SURJECTIVE ISOMETRIES ON REARRANGEMENT-INVARIANT SPACES

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1. Introduction

THE main result of this paper is the following theorem, which combines the statements of Theorems 6.4 and 7.2 below. We denote Lebesgue measure on $[0,1]$ by λ use the term rearrangement-invariant Banach function space in the sense of Lindenstrauss–Tzafriri [**21**].

THEOREM 1.1. *Let X be a (real) rearrangement-invariant Banach function space on $[0, 1]$. Suppose X is not (isometrically) equal to $L_2[0, 1]$. Let T: $X \to X$ be a surjective isometry. Then*

(1) *There exists a nonvanishing Borel function a: $[0,1] \to \mathbf{R}$ and an invertible Borel map σ: $[0,1] \to [0,1]$ such that, for any Borel set $B \subset [0,1]$, we have $\lambda(\sigma^{-1}B)=0$ if and only if $\lambda(B)=0$ and so that $Tf(s)=a(s)f(\sigma(s))$ a.e. for any $f \in X$.*

(2) *If X is not equal to L_p for some $1 \leq p \leq \infty$ up to renorming, then $|a|=1$ a.e. and σ is measure-preserving.*

The first part of this theorem is known for the case of spaces of complex functions and is due to Zaidenberg [**34**], [**35**]; we discuss the relationship between our result and Zaidenberg's below. The second part was announced by Zaidenberg [34] (for the complex case).

The study of isometries on classical function spaces goes back to Banach [**1**] who proved that the isometries of $L_p[0, 1]$ are disjointness-preserving when $p \neq 2$ (see [**1**] p. 175). Lamperti [**20**] later characterized the isometries on L_p. Since then there has developed an extensive literature on isometries of particular function spaces: see [**4**], [**5**], [**6**] and [**15**] for example.

In the case of (not necessarily rearrangement-invariant) complex function spaces, a technique developed by Lumer [**22**], [**23**], [**24**] has proved particularly effective. This technique was used by Lumer [**23**], [**24**] to study isometries on reflexive Orlicz spaces and later by Zaidenberg [**34**] and [**35**] to study isometries on general r.i. spaces, X. The idea is to characterize first the hermitian operators H: $X \to X$. H is hermitian if $\exp(itH)$ is an isometry for every real t. One shows that hermitian

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operators are simply multiplication operators by real functions, unless $X = L_2$. Then, if U is a surjective isometry on X we have that UHU^{-1} is hermitian for every hermitian H. Combining these ideas leads to Theorem 1.1 (1) in the complex case. See also, for example, **[8]**, **[9]**, **[12]** and **[33]**. For a fuller discussion of the existing literature we refer the reader to the forthcoming survey of Fleming and Jamison **[10]**.

This line of argument simply does not work for real spaces, and most of the known results use geometric techniques (e.g. extreme point arguments) for special spaces. In this paper, we follow a line of reasoning which is distantly related to the Lumer technique. We use the notion of a numerically positive operator **[28]**; this is an operator T such that $\|\exp(-tT)\| \leqslant 1$ for all $t \geqslant 0$ (see also **[25]** where $-T$ is called dissipative). Unfortunately, this is far too weak a notion to allow us to characterize such operators on an r.i. space, but by studying rank-one numerically positive operators and using results of Flinn (see **[28]**) we are able to prove a representation theorem for surjective isometries (Proposition 6.3), which is a partial step towards our main result. Then by a probabilistic technique we obtain Theorem 6.4 which is equivalent to Theorem 1.1 (1). Finally in Theorem 7.2 we show that if X is not L_p up to renorming then the representation in Theorem 6.4 can be further narrowed to the trivial case as in Theorem 1.1 (2).

Some remarks on the nature of our results are in order. First notice that we must restrict ourselves (as do Lumer and Zaidenberg) to surjective isometries. Many of the special results quoted above apply equally to isometries which are not surjective. Secondly, it will be seen that there appear to be obstacles to extending the main result to r.i. spaces on $[0, \infty)$, (see, for example, **[13]**, **[14]**). However our results do apply equally to separable and nonseparable r.i. spaces on $[0, 1]$. The proof can be simplified a little in the separable case and we indicate such simplifications at various points in the paper.

2. Introductory remarks on Köthe function spaces

Let us suppose that Ω is a Polish space and that μ is a σ-finite Borel measure on Ω. We use the term Köthe space in the sense of **[21]** p. 28. Thus a Köthe function space X on (Ω, μ) is a Banach space of (equivalence classes of) locally integrable Borel functions f on Ω such that:

(1) If $|f| \leqslant |g|$ a.e. and $g \in X$ then $f \in X$ with $\|f\|_X \leqslant \|g\|_X$.

(2) If A is a Borel set of finite measure then $\chi_A \in X$.

We say that X is *order-continuous* if whenever $f_n \in X$ with $f_n \downarrow 0$ a.e. then $\|f_n\|_X \downarrow 0$. X has the *Fatou property* if whenever $0 \leqslant f_n \in X$ with $\sup \|f_n\|_X < \infty$ and $f_n \uparrow f$ a.e. then $f \in X$ with $\|f\|_X = \sup \|f_n\|_X$.

The Köthe dual of X is denoted X'; thus X' is the Köthe space of all g such that $\int |f|\,|g|\,d\mu < \infty$ for every $f \in X$ equipped with the norm $\|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int |f|\,|g|\,d\mu$. Then X' can be regarded as a closed subspace of the dual X^* of X. If X is order-continuous then $X' = X^*$; if X has the Fatou property then X' is a norming subspace of X^*.

A rearrangement-invariant function space (r.i. space) is a Köthe function space on $([0,1], \lambda)$ where λ is Lebesgue measure which satisfies the conditions:

(1) Either X is order-continuous or X has the Fatou property.

(2) If τ: $[0,1] \to [0,1]$ is any measure-preserving invertible Borel automorphism then $f \in X$ if and only if $f \circ \tau \in X$ and $\|f\|_X = \|f \circ \tau\|_X$.

(3) $\|\chi_{[0,1]}\|_X = 1$.

In this section we will make some introductory remarks about operators and isometries on Köthe function spaces. Let us suppose that X is a Köthe function space on (Ω, μ). We first consider those operators T: $X \to X$ which are continuous for the topology $\sigma(X, X')$. This is equivalent to requiring the existence of adjoint T': $X' \to X'$. Of course, if X is order-continuous, every operator T: $X \to X$ is $\sigma(X, X')$-continuous.

Our first result is well-known, but we know no explicit reference.

PROPOSITION 2.1. *Let X be a Köthe function space on (Ω, μ). The following conditions on T: $X \to X$ are equivalent*:

(1) *T is $\sigma(X, X')$-continuous.*

(2) *If $0 \leq f_n \in X$ and $f_n \uparrow f$ a.e. then $\lim_{n\to\infty} \int |h|\,|Tf_n - Tf|\,d\mu = 0$ for every $h \in X'$.*

(3) *If $0 \leq g \in X$ and $|f_n| \leq g$ with $f_n \to f$ a.e. then $\lim_{n\to\infty} \int |h|\,|Tf_n - Tf|\,d\mu = 0$ for every $h \in X'$.*

Remark. Note that (2) says that T: $X \to L_1(|h|\,d\mu)$ is an order-continuous operator for every $h \in X'$; see Weis **[31]**.

Proof. (1)$\to$(3): Consider the operator S: $L_\infty(\mu) \to L_1(\mu)$ defined by $S\phi = h(T\phi g)$. Then S is $\sigma(L_\infty, L_1) \to \sigma(L_1, L_\infty)$-continuous and hence weakly compact. Now we may choose $\phi_n \to \phi$ a.e. such that $\phi_n g = f_n$ and $\phi g = f$. Consider the adjoint S': $L_\infty \to L_1$. Then $S'(B_{L_\infty})$ is weakly compact and hence uniformly integrable in L_1 (**[32]**, p. 137). Thus

$$\lim_{n\to\infty} \sup_{\|h\|_\infty \leq 1} \left| \int \phi_n S'h\,d\mu \right| = 0$$

which quickly gives (3).

(3)$\to$(2): Obvious.

(2)$\to$(1): We must show that the adjoint T^*: $X^*\to X^*$ maps X' into X'. Consider any $h\in X'$. It follows quickly that the set-function $A\to\int hT_{\chi_A}\,d\mu$ is countably additive when restricted to any Borel set A of finite measure. Thus there exists $\phi\in L_0$ so that $\int h(Tf)\,d\mu=\int f\phi\,d\mu$ for any simple function f supported on a set of finite measure. Now for general positive $f\in X$ we find a sequence f_n of such simple functions so that $f_n\uparrow f$ a.e. and then

$$\int h(Tf)=\lim_{n\to\infty}\int h(Tf_n)\,d\mu=\lim_{n\to\infty}\int \phi f_n\,d\mu=\int \phi f\,d\mu.$$

We conclude that $T^*h\in\phi\in X'$.

An operator T: $X\to X$ will be called *elementary* if there is a Borel function a and a Borel map σ: $\Omega\to\Omega$ such that $Tf(s)=a(s)f(\sigma(s))$ a.e. for every $f\in X$. Observe that a necessary condition on a and σ is that if B is a Borel set with $\mu(B)=0$ then $\mu(\sigma^{-1}B\cap\{|a|>0\})=0$. T is called *disjointness-preserving* if $\min(|f|,|g|)=0$ a.e. implies $\min(|Tf|,|Tg|)=0$, a.e.

