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Cesaro mean convergence of martingale differences in rearrangement invariant spaces

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Abstract. We study the class of r.i. spaces in which Cesaro means of any weakly null martingale difference sequence is strongly null. This property is related to the Banach-Saks property. We show that in classical (separable) r.i. spaces (such as Orlicz, Lorentz and Marcinkiewicz spaces) these properties coincide but this is no longer true for general r.i. spaces. We locate also a class of r.i. spaces having this property where an analogue of the classical Dunford-Pettis characterization of relatively weakly compact subsets in L_1 holds.

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1. Introduction

A Banach space is said to have the Banach-Saks property if and only if each weakly null sequence contains a subsequence whose arithmetic (Cesaro) means converge strongly to zero (sometimes this is called the weak Banach-Saks property). It is a classical result [5], [6], that the spaces $L_p[0,1), 1 < p < \infty$ have this property. Later, it was shown by Szlenk [33] that the non-uniformly convex Banach space $L_1[0,1)$ also has the Banach-Saks property. More recently, Banach-Saks property and Banach-Saks type properties have been actively studied in the class of separable rearrangement invariant (r.i.) function spaces X with the Fatou property [10], [9], [32], [31], [3], [4] (all relevant definitions are given below). A general approach employed in these articles goes back to the paper [15] (see also the paper of Gaposhkin [13]) which studied weakly null sequences and martingale differences in classical L_p-spaces. It consists in the decomposition of some subsequence of any bounded sequence in X into the sum of a norm perturbation of an equimeasurable sequence (or a sequence of martingale differences) and a disjointly supported sequence which converges to zero for the measure topology. In this context, an interesting result for weakly null martingale difference sequences may be found in

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Freniche's paper [12]. It is shown there that Cesaro means of any such sequence in $L_p[0,1], 1 \leq p < \infty$ converge strongly. Later, an extension of the Freniche's result to a special class of Orlicz spaces L_F had been obtained in [1]. The class of Orlicz spaces considered there was denoted as ∇_3 and may be described as follows: $L_F \in (\nabla_3)$ if and only if $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ for some $c > 1$, where G is the complementary function to F.

The main objective of the present paper consists in studying of the question of identifying the class of r.i. spaces X such that Cesaro means of any weakly null martingale difference sequence in X converge strongly (we will say, in this case, that X has the martingale difference Cesaro mean property (MDCMP)). We show that if a separable r.i. space X does not contain a subspace isomorphic to ℓ_1, then X has (MDCMP) if and only if X is p-convex for some $p > 1$ and its lower Boyd index $\alpha_X > 0$. At the same time, we identify a wide class of spaces which contain a subspace isomorphic to ℓ_1 and, however, also have such a property. The class of these spaces (following to [21], we call them (Wm)-spaces) may be described as follows: these are the spaces where an analogue of the classical Dunford-Pettis characterization [8] of relatively weakly compact subsets in L_1 holds. More precisely, an r.i. space $X \in (Wm)$ if and only if every relatively weakly compact subset of X consists of elements having equicontinuous norms in X (see Theorem 5.5 below). The fact that every Orlicz space $L_F \in (\nabla_3)$ also possesses the (Wm)-property easily follows from [26] (see also [2] [22]). It should be pointed out that the (Wm)-property is close (though not equivalent) to the so-called weak Dunford-Pettis property (see e.g. [34], [17]). We shall show in this paper that every Lorentz space has the (Wm)-property and prove that an Orlicz space L_F has the (Wm)-property if and only if $L_F \in (\nabla_3)$. We shall also fully characterize the class of Orlicz and Marcinkiewicz spaces which have (MDCMP).

Note that every separable r.i. space X having (MDCMP) automatically has the Banach-Saks property (see Proposition 2.1 below). In the final section of the present paper, we shall show that the converse implication fails in the class of all r.i. spaces. More precisely, using constructions of the paper [16], we shall build an example of an r.i. space X which possesses the Banach-Saks property and which fails (MDCMP). In fact, the space X fails analogous property even for weakly null sequences of disjointly supported functions. This is in marked contrast with the situation in classical (Orlicz, Lorentz, Marcinkiewicz) r.i. spaces, where these two properties are equivalent.

