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SADDLE POINTS FOR LINEAR DIFFERENTIAL GAMES*

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Abstract. It is proved that there is a saddle point over the relaxed controls, and so over the strategies defined on the relaxed controls, for differential games in which the trajectory variable appears linearly in the dynamical equation and payoff. This is a strong saddle point property, but the example of Berkovitz [1], of a game that does not have a saddle point in pure strategies, does have a saddle point in this sense. Saddle points over the chattering controls are obtained for linear games in which the opposing control variables appear separated. The introduction of relaxed controls into differential games is analogous to the introduction by von Neumann of mixed strategies into two person, zero sum games.

1.1. Introduction. Motivated by work in the calculus of variations and control theory (see [6], [12], [13] and [14]), we introduce relaxed controls to study two person, zero sum differential games. For differential games which possess a certain linearity property in the trajectory variable (in the differential equation describing the game and in the payoff function), we prove there is a saddle point over the relaxed controls, and so over the strategies defined on the relaxed controls. As pointed out below this concept of a saddle point over the (relaxed) controls is a strong property because a player can use his saddle point control, and, independently of what the other player does, not loose in comparison with the saddle point payoff.

1.2. Outline. We first define a two person, zero sum differential game and describe the notions of strategy and saddle point. In § 3 and § 4 we introduce relaxed controls in a rigorous manner as elements of the dual of a Banach space of integrable vector-valued functions. The presentation follows Warga [13]. The relaxed controls are shown to be a compact convex set and we prove that the "piecewise constant" relaxed controls are dense.

The games considered have a payoff which, though bilinear, is only separately continuous on the product space of relaxed controls for the two players. However, a general theorem of Sion [10] then states there is a saddle point, so our principal result concerns the existence of a saddle point among the relaxed controls, for linear differential games. We observe that the well-known example of Berkovitz [1] (see also [3]) of a differential game that does not have a saddle point in pure strategies does have a saddle point in our sense in the form of constant strategies on the relaxed controls.

Finally, for linear games in which the control variables appear separated, saddle points are obtained over the particularly simple relaxed controls known as chattering controls.

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2. Differential games. The situation to be discussed is a two person, zero sum differential game described as follows.

Notation 2.1. We have a dynamical system

$$\dot{x}(t) = f(t, x, u, v), \tag{2.1}$$

where the trajectory $x(t) \in R^m$, an initial position $x(0) = (x_1(0), \cdots, x_m(0)) = x_0$ is given, and the time t belongs to some closed bounded interval of T of R—say $T = [0, 1]$. There are two sets of control variables: $\{u\} \subset Y$, a compact subset of R^p, and $\{v\} \subset Z$, a compact subset of R^q.

To ensure integrability of the differential equation we shall assume that the vector function

$$f(t, x, u, v) = (f_1(t, x, u, v), \cdots, f_m(t, x, u, v))$$

is continuous in t, x, u and v and satisfies a Lipschitz condition in x of the form:

$$|f(t, x, u, v) - f(t, x', u, v)| \leqq \Psi(t)|x - x'| \quad \text{for } x, x' \in R^m$$

(or at least for x, x' in some compact subset of R^m wherein all trajectories $x(t)$ are known to lie), $t \in T$, $u \in Y$ and $v \in Z$. Here $\Psi(t)$ is an integrable real-valued function of t and $|\cdot|$ denotes the usual distance in R^m.

Together with the above dynamical system we have a payoff of the form:

$$P(u, v) = \mu(x(t)) + \int_0^1 h(t, x, u, v)\, dt. \tag{2.2}$$

Here μ is a (not necessarily linear) real-valued function on the Banach space $C([0, 1])$ of curves in R^m over $[0, 1]$, and h is a continuous real-valued function on $T \times R^m \times Y \times Z$.

DEFINITION 2.2. A *two player, zero sum differential game* is a dynamical system described by differential equations of the above form, together with the compact sets Y and Z and the payoff P.

The first player chooses $u \in Y$ at each time $t \in T$ in a measurable way, thus generating a function $u(t)$, so that the final payoff $P(u(t), v(t))$ is as large as possible. At the same time the second player chooses v at each time $t \in T$ so that $P(u(t), v(t))$ is as small as possible.

Remark 2.3. Denote by U (resp. V) the set of measurable functions from T to Y (resp. Z).

A (pure) strategy for the first player would ideally be defined as some rule which, for each time $t \in T$, determines for him his choice of $u(t)$ on the basis of what has happened in the game so far, that is, from the knowledge of $x(\tau)$, $u(\tau)$ and $v(\tau)$ for $0 \leqq \tau < t$. Similarly, there would be a notion of strategy for the second player.

