COMPACT p-CONVEX SETS

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LET A be a subset of a topological vector space X, and let Y be another topological vector space. We shall say that A can be linearly embedded in Y if there is a linear map $T:\text{lin}(A) \to Y$ (not necessarily continuous) whose restriction to A is a homeomorphism. Until recently it was unknown whether every compact convex subset of a topological vector space could be linearly embedded in a locally convex space. However, J. Roberts [**6**] has now constructed a non-empty compact convex subset of $L_p = L_p\,(0, 1)$ $(0<p<1)$ which has no extreme points and hence cannot be linearly embedded in a locally convex space (see [**4**] for some results in the other direction).

In this note we consider a similar problem for p-convex sets where $0<p<1$ (see the definition below). In view of the example of Roberts it is perhaps somewhat surprising that we are able to show that a compact p-convex set can be linearly embedded in a locally p-convex topological vector space, and always has p-extreme points. We are also able to prove an appropriate version of Choquet's theorem which for $p<1$ takes a rather trivial form.

Throughout the paper we shall assume that all vector spaces are over the real field and that all topologies are Hausdorff. A subset C of a vector space is p-convex if whenever $x, y \in C$ and $a, b \in R$ with $0 \leqslant a, b \leqslant 1$ and $a^p + b^p = 1$ then $ax + by \in C$. C is absolutely p-convex if it is p-convex and $x \in C$ implies $-x \in C$. A *p-extreme point* of a set C is any point $x \in C$ such that whenever $x = ay_1 + by_2$ with $y_1,\ y_2 \in C$, $0 \leqslant a,\ b \leqslant 1$ and $a^p + b^p = 1$ then $x = y_1$ or $x = y_2$; the set of p-extreme points of C is denoted by $\partial_p C$. If C is any set we denote by $\Gamma_p(C)$ and $\Delta_p(C)$ the smallest p-convex and absolutely p-convex sets containing C. Note that in a topological vector space, if $p<1$, a closed p-convex set always contains 0.

Let K be a compact Hausdorff topological space, and let $C(K)$ be the Banach space of all real-valued continuous functions on K. Let $\mathcal{M}(K) = C(K)^*$ be the dual of $C(K)$, i.e. the space of all regular Borel measures on K, with the usual dual norm denoted by $\|\cdot\|_1$; we shall denote by w^* the weak*-topology on $\mathcal{M}(K)$ induced by $C(K)$. For $x \in K$ we denote by δ_x or $\delta(x) \in \mathcal{M}(K)$ the unit mass concentrated at x; let $\delta(K) = \{\delta(x) : x \in K\}$.

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Now suppose $0<p<1$, and let $\mathcal{M}_p(K)$ be the subspace of $\mathcal{M}(K)$ of all μ of the form

$$\mu=\sum_{n=1}^{\infty} a_n\delta(x_n)$$

where $(x_n : n\in\mathbb{N})$ is a sequence of distinct points K and

$$\|\mu\|_p=\sum_{n=1}^{\infty}|a_n|^p<\infty.$$

Let $U_p=\{\mu:\|\mu\|_p\leqslant 1\}$ and $U_p^+=\{\mu\geqslant 0:\|\mu\|_p\leqslant 1\}$. Observe that $\|\ \|_p$ is a p-norm on $\mathcal{M}_p(K)$ (see [7] p. 3).

We shall define the topology θ_p on $\mathcal{M}_p(K)$ to be the finest vector topology on $\mathcal{M}_p(K)$ which agrees with the w^*-topology on each set $nU_p(n\in\mathbb{N})$. We can give an explicit basic set of neighbourhoods for θ_p (see Wiweger [8]), namely sets of the form

$$\bigcup_{n=1}^{\infty}\sum_{k=1}^{n} kU_p\cap W_k$$

where $\{W_k : k\in\mathbb{N}\}$ is a sequence of w^*-neighbourhoods of 0. Since each U_p is p-convex and the w^*-topology is locally convex, we conclude:

LEMMA 1. *θ_p is a locally p-convex topology on $\mathcal{M}_p(K)$.*

In the next lemma we combine two results which have essentially the same proof.