LEMMA 2.2. *T is elementary if and only if T is disjointness preserving and $\sigma(X,X')$-continuous.*

Proof. It is trivial that an elementary operator is disjointness-preserving and continuous for $\sigma(X,X')$. For the converse we check that if $0\leqslant f\leqslant g\in X$ then $0\leqslant|Tf|\leqslant|Tg|$. It suffices by a density argument to establish this when f, g are both countably simple. Pick a maximal family of Borel sets $\{A_i\colon i\in I\}$ of finite positive measure such that $|T(f\chi_{A_i})|\leqslant|T(g\chi_{A_i})|$ a.e. This family is countable and so its union B is a Borel set. If A is a Borel set of positive measure disjoint from B then we may find a further set of positive measure $A'\subset A$ so that $f\chi_{A'}=\alpha g\chi_{A'}$ for some $0\leqslant\alpha\leqslant1$; thus $|Tf_{\chi_{A'}}|\leqslant|Tg_{\chi_{A'}}|$ contrary to our maximality assumption. Hence $\mu(\Omega\backslash B)=0$. Now by Proposition 2.1, $Tf=\sum_{i\in I}T(f\chi_{A_i})$ and $Tg=\sum_{i\in I}T(f\chi_{A_i})$ in $L_1(h\,d\mu)$ where h is any strictly positive function in X'. Thus $|Tf|\leqslant|Tg|$. It follows that T is regular and order-continuous as an operator from X into $L_1(h\,d\mu)$. As shown by Weis **[31]** this means that T is elementary.

LEMMA 2.3. *Suppose Ω is uncountable. If X is a Köthe function space on (Ω,μ) and T: $X\to X$ is an invertible elementary operator then T^{-1} is elementary and T can be represented in the form $Tf(s)=a(s)f(\sigma(s))$ where a is a nonvanishing Borel function and σ: $\Omega\to\Omega$ is an invertible Borel map such that $\mu(B)>0$ if and only if $\mu(\sigma^{-1}B)>0$.*

Proof. As noted above, T can be represented in the form $Tf = af \circ \sigma_0$ where a is a Borel function and σ_0: $\Omega \to \Omega$ is a Borel map. Since T is onto it is clear that a can vanish only on a set of measure zero and so we may assume that it is nonvanishing. Then for any f, supp $Tf = \sigma_0^{-1}$ (supp f.) Thus T^{-1} is disjointness-preserving. Now suppose $0 \leqslant g_n \uparrow g$ a.e.; we will verify condition (2) of Proposition 2.1 for T^{-1}. We can suppose $g_n = af_n \circ \sigma_0$ and $g = af \circ \sigma_0$. Then $T^{-1}g_n = f_n$; we will show that, almost everywhere, we have both $f_n(\omega) \to f(\omega)$ and $|f_n(\omega)| \leqslant |f(\omega)|$ for all n. Once this is established then the Dominated Convergence Theorem establishes Proposition 2.1 (2). Suppose E is any Borel set of finite measure such that for every $\omega \in E$ we have $\sup_n |f_n(\omega)| > |f(\omega)|$ or $f_n(\omega)$ does not converge to $f(\omega)$. Then $\sigma_0^{-1}E$ is contained in the set where $g_n(\omega)$ fails to converge monotonically to $g(\omega)$ and so has measure zero. This implies that $T\chi_E = 0$ (a.e.) and so $\mu(E) = 0$. Hence T^{-1} satisfies (2) of Proposition 2.1 and hence is $\sigma(X, X')$-continuous. It follows that T^{-1} is elementary and so can be represented in the form $T^{-1}f = bf \circ \tau$ where b is a nonvanishing Borel function and τ: $\Omega \to \Omega$ is a Borel map. Thus the identity map can be written in the form $f \to a(b \circ \sigma_0)f \circ \tau \circ \sigma_0$ and so $\tau\sigma_0 s = s$ a.e.; similarly $\sigma_0 \tau s = s$ a.e.

Let $E = \{s\colon \tau\sigma_0(s) = s\}$. By Lusin's theorem there is an increasing sequence of compact subsets K_n of E so that σ_0 is continuous on K_n and $\mu(\Omega \backslash K) = 0$ where $K = \cup K_n$. Then σ_0 is a Borel isomorphism of K onto $\sigma_0(K)$ and both sets are F_σ's. Let F be an uncountable compact subset of K of measure zero. Then we define $\sigma = \sigma_0$ on $K \backslash F$ and $\sigma = \rho$ on $F \cup (\Omega \backslash K)$ where ρ is any Borel isomorphism between the two uncountable Borel sets $F \cup (\Omega \backslash K)$ and $\sigma_0(F) \cup (\Omega \backslash \sigma(K))$. Then $\sigma = \sigma_0$ a.e. and is a Borel automorphism. We thus can replace σ_0 by σ and assume that σ is a Borel automorphism. Finally to show the measure properties of σ note that $\mu(B) = 0$ if and only if $T\chi_B = 0$ a.e. i.e. if and only if $\mu(\sigma^{-1}B) = 0$.

LEMMA 2.4. *If T: $X \to X$ is an invertible elementary operator then T': $X' \to X'$ is an elementary operator.*

Proof. We can represent T in the form $Tf = af \circ \sigma$ where a is nonvanishing and σ is an invertible Borel map with $\mu(\sigma^{-1}B) = 0$ if and only if $\mu(B) = 0$. Let w be the Radon–Nikodym derivative of the σ-finite measure $\nu(B) = \mu(\sigma^{-1}B)$. Then for $f \in X$, $g \in X'$ we have

$$\begin{aligned}\int (T'g)f \, d\mu &= \int gaf \circ \sigma \, d\mu \\ &= \int (g \circ \sigma^{-1})(a \circ \sigma^{-1})f \, d\nu \\ &= \int (g \circ \sigma^{-1})(a \circ \sigma^{-1})fw \, d\mu.\end{aligned}$$

Thus $T'g = a \circ \sigma^{-1} wg \circ \sigma^{-1}$ a.e. and thus is elementary.

Of course if X is order-continuous every operator $T: X \to X$ is $\sigma(X, X')$-continuous. However, for isometries we can prove a similar result even without this assumption.

PROPOSITION 2.5. *Let X be a Köthe function space with the Fatou property and suppose $T: X \to X$ is a surjective isometry. Then T is $\sigma(X, X')$-continuous.*

Proof. We will use ideas developed in [**11**]. We recall that the ball topology on X is the weakest topology b_X for which every closed ball (with any center and radius) is closed. Then $T: (X, b_X) \to (X, b_X)$ is continuous. The topology is not a Hausdorff topology, but ([**11**], Theorem 3.3) its restriction to any absolutely convex Rosenthal set is Hausdorff. Here a set is a Rosenthal set if every sequence contains a weakly Cauchy subsequence.

Now suppose h is any strictly positive function in X'.

We show that if $0 \leq f_n \in X$ and $f_n \uparrow f$ a.e. where $f \in X$ then Tf_n converges to Tf in $L_1(h\, d\mu)$. In fact, setting $f_0 = 0$, $\sum_{n \geq 1} (f_n - f_{n-1})$ is weakly unconditionally Cauchy in X and so $\sum_{n \geq 1} (Tf_n - Tf_{n-1})$ is weakly unconditionally Cauchy in X. Thus $\sum_{n \geq 1} (Tf_n - Tf_{n-1})$ converges unconditionally to some g in $L_1(h\, d\mu)$.

In particular Tf_n converges in $L_1(h\, d\mu)$ to g. Since (Tf_n) is bounded in X and X has the Fatou property (i.e. B_X is L_0-closed) it follows that $g \in X$.

Now consider the absolutely convex hull of (Tf_n) together with the points g and Tf. This is a Rosenthal set. Since b_X is weaker than the L_0-topology it follows that Tf_n converges to g in b_X. However f_n converges to f in b_X and Tf_n also converges to Tf. We conclude that $Tf = g$ and so (Tf_n) converges to Tf in $L_1(h\, d\mu)$.

We now can conclude the argument by appealing to Proposition 2.1.

Remark. This result will only be needed to prove the main result for nonseparable r.i. spaces. The reader who is only concerned with the separable case can observe that if X is separable it must be order-continuous and then any isometry T is $\sigma(X, X')$-continuous. Also its adjoint $T': X' \to X'$ can be shown directly to be $\sigma(X', X'')$ continuous by identifying X'' as the sequential closure of X in X^{**}.

3. Flinn elements

Let X be a real Banach space and suppose $T: X \to X$ is a linear operator. We define $\Pi(X)$ to be the subset of $X \times X^*$ of all (x, x^*) such that $\|x\| = \|x^*\| = x^*(x) = 1$. We recall that an operator $T\colon X \to X$ is *numerically positive* (Rosenthal [**28**]) if $x^*(Tx) \geq 0$ whenever $(x, x^*) \in \Pi(X)$. This is equivalent to requiring the slightly weaker condition that given x with $\|x\| = 1$ there exists x^* so that $(x, x^*) \in \Pi(X)$ and $x^*(Tx) \geq 0$ (see Lumer [**22**], [**3**]). By results of Lumer [**22**] and Lumer and Phillips [**25**] (see also [**3**]) it is equivalent to the requirement that $\|\exp(-\alpha T)\| \leq 1$ for $\alpha \geq 0$. In the case when T is a projection it is easily seen that T is numerically positive if and only if $\|I - T\| = 1$.

We next introduce an idea which is a real analogue of the notion of hermitian elements [**19**]. Based on ideas of P. H. Flinn [**28**] we say that $u \in X$ is a *Flinn element* if there is a numerically positive projection $P\colon X \to [u]$. The set of Flinn elements will be denoted $\mathscr{F}(X)$. Note that $0 \in \mathscr{F}(X)$ and that $u \in \mathscr{F}(X)$ and $\alpha \in \mathbf{R}$ imply $\alpha u \in \mathscr{F}(X)$. If $0 \neq u \in \mathscr{F}(X)$ then there exists $f \in X^*$ so that $f \otimes u$ is a numerically positive projection onto $[u]$. We say then that (u, f) is a *Flinn pair*. Clearly (u, f) is a Flinn pair if and only if $f(u) = 1$ and $f(x)x^*(u) \geq 0$ for $(x, x^*) \in \Pi(X)$.