On the grounds that a player knows his own previous choice of controls, Roxin [5] has defined a strategy for the first player as a function α from V to U which is nonanticipatory in the sense that:

if $v_1(t), v_2(t) \in V$ and $v_1(\tau) = v_2(\tau)$ for $0 \leqq \tau \leqq t_0$, then $\alpha(v_1)(\tau) = \alpha(v_2)(\tau)$ for $0 \leqq \tau \leqq t_0$.

The difficulty with this definition is that, in general, given a strategy α for the first player and a strategy β for the second player we do not know that there is an "outcome" for α and β, that is, a pair of control functions $u(t)$ and $v(t)$ such that

$$\alpha v = u \quad \text{and} \quad \beta u = v.$$

(We are looking for a fixed point of $\alpha \circ \beta : V \to V$, but α and β are not necessarily continuous.)

If a pair of strategies α, β do have an outcome $u(t)$, $v(t)$, we can write $P(\alpha, \beta)$ for the resulting payoff $P(u, v)$. A solution to the game is a pair of strategies α^*, β^*, which have an outcome which is "simultaneously best" for both players. This means that if α, β are other strategies such that α^*, β and α, β^* both have outcomes, then

$$P(\alpha^*, \beta) \geqq P(\alpha^*, \beta^*) \geqq P(\alpha, \beta^*). \tag{2.3}$$

DEFINITION 2.4. A pair of strategies α^*, β^* which satisfy (2.3) are said to form a *saddle point* for the differential game (*over the pure strategies*).

A much stronger notion of saddle point is now described.

DEFINITION 2.5. A pair of control functions $u^*(t) \in U$, $v^*(t) \in V$ are said to form a *saddle point* (*over the controls*) if for any other controls $u(t) \in U$, $v(t) \in V$,

$$P(u^*, v) \geqq P(u^*, v^*) \geqq P(u, v^*).$$

LEMMA 2.6. *If* (u^*, v^*) *are a saddle point over the controls, then there are identically constant strategies* (α^*, β^*) *which are a saddle point in the sense of Definition* 2.4.

Proof. For any $v(t) \in V$ and $u(t) \in U$ we define $\alpha^* v = u^*$, $\beta^* u = v^*$. That is, α^* is described by saying that, whatever the second player does, the first player continues to play his control $u^*(t)$. β^* is described similarly. If α, β are any other strategies, then α^*, β have an outcome u^*, v, say, and α, β^* have an outcome u, v^*. Thus, $P(\alpha^*, \beta) \geqq P(\alpha^*, \beta^*) \geqq P(\alpha, \beta^*)$.

Remarks 2.7. A saddle point over the controls is, therefore, a strong concept, because if one player uses his saddle point control then independently of what the other player does, he cannot lose. However, a saddle point over the strategies does not give us a saddle point over the controls.

If (u^*, v^*) form a saddle point over the controls and if u^* is played, then v must play to minimize $P(u^*, v)$. Hence $v^*(t)$ is an optimal controller subject to the maximum principle. Similarly, $u^*(t)$ is an optimal controller maximizing $P(u, v^*)$.

Also, note that any other saddle point $(\bar{u}, \bar{v})$ over the controls must give the same payoff, but nonsaddle plays might give the same value of P without satisfying the saddle point inequalities.

Background 2.8. We have already noted that there are difficulties surrounding the notion of a strategy for a player in a differential game. Friedman [3] circumvents these difficulties by considering "upper" and "lower" approximating games in which one player or the other has advance information. To obtain the convergence of the resulting "upper" and "lower" values of the game he requires in effect that the payoff $P(u, v)$ be jointly continuous in both control functions. This he ensures by considering only dynamical systems of the form

$$\dot{x} = f_1(t, x, u) + f_2(t, x, v)$$

and payoffs of the form

$$P(u, v) = \mu(x(t)) + \int_0^1 h_1(t, x, u) + h_2(t, x, v)\, dt,$$

so that u and v are "separated."

Somewhat earlier (cf. [1]), Berkovitz studied differential games by a "variational approach," introducing related Hamilton–Jacobi equations which are, unfortunately, hard to analyze.

2.1. A game with no pure strategy solutions. In [1] Berkovitz considers a differential game described by an equation

$$\dot{x}(t) = 4(u - v)^2, \qquad x \in R^1, \qquad t \in [0, 1], \qquad x(0) = 0,$$

and with payoff

$$P(u(t), v(t)) = \int_0^1 x(t)\, dt.$$

The control sets are

$$Y = \{u : 0 \leqq u \leqq 1\} \quad \text{and} \quad Z = \{v : 0 \leqq v \leqq 1\}.$$

In terms of his "variational approach," Berkovitz shows this game has no solution in pure strategies.

A. Friedman also cites this example in [3] and shows that for this game his upper value $V^+ = \frac{1}{2}$ while his lower value $V_- = 0$. Thus, as $V^+ \neq V_-$, this game does not have a value in his theory.