LEMMA 2. (i) *U_p and U_p^+ are θ_p-compact.*

(ii) *Suppose $T:\mathcal{M}_p(K)\to X$ is a linear map into a topological vector space satisfying (a) T is continuous for the weak*-topology on $\delta(K)$ and (b) whenever $\mu_n\in U_p$ and $\|\mu_n\|_1\to 0$ then $T\mu_n\to 0$; then T is continuous for the topology θ_p.*

Proof. The operator T in (ii) will be continuous if its restriction to U_p is continuous for the w^*-topology. Using this and the observation that the unit ball of $\mathcal{M}(K)$ is w^*-compact and contains U_p, we see that if either (i) or (ii) is false we can construct a net $\{\mu_\alpha\}$ in U_p such that $\mu_\alpha\to\mu\ w^*$ and either (1) $\mu\notin U_p$ or (2) $\mu\in U_p$ and there is a neighbourhood V of 0 in X such that $T(\mu-\mu_\alpha)\notin V$ for all α.

In either case, by replacing μ_α by a subnet, we *may assume* that when each μ_α is written in the form

$$\mu_\alpha=\sum_{n=1}^{\infty} a_{\alpha,n}\delta(x_{\alpha,n})$$

where $(x_{\alpha,n} n\in\mathbb{N})$ is a sequence of distinct points of K and $|a_{\alpha,n}|\geqslant$

$|a_{\alpha,n+1}|$ $(n \in \mathbb{N})$ then the limits $\lim_\alpha a_{\alpha,n}$ $(= a_n$, say) and $\lim_\alpha x_{\alpha,n}$ $(= x_n)$ exist. To see this consider the net $(a_{\alpha,n};\, x_{\alpha,n})$ in the compact space $[-1, 1]^{\mathbb{N}} \times K^{\mathbb{N}}$.

Now

$$n\,|a_{\alpha,n}|^p \leqslant \sum_{k=1}^{n} |a_{\alpha,k}|^p \leqslant 1$$

so that $|a_{\alpha,n}| \leqslant n^{-1/p}$ and hence

$$\begin{aligned}\Big\|\mu_\alpha - \sum_{k=1}^{n} a_{\alpha,k}\delta(x_{\alpha,k})\Big\|_1 &= \sum_{n+1}^{\infty} |a_{\alpha,k}| \\ &\leqslant (n+1)^{1-(1/p)} \sum_{n+1}^{\infty} |a_{\alpha,k}|^p \\ &\leqslant (n+1)^{1-(1/p)}.\end{aligned}$$

By the lower-semi-continuity of $\|\cdot\|_1$ with respect to w^* we have

$$\Big\|\mu - \sum_{k=1}^{n} a_k\delta(x_k)\Big\|_1 \leqslant (n+1)^{1-(1/p)}$$

and hence $\Sigma a_k\delta(x_k) = \mu$ in $\|\cdot\|_1$. However we clearly have $\Sigma\,|a_k|^p \leqslant 1$ and hence (after combining terms where $x_k = x_l$) it is clear that $\mu \in U_p$ contradicting (1).

For (2) pick a symmetric neighbourhood W of 0 in X such that $W + W + W \subset V$. Then there exists $n \in \mathbb{N}$ such that $\|\mu\|_p \leqslant 1$ and $\|\mu\|_1 \leqslant (n+1)^{1-(1/p)}$ implies $T\mu \in W$. Since T is continuous for the w^*-topology on $\delta(K)$ we may choose α such that

$$T\Big(\sum_{k=1}^{n} a_{\alpha,k}\delta(x_{\alpha,k}) - \sum_{k=1}^{n} a_k\delta(x_k)\Big) \in W.$$

Then

$$\begin{aligned}T(\mu_\alpha - \mu) = T\Big(\sum_{n+1}^{\infty} a_{\alpha,k}\delta(x_{\alpha,k})\Big) &+ T\Big(\sum_{k=1}^{n} a_{\alpha,k}\delta(x_{\alpha,k}) - \sum_{k=1}^{n} a_k(x_k)\Big) \\ &- T\Big(\sum_{n+1}^{\infty} a_k\delta(x_k)\Big) \\ &\in W + W + W \subset V\end{aligned}$$

contradicting (2). This completes the proof.