PROPOSITION 3.1. *The set $\mathscr{F}(X)$ is closed.*

Proof. Suppose $u_n \in \mathscr{F}(X)$ and $\lim \|u_n - u\| = 0$. It suffices to consider the case when $\|u_n\| \neq 0$ and $\|u\| \neq 0$. Then there exist $f_n \in X^*$ so that $f_n \otimes u_n$ is a numerically positive projection. Thus $\|f_n \otimes u_n\| = \|f_n\|\,\|u_n\| \leq 2$. Thus $\|f_n\| \leq 2 \sup(1/\|u_n\|)$. By Alaoglu's theorem (f_n) has a weak*-cluster point f and clearly (u, f) is a Flinn pair.

The next proposition is trivial, but we record it for future use.

PROPOSITION 3.2. *Suppose $U\colon X \to Y$ is a surjective isometry. Then $U(\mathscr{F}(X)) = \mathscr{F}(Y)$; furthermore if (u, f) is a Flinn pair then $(U(u), (U^*)^{-1}f)$ is a Flinn pair.*

The next theorem is due to Flinn (see [**28**], Theorem 1.1).

THEOREM 3.3. *Let X be a Banach space and π be a contractive projection on X with range Y. Suppose (u, f) is a Flinn pair in X. Suppose $f \notin Y^\perp$. Then $\pi(u) \in \mathscr{F}(Y)$.*

Proof. Let g be the restriction of f to Y. We may assume $\pi(u) \neq 0$. Then let $S = g \otimes \pi u$ be a rank one operator on Y. If $(y, y^*) \in \Pi(Y)$ then $(y, y^* \circ \pi) \in \Pi(X)$. Now $f(y)y^*(\pi u) \geq 0$ and so S is numerically positive. But $S^2 = \beta S$ where $\beta = g(\pi u)$. By considering $y = \pi u / \|\pi u\|$ and choosing y^* to norm πu it is immediately clear that $\beta \geq 0$. If $\beta = 0$ then $\exp(-\alpha S) = I - \alpha S$; since by assumption S is non-zero this contradicts

$\|\exp(-\alpha S)\| \leqslant 1$ for all $\alpha \geqslant 0$. Hence $\beta > 0$ and $(\pi u, \beta^{-1} g)$ is a Flinn pair.

4. Flinn elements in lattices

Now suppose that Ω is a Polish space and that μ is a σ-finite Borel measure on Ω.

PROPOSITION 4.1. *Let X be an order-continuous Köthe function space on Ω.*
(a) Suppose that (u, f) is a Flinn pair with $u \in X$ and $f \in X' = X^*$. Then $fu \geqslant 0$ a.e.
(b) Suppose $u \in \mathcal{F}(X)$. Then there exists $f \geqslant 0$ such that $(|u|, f)$ is a Flinn pair.

Proof. Let A be a Borel subset of the set $\{f > 0\} \cap \{u < 0\}$ of finite measure. Suppose $\mu(A) > 0$ and let $x = \chi_A / \|\chi_A\|$. Pick x^* so that $(x, x^*) \in \Pi(X)$ and $\operatorname{supp} x^* \subset A$. Then $x^* \geqslant 0$ (a.e.) and $\int u x^* \, d\mu < 0$ but $\int f x \, d\mu > 0$. This contradiction shows that $\mu(A) = 0$ and so the set $\{f > 0\} \cap \{u < 0\}$ has measure zero. Similar reasoning shows that the set $\{f < 0\} \cap \{u > 0\}$ has measure zero.

(b) There is an isometry of X onto X which carries u to $|u|$ so that $|u| \in \mathcal{F}(X)$ by Proposition 3.2. Now suppose $(|u|, f)$ is a Flinn element. Let $A = \{f < 0\}$ and consider the isometry $Ux = x - 2\chi_A x$. Clearly by (a), $U(|u|) = |u|$ and of course $(U^*)^{-1} f = |f|$ so that $(|u|, |f|)$ is a Flinn pair.

LEMMA 4.2. *Suppose μ is nonatomic and suppose $f, g \in L_1(\mu)$ with $\int |f| \, d\mu > 0$ satisfy the criterion that*

$$\left(\int hf \, d\mu\right)\left(\int hg \, d\mu\right) \geqslant 0$$

whenever $|h| = 1$ a.e. Then there is a nonnegative constant c so that $g = cf$ a.e.

Proof. Consider the subset Γ of $\mathbf{R}^2$ of all (a, b) such that for some $h \in L_\infty(\mu)$ with $|h| = 1$ a.e. we have $\int hf \, d\mu = a$ and $\int hg \, d\mu = b$. Then it is an immediate consequence of Liapunoff's theorem **[29]** that Γ is closed and convex. However $\Gamma = -\Gamma$ and the criterion is that Γ is contained entirely in the union of the first and third quadrants. This trivially implies that Γ is contained in a line through the origin; the hypothesis on f implies this line is not the y-axis and so we deduce the existence of $c \geqslant 0$ so that $\int hg \, d\mu = c \int hf \, d\mu$ for all such h and the lemma follows.

We now establish the analogue of Theorem 6.5 of **[19]**.

THEOREM 4.3. *Suppose μ is nonatomic and suppose X is an order-continuous Köthe function space on (Ω, μ). Then $u \in X$ is a Flinn element if and only if there is a nonnegative function $w \in L_0(\mu)$ with* $\operatorname{supp} w = \operatorname{supp} u = B$, *so that:*

(a) *If $x \in X(B)$ then* $\|x\| = \left(\int |x|^2 w \, d\mu\right)^{\frac{1}{2}}$.
and
(b) *If $v \in X(\Omega \backslash B)$ and $x, y \in X(B)$ satisfy $\|x\| = \|y\|$ then* $\|v+x\| = \|v+y\|$.

Proof. Assume first that $0 \neq u \in \mathscr{F}(X)$. We can assume there exists $f \in X^*$ so that (u, f) is a Flinn pair. Suppose first that $(x, x^*) \in \Pi(X)$. Then if $|h| = 1$ a.e. we also have $(hx, hx^*) \in \Pi(X)$ and so $\left(\int uhx^* \, d\mu\right)\left(\int fhx \, d\mu\right) \geqslant 0$. By Lemma 4.2, there is a constant $k_x > 0$ so that $ux^* = k_x fx$ almost everywhere. It follows immediately that we must have $f\chi_{\Omega \backslash B} = 0$ almost everywhere. Thus we can define a function w by $w = f/u$ on B and $w = 0$ otherwise. Then if $(x, x^*) \in \Pi(X)$ we have $x^*\chi_B = k_x wz\chi_B$.

Next let us suppose that e_1, $e_2 \in X(B)$ satisfy the conditions $\int e_1^2 w \, d\mu = \int e_2^2 w \, d\mu = 1$ and $\int e_1 e_2 w \, d\mu = 0$. Consider the function $F(\varphi) = e_1 \cos \varphi + e_2 \sin \varphi$ for $0 \leqslant \varphi \leqslant 2\pi$. Suppose $v \in X(\Omega \backslash B)$ and consider the function $H(\varphi) = \|v + F(\varphi)\|$.

We note that the function H is Lipschitz on $[0, 2\pi]$. We will show that $H'(\varphi) = 0$ a.e. and deduce that H is constant.

Let us suppose that θ is a point of differentiability of H. Let $g \in X^*$ be a norming function for $v + F(\theta)$. Then $H(\varphi) - \langle v + F(\varphi), g\rangle$ has a minimum at $\varphi = \theta$ and so we can deduce that $H'(\theta) = \langle F'(\theta), g\rangle$.

Since g norms $v + F(\theta)$ we conclude that $g\chi_B = cwF(\theta)$ for some nonnegative constant c. Thus

$$\begin{aligned}\langle F'(\theta), g\rangle &= c\int wF(\theta)F'(\theta) \, d\mu \\ &= c\cos 2\theta \int we_1e_2 \, d\mu - \frac{c}{2}\sin 2\theta \int w(e_1^2 - e_2^2) \, d\mu \\ &= 0.\end{aligned}$$

Thus H is constant as promised.

It follows immediately that if $x, y \in X(B)$ satisfy $\int x^2 w \, d\mu =$

$\int y^2 w \, d\mu = 1$ then $\|v+x\| = \|v+y\|$; simply determine e_2 so that $\int e_2 x w \, d\mu = 0$, $\int e_2^2 w \, d\mu = 1$ and $y = x\cos\varphi + e_2 \sin\varphi$.

Taking the special case $v=0$ this leads easily to (a). (b) is then the general case.

The converse is easy. First note that (b) easily implies that if $x, y \in X(B)$ with $\|x\| \leqslant \|y\|$ then for $v \in X(\Omega\backslash B)$ we have $\|v+x\| \leqslant \|v+y\|$. Suppose $\|u\|=1$ and (a) and (b) hold. We show that the pair (u, uw) is Flinn. Clearly $\langle u, uw\rangle = 1$. Suppose $x \in X$; then $I - uw \otimes u(x) = x\chi_{\Omega\backslash B} + y$ where $\|y\| \leqslant \|x\chi_B\|$ and so $\|I - uw\otimes u\| \leqslant 1$.

We now apply this theorem to the case when X is a separable r.i. space on $[0,1]$; for L_p-spaces see **[2]**, Proposition 5, p. 123 and Proposition 1, p. 135, and **[7]**, Proposition 3.

THEOREM 4.4. *Suppose X is a separable r.i. space on* $[0,1]$. *If* $\mathscr{F}(X) \neq \{0\}$ *then* $X = L_2[0,1]$.

Proof. If $\mathscr{F}(X) \neq 0$ then Theorem 4.3 shows that there is a Borel set $B \subset [0,1]$ of positive measure and a weight function $w \in L_0$ such that if $\operatorname{supp} f \subset B$ then $\|f\|_X = \left(\int |f|^2 w \, d\lambda\right)^{\frac{1}{2}}$. Further if $g\chi_B = 0$ and $f_1, f_2 \in X(B)$ then $\|g+f_1\|_X = \|g+f_2\|_X$. It follows immediately from rearrangement invariance that w is constant and we obtain the existence of $c, \delta > 0$ so that if $\lambda(\operatorname{supp} f) \leqslant \delta$ then $\|f\|_X = c\,\|g\|_2$.

