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Embedding l^n_∞-cubes in finite-dimensional 1-subsymmetric spaces

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ABSTRACT. In this paper we prove that the l^n_∞-cube can be $(1+\varepsilon)$-embedded into any 1-subsymmetric $C(\varepsilon)n$-dimensional normed space.

Marcus and Pisier in [5] iniciated the study of the geometry of finite metric spaces. Bourgain, Milman and Wolfson introduced a new notion of metric type and developed the non-linear theory of Banach spaces (see [2] and [7]). All these themes have been studied more intensively over the last years.

Johnson and Lindenstrauss proved that, given N points in the Euclidean space, they can be $(1+\varepsilon)$-embedded into a subspace of dimension $K(\varepsilon)$ *log N* (see lemma 1 in [3]). The method they use is based in the isoperimetric inequality of P. Levy. Another proof of the same fact was given by Pisier, using Gaussian processes ([8]). Bourgain, Milman and Wolfson, in the paper before mentioned, studied the l^n_p-cubes and their $(1+\varepsilon)$-embeddings in finite metric spaces. More recently, Schechtman obtained estimates for $(1+\ \varepsilon)$-embeddings of finite subsets of L^r into l^n_p-spaces (see [9]).

In this paper we will study $(1+\varepsilon)$-embeddings of the l^n_∞-cube in finite-dimensional subsymmetric spaces. The result we prove for the l_p-case $1\ \leq p\leq$ 2, can be deduced from Johnson and Lindenstrauss's lemma plus a refinement of Dvoretzky's theorem (see for instance [7], Theorem 3.9), but, as far as we know, it is new in other cases. The method we use is in essence of probabilistic nature and the main tool is a well known deviation inequality.

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We begin by recalling some definitions. Given two metric spaces (M,d) and (M',d'), we say that (M,d) $(1+\varepsilon)$-embeds into (M',d') if there is a one-to-one map f from M into M' such that $\|f\|_{Lip}\|f^{-1}\|_{Lip} \leq 1+\varepsilon$, where

$$\|f\|_{Lip} = sup_{x\neq y} \frac{d'(f(x),f(y))}{d(x,y)}$$

The l^n_∞-cube is the metric space (C^n_2, ρ_∞) where $C^n_2 = \{-1, +1\}^n$ and $\rho_\infty(\varepsilon,\varepsilon') = \max\limits_{1\leq i\leq n} |\varepsilon_i - \varepsilon'_i|$, for any par of elements ε,ε' belonging to C^n_2.
Since $\rho_\infty(\varepsilon,\varepsilon')=2$, whenever $\varepsilon\neq\varepsilon'$, the problem we are considering may be related with the sphere-packing problem, i.e., how many balls, with radius $\frac{1-\varepsilon}{2}$, can be packed into the unit ball of a finite dimensional Banach space, in an asymptotic way? (See the paper by Ball [1] for infinite dimensional sphere-packing problem)

In the sequel E_n will denote a finite-dimensional Banach space with a 1-subsymmetric normalized basis $\{e_1,...,e_n\}$. We use standard Banach space theory notation as may be found in [4].

The theorem we will prove here is the following

Theorem.–*There exists a numerical constant $C>0$ such that, for any $\varepsilon>0$ we can find a subset of N points $\{x_1,...,x_N\}$ in E_n verifying*

$$1-\varepsilon\leq\|x_i-x_j\|\leq 1+\varepsilon, \qquad i\neq j$$

provided that

$$n > \frac{C}{\varepsilon^2} \log N$$

Proof.– Let ε a given positive number verifying $0<\varepsilon<1$. Let n be a natural number to be determined after. Consider the function ψ defined by

$$\psi\left(\frac{m}{n}\right) = \frac{\|\sum_1^m e_i\|}{\|\sum_1^n e_i\|}, \qquad \text{if } 0\leq m\leq n,$$

and by a nondecreasing continuous extension in the other points of the unit interval [0,1]. The function ψ depends on n, but in some particular cases we can choose the same fixed function for all n. This happens, for instance, in the l^p-spaces where we may define $\psi(t) = t^{1/p}$, $0 \leq t \leq 1$.

We note that function ψ verifies $\psi(0) = 0$, $\psi(1) = 1$ and

$$\psi(2^{-j}) \geq 2^{-(j+1)}, j = 0, 1, \qquad (*)$$

Indeed, if $\frac{m}{n} \leq \frac{1}{2^j} \leq \frac{m+1}{n}$ we have

$$\|\sum_1^n e_i\| \leq \|\sum_1^{2^j(m+1)} e_i\| \leq 2^j \|\sum_1^{m+1} e_i\| \leq 2^{j+1}\|\sum_1^m e_i\|$$

In general we don't know the behaviour of the derivative of ψ in [0,1], but, by averaging in the interval [1/4, 1/2], given $\delta = \varepsilon/128$

$$\int_{1/4}^{1/2} [\psi(t+\delta)-\psi(t-\delta)]dt = \int_{1/4+\delta}^{1/2+\delta} \psi(t) - \int_{1/4-\delta}^{1/2-\delta} \psi(t) \leq \int_{1/2-\delta}^{1/2+\delta} \psi \leq 2\delta$$

and then, we can pick a number a in the interval (1/4, 1/2) such that $\psi(a+\delta)-\psi(a-\psi) \leq 8\delta$. Hence, for every $x,y \in [a-\delta, a+\delta]$, we have

$$|\psi(x)-\psi(y)| \leq 8\delta = \varepsilon/16. \qquad (**)$$

Let k be the integer part of $2an$, ($k \leq 2an < k+1$). Then, by (*)

$$\psi(\frac{k}{2n}) \geq \psi(\frac{1}{8}) \geq \frac{1}{16} \qquad \text{if } n \geq 4 \qquad (***)$$