We shall see below, however, that by introducing the idea of relaxed controls, this game, for example, does have a saddle point over the relaxed controls.

3. Relaxed controls. The notion of relaxed, or generalized, curve was introduced into the calculus of variations by Young [14], and applied to control theory by Warga [13], McShane [6] and Young [15]. For further introduction of relaxed controls into differential games see the paper by Smoljakov [11]. The method and content of Smoljakov's paper, however, is quite different from the treatment below. In the discussion of relaxed controls described below we shall follow the setting described by Warga [13].

Suppose we have a dynamical system and payoff as described in Notation 2.1. We have already introduced U and V for the spaces of measurable functions from T to Y and Z respectively. U and V are, of course, just the spaces of (classical) control functions.

DEFINITION 3.1. Denote by (PY) and (PZ) the space of all regular probability measures defined on the Borel subsets of $Y \subset R^p$ and $Z \subset R^q$, respectively. A *relaxed control* for the first player is a function σ from $T = [0, 1]$ to PY. A relaxed control is *continuous* (resp. *measurable*) if $\int_Y f(u)\sigma(du; t)$ is a continuous (resp. measurable) function of $t \in T$, for every continuous real-valued function f on Y. A relaxed control for the second player: $\tau : T \to PZ$ is defined similarly. We shall identify relaxed controls which differ only on a set of measure zero.

Remarks 3.2. Here $\sigma(A, t)$ denotes the $\sigma(t)$-measure of any Borel set $A \subset Y$. By approximating the characteristic function of A with continuous functions on

Y, for example, it is easy to see that, if σ is a measurable relaxed control and A is a Borel subset of Y, then $\sigma(A, t)$ is measurable and integrable over T.

Notation 3.3. Denote by $\mathscr{S}(Y)$ the set of measurable relaxed controls on Y.

If $u(t) \in U$ is a classical control, then $\delta(u(t))$ can be thought of as an associated relaxed control giving, in effect, the same control. (By $\delta(u(t))$ we mean the probability measure on Y which at time t has total unit mass at $u(t)$—that is, a Dirac δ-function at $u(t)$.)

DEFINITION 3.4. If $u_1(t), \cdots, u_k(t)$ are classical controls and $\alpha_1(t) \geqq 0, \cdots, \alpha_k(t) \geqq 0$ are measurable functions on T such that $\sum \alpha_j(t) = 1$ almost everywhere, then we shall say that $\sum_{j=1}^{k} \alpha_j(t)\, \delta(u_j(t))$ is a *chattering control of degree k*. Such controls are special cases of relaxed controls and are discussed in Lee and Markus [4].

DEFINITIONS 3.5. $C(Y)$ will denote the Banach space of continuous real-valued functions f on Y with the usual norm: $\|f\| = \sup_{u \in y} |f(u)|$. $L^1_{[0,1]}(C(Y))$ will denote the Lebesgue space of integrable $C(Y)$-valued functions $\{\varphi\}$ defined on $[0, 1]$ with the norm

$$\|\varphi\| = \int_0^1 \sup_{u \in y} |\varphi(u, t)|\, dt.$$

For a discussion of Lebesgue spaces of Banach-space-valued functions, see Dunford and Schwartz [2] and Schwartz [9].

A real-valued function $\varphi(u, t)$ defined on $Y \times [0, 1]$ defines a function in $L^1_{[0,1]}(C(Y))$ if:

(i) $\varphi(u, t)$ is measurable in t for each $u \in Y$;

(ii) $\varphi(u, t)$ is continuous in u for each $t \in [0, 1]$ and

(iii) there exists an integrable real-valued function $\Phi(t)$ on $[0, 1]$ such that $|\varphi(u, t)| \leqq \Phi(t)$ on $Y \times [0, 1]$.

Conditions (i), (ii) and (iii) ensure that $\sup_{u \in Y} |\varphi(u, t)|$ is integrable and so $\|\varphi\|$ is finite.

Notation 3.6. Write B for the Banach space $L^1_{[0,1]}(C(Y))$.

DEFINITION 3.7. Denote by B^* the dual of B and by $\langle \varphi, \lambda \rangle$ the value of $\lambda \in B^*$ at $\varphi \in B$. We shall consider B^* to have the weak star topology. A sequence $\{\lambda_i\}$ converges to $\lambda \in B^*$ in this topology if

$$\lim_{i \to \infty} \langle \varphi, \lambda_i \rangle = \langle \varphi, \lambda \rangle \quad \text{for all } \varphi \in B.$$

Elements of B^* are described by the following lemma, which follows from results in [9, Exposé 4, p. 3].

