $n = 1, 2, \ldots$, and $\lim a_n = a$, where $a < -1/4$; then the continued fraction

(8.1) $$\frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \ldots$$

cannot converge to a finite limit if

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

Proof: The denominators B_n of (8.1) satisfy the difference equation

$$B_0 = B_1 = 1; \; B_n = B_{n-1} + a_n B_{n-2}, \; n \geq 2.$$

Let the sequence $\{C_n\}$ be defined by

$$B_n = C_n P_n^{1/2},$$

where

$$P_n = \prod_{k=1}^{n}(-a_k).$$

Then

$$C_{n+2}P_{n+2}^{1/2} = C_{n+1}P_{n+1}^{1/2} + a_{n+2}C_nP_n^{1/2}$$

from which we obtain

$$C_{n+2} = C_{n+1}/(-a_{n+2})^{1/2} - C_n(a_{n+2}/a_{n+1})^{1/2}.$$

Writing the last equation as a matrix equation, we have

$$\begin{pmatrix} C_{n+2} \\ C_{n+1} \end{pmatrix} = A_{n+1} \begin{pmatrix} C_{n+1} \\ C_n \end{pmatrix},$$

where

$$A_{n+1} = \begin{pmatrix} (-a_{n+2})^{-1/2} & -(a_{n+2}/a_{n+1})^{1/2} \\ 1 & 0 \end{pmatrix}$$

The eigenvalues λ_i, $i = 1, 2$, of A_{n+1} satisfy

$$\lambda^2 - \lambda(-a_{n+2})^{-1/2} + (a_{n+2}/a_{n+1})^{1/2} = 0.$$

It follows from this equation that

$$|\lambda_i| = (a_{n+2}/a_{n+1})^{1/4}, \; i = 1, 2.$$

Hence the spectral radius r_{n+1} of A_{n+1} is given by

$$r_{n+1} = (a_{n+2}/a_{n+1})^{1/4}.$$

Thus

$$\prod_{k=1}^{n} r_k = (a_{n+1}/a_1)^{1/4}.$$

So

$$\sup_n \left(\prod_{k=1}^{n} r_k\right) < \infty,$$

since $\{a_n\}$ is a convergent (hence bounded) sequence.

Clearly $A_n \to A$ as $n \to \infty$, where

$$A = \begin{pmatrix} (-a)^{-1/2} & -1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of A have modulus 1 and are unequal since $a < -1/4$. Since

(8.2)
$$\sum_{n=1}^{\infty} |(-a_{n+1})^{-1/2} - (-a_n)^{-1/2}| = \sum_{n=1}^{\infty} \frac{|a_{n+1} - a_n|}{(a_n a_{n+1})^{1/2}((-a_n)^{1/2}+(-a_{n+1})^{1/2})},$$

it is easy to see that (8.2) converges because of our assumptions that $a_n \to a < -1/4$ and the series

$$(8.3) \qquad \sum_{n=1}^{\infty} |a_{n+1} - a_n|$$

converges. The hypotheses that $a_n \to a < -1/4$ and (8.3) converges also guarantee the convergence of the series

$$(8.4) \qquad \sum_{n=1}^{\infty} |(a_{n+2}/a_{n+1})^{1/2} - (a_{n+1}/a_n)^{1/2}|.$$

This is easier seen once we observe the relation

$$\left|(a_{n+2}/a_{n+1})^{1/2} - (a_{n+1}/a_n)^{1/2}\right| = \frac{|a_{n+1} - a_{n+2}|}{(-a_{n+1})^{1/2}((-a_{n+2})^{1/2} + (-a_{n+1})^{1/2})}$$
$$+ \frac{|a_{n+1} - a_{n+2}|}{(-a_{n+1})^{1/2}((-a_{n+1})^{1/2} + (-a_n)^{1/2})}$$
$$+ \frac{|a_{n+1} - a_n|}{(a_{n+1}a_n)^{1/2}}.$$

Hence the convergence of (8.3) implies the convergence of

$$\sum_{n=1}^{\infty} \|A_n - A_{n+1}\|.$$

Therefore, by Theorem 7.1 the sequence $\{C_n\}$ is bounded.