We remark that it is now clear that θ_p is the finest topology agreeing with the w^*-topology on U_p (by a result of Waelbroeck [7] p. 48).

LEMMA 3. *Let X be a topological vector space and suppose $x(t) \in X$ for $0 \leq t \leq 1$. Suppose that the set $\Delta_p(A)$ is relatively compact where*

$$A = \{(t-s)^{-(1/p)}(x(t)-x(s)) : 0 \leq s < t \leq 1\}.$$

Then $x(t) \equiv x(0)$ for $0 \leq t \leq 1$.

Proof. Let E be the space of real functions on $[0, 1]$ of the form

$$\varphi = \sum_{i=1}^{n} c_i \chi_i \tag{*}$$

where $\chi_1 \ldots \chi_n$ are characteristic functions of disjoint intervals. We may define a linear map $T : E \to X$ so that

$$T\chi_{(s,t)} = T\chi_{[s,t)} = T\chi_{(s,t]} = T\chi_{[s,t]} = x(t) - x(s).$$

If $\varphi \in E$ is given by (*), and

$$\int_0^1 |\varphi(t)|^p \, dt \leq 1$$

then

$$T\varphi = \sum_{i=1}^{n} c_i (t_i - s_i)^{1/p} [(t_i - s_i)^{-(1/p)}(x(t_i) - x(s_i))]$$

where $s_i < t_i$ are the endpoints of the interval whose characteristic function is χ_i.

As

$$\int_0^1 |\varphi(t)|^p \, dt = \sum_{i=1}^{n} |c_i|^p (t_i - s_i)$$

we have $T\varphi \in \Delta_p(A)$ and so T extends uniquely to a compact operator $T : L_p \to X$. Hence $T = 0$, by the results of [**5**] and so $x(t) \equiv x(0)$, $0 \leq t \leq 1$.

LEMMA 4. *Let K be a compact subset of a topological vector space X and suppose $\Delta_p(K)$ is relatively compact. Then the map $T : (\mathcal{M}_p(K), \theta_p) \to X$ defined by*

$$T\left(\sum_{n=1}^{\infty} a_n \delta(x_n)\right) = \sum_{n=1}^{\infty} a_n x_n$$

is continuous. (Note that the series necessarily converges since $\Delta_p(K)$ is bounded).

Proof. We use Lemma 2(ii). Clearly (a) is satisfied by T. To prove (b) suppose the contrary that there is a sequence $\mu_m \in U_p$ such that $\|\mu_m\|_p \leq 1$ and $\|\mu_m\|_1 \to 0$, but that for some neighbourhood V of 0 in X we have $T\mu_m \notin V$.

Let

$$\mu_m = \sum_{n=1}^{\infty} a_{m,n}\delta(x_{m,n})$$

where $(x_{m,n} : n \in \mathbb{N})$ is a sequence of distinct points of K. Define $y_m(t)$, $0 \leqslant t \leqslant 1$ as follows:

$$\begin{aligned} y_m(t) &= 0, & & 0 \leqslant t < |a_{m,1}|^p \\ &= \sum_{n=1}^{k} a_{m,n}x_{m,n}, & & \sum_{n=1}^{k} |a_{m,n}|^p \leqslant t < \sum_{n=1}^{k+1} |a_{m,n}|^p \\ &= \sum_{n=1}^{\infty} a_{m,n}x_{m,n}, & & \sum_{n=1}^{\infty} |a_{m,n}|^p \leqslant t \leqslant 1. \end{aligned}$$

