RADIAL BANACH ALGEBRAS

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1. Introduction

In this paper, we are concerned with Banach algebras which contain elements whose norm is the same as the spectral radius. These elements we shall call radial. We say that a Banach algebra is radial if every closed right ideal contains a radial element. This definition includes all B^* algebras, but more significantly the operator ideals of Pietsch [see e.g. (**12**)] and others. In fact, we show that operator algebras are really the canonical examples in that this gives an abstract characterization of them.

The theory has a geometric flavour similar to numerical ranges (**6**); in fact, Sinclair (**15**) has proved that hermitian elements are radial. But it is more closely related to the work of Bonsall and Duncan in (**4**) and (**5**) on dual representations. We shall need a stronger notion of Banach space pair in duality, and this is developed in § 2. The representation of these algebras takes place on minimal ideals, and in § 3 we give conditions under which radial algebras have socle. The natural class of algebras for this is the class of weakly compact algebras, which includes the compact algebras of Alexander in (**1**) and all algebras that are reflexive Banach spaces. In § 4, we prove the main representation theorem for radial algebras and give some applications. Amongst these we give the characterization of operator algebras on a dual pair, and also an application to the problem of when norm-decreasing isomorphisms of Banach algebras are isometries. This problem has been considered by Oshobi in (**11**), and by Bonsall in (**3**) since it is essentially the problem of which Banach algebras have minimal norm. Finally, in § 5, we show that the radial property characterizes H^*-algebras amongst Hilbert algebras, and we give some other consequences for *-algebras.

2. Basic definitions

Let A be a complex Banach algebra. We denote by $\rho(a)$ the spectral radius of an element $a \in A$; thus $\rho(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Definition 2.1. a is *radial* if $\|a\| = \rho(a)$.

Definition 2.2. A is *radial* (*co-radial*) if every non-trivial closed right-ideal (left-ideal) contains a non-zero radial element.

There are many examples of radial Banach algebras. It is obvious, for example, that every B^*-algebra is radial and co-radial; a $B^\#$-algebra is co-radial, but not necessarily radial. However, the examples which concern us in this section are generalizations of the operator ideals [see (**12**), (**9**)]. These examples will be discussed in the remainder of the section.

Let $\langle X, Y\rangle$ be a pair of Banach spaces in normed duality [(**14**) 62], such that for any x, y

$$|\langle x, y\rangle| \leqslant \|x\| . \|y\|.$$

(Thus, in the definition of Rickart we shall always assume $\beta = 1$.) We shall say that $\langle X, Y\rangle$ is *strictly norming* if given $x \in X$ there exists $y \in Y$ with $\|y\| = 1$ and

$$\langle x, y\rangle = \|x\|.$$

We denote by $B\langle X, Y\rangle$ [see (**4**) 82] the space of all bounded operators T on X which possess adjoints T' on Y such that

$$\langle Tx, y\rangle = \langle x, T'y\rangle \; x \in X, y \in Y.$$

As shown by Bonsall and Duncan (**4**) Proposition 7, $B\langle X, Y\rangle$ is a Banach algebra under the norm

$$|||T||| = \max(\|T\|, \|T'\|).$$

If $\langle X, Y\rangle$ is strictly norming and $\|x\| = 1$

$$\|Tx\| = \langle Tx, y\rangle$$

for some $y \in Y$ with $\|y\| = 1$; and so

$$\begin{aligned}\|Tx\| &= \langle x, T'y\rangle \\ &\leqslant \|T'y\| \leqslant \|T'\|.\end{aligned}$$

Thus $\|T\| \leqslant \|T'\|$ and $B\langle X, Y\rangle$ is a Banach algebra under the norm $|||T||| = \|T'\|$.

For any $T \in B\langle X, Y\rangle$, we define $\nu(T)$ to be the infimum of $\Sigma\|x_n\| \, \|y_n\|$ over all sequences (x_n) in X and (y_n) in Y such that

$$Tx = \sum_{n=1}^{\infty} \langle x, y_n\rangle x_n \; (x \in X).$$

(If there is no such representation, then $\nu(T) = \infty$.) It can be shown by direct calculation that $\nu(ST) \leqslant \nu(S)\nu(T)$ for $S, T \in B\langle X, Y\rangle$. If $N\langle X, Y\rangle$ denotes the spaces of all T such that $\nu(T) < \infty$, then $N\langle X, Y\rangle$ can be shown to be a Banach space. For suppose $T_k \in N\langle X, Y\rangle$ and $\Sigma\nu(T_k) < \infty$. Suppose

$$T_k x = \sum_{n=1}^{\infty} \langle x, y_{nk}\rangle x_{nk} \ (x \in X),$$

where $\sum_n \|y_{nk}\| \, \|x_{nk}\| \leqslant 2\nu(T_k)$.

Then let

$$Tx = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \langle x, y_{nk}\rangle x_{nk} \ (x \in X).$$

Since $\sum_k \sum_n \|x_{nk}\| \, \|y_{nk}\| < \infty$, T is well-defined; $T \in B\langle X, Y\rangle$ since its adjoint

$$T'y = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \langle x_{nk}, y\rangle y_{nk} \ (y \in Y)$$

is also well defined. It is easy to see that $T \in N\langle X, Y\rangle$ and that ΣT_k converges to T in the norm ν.

Thus $N\langle X, Y\rangle$ is a Banach algebra under the norm ν, which we may call the *nuclear algebra* on $\langle X, Y\rangle$. Clearly $\nu(T) \geqslant |||T|||$ for $T \in N\langle X, Y\rangle$.

We denote by $y \otimes x$ the operator on X given by

$$(y \otimes x)(z) = \langle z, y\rangle x$$

and $x \otimes y$ is the corresponding operator on Y.

LEMMA 2.3. *If $\langle X, Y\rangle$ is strictly norming, then for any $x \in X$ and $y \in Y$, $|||y \otimes x||| = \nu(y \otimes x) = \|y\| . \|x\|$.*

Proof. Clearly $\nu(y \otimes x) \leqslant \|y\| . \|x\|$. Now choose $y_1 \in Y$ with $\|y_1\| = 1$ and $\langle x, y_1\rangle = \|x\|$. Then $\|(y \otimes x)'y_1\| = \|(x \otimes y)y_1\| = \langle x, y_1\rangle\|y\| = \|x\| \, \|y\|$, so that $|||y \otimes x||| \geqslant \|y\| . \|x\|$.