Now (8.1) can converge to a finite limit only if the series

$$(8.5) \qquad \sum_{n=N}^{\infty} (-1)^n (B_n B_{n-1})^{-1} \prod_{k=1}^{n} a_k$$

converges for N large enough. But

$$(-1)^{n+1}(B_{n+1}B_n)^{-1}\prod_{k=1}^{n} a_k = (C_{n+1}C_n)^{-1}(-a_{n+1})^{1/2} \not\to 0$$

as $n \to \infty$ since $\{C_n\}$ is bounded and $a_{n+1} \to a < -1/4$. Therefore, (8.5) cannot converge to a finite limit, and we have completed our proof.

<u>Theorem</u> 8.2: <u>If</u> $a_n < 0$, $n = 1, 2, \ldots$, $\lim a_{2n-1} = a < 0$, <u>and</u> $\lim a_{2n} = b < 0$, <u>where</u>

$$(8.6) \qquad ((-a)^{1/2} - (-b)^{1/2})^2 < 1 < ((-a)^{1/2} + (-b)^{1/2})^2,$$

<u>then the continued fraction</u> (8.1) <u>cannot converge to a finite limit if both series</u>

$$(8.6) \qquad \sum_{n=1}^{\infty} |a_{2n-1} - a_{2n+1}|, \quad \sum_{n=1}^{\infty} |a_{2n} - a_{2n+2}|$$

<u>converge</u>.

<u>Proof:</u> We define $\{B_n\}$ and $\{C_n\}$ as in the proof of Theorem 8.1. Then

$$\begin{pmatrix} C_{2n+2} \\ C_{2n+1} \end{pmatrix} = Q_n \begin{pmatrix} C_{2n} \\ C_{2n-1} \end{pmatrix}, \quad n = 1, 2, \ldots,$$

where

$$Q_n = \begin{pmatrix} (a_{2n+2}a_{2n+1})^{-1/2} - \left(\frac{a_{2n+2}}{a_{2n-1}}\right)^{1/2} & -\left(\frac{a_{2n+1}}{a_{2n}}\right)^{1/2}(-a_{2n+2})^{-1/2} \\ (-a_{2n+1})^{-1/2} & -\left(\frac{a_{2n+1}}{a_{2n}}\right)^{1/2} \end{pmatrix}.$$

It follows from our hypotheses that $\lim Q_n = Q$, where

$$Q = \begin{pmatrix} (ab)^{-1/2} - (b/a)^{1/2} & -(a/b)^{1/2}(-b)^{-1/2} \\ (-a)^{-1/2} & -(a/b)^{1/2} \end{pmatrix}.$$

The characteristic equation of Q is

$$\lambda^2 + \lambda((a/b)^{1/2} + (b/a)^{1/2} - (ab)^{-1/2}) + 1 = 0.$$

Condition (8.6) guarantees that the discriminant of this equation is negative so that its roots are unequal, though they have equal absolute value 1. The characteristic equation of Q_n will have a negative discriminant provided

$$\text{(8.8)} \quad \begin{aligned} &((a_{2n+1}^2 a_{2n+2}/a_{2n})^{1/4} + (a_{2n+2}^2)^{1/4})^2 > 1 \\ &((a_{2n+1}^2 a_{2n+2}/a_{2n})^{1/4} - (a_{2n+2}^2)^{1/4})^2 < 1. \end{aligned}$$

Since $a_{2n-1} \to a < 0$, $a_{2n} \to b < 0$ and (8.6) holds, there exists a positive integer N such that (8.8) holds for $n \geq N$. Hence for $n \geq N$, it can be seen that the spectral radius r_n of Q_n is given by

$$r_n = (a_{2n+2}/a_{2n})^{1/4}.$$

Thus the product $\prod_{k=n}^{\infty} r_k$ converges to $(-b/a_{2N})^{1/4}$, so that $\{\prod_{k=1}^{n} r_k\}$ is bounded. After making some tedious estimates it can be seen that $\Sigma \|Q_n - Q_{n+1}\| < \infty$. Thus it follows from Theorem 7.1 that the sequence $\{C_n\}$ is bounded. As in the last part of the proof of Theorem 8.1, we can now come to the conclusion that (8.1) cannot converge to a finite limit under the given hypotheses.