Now pick an integer N so that $1/N < \delta$. It follows easily from condition (b) of the previous theorem that there is a constant $a > 0$ so that if $f_1, f_2, \ldots, f_N$ are disjoint functions satisfying $\lambda(\operatorname{supp} f_k) \leqslant 1/N$ and $\|f_k\|_2 = 1$ then $\|f\|_X = a$. Consider then any simple function f and write $f = f_1 + \cdots + f_N$ where f_k are identically distributed and disjointly supported. Then $\|f\|_X = a\,\|f_1\|_2 = aN^{-\frac{1}{2}}\,\|f\|_2$. By considering $\chi_{[0,1]}$ it is clear that $aN^{-\frac{1}{2}} = 1$ and the theorem follows easily.

5. Flinn elements of finite-dimensional r.i. spaces

Suppose N is a natural number. Let $e_i^N = \chi_{((i-1)2^{-N}, i2^{-N}]}$ for $1 \leqslant i \leqslant 2^N$. Let $X_N = [e_i^N\colon 1 \leqslant i \leqslant 2^N]$. We denote the averaging projection (conditional expectation operator) of X onto X_N by $\mathscr{E}_N$. Notice that X_N^* can be identified naturally with X'_N. We will also let $X_N^- = [e_i^N\colon 1 \leqslant i \leqslant 2^N - 1]$.

LEMMA 5.1. *Suppose X is an r.i. space on* $[0,1]$ *so that* $X \neq L_2$. *Then there exists* $N \in \mathbf{N}$ *so that* $\sum_{i<2^N} e_i^N \notin \mathscr{F}(X_N^-)$.

Proof. Suppose for every $n \in \mathbf{N}$ we have $\chi_n = \sum_{i<2^n} e_i^n = \chi_{[0,1-2^{-n}]} \in \mathscr{F}(X_n^-)$. Since $(X_n^-)^*$ can be identified with $(X'_n)^-$ there exists $f_n =$

$\sum_{i<2^n} a_{ni}e_i^n$ so that (χ_n, f_n) is a Flinn pair for X_n^- i.e. $\int f_n \, d\lambda = 1$ and $\|I - f_n \otimes \chi_n\| = 1$. Then for every permutation σ of $1, 2, \ldots, 2^n - 1$ we have that (χ_n, f_n^σ) is a Flinn pair where $f_n^\sigma = \sum_{i<2^n} a_{n\sigma(i)}e_i^n$. By averaging we conclude that $(\chi_n, (1-2^{-n})^{-1}\chi_n)$ is a Flinn pair.

Now suppose $x \in X$. We conclude that

$$\left\| \mathscr{E}_n(x\chi_n) - (1-2^{-n})^{-1}\left(\int_0^{1-2^{-n}} x(t) \, dt \right)\chi_n \right\|_X \leqslant \|\mathscr{E}_n(x\chi_n)\|_X .$$

Letting $n \to \infty$ we obtain (by the Fatou property of the norm when X is not separable) that

$$\left\| x - \left(\int_0^1 x(t) \, dt \right)\chi_{[0,1]} \right\|_X \leqslant \|x\|_X$$

and so $(\chi_{[0,1]}, \chi_{[0,1]})$ is a Flinn pair in $X \times X'$. Now if X is separable (i.e. order-continuous) Theorem 4.4 gives the conclusion that X is isometric to L_2. If not we consider X_0, the closure of the simple functions in X; it is immediate that $\chi_{[0,1]}$ is Flinn in X_0 and so if X_0 is separable, we can again apply Theorem 4.4 to get the conclusion that X is isometric to L_2. There remains one case, when X_0 is not order-continuous and so (**[21]**) $X_0 = L_\infty[0, 1]$ up to renorming. But then we conclude that $\chi_{[0,1]}$ is Flinn in X' which is L_1 up to renorming and get a contradiction.

We now need to introduce a technical definition. We will say that an r.i. space X has property (P) if for every $t > 0$,

$$\|e_1^1\|_X < \|e_1^1 + te_2^1\|_X .$$

We say that X has property (P') if X' has property (P).

LEMMA 5.2. *Any r.i. space X has at least one of the properties (P) or (P').*

Proof. Assume X fails both (P) and (P'). Then for small enough $\eta > 0$ we have $\|e_1^1 + \eta e_2^1\|_X = \|e_1^1\|_X$ and $\|e_1^1 + \eta e_2^1\|_{X'} = \|e_1^1\|_{X'}$. But then

$$\begin{aligned} \tfrac{1}{2}(1+\eta^2) &= \int (e_1^1 + \eta e_2^1)^2 \, d\lambda \\ &\leqslant \|e_1^1\|_X \, \|e_1^1\|_{X'} \\ &= \tfrac{1}{2}. \end{aligned}$$

This contradiction establishes the lemma.

Remark. If X is strictly convex then it has property (P).

LEMMA 5.3. *Assume X has property (P). Suppose (e_j^N, u) is a Flinn pair in $X_N \times X_N'$. Then $u = 2^N e_j^N$.*

Proof. It suffices to conside the case $j = 1$. We can write $u = 2^N e_1^N + \sum_{j>1} a_j e_j^N$. By using Proposition 3.2 it follows that $(e_1^N, |u|)$ is also a Flinn pair. Then by an averaging procedure as in the preceding Lemma 5.1 we can show that (e_1^N, v) is a Flinn pair where $v = 2^N e_i^N + \eta \sum_{j \geq 2} e_j^N$, where $(2^N - 1)\eta = \sum_{j \geq 2} |a_j|$. We now project by $\mathscr{E}_1$ onto X_1. By Theorem 3.3, $(\mathscr{E}_1 e_1^N, w)$ is a Flinn pair where w is a multiple of $\mathscr{E}_1 v$. Thus $(e_1^1, 2(e_1^1 + \tau e_2^1))$ is a Flinn pair for some $\tau > 0$.

Now consider $g = \frac{1}{2}\tau e_1^1 - e_2^1 \in X_1$ and suppose this is normed by $h = \alpha e_1^1 - \beta e_2^1 \in X_1'$, where $\alpha, \beta \geq 0$. Thus $\|h\|_{X'} = 1$ and $\frac{1}{4}\tau\alpha + \frac{1}{2}\beta = \|g\|_X$. Now $\int 2g(e_1^1 + \tau e_2^1)\, d\lambda = -\frac{1}{2}\tau < 0$, and hence $\int h e_1^1\, d\lambda \leq 0$ i.e. $\alpha \leq 0$. Hence $\alpha = 0$ and so $h = -\|e_2^1\|_{X'}^{-1} e_2^1$ and $\|g\|_X = \|e_2^1\|_X$, which contradicts property (P).

LEMMA 5.4. *Suppose l and m are positive integers and that $N = lm$. Suppose $d_1 \geq d_2 \geq \cdots \geq d_N \geq 0$. If*

$$b_j = \sum_{i=1}^{m} d_{(i-1)l+j}$$

for $1 \leq j \leq l$ then $\max_{j,k} |b_j - b_k| \leq d_1$.

Proof. If $1 \leq j < k \leq l$, then $b_j \geq b_k$ while

$$b_j - b_k = d_j + \sum_{i=1}^{m-1} (d_{il+j} - d_{(i-1)l+k}) - d_{(m-1)l+k}.$$

Thus $|b_j - b_k| \leq d_1 - d_N \leq d_1$.

PROPOSITION 5.5. *Suppose X is an r.i. space on $[0, 1]$ with property (P') and such that $X \neq L_2$. Then for any $0 < p < \infty$ there is a constant $A_p = A_p(X)$ so that for every $n \in \mathbf{N}$ and every $u = \sum_{i=1}^{2^n} a_i e_i^n \in \mathscr{F}(X_n)$ we have*

$$\left(\sum_{i=1}^{2^n} |a_i|^p\right)^{1/p} \leq A_p \max_{1 \leq i \leq 2^n} |a_i|.$$

Proof. We start with the simple observation that if $\sum a_i e_i^n$ is Flinn then so is $\sum |a_i| e_i^n$ and so it suffices to consider only the case when $u \geq 0$. Similarly we are free to permute the (a_i). We therefore consider the case when $a_1 \geq a_2 \geq \cdots \geq a_{2^n} \geq 0$.

Now according to Lemma 5.1 there exists m so that $\sum_{i<2^m} e_i^m \notin \mathcal{F}(X^{-m})$. In fact, by Proposition 3.1, this means that there exists $\delta > 0$ so that if $w \in \mathcal{F}(X_m^-))$ then $\left\| w - (2^m - 1)^{-1} \sum_{i<2^m} e_i^m \right\|_\infty \geq \delta/2$. This implies that if $w = \sum_{i<2^m} b_i e_i^m$ and $\sum b_i = 1$ then $\max_{i,j} |b_i - b_j| \geq \delta$.

Now let us suppose $n > m$ and that $u = \sum_{j=1}^{2^n} a_j e_j^n$ is Flinn in X_n, where $a_1 \geq a_2 \geq \cdots \geq a_{2^n} \geq 0$. Let us set $S_k = \sum_{j \leq k} a_j$ for $1 \leq k \leq 2^n$. Let $S_0 = 0$ and $S = S_{2^n - 2^m}$.

Fix $1 \leq k \leq 2^{n-m}$. We consider a permutation σ of $\{1, 2, \ldots, 2^n\}$ so that $\sigma\{2^n - 2^{n-m} + 1, \ldots, 2^n\} = \{i\colon i < k\} \cup \{i\colon i \geq 2^n - 2^{n-m} + k\}$ and such that if

$$b_j = \sum_{i=1}^{2^{n-m}} a_{\sigma((j-1)2^{n-m}+i)}$$

for $1 \leq j \leq 2^m - 1$ then $\max |b_i - b_j| \leq a_k$. Such a permutation exists by Lemma 5.4.