Definition 2.4. An *operator algebra* on a strictly norming dual pair $\langle X, Y\rangle$ is a subalgebra A of $B\langle X, Y\rangle$, containing $N\langle X, Y\rangle$ equipped with a norm $|\cdot|$ such that $(A, |\cdot|)$ is a Banach algebra and for $T \in A$, $|||T||| \leqslant |T| \leqslant \nu(T)$.

The most commonly arising operator algebras are the operator ideals. Take $Y = X^*$; then by the Hahn-Banach theorem $\langle X, X^*\rangle$ is a strictly norming dual pair. Then the algebra $N\langle X, X^*\rangle$ is the algebra $N(X)$ of nuclear operators on X; other examples of operator ideals are the algebras of absolutely p-summing operators $A_p(X)$ $(1 \leqslant p < \infty)$ and of compact operators $K(X)$ [see (**9**) and (**12**)]. We remark that $\langle X^*, X\rangle$ is only strictly norming if X is reflexive [see (**13**)].

THEOREM 2.5. *An operator algebra A on a strictly norming dual pair $\langle X, Y\rangle$ is radial. A is co-radial if and only if $\langle Y, X\rangle$ is strictly norming.*

Proof. Suppose R is a non-trivial closed right-ideal in A and $T \in R$ with $T \neq 0$. Choose $x_0 \in X$ with $Tx_0 \neq 0$ and $y_0 \in Y$ such that $\|y_0\| = 1$ and

$$\langle Tx_0, y_0\rangle = \|Tx_0\|.$$

Then $T.(y_0 \otimes x_0) \in R$, i.e. $y_0 \otimes Tx_0 \in R$. Then

$$(y_0 \otimes Tx_0)^2 = \|Tx_0\|(y_0 \otimes Tx_0)$$

so that $\rho(y_0 \otimes Tx_0) = \|Tx_0\| = \|Tx_0\|\,\|y_0\| = |y_0 \otimes Tx_0|$ by Lemma 2.3.

Suppose now A is co-radial and $y \in Y$. Let L be the left-ideal in A of all operators $y \otimes x$, $x \in X$. Then since $|y \otimes x| = \|y\|\,\|x\|$, L is closed and contains a radial element $y \otimes x_0$ of norm $\|y\|$. Then $\rho(y \otimes x_0) = |\langle x_0, y\rangle| = \|y\|$ and hence if $x_1 = \|y\|\langle x_0, y\rangle^{-1}x_0$, then $\langle x_1, y\rangle = \|y\|$ while $\|x_1\| = 1$.

As pointed out above the usual examples of operator algebras are modelled on $\langle X, X^*\rangle$. In the remainder of this section we give some conditions enabling us to deduce that $\langle X, Y\rangle$ is isometrically isomorphic to $\langle X, X^*\rangle$.

If $\langle X, Y\rangle$ are in normed duality then there is a natural map J: $y \to \hat{y}$ from Y into X^* given by

$$\hat{y}(x) = \langle x, y\rangle.$$

J is norm decreasing and one-one. Suppose U is the unit ball of Y and V the unit ball of X^*; then $\hat{U} = J(U) \subset V$.

LEMMA 2.6. *If $\langle X, Y\rangle$ is strictly norming, $\hat{U}$ is weak*-dense in V.*

Proof. We use the theorem of bipolars. The weak*-closure of $\hat{U}$ is $\hat{U}^{00}$. If $x \in \hat{U}^0$ then $\|x\| \leqslant 1$ by the strictly norming condition. Hence $\hat{U}^{00} \supset V$.

LEMMA 2.7. *If $\hat{U}$ is norm dense in V, then J is an isometry of Y onto X^*.*

Proof. If $\hat{U}$ is norm dense in V, J is almost open (or nearly open) and therefore open and hence an isomorphism. Then $\hat{U}$ is closed and $\hat{U} = V$, i.e. J is an isometry.

LEMMA 2.8. *Let $\langle X, Y\rangle$ be a dual pair and suppose $\{x_n\}$ is a bounded sequence in X such that $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in Y$. Then the absolutely convex hull of $\{x_n : n \in N\}$ is $\sigma(X, \hat{Y})$-relatively compact.*

Proof. Let c be the Banach space of convergent sequences; the dual space of c may be identified with the space l_1 of absolutely converging series $(\alpha_n : n \geqslant 0)$ under the identification

$$\langle \alpha, \beta\rangle = \alpha_0 \lim_{n\to\infty} \beta_n + \sum_{i=1}^{\infty} \alpha_i \beta_i.$$

Define $S: Y \to c$ by $Sy = \{\langle x_n, y\rangle\}$. Then S possesses an adjoint $S': l_1 \to X$ defined by

$$S'(\alpha) = \alpha_0 x + \sum_{i=1}^{\infty} \alpha_i x_i,$$

where the series converges in X since $\sup\|x_n\| < \infty$. S' is continuous for the weak*-topology on c^* and the topology $\sigma(X, \hat{Y})$ on X. Hence the image of the unit ball of c^*, the $\sigma(X, \hat{Y})$-closed absolutely-convex cover of $\{x_n : n \in N\}$, is $\sigma(X, \hat{Y})$-compact.

THEOREM 2.9. *Suppose $\langle X, Y\rangle$ is a strictly norming dual pair with X reflexive, then J is an isometry of Y onto X^*.*

Proof. It is enough to show that $\hat{U}$ is norm dense in V by Lemma 2.7. By Lemma 2.6 $\hat{U}$ is weak*-dense and hence weakly dense in V. Therefore $\hat{U}$ is norm dense.