LEMMA 3.8. *Suppose $\lambda \in B^*$. Then there is a measurable map μ from $[0, 1]$ to the class of regular signed Borel measures on Y such that*

$$\langle \varphi, \lambda \rangle = \int_0^1 \int_Y \varphi(u, t) \mu(du; t)\, dt \quad \textit{for all } \varphi \in B.$$

Furthermore, $|\mu|(Y, t) \in L^\infty([0, 1])$.

Note that the norm of $\lambda \in B^*$ is just $\operatorname{ess\,sup}_{t \in [0,1]} |\mu|(Y, t)$. Hence the norm of any relaxed controller $\sigma \in \mathscr{S}(Y)$ is just 1. From this lemma we have the following theorem.

THEOREM 3.9. *The set $\mathscr{S}(Y)$ of relaxed controls can be considered as a closed convex subset of the unit ball of B^*, and so with the weak star topology $\mathscr{S}(Y)$ is compact.*

For the proof again see [13].

For simplicity of exposition consider, instead of a differential game, a control system with just one control variable $u \in Y \subset R^p$; that is, a dynamical system

$$\dot{x}(t) = f(t, x, u), \qquad x(t) \in R^m, \tag{3.1}$$

with initial condition $x(0) = x_0$ and f satisfying a Lipschitz condition

$$|f(t, x, u) - f(t, x', u)| \leqq \Psi(t)|x - x'|$$

with $\Psi(t)$ integrable.

Under these hypotheses we quote from [13].

THEOREM 3.10. *Suppose $\sigma \in \mathscr{S}(Y)$ is a measurable relaxed control. Then there is a unique absolutely continuous solution $x(t)$ of the differential equation*

$$\dot{x}(t) = \int_Y f(t, x, u)\sigma(du; t) \tag{3.2}$$

differentiable and satisfying (3.2) *almost everywhere, with initial condition*

$$x(0) = x_0 \in R^m.$$

DEFINITION 3.11. Such a solution is called a *relaxed trajectory.*

From this Warga [13] proves the next result.

THEOREM 3.12. *Denote by $x(t; \sigma)$ the relaxed trajectory solution of* (3.2). *Suppose $\{\sigma_i\}$ is a sequence of measurable relaxed controls such that $\sigma_i \to \sigma$ in the weak star topology of $\mathscr{S}(Y)$. Then $x_i(t; \sigma_i)$ converges to $x(t; \sigma)$ in the uniform topology on* $[0, 1]$.

A corollary of this result is that the space of relaxed trajectories is compact in the uniform topology.

In preparation for our discussion of differential games let us return to our discussion of dynamical systems with two sets of control variables as described in Notation 2.1.

LEMMA 3.13. *If σ is a measurable relaxed control on Y and τ is a measurable relaxed control on Z, then $\sigma \times \tau$ is a measurable relaxed control on $Y \times Z$, and $\sigma \times \tau$ can be considered to belong to the unit sphere of the dual of $L^1(C(Y \times Z))$.*

Proof. The first statement is a simple consequence of results on product measures.

Given a function $f(t, u, v)$ in $L^1(C(Y \times Z))$ we have that for each $v' \in Z$, $f(t, u, v') \in L^1(C(Y))$. Therefore, $\int_Y f(t, u, v)\sigma(du; t)$ is continuous in v for each $t \in [0, 1]$ and is measurable and dominated by an integrable function in t (uniformly for all $v \in Z$). Thus we can consider

$$\int_0^1 \int_Y \int_Z f(t, u, v)\sigma(du; t)\tau(dv; t)\, dt = \langle f, \sigma \times \tau \rangle.$$

It is clear that $\sigma \times \tau$ is a probability measure on $Y \times Z$ for each $t \in [0, 1]$ and that $\sigma \times \tau$ has unit norm as a linear functional on $L^1(C(Y \times Z))$. Thus $\sigma \times \tau$ belongs to $\mathscr{S}(Y \times Z)$.

Remark 3.14. Note that by Fubini's theorem,

$$\int_Z \int_Y f(t, u, v)\sigma(du; t)\tau(dv; t) = \int_Y \int_Z f(t, u, v)\tau(dv; t)\sigma(du; t).$$

Discussion 3.15. Suppose $Y \subset R^p$, $Z \subset R^q$ are compact sets as above, and write

$$B_1 = L^1(C(Y)), \quad B_2 = L^1(C(Z)), \quad B_3 = L^1(C(Y \times Z)).$$

The dual spaces B_j^* will as usual be given the weak star topology. Denote by $\mathscr{S}(Y)$, $\mathscr{S}(Z)$, $\mathscr{S}(Y \times Z)$ the spaces of relaxed controls over $[0, 1]$ on Y, Z and $Y \times Z$ respectively, so that we have

$$\mathscr{S}(Y) \subset B_1^*, \quad \mathscr{S}(Z) \subset B_2^*, \quad \mathscr{S}(Y \times Z) \subset B_3^*.$$

Lemma 3.13 above tells us that we have a natural mapping from $\mathscr{S}(Y) \times \mathscr{S}(Z)$ to $\mathscr{S}(Y \times Z)$. Of course this map is not surjective, but more surprising this (bilinear on convex combinations) mapping is *not* jointly continuous in both variables, as the following example shows.