We shall show that if $1 \geqslant t > s \geqslant 0$ and $t - s \geqslant 2\|\mu_m\|_1^p$ then $(t-s)^{-(1/p)}(y_m(t) - y_m(s)) \in (\frac{3}{2})^{(1/p)}\Delta_p(K)$. This will be trivially true if either $t < |a_{m,1}|^p$ or $s \geqslant \sum_{n=1}^{\infty} |a_{m,n}|^p$. Hence we assume $t \geqslant |a_{m,1}|^p$ and $s < \sum_{n=1}^{\infty} |a_{m,n}|^p$. Then

$$y_m(t) - y_m(s) = \sum_{n=l+1}^{k} a_{m,n}x_{m,n}$$

where $0 \leqslant l < \infty$ and $1 \leqslant k \leqslant \infty$, and

$$\begin{aligned} t - s &\geqslant \sum_{n=l+2}^{k} |a_{m,n}|^p \\ &\geqslant \sum_{n=l+1}^{k} |a_{m,n}|^p - \|\mu_m\|_1^p \\ &\geqslant \sum_{n=l+1}^{k} |a_{m,n}|^p - \tfrac{1}{2}(t-s). \end{aligned}$$

Hence

$$t - s \geqslant \tfrac{2}{3} \sum_{n=l+1}^{k} |a_{m,n}|^p$$

and so

$$\begin{aligned} (t-s)^{-(1/p)}(y_m(t) - y_m(s)) &= \sum_{n=l+1}^{k} a_{m,n}(t-s)^{-(1/p)}x_{m,n} \\ &\in \lambda\Delta_p(K) \end{aligned}$$

where

$$\lambda^p = (t-s)^{-1} \sum_{l+1}^{k} |a_{m,n}|^p \leqslant \tfrac{3}{2}.$$

Now considering $(y_m : m \in \mathbb{N})$ as a sequence in the compact space of all $\Delta_p(K)$-valued functions on $[0, 1]$ with pointwise convergence we may find a cluster point $y(t)$. Then since $\|\mu_m\|_1 \to 0$ we will have

$$(t-s)^{-(1/p)}(y(t)-y(s)) \in (\tfrac{3}{2})^{(1/p)}\Delta_p(K)$$

whenever $0 \leqslant s < t \leqslant 1$. Hence by the preceding lemma, $y(t) \equiv y(0) = 0$ for all t. However $y(1)$ is a cluster point of the sequence $T\mu_m$ and $T\mu_m \notin V$; thus we have arrived at a contradiction.

THEOREM 1. *Let K be a compact p-convex subset $(0<p<1)$ of a topological vector space X. Then K can be linearly embedded in a locally p-convex topological vector space.*

Proof. Clearly $\Delta_p(K) = \{ax - by : 0 \leqslant a,\ b \leqslant 1,\ a^p + b^p \leqslant 1,\ x, y \in K\}$ is compact and hence we may construct the continuous operator $T : (\mathcal{M}_p(K), \theta_p) \to X$ as in Lemma 4. Let $N = T^{-1}(0)$ and consider the quotient space $\mathcal{M}_p(K)/N$ with quotient θ_p-topology, which is locally p-convex. Then there is an induced injective map $\tilde{T} : \mathcal{M}_p(K)/N \to X$. Restricted to $q(\delta(K))$ (where $q : \mathcal{M}_p(K) \to \mathcal{M}_p(K)/N$ is the quotient map), $\tilde{T}$ is a homeomorphism onto K; $\tilde{T}^{-1}$ is the required embedding.

In view of Theorem 1, we could appeal to the results of Fuchssteiner ([**1**], [**2**]) to demonstrate the existence of p-extreme points, and an analogue of the Krein-Milman theorem. In fact we may go further and establish a version of Choquet's theorem (improving Theorem 2 of [**3**]).