LEMMA 2.10. *If $\langle X, Y\rangle$ is a strictly norming dual pair, with Y separable, then Y is norm dense in X^*.*

Proof. We embed X^* in $X^\#$, the space of all linear functionals on X. Let $\mathscr{A}$ be the collection of all $\sigma(X, \hat{Y})$-compact and absolutely convex subsets of X. If $C \in \mathscr{A}$ then

$$\sup_{x\in C} |\langle x, y\rangle| < \infty$$

for each $y \in Y$ and so by the Principle of Uniform Boundedness

$$\sup_{x\in C} \sup_{\|y\|\leqslant 1} |\langle x, y\rangle| < \infty,$$

and so C is bounded in X (since $\langle X, Y\rangle$ is strictly norming). Let Z be the subspace of $X^\#$ of all linear functionals which are bounded on each $C \in \mathscr{A}$; then $X^* \subset Z$. Let $\tau_{\mathscr{A}}$ be the topology of uniform convergence on each $C \in \mathscr{A}$; then $(Z, \tau_{\mathscr{A}})$ is a complete locally convex space.

On $\hat{Y}$, $\tau_{\mathscr{A}}$ coincides with the Mackey topology $\mu(\hat{Y}, X)$ and hence every $\tau_{\mathscr{A}}$-continuous linear functional ψ on $\hat{Y}$ is of the form

$$\psi(\hat{y}) = \langle x, y \rangle.$$

Hence, since $\langle X, Y \rangle$ is strictly norming, for every $\psi \in (\hat{Y}, \tau_{\mathscr{A}})'$ there exists $y_0 \in U$ such that

$$\psi(\hat{y}_0) = \|x\| \geqslant |\langle x, y \rangle| \qquad \forall\, y \in U.$$

Thus by Pryce's generalization of the James theorem (**13**), (we require an obvious extension to complex vector spaces), $\hat{U}$ is relatively weakly compact in $(Z, \tau_{\mathscr{A}})$. Let W be the $\tau_{\mathscr{A}}$-closure of $\hat{U}$; then W is weakly compact and therefore $\sigma(Z, X)$-compact. As $W \supset \hat{U}$, by Lemma 2.6, $W \supset V$ and hence $\hat{Y}$ is $\tau_{\mathscr{A}}$-dense in X^*. In particular $\tau_{\mathscr{A}}$ on X^* reduces to the Mackey topology $\mu(X^*, X)$ and so every $C \in \mathscr{A}$ is weakly compact in X.

Now suppose $\phi \in X^*$ and $m \in N$. Let S be the unit ball of X; the set $mS \cap \phi^{-1}(0)$ is weakly closed in X. We show that $mS \cap \phi^{-1}(0)$ is $\sigma(X, \hat{Y})$-closed; as Y is separable it is enough to check that $mS \cap \phi^{-1}(0)$ is $\sigma(X, \hat{Y})$-sequentially closed. Suppose $x_n \in mS \cap \phi^{-1}(0)$ and $x_n \to x\ \sigma(X, \hat{Y})$. Then by Lemma 2.8 the absolutely closed convex cover of $\{x_n : n \in N\}$ is $\sigma(X, \hat{Y})$-compact and hence weakly compact. Hence $x_n \to x$ weakly and so $\phi(x) = 0$. Clearly mS is $\sigma(X, \hat{Y})$-closed (as $\langle X, Y \rangle$ is strictly norming) and hence $mS \cap \phi^{-1}(0)$ is $\sigma(X, \hat{Y})$-closed. Suppose $x_0 \in X$ is any point such that $\phi(x_0) = 1$; then there exists $y_m \in Y$ such that

$$|\hat{y}_m(x)| < 1, \quad x \in mS \cap \phi^{-1}(0), \quad \hat{y}_m(x_0) = 1.$$

Then for any $x \in X$

$$\hat{y}_m(x) = \phi(x) + \hat{y}_m(x - \phi(x)x_0),$$

and so

$$\|\hat{y}_m - \phi\| \leqslant \frac{1}{m}(1 + \|\phi\|\,\|x_0\|),$$

and so $\hat{y}_m \to \phi$. Thus $\hat{Y}$ is norm-dense in X^*.

THEOREM 2.11. *Suppose $\langle X, Y \rangle$ and $\langle Y, X \rangle$ are both strictly norming and X (or Y) is separable, then X is reflexive and Y is isometric to X^*.*

Proof. By 2.10 with X and Y interchanged, $\hat{X}$ is norm dense in Y^*. The map $x \to \hat{x}$ is an isometry since $\langle X, Y \rangle$ is strictly norming and hence $\hat{X} = Y^*$. Hence $\langle Y^*, Y \rangle$ is strictly norming and by the James-Pryce theorem [see] (**13**) Y is reflexive.

Example 2.12. Let $X = C[0, 1]$ and let $Y_0 = l_1[0, 1]$ – the space of functions f on $[0, 1]$ such $\sum_t |f(t)| < \infty$, under the norm

$$\|f\| = \sum_t |f(t)|.$$

Let $Y = Y_0 \oplus C$ with the norm $\|(f, \alpha)\| = \|f\| + |\alpha|$. Then $\langle X, Y\rangle$ is strictly norming under the duality

$$\langle x, (f, \alpha)\rangle = \sum_t f(t)x(t) + \frac{\alpha}{2}\int_0^1 x(t)\,dt.$$

However $J: Y \to X^*$ is not even an isometry onto its image.

3. The existence of the socle

We recall that if a Banach algebra A has minimal left-ideals, the left-socle of A is the linear span of all the minimal left-ideals; similarly the right-socle is the linear span of the minimal right-ideals. A is called semi-prime [(7) 155] if $\{0\}$ is the only two-sided ideal J in A for which $J^2 = \{0\}$. In a semi-prime Banach algebra with minimal left- or right-ideals the left-socle and the right-socle exist and are equal; this set is then called the socle and is a two-sided ideal [(**14**) 45–47 or (**7**) 155–156]. In this section we study conditions under which a radial Banach algebra possesses a socle. First we make an important preliminary observation.

PROPOSITION 3.1. *A radial Banach algebra A is semi-simple and hence semi-prime.*

Proof. Let R be the radical of A. Then R is a closed two-sided ideal. If $R \neq \{0\}$ then R contains a non-zero radial element, contradicting Proposition 1 of (**7**) 126. A semi-simple Banach algebra is semi-prime [Proposition 5 of (**7**) 155].

Definition 3.2. A Banach algebra A is *weakly compact* if for every $a \in A$ the map $b \to aba$ is weakly compact.