Example 3.16. Suppose $Y = [0, 1]$ and also $Z = [0, 1]$. Consider a partition of the time interval $T = [0, 1]$ into 2^n equal intervals $T_1 = [0, 1/2^n]$, $T_j = (j - 1)/2^n, j/2^n]$, $j = 2, \cdots, 2^n$. Corresponding to the 2^n partition of T consider a relaxed control σ_n on Y and a relaxed control τ_n on Z which are piecewise constant on each T_j, $j = 1, \cdots, 2^n$, and which are such that

$$\sigma_n(\cdot\,; t) \quad \text{and} \quad \tau_n(\cdot\,; t)$$

are the unit mass at the point $1 \in Y$ (resp. $1 \in Z$) if $t \in T_1 \cup T_3 \cup T_5 \cup \cdots \cup T_{2^n - 1}$, and $\sigma_n(\cdot\,; t)$ and $\tau_n(\cdot\,; t)$ are the unit mass at the point $0 \in Y$ (resp. $0 \in Z$) if $t \in T_2 \cup T_4 \cup T_6 \cup \cdots \cup T_{2^n}$.

Then it is easy to see that both σ_n and τ_n converge in B_1^* (resp. in B_2^*) to the constant relaxed control σ on Y (resp. τ on Z) which consists of a mass $\frac{1}{2}$ at 0 and mass $\frac{1}{2}$ at 1.

However, the product relaxed control $\sigma_n \times \tau_n$ on $Y \times Z = [0, 1] \times [0, 1]$ converges in B_3^* to the constant relaxed control π which consists of a mass $\frac{1}{2}$ at $(0, 0) \in Y \times Z$ and mass $\frac{1}{2}$ at $(1, 1) \in Y \times Z$.

Clearly $\pi \neq \sigma \times \tau$, so the map is not jointly continuous. (To check the above statements about the weak star convergence of σ_n, τ_n and $\sigma_n \times \tau_n$, it is sufficient to check how these relaxed controls act on products of functions $f(t)$, $\varphi(u)$, $\psi(v)$, where f is continuous on T, φ is continuous on Y and ψ is continuous on Z. This is because, for example, sums of products of the form $f(t)\,\varphi(u)\,\psi(v)$ are dense in B_3.)

DEFINITION 3.17. In Definition 3.1, we introduced the idea of measurable relaxed controls. Returning to a differential game described as in Notation 2.1, following Theorem 3.10, if the first player uses a relaxed control $\sigma(\cdot\,; t)$ and the second player uses a relaxed control $\tau(\cdot\,; t)$, then we define the dynamical equations to be given by the system

$$\dot{x}(t) = \int_Y \int_Z f(t, x, u, v)\sigma(du; t)\tau(dv; t). \tag{3.3}$$

Furthermore, the payoff corresponding to the relaxed controls σ and τ is defined to be

$$(3.4) \qquad P(\sigma, \tau) = \mu(x(t)) + \int_0^1 \int_Y \int_Z h(t, x, u, v)\sigma(du; t)\tau(du; t)\, dt.$$

Remarks 3.18. It is a consequence of Example 3.16 and Definition 3.17 that $P(\sigma, \tau)$ is not in general jointly continuous in σ and τ. However, in special situations, for example those discussed by Friedman [3] in which the control variables are separated, the payoff is jointly continuous and the definition of $P(\sigma, \tau)$ can be motivated by continuity because (cf. [13, Thm. 2.4]) the classical control functions are dense in the relaxed controls.

Similar to Definition 2.5 we have the strong concept of a saddle point over the relaxed controls.

DEFINITION 3.19. A pair of relaxed controls $\sigma^* \in \mathscr{S}(Y)$, $\tau^* \in \mathscr{S}(Z)$ is said to form a *saddle point over the relaxed controls* if for any other relaxed controls $\sigma \in \mathscr{S}(Y)$, $\tau \in \mathscr{S}(Z)$,

$$P(\sigma^*, \tau) \geqq P(\sigma^*, \tau^*) \geqq P(\sigma, \tau^*).$$

Remarks 3.20. Having made the above definitions we remark that, as is easily seen, one reason the Berkovitz game (cf. § 2.1) is difficult to analyze is that having introduced relaxed controls its payoff is not jointly continuous on $\mathscr{S}(Y) \times \mathscr{S}(Z)$.