Now we argue that if $v = \sum_{j=1}^{2^n} a_{\sigma(j)} e_j^n$ then $v \in \mathcal{F}(X_n)$ and so $\mathcal{E}_m(v) \in \mathcal{F}(X_m)$. To see this observe that there exists $g \in X_n'$ with $g \geq 0$ so that (v, g) is a Flinn pair by Proposition 4.1; clearly $\mathcal{E}_m(g) \neq 0$ and so by Theorem 3.3, $\mathcal{E}_m(v) \in \mathcal{F}(X_m)$. Thus $w = \sum_{i \leq 2^m} b_i e_i^m \in \mathcal{F}(X_m)$.

Next we claim that $w_0 = \sum_{i<2^m} b_i e_i^m \in \mathcal{F}(X_m^-)$. If $w_0 = 0$ this is trivial. If not, select $h \geq 0$ in X_m' so that (w, h) is a Flinn pair. If $h = \sum_{i \leq 2^m} c_i e_i^m$ we argue that there exists $i < 2^m$ so that $c_i > 0$. For, if not, h is a multiple of $e_{2^m}^m$ and by Lemma 5.3, since X' has (P), we get that $b_i = 0$ for $i < 2^m$, i.e. $w_0 = 0$. Now we can apply Theorem 3.3 to deduce that $w_0 \in \mathcal{F}(X_m^-)$.

Recalling the original choice of δ this implies that

$$\max_{i,j<2^m} |b_i - b_j| \geq \delta \sum_{j=1}^{2^m - 1} b_j.$$

In view of the selection of σ we have

$$a_k \geq \delta(S_{2^n - 2^{n-m} + k - 1} - S_{k-1}) \geq \delta(S - S_{k-1})$$

and this holds for $1 \leqslant k \leqslant 2^{n-m}$. For convenience, let us put $\alpha = 1 - \delta$. Then, for $1 \leqslant k \leqslant 2^{n-m}$ we have

$$(S - S_k) \leqslant \alpha(S - S_{k-1}).$$

By induction, we have

$$(S - S_k) \leqslant \alpha^k S$$

for $1 \leqslant k \leqslant 2^{n-m}$. This gives an estimate on a_k, i.e.

$$a_k \leqslant S - S_{k-1} \leqslant \alpha^{k-1} S \leqslant \delta^{-1} \alpha^{k-1} a_1,$$

for $1 \leqslant k \leqslant 2^{n-m}$.

If $0 < p < \infty$, this implies that

$$\begin{aligned}\sum_{i=1}^{2^n} a_i^p &\leqslant 2^m \sum_{i=1}^{2^{n-m}} a_i^p \\ &\leqslant 2^m a_1^p \delta^{-p} (1 - \alpha^p)^{-1} \\ &= a_1^p B_p^p,\end{aligned}$$

say. This estimate holds if $n > m$. If we take $A_p = \max(2^{m/p}, B_p)$ we obtain the Proposition as stated.

6. Isometries on r.i. spaces

THEOREM 6.1. *Let X be an r.i. space on $[0, 1]$ with $X \neq L_2$. Suppose X has property (P). Then for any $0 < p \leqslant 1$ there is a constant $C_p = C_p(X)$ with the following property. Suppose Y is any Köthe function space on some Polish space (Ω, μ) for which Y' is norming. Suppose $T\colon X \to Y$ is an isometric isomorphism of X onto Y. Then*

$$\sup_n \left\| \left(\sum_{i=1}^{2^n} |Te_i^n|^p \right)^{1/p} \right\|_Y \leqslant C_p.$$

Remark. The sequence $\left(\sum_{i=1}^{2^n} |Te_i^n|^p \right)^{1/p}$ is increasing. If Y has the Fatou property it will follow that $\sup_n \left(\sum_{i=1}^{2^n} |Te_i^n|^p \right)^{1/p} \in Y$.

Proof. We note first that by Proposition 2.5, T^{-1} is $\sigma(X, X')$-continuous and so has an adjoint $S = (T^{-1})'\colon X' \to X'$. We define $f_i^n = Te_i^n$ and $g_i^n = Se_i^n$. Suppose $(x, x^*) \in \Pi(X_n)$ where $x = \sum a_i e_i^n$ and $x^* = \sum a_i^* e_i^n$. Then $(Tx, Sx^*) \in \Pi(Y)$ and this implies that

$$(*) \qquad \left(\sum_{i=1}^{2^n} a_i f_i^n(\omega) \right) \left(\sum_{i=1}^{n} a_i^* g_i^n(\omega) \right) \geqslant 0$$

for μ – a.e. $\omega \in \Omega$.

Using the fact that $\Pi(X_n)$ is separable it follows that there is a set of measure zero Ω_0^n so that if $\omega \notin \Omega_0^n$, (*) holds for every $(x, x^*) \in \Pi(X_n)$. Let $\Omega_0 = \bigcup_{n\geq 1} \Omega_0^n$.

Now define $F_n(\omega) = \sum_{i=1}^{2^n} f_i^n(\omega)e_i^n \in X_n'$ and $G_n(\omega) = \sum_{i=1}^{2^n} g_i^n(\omega)e_i^n \in X_n$. The above remarks show the operator $G_n(\omega)\otimes F_n(\omega)$ is numerically positive on X_n' if $\omega \notin \Omega_0$.

Now let $B_n = \{\omega\colon\ G_n(\omega) = 0\}$. Clearly (B_n) is a decreasing sequence of Borel sets. Let $B = \bigcap B_n$. If $\mu(B) > 0$ then there exists a nonzero $h \in Y$ supported on B and $\langle h, Sx'\rangle = 0$ for every $x' \in X'$. Thus $T^{-1}h = 0$, which is absurd.

Let $D_n = \Omega\backslash(\Omega_0 \cup B_n)$. If $\omega \in D_n$ then $G_n(\omega) \neq 0$ and so it follows that $F_n(\omega) \in \mathcal{F}(X_n')$. We recall that X has property (P) and so X' has property (P'). Hence, letting $A_p = A_p(X')$ be the constant from Proposition 5.5,

$$\left(\sum_{i=1}^{2^n} |f_i^n(\omega)|^p\right)^{1/p} \leqslant A_p \max_{1\leqslant i\leqslant 2^n} |f_i^n(\omega)|.$$

Hence

$$\begin{aligned}\left\|\chi_{D_n}\left(\sum_{i=1}^{2^n} |f_i^n|^p\right)^{1/p}\right\|_Y &\leqslant A_p \left\|\max_{1\leqslant i\leqslant 2^n} |f_i^n|\right\|_Y \\ &\leqslant A_p \left\|\left(\sum_{i=1}^{2^n} |f_i^n|^2\right)^{\frac12}\right\|_Y \\ &\leqslant K_G A_p \left\|\left(\sum_{i=1}^{2^n} |e_i^n|^2\right)^{\frac12}\right\|_X \\ &= K_G A_p\end{aligned}$$

by Krivine's theorem ([**19**] 1.f.14, p. 93.) Now the sequence $\chi_{D_n}\left(\sum_{i=1}^{2^n} |f_i^n|^p\right)^{1/p}$ is increasing, as $0 < p \leqslant 1$. If $g \geqslant 0$ and $\|g\|_{Y'} \leqslant 1$ we have

$$\int_{D_n} g\left(\sum_{i=1}^{2^n} |f_i^n|^p\right)^{1/p} \mathrm{d}\mu \leqslant K_G A_p$$

and so

$$\int_{\Omega} g\left(\sup_n \left(\sum_{i=1}^{2^n} |f_i^n|^p\right)^{1/p}\right) \mathrm{d}\mu \leqslant K_G A_p.$$

We now quickly obtain the Theorem since Y' is norming.

Let $\mathcal{M} = \mathcal{M}[0,1] = C[0,1]^*$ denote the space of regular Borel measures on $[0,1]$. If $0<p\leq 1$ and $\mu \in \mathcal{M}$ we define the p-variation of μ by

$$\|\mu\|_p = \sup\left\{\left(\sum_{k=1}^{n} |\mu(B_k)|^p\right)^{1/p} : n \in \mathbf{N}, B_1, \ldots, B_n \in \mathcal{B} \text{ disjoint}\right\}.$$

If $p=1$ this reduces to the usual variation norm. For $p<1$ it is easily seen that $\|\mu\|_p < \infty$ if and only if $\mu = \sum_{n=1}^{\infty} a_n \delta(t_n)$ for some sequence of distinct elements (t_n) in $[0,1]$ and $a_n \in \mathbf{R}$ such that $\sum |a_n|^p = \|\mu\|_p^p$ (see **[14]**, **[24]**). The following lemma is standard and we omit the proof.

Lemma 6.2. *For $\mu \in \mathcal{M}$ we have*

$$\|\mu\|_p = \sup_n \left(\sum_{k=1}^{2^n} |\mu(D(n,k))|^p\right)^{1/p}$$

where $D(n,1) = [0, 2^{-n}]$ and $D(n,k) = ((k-1)2^{-n}, k2^{-n}]$ for $2 \leq k \leq 2^n$.

We now use the machinery developed in **[15]**. Suppose X is an r.i. space. Let $T\colon X \to L_0[0,1]$ be a continuous linear operator. We say that T is *controllable* if there exists $h \in L_0$ so that $|Tx| \leq h$ a.e. when $\|x\|_\infty \leq 1$. T is said to be *measure-continuous* if it satisfies the criterion that $|Tx_n|$ converges to zero in L_0 whenever $\sup \|x_n\|_\infty \leq 1$ and $|x_n|$ converges to zero in L_0. Thus it follows from Theorem 3.1 of **[15]** (cf. Sourour **[28]**) that T is controllable and measure-continuous if and only if there is a weak*-Borel map $s \to \nu_s^T$ from $[0,1]$ into $\mathcal{M}$ satisfying $|\nu_s^T|(B) = 0$ almost everywhere when B has measure zero, and such that we have for any $x \in L_\infty$

$$Tx(s) = \int x(t)\, d\nu_s^T(t).$$

The map $s \to \nu_s^T$ is called the *representing kernel* or *representing random measure* for T and it is unique up to sets of measure zero.