Weakly compact Banach algebras as such do not appear to have been studied in detail. They include the compact Banach algebras of Alexander (**1**) and Banach algebras which are reflexive as Banach spaces; in the latter category we should point out the result of Gordon, Lewis and Retherford (**9**) on the reflexivity of the operator ideal $A_1(E, E)$ of absolutely summing operators.

PROPOSITION 3.3. *Let A be a weakly compact radial Banach algebra. Then every closed right-ideal in A contains a non-zero radial idempotent.*

Proof. If J is a closed right-ideal then there exists $a \in J$ with $\|a\| =$

$\rho(a) = 1$. Then $\|a^n\| = 1$ for all n and since the map $b \to aba$ is weakly compact the set $\{a^n : n \geqslant 3\}$ is relatively weakly compact. Its weak closure is a compact semigroup (with separately continuous multiplication) in the weak topology; hence there is an idempotent e in the weak closure of $\{a^n : n \geqslant 3\}$, with $\|e\| \leqslant 1$. [See (**8**) 19.]

Let A_0 be the commutative Banach algebra generated by $\{a\}$; then there is a multiplicative linear functional ϕ on A_0 with $|\phi(a)| = \|\phi\| = 1$. Then $\phi(e)$ is a cluster point of $(\phi(a))^n$. Hence $|\phi(e)| = 1$ and so $e \neq 0$.

THEOREM 3.4. *Let e be a non-zero idempotent in a radial Banach algebra A. Then eAe is a radial Banach algebra.*

Proof. Suppose $J \subset eAe$ is a closed right-ideal and suppose $a \in J$ with $a \neq 0$. Then $\overline{aA}$ contains a non-zero radial element b with $\|b\| = 1$. Then $eb = b$, and hence $b^n = (eb)^n = (ebe)^{n-1}b$ and hence $\|(ebe)^{n-1}\| \geqslant 1$. Therefore ebe is radial (since $\|e\| = 1$) and $ebe \in \overline{aAe} = \overline{aeAe} \subset J$.

We recall that an idempotent e is minimal if $e \neq 0$ and $eAe = \mathscr{C}e$.

THEOREM 3.5. *Let A be a weakly compact radial Banach algebra and let J be a non-zero closed right-ideal in A. Then J contains a radial minimal idempotent.*

By 3.3, J contains a non-zero radial idempotent f. If g and h are radial idempotents we write $g \geqslant h$ if $gh = hg = h$. We show by Zorn's lemma that the set M of non-zero radial idempotents g such that $g \leqslant f$ has minimal elements with respect to this order relation. Let $C = \{g_\alpha : \alpha \in \mathscr{A}\}$ be a chain in M; we consider C as a monotone decreasing net. Then fAf is reflexive (since the map $a \to faf$ is a weakly compact projection) and $\|g_\alpha\| = 1$ for $\alpha \in \mathscr{A}$; hence there is a subnet of C, $(g_{\alpha(\beta)} : \beta \in \mathscr{B})$ say, which converges weakly to some g.

Then for $\alpha_0 \in \mathscr{A}$,

$$gg_{\alpha_0} = \lim_\beta g_{\alpha(\beta)}\, g_{\alpha_0} = g \quad \text{(weakly)}$$

since $g_{\alpha(\beta)} \leqslant g_{\alpha_0}$ eventually. Similarly $g_{\alpha_0}g = g$ and

$$g^2 = \lim_\beta gg_{\alpha(\beta)} = g,$$

and so g is an idempotent and $g \leqslant g_\alpha$, $\alpha \in \mathscr{A}$. It remains to show that $\|g\| = 1$; clearly $\|g\| \leqslant 1$. Now $g_{\alpha(\beta)} \to g$ weakly, and hence there is a convex combination $\sum_{n=1}^k c_n g_{\alpha(\beta_n)}$ with $\sum_{n=1}^k c_n = 1$, $c_n \geqslant 0$ and

$$\Big\| \sum_{n=1}^{k} c_n g_{\alpha(\beta_n)} - g \Big\| \leqslant \tfrac{1}{2}.$$

Now suppose, without loss of generality, that $g_{\alpha(\beta_k)} \leqslant g_{\alpha(\beta_n)}$ for $n \leqslant k$. Then

$$\|(\sum_{n=1}^{k} c_n g_{\alpha(\beta_n)} - g) g_{\alpha(\beta_k)}\| \leqslant \tfrac{1}{2},$$

i.e.

$$\|g_{\alpha(\beta_k)} - g\| \leqslant \tfrac{1}{2},$$

and so $g \neq 0$. Thus g is a radial idempotent, and hence by Zorn's Lemma J contains a radial idempotent e such that if $eg = ge = g$ and g is a radial idempotent, $g = e$.

Now $eAe = B$ is a radial Banach algebra (3.4). Suppose $b \in B$. Then by 3.3, if $b \neq 0$, $\overline{bB}$ contains a radial idempotent. Hence $e \in \overline{bB}$ and since e is the identity of B, $bB = B$ and so every $b \in B$ with $b \neq 0$ is invertible. By the Gelfand-Mazur theorem $B = \mathscr{C}e$ and e is a minimal idempotent.

It follows that a weakly compact radial Banach algebra contains a minimal left- and right-ideal and hence has non-trivial socle S.

THEOREM 3.6. *Let A be a weakly compact radial Banach algebra. If $a \in A$ and $aS = \{0\}$ then $a = 0$.*

Proof. Let $J = \{a : aS = \{0\}\}$. Then J is a closed right-ideal and so if $J \neq \{0\}$, J contains a radial minimal idempotent e. Then $e \in S$ and so $eS \neq \{0\}$. Thus $J = \{0\}$.

Thus weakly compact radial Banach algebras are, in a sense, determined by the socle. This also is the case under a geometrical condition. A is called strictly convex if $\|a\| = \|b\| = 1$ and $\|a+b\| = 2$ together imply that $a = b$.