4. Certain linear games. Suppose the system of equations describing the game has the form

$$(4.1) \qquad \dot{x}(t) = A(t)x(t) + f(t, u, v)$$

with initial condition $x(0) = x_0 \in R^m$. Here $A(t)$ is a continuous linear function of $t \in [0, 1]$, that is, $A(t)$ is an $m \times m$ matrix whose entries are continuous functions of time. $f(t, u, v)$ is a continuous function on $T \times Y \times Z$.

Furthermore, suppose the payoff has the form

$$(4.2) \qquad P(u, v) = \mu(x(t)) + \int_0^1 h(t, u, v)\, dt,$$

where μ is a continuous real-valued linear function on the Banach space $C([0, 1])$ of continuous R^m-valued functions on $[0, 1]$.

We are now in a position to prove our final result.

THEOREM 4.1. *Consider the differential game with dynamics and payoff given by equations of the above form* (4.1) *and* (4.2). *Then there is a pair of relaxed controls* $\sigma^*(\cdot, t)$, $\tau^*(\cdot, t)$ *which give a saddle point for the game when each player can play over his set of all relaxed controls. That is, if* $\sigma(\cdot, t)$ (*resp.* $\tau(\cdot, t)$) *is any other relaxed control for the first* (*resp. second*) *player,*

$$P(\sigma^*, \tau) \geqq P(\sigma^*, \tau^*) \geqq P(\sigma, \tau^*).$$

Proof. Since the system equations are linear in x, it follows that for fixed τ in $\mathscr{S}(Z)$ the mapping $\sigma \to \mu(x(\cdot))$ from $\mathscr{S}(Y)$ to the real numbers is continuous and linear on $\mathscr{S}(Y)$, where $\mathscr{S}(Y)$ is a subset of B_1^*, endowed with the weak star topology. Similarly, for fixed σ in $\mathscr{S}(Y)$ the mapping $\tau \to \mu(x(\cdot))$ from $\mathscr{S}(Z)$ to the real numbers is continuous and linear. Hence for fixed τ in $\mathscr{S}(Z)$ the mapping $\sigma \to P(\sigma, \tau)$ is continuous and linear on $\mathscr{S}(Y)$. Similarly for fixed σ in $\mathscr{S}(Y)$ the mapping $\tau \to P(\sigma, \tau)$ is continuous and linear on $\mathscr{S}(Z)$.

Since $\mathscr{S}(Y)$ and $\mathscr{S}(Z)$ are convex and compact, the existence of a saddle point follows from the general theorem of M. Sion [10]. The results in Sion give inf sup = sup inf, but since we have P continuous in each variable and $\mathscr{S}(Y)$ and $\mathscr{S}(Z)$ compact, it is easy to see that we also have max min = min max and the existence of a saddle point, that is, there are relaxed controls $\sigma^* \in \mathscr{S}(Y)$ and $\tau^* \in \mathscr{S}(Z)$ such that (4.3) is satisfied.

Remark 4.2. We note again that the example of Berkovitz [1] described in § 2.1 is a differential game described by an equation of the form (4.1) and with payoff of the form (4.2). This game, therefore, has a saddle point over the relaxed controls.

In fact, Smoljakov [11] proves by variational methods that a saddle point is obtained over the relaxed controls in the Berkovitz game if u (trying to maximize the payoff) "plays" a constant probability measure $\sigma^* =$ (mass $\frac{1}{2}$ at 0 and mass $\frac{1}{2}$ at 1) throughout the time interval, while v plays the constant control $v(t) = \frac{1}{2}$ throughout the interval.

5. Chattering control saddle points. In this section we examine several special cases of Theorem 4.1; in particular, we consider when the saddle point (σ^*, τ^*) over the relaxed controls can be reduced to a saddle point over classical controls or, perhaps, chattering controls (see Definition 3.4) of a specified degree.

THEOREM 5.1. *Consider a game with dynamics*

$$\dot{x} = A(t)x + B(u, t) + C(v, t),$$

$$x(0) = x_0 \in R^m,$$

and payoff

$$P(u, v) = \mu(x(t)) + \int_0^1 (F(u, t) + G(v, t))\, dt,$$

where

$$B : Y \times [0, 1] \to R^m, \quad C : Z \times [0, 1] \to R^m,$$

$$F : Y \times [0, 1] \to R, \quad G : Z \times [0, 1] \to R$$

are each continuous, and A and μ are as in (4.1). *Then if for each $t \in [0, 1]$ the sets*

$$L_t = \left\{ \begin{pmatrix} B(u, t) \\ F(u, t) \end{pmatrix} u \in Y \right\}, \qquad M_t = \left\{ \begin{pmatrix} C(v, t) \\ G(v, t) \end{pmatrix} v \in Z \right\}$$

in R^{m+1} are convex, there is a saddle point $(u(t), v(t))$ over the classical controls.