THEOREM 2. *Let C be a compact p-convex subset $(0<p<1)$ of a topological vector space X and let K be a closed subset of C such that C is the closure of $\Gamma_p(K)$. Then*

(1) $\partial_p C \subset K$.

(2) If $x \in C$ there is a sequence of distinct points $x_n \in \partial_p C$ and $a_n \geqslant 0$ with $\sum a_n^p = 1$ such that $x = \sum_{n=1}^{\infty} a_n x_n$.

Proof. Construct as in Lemma 4 the map $T : \mathcal{M}_p(K) \to X$. Then $T(U_p^+)$ is a compact p-convex set containing K, and is clearly the smallest such. Hence $T(U_p^+) = C$. If $x \in \partial_p C$ then

$$x = \sum a_n x_n$$

where $x_n \in K$ and $\Sigma a_n^p \leqslant 1$. Since it is p-extreme, and using the fact that $0 \in C$ we see that this representation must be trivial, i.e. $x \in K$.

For (2) consider the map T as in (1) but in the case $K = C$. For $x \in C$, the set $T^{-1}\{x\} \cap U_p^+$ is w^*-compact and hence there exists

$\nu \in T^{-1}\{x\} \cap U_p^+$ such that $\nu(C) \leq \mu(C)$ whenever $\mu \in T^{-1}\{x\} \cap U_p^+$. Let

$$\nu = \sum b_n \delta(y_n)$$

where the y_n are distinct and each $b_n \neq 0$. Then if some $y_k \notin \partial_p C$ we have $y_k = c_1 z_1 + c_2 z_2$ where $z_1, z_2 \in C$, $0 < c_1, c_2 < 1$ and $c_1^p + c_2^p = 1$. Consider

$$\nu' = \sum_{n \neq k} b_n \delta(y_n) + b_k c_1 \delta(y_1) + b_k c_2 \delta(y_2)$$

Then $\nu' \in U_p^+ \cap T^{-1}\{x\}$ but

$$\nu'(C) = \sum_{n \neq k} b_n + b_k(c_1 + c_2) < \nu(C)$$

and we have a contradiction.

Now we have $\Sigma b_n^p \leq 1$; if $\Sigma b_n^p = 1$ we are home. Suppose $0 \in \partial_p C$; then by the minimality of $\nu(C)$, $0 \notin \{y_n : n \in \mathbb{N}\}$. Hence if $\Sigma b_n^p < 1$, x may be represented in the required form:

$$x = \sum_{n=1}^{\infty} b_n y_n + \left(1 - \sum b_n^p\right)^{(1/p)} (0)$$

Next suppose $0 \notin \partial_p C$; then $0 = c_1 z_1 + c_2 z_2$ where $z_1 \neq z_2 \in C$ and $0 < c_1, c_2 < 1$ and $c_1^p + c_2^p = 1$. By the preceding argument there exist non-zero measures $\nu_1, \nu_2 \in U_p^+$ such that $\nu_1(C \backslash \partial_p C) = \nu_2(C \backslash \partial_p C) = 0$ and $T\nu_1 = z_1$, $T\nu_2 = z_2$. Then $T(c_1\nu_1 + c_2\nu_2) = 0$, and hence for any $\lambda \geq 0$,

$$T(\nu + \lambda(c_1\nu_1 + c_2\nu_2)) = x.$$

Then $\lambda \to \|\nu + \lambda(c_1\nu_1 + c_2\nu_2)\|_p$ is continuous in λ and tends to infinity as $\lambda \to \infty$. Hence for suitable $\lambda > 0$,

$$\|\nu + \lambda(c_1\nu_1 + c_2\nu_2)\|_p = 1.$$

Letting

$$\nu + \lambda(c_1\nu_1 + c_2\nu_2) = \sum a_n \delta(x_n)$$

with the x_n distinct, we are home.

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