THEOREM 3.7. *Let A be a strictly convex radial Banach algebra. Then*

(i) *every radial element is a scalar multiple of a minimal idempotent;*

(ii) *the socle S of A exists and if $aS = \{0\}$ then $a = 0$.*

Proof. (i) Suppose $\|a\| = \rho(a)$, and suppose $\lambda \in Sp(a)$ with $|\lambda| = \rho(a)$. Then let $f = \lambda^{-1}a$ so that $1 \in Sp(f)$ and $\|f\| = \rho(f) = 1$. Then $2 \in Sp(f+f^2)$ and so $\|f+f^2\| = 2$ with $\|f\| = \|f^2\| = 1$. Hence $f^2 = f$ and f is an idempotent. If g is a non-zero radial idempotent with $g \leqslant f$ then $\|f+g\| = 2$ (since $(f+g)g = 2g$) and hence $f = g$. It follows easily as in 3.5 that f is a minimal idempotent.

(ii) is similar to 3.6.

Some similar but deeper results on the existence of minimal idempotents of norm one are obtained by Bonsall and Duncan (**5**).

In the next section we will consider the geometric structure of the socle in a radial Banach algebra. However first we collect together for convenience some remarks on the socle in Banach algebras.

PROPOSITION 3.8. *Let A be a semi-prime Banach algebra, with socle S. Then*

(i) *if e is a minimal idempotent, then the smallest closed two-sided ideal I containing e is a minimal closed two-sided ideal,*

(ii) *$\bar{S}$ is the closed linear span of the minimal closed two-sided ideals contained in it.*

If A is radial and either weakly compact or strictly convex, we have

(iii) *every minimal-closed two-sided ideal is contained in $\bar{S}$.*

Proof. (i) [cf (7) 165, Proposition 16]. Suppose $J \subset I$ is a closed two-sided ideal. If $e \in J$ then $J = I$; if $e \notin J$ then eJ is strictly contained in eA and hence $eJ = \{0\}$. Hence $e \in \operatorname{lan} J$, the left-annihilator of J. As $\operatorname{lan} J$ is a closed two-sided ideal, $J \subset I \subset \operatorname{lan} J$ and hence $J^2 = \{0\}$. Therefore $J = \{0\}$.

(ii) If L is a minimal left-ideal, then $L = eA$ for some minimal idempotent e. Let I be the smallest closed two-sided ideal containing e; then $L \subset I \subset \bar{S}$, and the result follows.

(iii) If I is a minimal-closed two-sided ideal, then I contains a radial minimal idempotent e by 3.5 or 3.7. Hence I is the smallest closed two-sided ideal containing e and $I \subset \bar{S}$.

4. Structure of the minimal ideals

LEMMA 4.1. *Let A be a radial Banach algebra and let L be a minimal left-ideal of A. Suppose $a \in L$ and $a \neq 0$; then there is a radial minimal idempotent p in A such that $pb = \pi(b)a$ for $b \in L$, where π is a linear functional on L with $\pi(a) = 1$.*

Proof. $\overline{aA}$ contains a non-zero radial element p; we may suppose $\|p\| = 1$ and $1 \in Sp(p)$. Suppose $ac_n \to p$ and that $L = Af$ where f is a minimal idempotent. Then, for any $x \in A$, $fxf = \psi(x)f$ where $\psi \in A^*$.

For any n,

$$(ac_n)^2 = afc_n afc_n = \psi(c_n a)ac_n,$$

so that $p^2 = \lambda p$ for some $\lambda \in \mathscr{C}$. Clearly since $1 \in Sp(p)$, $\lambda = 1$ and $\psi(c_n a) \to 1$.

For any $b \in A$,

$$ac_n bac_n = \psi(c_n ba)ac_n,$$

and $pbp \in \mathscr{C}p$, so that p is a minimal idempotent.

If $b \in L$,

$$ac_n b = \psi(c_n b)a,$$

so that

$$pb = \pi(b)a,$$

where $\pi(b) = \lim_{n\to\infty} \psi(c_n b)$. If $b = a$ then $\lim_{n\to\infty} \psi(c_n a) = 1$, so that $\pi(a) = 1$.

LEMMA 4.2. *Let A be a radial Banach algebra and let $T: A \to B(X)$ be a norm-decreasing topologically irreducible representation of A on a normed space X. Suppose $a \neq 0$ and $b \neq 0$ belong to the same minimal left-ideal L of A; then for any $x \in X$,*

$$\|a\| \, \|T_b x\| = \|b\| \, \|T_a x\|.$$

Proof. Again let $L = Af$ where f is a minimal idempotent. Suppose $T_a x = \lambda T_b x$ some $\lambda \in \mathscr{C}$. Then $T_{a-\lambda b} T_f x = 0$. By Lemma 1 of (**2**), T_f is a one-dimensional projection so that either $T_f x = 0$ or $T_{a-\lambda b} = T_{a-\lambda b} T_f = 0$.

If $T_f x = 0$ then $T_a x = T_a T_f x = T_b x = 0$ so that the lemma is trivial. If $T_{a-\lambda b} = 0$ then $a - \lambda b \in T^{-1}(0) \cap L$. As L is minimal, either $a = \lambda b$ in which case the lemma is again trivial, or $L \subset T^{-1}(0)$ so that $T_a = T_b = 0$.

Hence we may assume $T_a x$ and $T_b x$ linearly independent. Let

$$c_t = ta + (1-t)b \quad (0 \leqslant t \leqslant 1).$$

By Lemma 4.1, there is a minimal radial idempotent p_t such that

$$p_t c = \pi_t(c) c_t, \quad c \in L,$$

where $\pi_t \in L^*$ and $\pi_t(c_t) = 1$.