Proof. Let $(\sigma^*(t), \tau^*(t))$ be the saddle point over the relaxed controls obtained by Theorem 4.2. We determine $s(t) \in R^{m+1}$ by

$$s_i(t) = \begin{cases} \displaystyle\int_Y B_i(u, t)\, d\sigma^*(t, u), & 1 \leqq i \leqq m, \\ \displaystyle\int_Y F(u, t)\, d\sigma^*(t, u), & i = m + 1. \end{cases}$$

L_t is by assumption convex, and it is also compact as Y is compact and B and F are continuous. Hence it follows that $s(t) \in L_t$ and is a measurable function of $t \in [0, 1]$. By the Filippov implicit function theorem (cf. [5]) there is a measurable function $u^*:[0, 1] \to Y$ such that

$$\begin{pmatrix} B(u^*(t), t) \\ F(u^*(t), t) \end{pmatrix} = s(t) = \int_Y \begin{pmatrix} B(u, t) \\ F(u, t) \end{pmatrix} d\sigma^*(t, u).$$

Similarly we determine $v^*(t)$ such that

$$\begin{pmatrix} C(v^*(t), t) \\ G(v^*(t), t) \end{pmatrix} = \int_Z \begin{pmatrix} C(v, t) \\ F(v, t) \end{pmatrix} d\tau^*(t, v).$$

It is clear that $u^*(t)$ has the same "effect" on the game as the relaxed control $\sigma^*(t)$, and similarly $v^*(t)$ has the same effect as $\tau^*(t)$. Hence it follows easily that

$$P(u^*, v) \geqq P(u^*, v^*) \geqq P(u, v^*)$$

for any other pair of control functions $u(t)$, $v(t)$.

COROLLARY 5.2. *The above result holds when Y and Z are compact and convex and*

$$B(t, u) = B'(t)u, \qquad C(t, v) = C'(t)v,$$

$$F(t, u) = F'(t)u, \qquad G(t, v) = G'(t)v,$$

where $B'(t)$, $C'(t)$, $F'(t)$, $G'(t)$ are each matrix-valued.

The assumption that L_t and M_t are convex for each t may be dropped if we are only interested in establishing a saddle point over the chattering controls of suitable degree.

THEOREM 5.3. *Consider a game with the same form as in Theorem 5.1 except that we do not assume that L_t and M_t are convex. Then there is a saddle point (σ_*, τ_*) over the chattering controls of a degree $m + 2$. If Y and Z are connected, we may take (σ_*, τ_*) of degree $m + 1$.*

Proof. Let $\Gamma(L_t)$ be the closed convex cover of L_t in R^{m+1}; then by a theorem of Carathéodory, if $\xi \in \Gamma(L_t)$,

$$\xi = \sum_{i=1}^{m+2} \alpha_i \xi_i,$$

where $\sum \alpha_i = 1$, $\alpha_i \geqq 0$ and $\xi_i \in L_t$. Now consider the set $\Delta^{m+2} \times Y^{m+2} \times [0, 1]$, where $\Delta^{m+2} \subset R^{m+2}$ is the set of all $\{\alpha_i\}_{i=1}^{m+2}$ such that $\sum \alpha_i = 1$ and $\alpha_i \geqq 0$. The map

$$\theta : \Delta^{m+2} \times Y^{m+2} \times [0, 1] \to R^{m+1}$$

given by

$$\theta(\alpha_1, \cdots, \alpha_{m+2}, u_1, \cdots, u_{m+2}, t) = \sum_{i=1}^{m+2} \alpha_i \begin{pmatrix} B(u_i, t) \\ F(u_i, t) \end{pmatrix}$$

is continuous; hence if (σ^*, τ^*) is the saddle point over relaxed controls, we may apply Fillipov's theorem to deduce the existence of measurable functions

$$\alpha_i:[0,1] \to R, \qquad i = 1, 2, \cdots, m+2,$$
$$v_i:[0,1] \to Y, \qquad i = 1, 2, \cdots, m+2,$$

such that $\alpha_i(t) \geqq 0$, $\sum \alpha_i(t) = 1$, and

$$\sum_{i=1}^{m+2} \alpha_i \begin{pmatrix} B(u_i, t) \\ F(u_j, t) \end{pmatrix} = \int_Y \begin{pmatrix} B(u, t) \\ F(u, t) \end{pmatrix} d\sigma^*(t, u)$$

for all t. The chattering control $\sigma_* = \sum_{i=1}^{m+2} \alpha_i(t)\delta_{u_i(t)}$ has the same "effect" as σ^*. A similar argument to that of Theorem 5.1 concludes the proof.

If Y (and then L_t) is connected, Carathéodory's result may be improved to expressing $\xi \in \Gamma(L_t)$ as

$$\xi = \sum_{i=1}^{m+1} \alpha_i \xi_i$$

and the proof proceeds as before.