We further remark that if $\|\nu_s^T\|_p < \infty$ a.e. for some $p<1$ then ν_s^T is purely atomic for almost every s and so (cf. **[14]**, **[29]** Theorem 4.1) there is a sequence of Borel maps $\sigma_n\colon [0,1] \to [0,1]$ and Borel functions a_n on $[0,1]$ so that $|a_n(s)| \geq |a_{n+1}(s)|$ a.e. for every n and $\sigma_m(s) \neq \sigma_n(s)$ whenever $m \neq n$ and $s \in [0,1]$ and for which

$$\nu_s^T = \sum_{n=1}^{\infty} a_n(s) \delta(\sigma_n(s)).$$

We can now summarize our conclusions, restricting attention to surjective isometries on X.

Proposition 6.3. *Let X be an r.i. space on* $[0,1]$ *with property* (P), *and* *such that $X \neq L_2$ (isometrically). Then for any $0 < p \leqslant 1$ there is a constant C_p depending only on X such that the following holds. Suppose T: $X \to X$ is a surjective isometry. Then T is controllable and further its representing kernel ν_s^T satisfies*

$$\int_0^1 \|\nu_s^T\|_p^p \, ds \leqslant C_p^p.$$

Proof. This is an almost immediate consequence of Theorem 6.1. We use the same notation. If $F = \sup_n \left(\sum_{k=1}^{2^n} |Te_k^n| \right)$ then we have $\|F\|_{X''} \leqslant C_1$ and so $\|F\|_1 \leqslant C_1$. It is easy to deduce that if $\|x\|_\infty \leqslant 1$ then $|Tx| \leqslant F$. To conclude the argument we will need that T is measure-continuous. This is immediate if X is not equal to L_∞ with some equivalent renorming, since in this case $\|x_n\|_\infty \leqslant 1$ and $|x_n| \to 0$ in measure imply that $\|x_n\|_X \to 0$. In the exceptional case we use Propositions 2.1 and 2.5 to deduce that T is measure-continuous. We conclude that in every case T has a representing random measure ν_s^T.

Now if $0 < p \leqslant 1$, then by Lemma 6.2

$$\|\nu_s^T\|_p = \sup_n \left(\sum_{k=1}^{2^n} |Te_k^n(s)|^p \right)^{1/p}$$

almost everywhere. Hence

$$\left(\int \|\nu_s^T\|_p^p \, ds \right)^{1/p} \leqslant \| \, \|\nu_s^T\|_p \|_{X''} \leqslant C_p.$$

THEOREM 6.4. *Let X be an r.i. space on $[0, 1]$ which is not isometrically equal to $L_2[0, 1]$, and let T: $X \to X$ be a surjective isometry. Then there exists a Borel function a on $[0, 1]$ with $|a| > 0$ and an invertible Borel map σ: $[0, 1] \to [0, 1]$ such that $\lambda(\sigma^{-1}(B)) > 0$ if and only if $\lambda(B) > 0$ for $B \in \mathcal{B}$ and so that $Tx(s) = a(s)x(\sigma(s))$ a.e. for every $x \in X$.*

Proof. We start by assuming that X has property (P). According to the previous proposition every surjective isometry T is controllable and further for every $0 < p \leqslant 1$ there is a constant C_p depending only on X so that

$$\left(\int_0^1 \|\nu_s^T\|_p^p \, ds \right)^{1/p} \leqslant C_p.$$

Let us define K_p to be the least such constant i.e.

$$K_p = \sup \{ \| (\|\nu_s^T\|_p) \|_p : \ T \text{ is a surjective isometry} \}.$$

Suppose T is any fixed isometry. We can represent

$$\nu_s^T = \sum_{n=1}^{\infty} a_n(s)\delta(\sigma_n(s))$$

where a_n is a sequence of Borel functions, and σ_n: $[0,1]\to[0,1]$ is a sequence of Borel maps satisfying $\sigma_m(s)\neq\sigma_n(s)$ whenever $m\neq n$ and $0\leq s\leq 1$. In this representation we can assume that $\sigma_i(s)\neq 0$ for all i, s since the measure of set where $\nu_s^T(\{0\})\neq 0$ is clearly zero; thus we could simply redefine σ_i to avoid 0 without changing the kernel except on a set of measure zero. It follows that

$$\|\nu_s^T\|_p^p = \sum_{n=1}^{\infty} |a_n(s)|^p .$$

We define the function $H_p(s)=\sum |a_n(s)|^p$.

From now on we will fix $0<p\leq 1$. Let $M_N(s)$ be the greatest index such that $\sigma_1(s),\dots,\sigma_M(s)$ belong to distinct dyadic intervals $D(N,k)$. Then $M_N(s)\to\infty$ for all s and it follows easily that given $\epsilon>0$ we can find M, N and a Borel subset E of $[0,1]$ with $\lambda(E)>1-\epsilon$ and such that $M_N(s)\geq M$ for $s\in E$, and

$$\int_{[0,1]\setminus E} H_p\, dt<\epsilon$$

$$\int_E \sum_{n=M+1}^{\infty} |a_n(s)|^p\, ds<\epsilon.$$

For notational convenience we will set $P=2^N$. Let us identify the circle group $\mathbf{T}$ with $\mathbf{R}/\mathbf{Z}=[0,1)$ in the natural way. For $\theta\in[0,1)^P$ we define a measure preserving Borel automorphism $\gamma=\gamma(\theta_1,\dots,\theta_P)$ given by $\gamma(0)=0$ and then

$$\gamma(s)=s+(\theta_k-\rho)2^{-N}$$

for $(k-1)2^{-N}<s\leq k2^{-N}$ where $\rho=1$ if $2^N s+\theta_k>k$ and $\rho=0$ otherwise. Thus γ leaves each $D(N,k)$ invariant. The set of all such γ is a group of automorphisms Γ which we endow with the structure of the topological group $\mathbf{T}^P=[0,1)^P$. We denote Haar measure on Γ by $d\gamma$. For each k let Γ_k be the subgroup of all $\gamma(\theta)$ for which $\theta_i=0$ when $i\neq k$. Thus $\Gamma=\Gamma_1\cdots\Gamma_P$.

We also let the finite permutation group Π_P act on $[0,1]$ by considering a permutation π as inducing an automorphism also denoted π by $\pi(0)=0$ and then $\pi(s)=\pi(k)-k+s$ for $(k-1)2^{-N}<s\leq k2^{-N}$. We again denote normalized Haar measure on Π_P by $d\pi$. Finally note that the set

$\Gamma\Pi_P = \mathscr{T}$ also forms a compact group when we endow this with the product topology and Haar measure $d\tau = d\gamma\, d\pi$ when $\tau = \gamma\pi$.

We now wish to consider the isometries V_τ: $X \to X$ for $\tau \in \mathscr{T}$ defined by $V_\tau x = x \circ \tau$. For every $\tau \in \mathscr{T}$ the operator $S(\tau) = TV_\tau T$ is a surjective isometry and so has an abstract kernel $v_s^{S(\tau)}$.

LEMMA 6.5. *For almost every $\tau \in \mathscr{T}$ we have that*

$$\int_0^1 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |a_j(s)|\, |a_n(\tau\sigma_j s)|\, ds < \infty \tag{1}$$

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} a_j(s) a_n(\tau\sigma_j s)\delta(\sigma_n \tau\sigma_j s) = v_s^{S(\tau)} a.e. \tag{2}$$

Proof. Let us prove (1). Note that for any fixed s and (n, j) we have

$$\int_{\mathscr{T}} |a_n(\tau\sigma_j s)|\, d\tau = \int_0^1 |a_n(t)|\, dt$$

and so it follows that

$$\int_0^1 \int_{\mathscr{T}} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |a_j(s)|\, |a_n(\tau\sigma_j s)|\, d\tau\, ds < \infty.$$

This proves the first assertion. Note, in particular, if (1) holds

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |a_j(s)|\, |a_n(\tau\sigma_j s)| < \infty$$

for almost every s.

To obtain (2) let us suppose that τ is such that (1) holds. Suppose $\|x\|_\infty \leq 1$. Then $V_\tau Tx$ may not be bounded but there is an increasing sequence F_m of Borel subsets of $[0, 1]$ with $\bigcup F_m = [0, 1]$, so that $\chi_{F_m} V_\tau Tx$ is bounded. Thus

$$\begin{aligned} T(\chi_{F_m} V_\tau Tx)(s) &= \sum_{j=1}^{\infty} a_j(s)\chi_{\sigma_j^{-1}F_m}(s) Tx(\tau\sigma_j s) \\ &= \sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} a_j(s)\chi_{\sigma_j^{-1}F_m}(s) a_n(\tau\sigma_j s) x(\sigma_n \tau\sigma_j s) \right). \end{aligned}$$

Now for almost every s since the double series absolutely converges we may obtain

$$\lim_{m\to\infty} T(\chi_{F_m} V_\tau Tx)(s) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} a_j(s) a_n(\tau\sigma_j s) x(\sigma_n \tau\sigma_j s).$$

If X is order-continuous of X the left hand side is simply $(S(\tau)x)(s)$ a.e. Thus by the uniqueness of the representing random measure we obtain (2). For the general case we use Propositions 2.1 and 2.5 to give the same conclusion. It follows that

$$TV_\tau Tx(s) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} a_j(s) a_n(\tau\sigma_j s) x(\sigma_n \tau\sigma_j s)$$

for almost every s. Again the uniqueness of the representing random measure gives (2) and completes the proof of the Lemma.