Let

$$\psi(t) = \|c_t\| \quad (0 \leqslant t \leqslant 1).$$

Then ψ is Lipschitz and so differentiable almost everywhere; suppose t is a point of differentiability with $0 < t < 1$. Then

$$\psi(t+s) = \psi(t) + s\psi'(t) + o(s),$$

and hence

$$\|p_t c_{t+s}\| \leqslant \|c_t\| + s\psi'(t) + o(s),$$

$$\|c_t + sp_t(a-b)\| \leqslant \|c_t\| + s\psi'(t) + o(s),$$

$$\|c_t\| \, |1 + s\pi_t(a-b)| \leqslant \|c_t\| + s\psi'(t) + o(s).$$

As this is true for both positive and negative values of s we conclude

$$\psi(t)[\mathrm{Re}\pi_t(a-b)] = \psi'(t).$$

Now let $\phi(t) = \|T_{c_t}x\|$; again ϕ is Lipschitz and differentiable almost everywhere. For $0 < t < 1$ and t a point of differentiability,

$$\|T_{p_t}T_{c_{t+s}}x\| \leqslant \|T_{c_t}x\| + s\phi'(t) + o(s),$$

so that

$$\|T_{c_t}x + sT_{p_t(a-b)}x\| \leqslant \|T_{c_t}x\| + s\phi'(t) + o(s),$$

and by similar reasoning

$$\phi(t)[\mathrm{Re}\,\pi_t(a-b)] = \phi'(t).$$

The function $\log(\psi(t))/(\phi(t))$ is also Lipschitz (neither ϕ nor ψ vanish) and so

$$\log\frac{\psi(1)}{\phi(1)} - \log\frac{\psi(0)}{\phi(0)} = \int_0^1 \frac{d}{dt}\log\frac{\psi(t)}{\phi(t)}\,dt,$$

(where the integrand is defined almost everywhere).

However, almost everywhere

$$\frac{d}{dt}\log\frac{\psi(t)}{\phi(t)} = \frac{\psi'(t)}{\psi(t)} - \frac{\phi'(t)}{\phi(t)}$$
$$= \mathrm{Re}\,\pi_t(a-b) - \mathrm{Re}\,\pi_t(a-b) = 0.$$

Hence

$$\frac{\psi(1)}{\phi(1)} = \frac{\psi(0)}{\phi(0)}$$

and the lemma follows.

We now use Lemma 4.2 to establish the main theorem on representations of radial Banach algebras.

THEOREM 4.3. *Let A be a radial Banach algebra and let $T: A \to B(X)$ be a norm decreasing topologically irreducible representation of A. Suppose L is a minimal left-ideal with $T(L) \neq \{0\}$. Then L is isometrically isomorphic to X and there is an isometry $\theta: L \to X$ such that for any $x \in X$ and $a \in A$,*

$$T_a x = \theta a \theta^{-1} x$$

Proof. Let $L = Af$, where f is a minimal idempotent. As $T(L) \neq \{0\}$ we conclude $T_f \neq 0$ and so there exists $x_0 \in X$, $x_0 \neq 0$, with $T_f x_0 = x_0$ and $\|x_0\| = \|f\|$.

Then define, for $a \in L$, $\theta(a) = T_a x_0$. Then

$$\|f\| \, \|\theta(a)\| = \|a\| \, \|T_f x_0\|,$$

so that $\|\theta(a)\| = \|a\|$, and θ is an isometry. Thus $\theta(L)$ is a closed invariant subspace of X and so $\theta(L) = X$.

Finally for any x,

$$T_a x = T_a \theta(\theta^{-1}x) = T_a T_{\theta^{-1}x} x_0 = \theta a . \theta^{-1} x.$$

Suppose now I is a minimal closed two-sided ideal and let K and L be any two minimal left-ideals contained in I. Then using the natural left-regular representation $S: A \to B(K)$ defined by

$$S_a . b = ab,$$

we see that S is one-one on I and hence on L. By Theorem 4.3 we see that L is isometrically isomorphic to K.

COROLLARY 4.4. *In a radial Banach algebra any two minimal left-ideals contained in the same minimal closed two-sided ideal are isometric.*

We now give an application to the problem of when a norm-decreasing isomorphism is necessarily isometric.

COROLLARY 4.5. *If A is a radial Banach algebra and T is a norm-decreasing homomorphism of A into an arbitrary Banach algebra B, then T is a scalar multiple of an isometry on each minimal left-ideal of A.*

Proof. Let J be a minimal left-ideal of A. If $T(J) = \{0\}$, the result is trivially true. If $T(J) \neq \{0\}$, then $T(J)$ is a minimal left-ideal of $T(A)$, and there is a norm-decreasing representation of $T(A)$ and hence of A into $B(T(J))$ which is topologically irreducible. Applying 4.3, we have that J and $T(J)$ are isometrically isomorphic with T a scalar multiple of the isometry.

We come now to the identification of the minimal closed two-sided ideals. Suppose I is such an ideal and L is a minimal left-ideal in I. Then by Lemma 4.1, I contains a radial minimal idempotent e. Then Ae and eA form a dual pair of normed spaces under the duality

$$\langle x, y \rangle e = yx, \; x \in Ae, \; y \in eA.$$

There is a dual representation $T: A \to B\langle Ae, eA \rangle$ defined by

$$T_a x = ax, \; T'_a y = ya.$$

Clearly

$$\max(\|T_a\|, \|T'_a\|) \leqslant \|a\|.$$

PROPOSITION 4.6. *If e is a radial minimal idempotent in a radial Banach algebra A then $\langle Ae, eA\rangle$ is a strictly norming pair.*

Proof. Let I be the smallest closed ideal containing e; I is a minimal-closed ideal (3.8). For $x \in Ae$, let

$$|x| = \sup_{\|y\|\leqslant 1} |\langle x, y\rangle|.$$

Clearly $|x| \leqslant \|x\|$. If $|x| = 0$, then $x \in \operatorname{ran}(eA)$, the right-annihilator of eA. However, $\operatorname{ran}(eA) \cap I$ is a closed two-sided ideal and since A is semi-prime, $I \not\subseteq \operatorname{ran}(eA)$. Therefore $\operatorname{ran}(eA) \cap I = \{0\}$ and $x = 0$. Hence $|\cdot|$ is a norm on Ae; we denote by X the normed space $(Ae, |\cdot|)$. Consider the representation $T: A \to B(X)$, where $T_a x = ax$. Then for $a \in A$, $x \in X$ and $y \in eA$ with $\|y\| \leqslant 1$,

$$|\langle T_a x, y\rangle| = |\langle ax, y\rangle| = |\langle x, ya\rangle| \leqslant \|a\|\, |x|.$$