This theorem may be extended to cases in which the u- and v-dependence in the dynamics does not split entirely, but becomes "polynomial-like" (in the terminology used in simple game theory). For simplicity we consider only the case where x is a real variable (i.e., we assume $m = 1$), with the dynamics of the game given by

$$\dot{x} = A(t)x + \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij}(t)\varphi_i(u, t)\psi_j(v, t) \tag{5.1}$$

(where we assume that $\varphi_0(u, t) \equiv 1$, $\psi_0(v, t) \equiv 1$), subject to the initial condition $x(0) = 0$; the payoff is also "polynomial-like":

$$P = \lambda(x(t)) + \int_0^1 \sum_{i=0}^{p} \sum_{j=0}^{q} b_{ij}(t)\varphi_i(u, t)\psi_j(\sigma, t)\, dt. \tag{5.2}$$

We assume that

$$\varphi_i: Y \times [0,1] \to R, \qquad i = 0, 1, 2, \cdots, p,$$
$$\psi_j: Z \times [0,1] \to R, \qquad j = 0, 1, 2, \cdots, q,$$
$$\left.\begin{aligned} a_{ij}&:[0,1] \to R \\ b_{ij}&:[0,1] \to R \end{aligned}\right\} \qquad \left\{\begin{aligned} i &= 0, 1, 2, \cdots, p, \\ j &= 0, 1, 2, \cdots, p, \end{aligned}\right.$$

are all continuous. Then we can state the following theorem.

THEOREM 5.4. *The game described by* (5.1) *and* (5.2) *has a saddle point in chattering controls* (σ^*, τ^*) *of degree* $p + 1$ *and* $q + 1$ *respectively. If* Y *and* Z *are connected,* σ^* *and* τ^* *may be taken of degrees* p *and* q.

Proof. Consider the map $\Phi: Y \times [0,1] \to R^p$,

$$\Phi(u, t) = \begin{pmatrix} \varphi_1(u, t) \\ \vdots \\ \varphi_p(u, t) \end{pmatrix}.$$

Let $\Phi_t(Y) = \Phi(Y \times \{t\})$, $0 \leqq t \leqq 1$; then

$$\int_Y \Phi(u, t)\, d\sigma^*(t, u) \in \Gamma(\Phi_t(Y)).$$

Consider the map

$$\theta : \Delta^{p+1} \times Y^{p+1} \times [0, 1] \to R^p,$$

$$\theta(\alpha_1, \cdots, \alpha_{p+1}, u_1, \cdots, u_{p+1}, t) = \sum_{i=1}^{p+1} \alpha_i \Phi(u_i, t).$$

Then by Carathéodory's result,

$$\theta(\Delta^{p+1} \times Y^{p+1} \times \{t\}) = \Gamma(\Phi_t(Y))$$

and θ is continuous. Hence by the Filippov theorem we may determine measurable functions $\alpha_1(t), \cdots, \alpha_{p+1}(t)$, $v_1(t), \cdots, v_{p+1}(t)$ such that

$$\sum_{i=1}^{p+1} \alpha_i(t)\Phi(u_i(t), t) = \int_Y \Phi(u, t)\, d\sigma^*(t, u).$$

We show as before that the chattering control

$$\sigma_* = \sum_{i=1}^{p+1} \alpha_i(t)\delta_{v_i(t)}$$

has the same effect as σ^*. Let $\tau(t)$ be any relaxed control for the second player. Then the trajectory described by (σ_*, τ) is given by

$$\begin{aligned} \dot{x} &= A(t)x + \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij}(t) \int_Z \int_Y \varphi_i(u, t)\psi_j(v, t)\, d\sigma_*(t, u)\, d\tau(t, \sigma) \\ &= A(t)x + \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij}(t) \int_Z \int_Y \varphi_i(u, t)\psi_j(\sigma, t)\, d\sigma^*(t, u)\, d\tau(t, \sigma) \end{aligned}$$

and is therefore the same as the trajectory described by (σ^*, τ); a similar argument may be used on the payoff. We determine τ_* for the second player and the result follows. Once again if Y and Z are connected, we may reduce the $(p + 1)$-degree to p, as in Theorem 5.3.

A similar theorem may be stated in R^m, where for each coordinate the dynamical equation is "polynomial-like." We conclude by observing that the Berkovitz game (see § 2.1) may be analyzed by Theorem 5.4. Thus if

$$\dot{x} = (u - v)^2 = u^2 - 2uv + v^2,$$

we may take $\varphi_1(u, t) = u$, $\varphi_2(u, t) = u^2$, while $\psi_1(v, t) = v$, $\psi_2(v, t) = v^2$. As the payoff

$$p = \int_0^1 x(t)\, dt$$

does not depend on u and v, these are the only functions required. Thus $p = q = 2$, and Y and Z are connected. We may therefore expect a saddle point over chattering controls of order 2 (see Remarks 4.2).

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