<u>Theorem</u> 8.3: <u>Let</u> $\{h_n\}_{n\geq 0}$ <u>be a sequence of real numbers satisfying</u>

$$\text{(8.9)} \qquad h_n > 1,\ h_{n+1} \geq h_n,\ h_n \to \infty \ \underline{\text{as}}\ n \to \infty.$$

<u>Let</u> $\{a_n\}_{n\geq 1}$ <u>be defined by</u>

$$\text{(8.10)} \qquad a_{2n-1} = -h_{2n-2}(-1+h_{2n-1});\ a_{2n} = -h_{2n-1}(1+h_{2n}).$$

<u>Then the continued fraction</u> (8.1) <u>diverges by oscillation if</u> $\Sigma\, 1/h_k < \infty$. <u>If</u> $\Sigma\, 1/h_k = \infty$ <u>and</u> $\Sigma\, 1/h_k^2 < \infty$, <u>then</u> (8.1) <u>converges</u> <u>to</u> h_0.

<u>Proof</u>: Set

$$t_n = -(1 + (-1)^n/h_n),\ n \geq 1.$$

Then it can be established that

$$\text{(8.11)} \qquad f_n - h_0 = -h_0/(1 + t_1 + t_1t_2 + \dots + t_1t_2 \dots t_n),$$

where f_n is the nth approximant of (8.1). Set

$$C_n = \prod_{k=1}^{n} t_k = (-1)^n \prod_{k=1}^{n} (1 + (-1)^k/h_k).$$

Then $\{|C_n|\}$ is bounded if the infinite product

$$\text{(8.12)} \qquad \prod_{k=1}^{\infty} (1 + (-1)^k/h_k)$$

converges. Clearly (8.12) converges absolutely if $\Sigma\, 1/h_k < \infty$. If $\Sigma\, 1/h_k = \infty$ and the h_k satisfy (8.9), then $\Sigma\, (-1)^k/h_k < \infty$ and (8.12) converges if and only if $\Sigma\, 1/h_k^2 < \infty$. If (8.12) converges, then

$\lim |c_n| \neq 0$, so that the series $\Sigma\, c_n$ cannot converge to a finite limit. In this case, it follows from (8.11) that the only way (8.1) can converge is that $\lim |\sum_{k=1}^{n} c_k| = \infty$. Let

$$T_n = \sum_{k=1}^{n} c_k.$$

Then

$$T_{2n} = \sum_{k=1}^{2n} c_k = \sum_{k=1}^{n} (c_{2k-1} + c_{2k}).$$

But

$$c_{2n} + c_{2n-1} = \prod_{k=1}^{2n} (1 + (-1)^k/h_k) - \prod_{k=1}^{2n-1} (1 + (-1)^k/h_k)$$

$$= c_{2n-1}/h_{2n}$$

So

$$(8.13) \qquad T_{2n} = \sum_{k=1}^{n} c_{2k-1}/h_{2k}.$$

Now if $\Sigma\, 1/h_k < \infty$, then c_{2n-1} tends to a negative limit and $\Sigma\, c_{2n-1}/h_{2n} < \infty$, so that $\{T_{2n}\}$ converges to a finite limit. Hence, in this case, the continued fraction (8.1) must diverge by oscillation.

If $\Sigma\, 1/h_k = \infty$ but $\Sigma\, 1/h_k^2 < \infty$, the product (8.12) is still convergent. In this case it follows (with the aid of (8.9) and (8.13)) that $T_{2n} \to -\infty$ as $n \to \infty$. Since $T_{2n+1} = c_{2n+1} + T_{2n}$, it is also true that $T_{2n+1} \to -\infty$. Hence, using (8.11), we have that (8.1) converges to h_0. This completes our proof of Theorem 8.3.

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