Hence T is norm-decreasing, and by Lemma 4.2 for $x \in Ae$

$$\|x\|\, |T_e e| = \|e\|\, |T_x e|, \ \|x\|\, |e| = |x|.$$

However, $1 = \|e\| \geqslant |e| \geqslant \langle e, e\rangle = 1$. Hence $|x| = \|x\|$ for $x \in Ae$. For fixed $x \in Ae$, there is a minimal idempotent p satisfying Lemma 4.1 and $p \in I$. Hence $T'_p \neq 0$ and by Lemma 1 of (**2**), $T'_p = \psi \otimes y$ where $\psi \in (eA)^*$ and $\|y\| = 1$. Given $\varepsilon > 0$, there exists $z \in eA$ such that $\|z\| = 1$ and

$$\langle x, z\rangle \geqslant \|x\| - \varepsilon.$$

Then

$$\langle x, z\rangle = \langle T_p x, z\rangle = \langle x, T'_p z\rangle = \psi(z)\langle x, y\rangle.$$

Hence

$$|\langle x, y\rangle| \geqslant |\psi(z)|^{-1}(\|x\| - \varepsilon) \geqslant \|\psi\|^{-1}(\|x\| - \varepsilon).$$

However $\|T'_p\| = \|\psi\| = 1$. Hence $|\langle x, y\rangle| = \|x\|$ and $\langle Ae, eA\rangle$ is strictly norming.

THEOREM 4.7. *A minimal closed two-sided ideal I which contains a minimal left-ideal in a radial Banach algebra is isometrically isomorphic to an operator algebra on a strictly norming dual pair.*

Proof. By Proposition 4.6 we set up a representation $T: I \to B\langle Ae, eA\rangle$. As I is minimal T is one-one. Now for $a \in I$ we define

$$|T_a| = \|a\|.$$

Then $T(I)$ is a Banach algebra under $|\cdot|$. It remains to show that $T(I)$

is an operator algebra as in § 2. Suppose $y \in eA$ and $x \in Ae$; let $u = xy$. Then for $\xi \in Ae$,

$$T_u\xi = xy\xi = \langle y, \xi\rangle x,$$

so that $T_u = y \otimes x$. Now $|T_u| = \|u\| \geqslant \|T'_u\| = |||T_u|||$. However, we may choose $\eta \in eA$ so that $\|\eta\| = 1$ and $\langle x, \eta\rangle = \|x\|$. Then $T'_u\eta = \|x\|y$, so that $|||T_u||| \geqslant \|x\|\ \|y\| \geqslant \|xy\| = \|u\|$. Hence $|T_u| = |||T_u|||$. It follows easily that for any $a \in I$, $|||T_a||| \leqslant |T_a| \leqslant \nu(T_a)$, and hence $T(I)$ is an operator algebra.

Corollary 4.8. *Suppose I is a reflexive Banach space. Then I is isometrically isomorphic to an operator algebra on a dual pair $\langle X, X^*\rangle$, with X a reflexive Banach space.*

Proof. Apply Theorem 2.9.

Corollary 4.9. *Suppose A is a radial Banach algebra which is a reflexive space. Then A is co-radial.*

Proof. Let L be a closed left-ideal, and let S be the socle of A (see § 3). If $Sa = 0$ then $a = 0$ (a similar argument to 3.6). Then $SL \neq \{0\}$ and so for some minimal-closed two-sided ideal $I \cap L \neq \{0\}$. However, I is isometric to an operator algebra on a dual pair $\langle X, X^*\rangle$ where X is reflexive. Hence I is co-radial and $I \cap L$ contains a radial element.

Corollary 4.10. *Suppose A is a separable Banach algebra which is radial and co-radial. Then every minimal left-ideal L of A is reflexive, and the minimal two-sided ideal generated by L is isometrically isomorphic to an operator algebra on the dual pair $\langle L, L^*\rangle$.*

Proof. Suppose e is a minimal radial idempotent. Then $\langle Ae, eA\rangle$ and $\langle eA, Ae\rangle$ are strictly norming and so by 2.11, Ae is reflexive. By 4.4 any minimal left- (or right-) ideal is reflexive. The rest follows from 4.7.

The corresponding result for norm-decreasing homomorphisms for reflexive algebras is as follows:

Proposition 4.11. *If A is a radial Banach algebra which is a reflexive Banach space, then any norm-decreasing isomorphism of A into an arbitrary Banach algebra B is an isometry on minimal left-ideals.*

We conclude this section with two easy applications of numerical ranges of operators [see (**6**)], which will be used in the next section.

Proposition 4.12. *Suppose A is a radial Banach algebra.*

(i) *Every minimal-closed two-sided ideal containing a reflexive minimal left-ideal is the closed linear span of its radial idempotents.*

(ii) *If A is weakly compact and $a \in A$ with $eae = 0$ for every radial idempotent e then $a = 0$.*

Proof. Let I be a minimal closed two-sided ideal intersecting the socle. Then I is isometrically isomorphic to an operator algebra on a strictly norming dual pair $\langle X, Y\rangle$. For $x \in X$ there exists $x^* \in Y$ such that $\|x^*\| = \|x\|$ and $\langle x, x^*\rangle = \|x\|^2$. If we define for $x_1, x_2 \in X$ $[x_1, x_2] = \langle x_1, x_2^*\rangle$, then $[\ ,\ ]$ is a semi-inner product on X.

(i) Suppose ψ is a bounded linear functional on I annihilating each radial idempotent. Then it is easy to see from the reflexivity of X that

$$\psi(y \otimes x) = \langle Vx, y\rangle$$

where $V: X \to X$ is a bounded linear map. Then for any $x \in X$, $x \neq 0$, $1/(\|x\|^2)x^* \otimes x$ is a radial idempotent on I so that $\psi(x^* \otimes x) = 0$, i.e. $[Vx, x] = 0$ for every $x \in X$. By a result of Lumer (**10**) Theorem 5, $V = 0$ and hence $\psi(y \otimes x) = 0$ for any $y \in Y$ and $x \in X$. As the finite-dimensional operators are dense in I, $\psi(I) = 0$.

(ii) Suppose e is a radial idempotent in I and take $\langle X, Y\rangle = \langle Ae, eA\rangle$. Let $T: A \to B\langle Ae, eA\rangle$ be the induced dual representation. Then for any $x \in X$, $x^* \otimes x . T_a . x^* \otimes x = 0$, i.e. $x^*(T_a x) = 0$ which reduces to $[T_a x, x] = 0$ for every x. Hence $T_a = 0$ as before and so $aS = 0$. By Theorem 3.6, $a = 0$.