We now define X a random E_n-valued vector by $X(\omega) = \sum_1^k \varepsilon_i(\omega)e_i$, where $\{\varepsilon_i\}_1^k$ is an i.i.d. sequence of symmetric $\{+1,-1\}$-valued random variables defined in some probality space. If Y is another i.i.d. copy of X, it is clear that the two random variables $\|X - Y\|$ and $2\|\sum_1^k \eta_i e_i\|$ (where $\{\eta_i\}_1^k$ is an i.i.d. sequence of random variables uniformly distributed on the set $\{0,1\}$) have the same distribution. Then, if we denote $\lambda(n) = \|\sum_1^n e_i\|$, the 1-subsymmetry of the norm implies that the distribution of the random variables $\psi(\frac{1}{n}\sum_1^k \eta_i)$ and $\|\frac{1}{2\lambda(n)}(X - Y)\|$ also coincides.

Since $E\,(\frac{1}{n}\sum_1^k \eta_i)=\frac{k}{2n}$ we will compute the probability of deviation of $\|\frac{1}{2\lambda(n)}(X-Y)\|$ from $\psi(\frac{k}{2n})$.

$$A=P\{\omega;|\|\frac{1}{2\lambda(n)}(X-Y)\|-\psi(\frac{k}{2n})|>\varepsilon\psi(\frac{k}{2n})\}\le^{(if\ n\geq 4)}$$

$$\le P\{\omega;|\psi(\frac{1}{n}\sum_1^k \eta_i)-\psi(\frac{k}{2n})|>\varepsilon\frac{1}{16}\}\ by\ (***).$$

Note that $a-\frac{1}{2n}\le\frac{k}{2n}$, and so $\frac{k}{2n}\in[a-\frac{\delta}{2},a+\frac{\delta}{2}]$ if $n>\frac{128}{\varepsilon}$. Thus, $|\frac{1}{n}\sum_1^k\eta_i-\frac{k}{2n}|\le\frac{\delta}{2}$ implies

$$|\psi(\frac{1}{n}\sum_1^k\eta_i)-\psi(\frac{k}{2n})|<\varepsilon\frac{1}{16}\ \text{by (**)}.$$

Since

$$\frac{1}{n}\sum_1^k\eta_i-\frac{k}{2n}\overset{D}{=}\frac{1}{2n}\sum_1^k\varepsilon_i$$

we have

$$A\le P\{\omega;|\psi(\frac{1}{n}\sum_1^k\eta_i)-\psi(\frac{k}{2n})|>\varepsilon\frac{1}{16}\}\le$$

$$\le P\{\omega;|\frac{1}{n}\sum_1^k\varepsilon_i(\omega)|>\varepsilon\frac{1}{128}\}\le 2\,exp\,(-\frac{\varepsilon^2n^2}{Ck})\le$$

$$\le 2\,exp\,(-\frac{\varepsilon^2 n}{C})$$

where C is a numerical constant. In this last step we have used the well known probabilistic deviation inequality,

$$P\{\omega;\sum_1^m\varepsilon_i(\omega)>\lambda\sqrt{m}\}\le exp\,(-\frac{\lambda^2}{2})\qquad \lambda>0, m\in\mathbb{N}$$

(see, for instance, [6] Theorem III.15).

Consider now a natural number N such that $n>\frac{2C}{\varepsilon^2}\log N$. If $\{X_i\}_1^N$ is an i.i.d. sequence of copies of X, then

$$P\{\omega;|\| \frac{1}{2\lambda(n)}(X_i-X_j)\|-\psi(\frac{k}{2n})|\leq \varepsilon\psi(\frac{k}{2n}), \text{for all } i\neq j\}\geq$$

$$\geq 1-\binom{N}{2}2exp(-\frac{\varepsilon^2 n}{2C})>0$$

Hence, there exists ω in the probability space, such that the corresponding points

$$x_i=\frac{X_i(\omega)}{2\lambda(n)\psi(\frac{k}{2n})} \qquad 1\leq i\leq N$$

satisfy the conclusion of the theorem.

Corollary.- *The l^n_∞-cube is $(1+\varepsilon)$-embedded in any finite-dimensional 1-subsymmetric space E, provided that dim E> $\frac{C}{\varepsilon^2}$ n. (C is an absolute constant)*

Remarks.-

i) Since

$$\|\sum_1^k e_i\|\leq 2\|\sum_1^{[k/2]} e_i\|+1$$

it is easy to prove that $\|x_i\|\leq\frac{3}{2}$ $\quad 1\leq i\leq N.$

ii) The asymptotic estimate $n>K\ log\ N$ is essentially best possible. Indeed, in a ball of radius r *of* E_n the number N of balls of radius $r/2$ we can pack into (with disjoint interior) is given by

$$r^n vol(B_1)\geq N(\frac{r}{2})^n vol(B_1)$$

(vol (B_1) is the n-dimensional volume of the unit ball)

iii) When $E=l_p$, $1\leq p<\infty$, we can improve slightly the numerical constant. Indeed, by taking $a=1/2$ and using the mean value theorem we obtain the following:

a) If $\varepsilon < \frac{2}{p2^{1/p}}$ then $n > \frac{C}{\varepsilon^2 p^2} \log N$

b) If $\varepsilon > \frac{1}{p2^{1/p}}$ then $n > C \log. N$

(C is a numerical constant). These expressions say that $p = \infty$ is the best possible situation, because, an isometric embedding ($\varepsilon = 0$) is possible in this case.

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