Let us now define $\mu(s, \tau) \in \mathcal{M}$ by setting

$$\mu(s, \tau) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} a_j(s) a_n(\tau\sigma_j s) \delta(\sigma_n \tau\sigma_j s)$$

whenever the series in (1) converges absolutely and setting $\mu(s, \tau) = 0$ otherwise. It is not difficult to see that the map $(s, \tau) \to \mu(s, \tau)$ is weak*-Borel. For a.e. τ we have $\mu(s, \tau) = v_s^{S(\tau)}$ a.e. and so

$$\int_0^1 \|\mu(s, \tau)\|_p^p \, ds \leqslant K_p^p.$$

It follows that

$$\int_{\mathcal{T}} \int_0^1 \|\mu(s, \tau)\|_p^p \, ds \, d\tau \leqslant K_p^p.$$

LEMMA 6.6. *For almost every $(s, \tau) \in E \times \mathcal{T}$ we have*

$$\|\mu(s, \tau)\|_p^p \geqslant \left(\sum_{j=1}^{M} - \sum_{j=M+1}^{\infty} \right) \sum_{n=1}^{\infty} |a_j(s)|^p \, |a_n(\tau\sigma_j s)|^p .$$

Proof. Assuming (s, τ) belongs to set where (1) holds it is clear the conclusion fails for (s, τ) if and only if there exist two distinct pairs (n, j), (m, i) where $m, n \in \mathbf{N}$ and $i, j \leqslant M$ so that $\sigma_n \tau\sigma_j s = \sigma_m \tau\sigma_i s$ and $a_n(\tau\sigma_j s) \neq 0$.

Assume that the conclusion of the lemma is false. Then there is a distinct pair (n, j), (m, i) as above and Borel subset B of $E \times \mathcal{T}$ so that $\int_B |a_n(\tau\sigma_j s)| \, ds \, d\tau > 0$ and $\sigma_n \tau\sigma_j s = \sigma_m \tau\sigma_i s$ for $(s, \tau) \in B$. Note first that we must have $i \neq j$.

It will now follow from Fubini's theorem that there is a Borel subset B' of Γ and a fixed $s \in E$ and $\pi \in \Pi_P$ for which $\int_{B'} |a_n(\gamma\pi\sigma_i s)| \, d\gamma > 0$ and so that $\sigma_n \gamma\pi\sigma_j s = \sigma_m \gamma\pi\sigma_i s$ for $\gamma \in B'$.

Now $\pi\sigma_j s \in D(N, k)$ for some k and $\pi\sigma_i s \in D(N, l)$ where $l \neq k$ since

$i, j \leq M$ and $s \in E$. We write $\Gamma = \Gamma_k \times \Gamma'$ where $\Gamma' = \prod_{r \neq k} \Gamma_r$. Again by Fubini's theorem there exists a fixed $\gamma' \in \Gamma'$ and a Borel subset B_0 of Γ_k so that $\int_{B_0} |a_n(\gamma_k \gamma' \pi \sigma_i s)| \, d\gamma_k > 0$ and $\sigma_n \gamma_k \gamma' \pi \sigma_j s = \sigma_m \gamma_k \gamma' \pi \sigma_i s$ for $\gamma_k \in B_0$.

Now we note that $\gamma' \pi \sigma_i s \in D(N, l)$ is fixed by every γ_k and so $\sigma_n \gamma_k \gamma' \pi \sigma_j s = s'$ is fixed for $\gamma_k \in B_0$. But B_0 has positive measure and $\gamma' \pi \sigma_i s \in D(N, k)$. Thus there is a subset A of $D(N, k)$ so that $\int |a_n(t)| \, dt > 0$ and $\sigma_n(A) = \{s'\}$. This means that $|\nu_t^T| (\{s'\}) > 0$ on a set of positive measure and we have a contradiction.

We now complete the proof. We have

$$K_p^p \geq \int_E \int_{\mathcal{T}} \|\mu(s, \tau)\|_p^p \, d\tau \, ds$$

$$\geq \int_E \int_{\mathcal{T}} \left(\sum_{j=1}^{\infty} - 2 \sum_{j=M+1}^{\infty} \right) \sum_{n=1}^{\infty} |a_j(s)|^p \, |a_n(\tau \sigma_j s)|^p \, d\tau \, ds$$

As before

$$\int_{\mathcal{T}} |a_n(\tau \sigma_j s)|^p \, d\tau = \int_0^1 |a_n(t)|^p \, dt.$$

Thus we obtain

$$K_p^p \geq \left(\sum_{n=1}^{\infty} |a_n|^p \, dt \right) \left(\int_E \left(\sum_{n=1}^{\infty} - \sum_{n>M} \right) |a_n(t)|^p \, dt \right)$$

and this implies that

$$K_p^p \geq \left(\int_0^1 H_p \, dt \right) \left(\int_E H_p \, dt - 2\epsilon \right).$$

We finally deduce that $J = \int_0^1 H_p \, dt$ then $J(J - 3\epsilon) \leq K_p^p$. But $\epsilon > 0$ is arbitrary and so we have $J^2 \leq K_p^p$. But as this applies to all such T we have the conclusion $K_p^2 \leq K_p$ i.e. $K_p \leq 1$. Now this applies to all $0 < p \leq 1$.

Returning to our original T we note that

$$\int_0^1 \sum_{n=1}^{\infty} |a_n(s)|^p \, ds \leq 1$$

for all p. Let $R(s)$ be the number of points in the support of ν_s^T. Then

$$R(s) = \lim_{p\to 0} \sum |a_n(s)|^p .$$

By Fatou's Lemma

$$\int_0^1 R(s)\, ds \leqslant 1.$$

To deduce that T is elementary we must show $R(s)=1$ a.e. If X is order-continuous this is obvious since the fact that T is surjective requires $R(s)\geqslant 1$ a.e.

For the general case we again use Proposition 2.1. We first note that $R(s)<\infty$ a.e. We then show that if $x\in X$ then

$$Tx(s) = \sum_{j=1}^{\infty} a_j(s)x(\sigma_j s) \tag{3}$$

almost everywhere. To see this it suffices to consider the case $x\geqslant 0$. We first find an increasing sequence of Borel sets F_n such that $x\chi_{F_n}\in L_\infty$ and $\bigcup F_n=[0,1]$. Then by Proposition 2.1 $Tx\chi_{F_n}$ converges in measure to Tx. However,

$$Tx\chi_{F_n}(s) = \sum_{j=1}^{\infty} a_j(s)x(\sigma_j s)\chi_{F_n}(\sigma_j s)$$

and this converges almost everywhere to the right-hand side of (3). Now as in the order-continuous case we can argue that if T is onto we must have $R(s)\geqslant 1$ a.e. and hence $R(s)=1$ a.e. We conclude that T is elementary in the case when X has property (P).

If X fails property (P) then by Lemma 5.2 X' has property (P). Further, Proposition 2.5 says that the adjoint $T'\colon X'\to X'$ is a surjective isometry. But then T' is elementary and by Lemma 2.2 T'' and hence T is elementary.

7. Isometries in spaces not isomorphic to L_p

We now recall the definition of the Boyd indices of an r.i. space X (cf. [19] p. 129). For $0<s<\infty$ define $D_s\colon X\to X$ by $D_s f(t)=f(t/s)$ where we let $f(t)=0$ for $t>1$. Then the Boyd indices p_X and q_X are defined by

$$\frac{1}{p_X} = \lim_{s\to\infty} \frac{\log \|D_s\|}{\log s}$$

$$\frac{1}{q_X} = \lim_{s\to 0} \frac{\log \|D_s\|}{\log s}.$$

PROPOSITION 7.1. *Let X be an r.i. space and suppose T: $X \to X$ is an elementary operator. Suppose $p_X \leq r \leq q_X$. Then T is bounded on $L_r[0, 1]$ and $\|T\|_{L_r} \leq \|T\|_X$.*

This Proposition is proved in **[16]** Theorem 5.1. In fact the hypotheses of **[16]** Theorem 5.1 suppose X is a quasi-Banach space and have an additional unnecessary restriction $r \leq \min(1, q_X)$. This restriction is not used in Theorem 5.1 of **[16]** but is important in the following Theorem 5.2.

We will, however, show a direct proof under the assumption that $Tx = ax \circ \sigma$, where σ is a Borel automorphism of $[0, 1]$. This is the case we need. For convenience we consider the case $r < \infty$, the other case being similar. Let us assume $\|T\|_X = 1$. We define a Borel measure μ by $\mu(B) = \lambda(\sigma^{-1}B)$ and it follows from the fact that T is bounded that μ is continuous with respect to λ and so has a Radon–Nikodym derivative w. Now for any x we have

$$\|Tx\|_r^r = \int_0^1 |a(s)|^r \, |x(\sigma(s))|^r \, ds$$

$$= \int_0^1 |a(\sigma^{-1}s)|^r \, w(s) \, |x(s)|^r \, ds$$

and so we need to show that $|a(\sigma^{-1}s)|^r w(s) \leq 1$ a.e. Suppose not. Then there is a Borel set E of positive measure δ and $0 < \alpha, \beta$ so that $\alpha^r \beta > 1$ and $|a(\sigma^{-1}s)| > \alpha$ and $w(s) > \beta$ for $s \in E$. Then if x is supported in E it quickly follows that $\|Tx\|_X \geq \alpha \, \|D_\beta x\|_X$ and so $\|D_\beta\|_{X[0,\delta]} \leq \alpha^{-1}$. However for any $\delta > 0$ we have the estimate $\|D_\beta\|_{X[0,\delta]} \geq \max(\beta^{1/p}, \beta^{1/q})$ where $p = p_X$ and $q = q_X$. Thus $\beta^{1/r} \leq \alpha^{-1}$ contrary to assumption.

In **[18]** Lamperti shows that if $1 < p < \infty$ then $L_p[0, 1]$ has an equivalent r.i. norm (not equal to the original norm) so that there are isometries of the form $Tf = af \circ \sigma$ with $|a| \neq 1$ on a set of positive measure. In the next theorem we show that if X is not equal to L_p up to equivalence of norm then the isometries of X can only be of the very simplest form (see [34]).