5. Hilbert spaces

A semi-simple H^*-algebra is radial if (and only if) every minimal self-adjoint idempotent is radial. In this section we prove a converse result that every radial Banach algebra which is a Hilbert space is an H^*-algebra. We then give some results on radial $*$-algebras.

LEMMA 5.1. *Let A be a radial Banach algebra which is a Hilbert space. Then if $x \in A$ and e is a minimal radial idempotent $(x, e)e = exe$.*

Proof. If $\psi(x)e = exe$ then $\psi \in A^*$ with $\|\psi\| \leqslant 1$. As $\psi(e) = 1$ it follows that $\psi(x) = (x, e)$.

LEMMA 5.2. *Let A be a Hilbert operator algebra modelled on a Hilbert space H (i.e. $N(H) \subset A \subset B(H)$). Then the ideal $\Sigma(H)$ of Hilbert-Schmidt operators is a closed ideal in A and $|T| = \sigma(T)$ for $T \in \Sigma(H)$.*

Proof. Let $(\ ,\ )$ be the inner product in A and $(\ ,\ )_\sigma$ be the inner product in $\Sigma(H)$. Then for any two minimal radial idempotents E_1, E_2 in A,

$$(E_1, E_2)E_2 = E_2E_1E_2 = (E_1, E_2)_\sigma E_2,$$

so that $(E_1, E_2) = (E_1, E_2)_\sigma$.

Suppose T is in the linear span of the radial idempotents of A; then $T = \sum_{i=1}^n \lambda_i E_i$ and

$$|T^2| = \sum_{i=1}^n \sum_{j=1}^n \lambda_i\bar{\lambda}_j(E_i, E_j) = \sum_{i=1}^n \sum_{j=1}^n \lambda_i\bar{\lambda}_j(E_i, E_j)_\sigma = \sigma(T)^2;$$

it follows that for T in the closed linear span of the radial idempotents, $|T| = \sigma(T)$.

By 4.12(i), the closure J of the finite-dimensional operators is the closed linear span of the radial idempotents. Hence $J \subset \Sigma(H)$ and $|T| = \sigma(T)$ for $T \in J$. Thus $J = \Sigma(H)$.

THEOREM 5.3. *A radial Banach algebra which is a Hilbert space is isometrically isomorphic to an H^*-algebra.*

Proof. By Theorem 3.6, the socle S of A is non-trivial and $aS = 0$ implies $a = 0$. Suppose $a \in S^\perp$; then for every minimal radial idempotent e, $(a, e)e = eae = 0$ (5.1). Hence by 4.12(ii), $a = 0$ and so S is dense in A.

Now let I_1, I_2 be any two distinct minimal-closed two-sided ideals in A. Then for $a \in I_1$ and $e \in I_2$ a minimal radial idempotent, $(a, e)e = eae = 0$ so that $(a, e) = 0$. By 4.12(i), $(a, b) = 0$ for any $b \in I_2$. Then the minimal-closed two-sided ideals are mutually orthogonal. Since A is the closed linear span of the minimal two-sided closed ideals, A is their l_2-sum.

Finally, by Lemma 5.2, every such minimal two-sided ideal is isometrically isomorphic to an algebra of Hilbert-Schmidt operators. Thus $A \cong l_2 - \oplus\, \Sigma(H_\alpha)$ which by introducing the obvious involution on each minimal ideal is an H^*-algebra.

We conclude by giving a result on the geometry of radial *-algebras. We suppose A is an algebra with a proper isometric involution (that is for any $x \neq 0$, $\|x^*\| = \|x\|$ and $x^*x \neq 0$). The isomorphic structure of *-algebras with minimal ideals is well developed (see (**14**) 260–276]. It is not surprising that if A is radial then we obtain some similar results on the isometric structure.

THEOREM 5.4. *Let A be a radial Banach *-algebra, with a proper isometric involution. Then*

(i) *every minimal left- or right-ideal is isometrically isomorphic to a Hilbert space,*

(ii) *if e is a minimal idempotent in A then e is radial if and only if e is self-adjoint.*

Proof. It is enough to consider a minimal-closed two-sided ideal I. If I contains a minimal left-ideal L then $L = If$ where f is a self-adjoint minimal idempotent [(**14**) Lemma 4.10.1]. By (**14**) Theorem 4.10.3 we may introduce an inner product $(\ ,\ )$ on L, defined by

$$(y, x)f = x^*y,$$

so if L_0 is L equipped with the inner-product norm then the left-regular representation $T: I \to B(L_0)$ satisfies

$$\|T_a\|^2 \leqslant \rho(a^*a).$$

Hence T is norm-decreasing (since $\|a^*\| = \|a\|$). Hence by Theorem 4.3 L_0 is isometric to L, and so L is isometric to a Hilbert space.

Now let e be a radial idempotent in I. Then $\|T_e\| = 1$ and T_e is a projection on L_0 so that $T_e^* = T_e$. However, T is a *-representation [(**13**) 4.10.3] and hence $T_{e^*} = T_e^*$. Finally T is faithful on I and as $I^* = I$ (I is minimal-closed), $e^* \in I$; hence $e^* = e$, and any radial idempotent in I is self-adjoint.

Now consider f; by Lemma 4.1 there is a radial minimal idempotent p in I such that $pf = f$. Then $p^* = p$ and hence $(pf)^* = fp = f$. Thus $pfp = f \in \mathscr{C}p$ since p is minimal and so $p = f$ and f is radial. Thus any self-adjoint minimal idempotent is radial.

Finally if R is a minimal right-ideal then $R = eI$ for some minimal radial and self adjoint idempotent e. Then $\langle Ie, eI \rangle$ form a strictly norming dual pair and as Ie is isometric to a Hilbert space, so is eI [Theorem 2.9].

Remarks. We make two comments on the preceding theorem. First we note that the assumption that $\|a^*\| = \|a\|$ for $a \in A$ may be replaced by $\rho(a^*a) \leqslant \|a\|^2$ without affecting the result. Secondly we observe that I is isometric to a *-invariant operator algebra on a Hilbert space by an application of the results of § 4.

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