THEOREM 7.2. *Suppose X is an r.i. space and that T is a surjective isometry. Then either $X = L_p[0, 1]$ up to equivalence of norm for some $1 \leq p \leq \infty$ or there is an invertible measure-preserving Borel map σ: $[0, 1] \to [0, 1]$ and a function $a \in L_0[0, 1]$ with $|a| = 1$ a.e. such that $Tx = ax \circ \sigma$ for $x \in X$.*

Remark. If $p_X < q_X$ this follows routinely from Proposition 7.1. The interesting case is thus when $p_X = q_X$.

Proof. We have that $Tx = ax \circ \sigma$ where $|a| > 0$ a.e. and σ: $([0, 1], \lambda) \to ([0, 1], \lambda)$ is a Borel automorphism by Theorem 6.4. Suppose $p_X \leq r \leq q_X$. By Proposition 6.1, $\|Tx\|_r \leq \|x\|_r$ whenever $x \in L_r$ and similarly $\|T^{-1}x\|_r \leq \|x\|_r$. Thus T also defines an isometry on L_r for $p_X \leq r \leq q_X$.

Let us consider first the case $p_X = q_X = \infty$. Then $|a| \leq 1$ a.e. In fact if $B = \{|a| < 1 - \epsilon\}$ for some $\epsilon > 0$ then $T\chi_{\sigma B} = a\chi_B$ since σ is invertible and so $\lambda(B) = 0$. Hence $|a| = 1$ a.e. Again suppose $\epsilon > 0$. Then, assuming X is not isomorphic to L_∞, there exists a least δ so that $\|\chi_{[0,\delta]}\|_X = \|\chi_{[0,\epsilon]}\|_X$. Then if $\lambda(B) = \delta$ we have $\lambda(\sigma^{-1}(B)) \geq \delta$. For an arbitrary Borel subset E of $[0, 1]$, we can split E into sets of measure δ and one remainder set to conclude $\lambda(\sigma^{-1}(E)) \geq \lambda(E) - \delta$. As ϵ was arbitrary $\lambda(\sigma^{-1}(E)) \geq \lambda(E)$ for all E. Since σ is invertible this forces $\lambda(\sigma^{-1}(E)) = \lambda(E)$ for every E i.e. σ is measure-preserving.

We turn to the case when $p_X = p < \infty$. It then follows that if $|a| = 1$ a.e. we must have σ measure-preserving. We thus assume that $|a| \neq 1$ on a set of positive measure; we will prove that the norm $\| \ \|_X$ is equivalent to $\| \ \|_p$. It suffices to consider the case when $a > 0$. It follows first that $\{a > 1\}$ and $\{a < 1\}$ both must have positive measure.

Let us now make an assumption.

Assumption. *There exist two disjoint closed intervals I_1 and I_2 contained in $(1, \infty)$ and so that $a^{-1}(I_1)$ and $a^{-1}(I_2)$ have positive measure.*

We will proceed under this assumption. We can then deduce that there is a constant $\kappa > 1$ and two disjoint Borel sets A_1, A_2 of positive measure such that $a(s) > \kappa$ for $s \in A_2$, while $a(s) > \kappa a(t)$ whenever $s \in A_1$, $t \in A_2$ but $a(s) \leq \kappa a(t)$ whenever s, t are either both in A_1 or both in A_2.

Let $\delta = \min(\lambda(\sigma(A_1)), \lambda(\sigma(A_2)))$. Let us first note that if x is supported in $\sigma(A_1) \cup \sigma(A_2)$ then since $a > \kappa$ on $A_1 \cup A_2$ we will have $\lambda(\operatorname{supp} Tx) \leq \kappa^{-p}\lambda(\operatorname{supp} x)$. We also have since a is bounded on $A_1 \cup A_2$ an estimate $\lambda(\operatorname{supp} Tx) \geq c\lambda(\operatorname{supp} x)$ for some $c > 0$.

Let us consider any nonnegative $x \in X$ with support E of measure at most δ, and such that $\|x\|_p = 1$. We define the distortion $H(x)$ by setting

$$H(x) = \operatorname{ess\,sup}\{x(s)/x(t):\ (s, t) \in E^2\}.$$

If the distortion $H(x) < \infty$ then it is clear we can define $\alpha(x) = \operatorname{ess\,inf}\{x(s):\ s \in E\}$ and $\beta(x) = \operatorname{ess\,sup}\{x(s):\ s \in E\}$ and then $0 < \alpha(x) < \beta(x) < \infty$ and $\beta(x) = H(x)\alpha(x)$. Further $\alpha(x)\chi_E \leq x \leq \beta(x)\chi_E$.

We now define a procedure. Assume $H(x) < \infty$. Given such x we define x' with the same distribution supported on $\sigma(A_1) \cup \sigma(A_2)$ so that $x' \leq (\alpha(x)\beta(x))^{\frac{1}{2}}$ on $\sigma(A_1)$ but $x' \geq (\alpha(x)\beta(x))^{\frac{1}{2}}$ on $\sigma(A_1) \cap \operatorname{supp} x'$.

Now compute $y = Tx'$. Then y is supported on $A_1 \cup A_2$. If $y(s)$, $y(t)$ are both nonzero and s, t are in the same A_j we have $y(s) \leq \beta(x)^{\frac{1}{2}}\alpha(x)^{-\frac{1}{2}}\kappa y(t)$. If $s \in A_1$ and $t \in A_2$ we have $y(s) \leq \beta(x)\alpha(x)^{-1}\kappa^{-1}y(t)$.

If $s \in A_2$ and $t \in A_1$ we have $y(s) \leqslant \kappa y(t)$. It follows that

$$H(y) \leqslant \max(\kappa H(x)^{\frac{1}{2}}, \kappa^{-1} H(x)).$$

Notice also that $c\lambda(\operatorname{supp} x) \leqslant \lambda(\operatorname{supp} y) \leqslant \kappa^{-p} \lambda(\operatorname{supp} x)$.

If we put $y = x_1$ we can then iterate the procedure to produce a sequence (x_n). Let $\delta_n = \lambda(\operatorname{supp} x_n)$; then $c\delta_n \leqslant \delta_{n+1} \leqslant \kappa^{-p}\delta_n$ and, in particular, $\lim_{n\to\infty} \delta_n = 0$. Since

$$H(x_n) \leqslant \max(\kappa H(x_{n-1})^{\frac{1}{2}}, \qquad \kappa^{-1} H(x_{n-1})),$$

we deduce that $\limsup H(x_n) < \kappa^5$.

Fix any n where $H(x_n) < \kappa^5$. Then for suitable $\alpha > 0$ and a Borel set E of measure δ_n we have $\alpha\chi_E \leqslant x_n \leqslant \kappa^5 \alpha \chi_E$. However $\|x_n\|_p = 1$ and so we obtain $\alpha\delta_n^{1/p} \leqslant 1 \leqslant \kappa^5 \alpha \delta_n^{1/p}$ or

$$\kappa^{-5} \delta_n^{-1/p} \leqslant \alpha \leqslant \delta_n^{-1/p}.$$

Now we introduce the notation $\phi(t) = \|\chi_{[0,t]}\|_X$. The above inequalities give us

$$\alpha\phi(\delta_n) \leqslant \|x_n\|_X = \|x\|_X \leqslant \kappa^5 \alpha \phi(\delta_n),$$

and hence

$$\kappa^{-5}\phi(\delta_n)\delta_n^{-1/p} \leqslant \|x\|_X \leqslant \kappa^5 \phi(\delta_n)\delta_n^{-1/p}.$$

Now since $\delta_{n+1} \geqslant c\delta_n$ we have that for $\delta_{n+1} \leqslant t \leqslant \delta_n$,

$$c^{1/p}\kappa^{-5}\phi(t)t^{-1/p} \leqslant \|x\|_X \leqslant c^{-1/p}\kappa^5 \phi(t) t^{-1/p}.$$

As $H(x_n) \leqslant \kappa^5$ eventually we can conclude that

$$0 < \liminf_{t\to 0} \phi(t)t^{-1/p} < \limsup_{t\to 0} \phi(t)t^{-1/p} < \infty.$$

In fact if we let $K = \limsup \phi(t)t^{-1/p}$ we obtain

$$c\kappa^{-5}K \leqslant \|x\|_X \leqslant c^{-1}\kappa^5 K.$$

But this estimate is independent of the original choice of x subject to $\lambda(\operatorname{supp} x) \leqslant \delta$, $H(x) < \infty$ and $\|x\|_p = 1$. Hence we obtain that $\|x\|_X$ is equivalent to $\|x\|_p$.

Thus our assumption yields the conclusion that $X = L_p[0, 1]$ up to an equivalent norm. Clearly it suffices to find one surjective isometry for which the assumption holds to give this conclusion.

If the assumption fails for T then a is essentially constant (with value α, say) on $\{a > 1\}$. If the assumption fails for T^{-1} it is easy to see that a is also essentially constant (with value β, say) on $\{a < 1\}$. Now the same reasoning must apply to any surjective isometry. However it is now easy to construct an isometry of the form $S = TV_{\tau_1}TV_{\tau_2}T$, where $V_\tau = x \circ \tau$ for some measure preserving Borel automorphism τ, and so that $S\chi_{[0,1]}$ takes each of the four distinct values α^3, $\alpha^2\beta$, $\alpha\beta^2$ and β^3 (of which three must be distinct from 1) with positive measure. Thus we can again conclude that X is isomorphic to L_p.

Remarks. This theorem can be cast as a statement about maximal norms (cf. **[27]**, **[19]**). A Banach space X has a *maximal norm* if no equivalent norm has a strictly bigger group of invertible isometries. The above theorem shows immediately that any r.i. space on $[0, 1]$ which is not isomorphic to $L_p[0, 1]$ has a maximal norm; Rolewicz **[27]** showed that the spaces $L_p[0, 1]$ have maximal norms. However if X is isomorphic but not isometric to L_p its norm cannot be maximal; this follows rather easily from Proposition 7.1 and the almost transitivity of the norm in L_p.

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