2. Preliminaries

In this section, we shall briefly list the definitions and notions used throughout this paper.

A Banach space $(X, \|\cdot\|_X)$ of real-valued Lebesgue measurable functions (with identification m-a.e.) on the interval $[0,1]$ will be called *rearrangement invariant* if

(i) X is an ideal lattice, that is, if $y \in X$, and if x is any measurable function on $[0,1]$ with $0 \le |x| \le |y|$ then $x \in X$ and $\|x\|_X \le \|y\|_X$;
(ii) X is rearrangement invariant in the sense that if $y \in X$, and if x is any measurable function on $[0,1]$ with $x^* = y^*$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

Here, m denotes Lebesgue measure and x^* denotes the non-increasing, right-continuous rearrangement of x given by

$$x^*(t) = \inf\{\, s \ge 0 : m(\{|\, x \,|> s\}) \le t \,\}, \quad t > 0.$$

For basic properties of rearrangement invariant spaces, we refer to the monographs [20,24]. We note that for any rearrangement invariant (=r.i.) space $X = X[0,1]$

$$L_\infty[0,1] \subseteq X \subseteq L_1[0,1].$$

The Köthe dual $X^\times$ of an r.i. space X consists of all measurable functions y for which

$$\|y\|_{X^\times} := \sup\left\{ \int_0^1 |x(t)y(t)|dt : \ x \in X, \ \|x\|_X \le 1 \right\} < \infty.$$

The basic properties of Köthe duality may be found in [24] and [20] (where the Köthe dual is called the *associate* space). If X^* denotes the Banach dual of X, it is known that $X^\times \subset X^*$ and $X^\times = X^*$ if and only if the norm $\|\cdot\|_X$ is order-continuous, i.e. from $\{x_n\} \subseteq X, x_n \downarrow_n 0$, it follows that $\|x_n\|_X \to 0$. We note that the norm $\|\cdot\|_X$ of the r. i. space X is order-continuous if and only if X is separable. An r.i. space X is said to have the *Fatou property* if whenever $\{f_n\}_{n\ge1} \subseteq X$ and f measurable on $[0,1]$ satisfy $f_n \to f$ a.e. on $[0,1]$ and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f\|_X \le \liminf_{n\to\infty} \|f_n\|_X$. It is well known that an r.i. space X has the Fatou property if and only if the natural embedding of X into its Köthe bidual $X^{\times\times}$ is a surjective isometry. Such spaces are called *maximal*. If X is separable or maximal r.i. space, then the natural embedding $X \hookrightarrow X^{\times\times}$ is isometric. We denote by $(X)_0$ the closure of L_∞ in X. The space $(X)_0$ is a r.i. subspace of X.

Recall that for $\tau > 0$, the dilation operator σ_τ is defined by setting $\sigma_\tau x(t) = x(t/\tau)\chi_{(0,\min(1,\tau))}(t)$. Operators σ_τ are bounded in every r.i. space X. The numbers α_X and β_X given by

$$\alpha_X := \lim_{\tau\to0} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}, \quad \beta_X := \lim_{\tau\to\infty} \frac{\ln \|\sigma_\tau\|_X}{\ln \tau}$$

belong to the closed interval $[0,1]$ and are called the Boyd indices of X (see [20]). The Boyd indices of a given r.i. space X are said to be non-trivial if $0 < \alpha_X \le \beta_X < 1$.

Let us recall some classical examples of r.i. spaces on $[0,1]$. Each increasing concave function φ on $[0,1], \varphi(0) = 0$, generates the Lorentz space $\Lambda(\varphi)$ endowed with the norm

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t)d\varphi(t),$$

and the Marcinkiewicz space $M(\varphi)$ endowed with the norm

$$\|x\|_{M(\varphi)} = \sup_{0<\tau\le 1} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t)dt. \tag{2.1}$$

The space $M(\varphi)$ is not separable, but the space

$$\left\{ x \in M(\varphi) : \lim_{\tau\to 0} \frac{1}{\varphi(\tau)} \int_0^\tau x^*(t)dt = 0 \right\}$$

endowed with the norm (2.1) is a separable r.i. space which, in fact, coincides with the space $(M(\varphi))_0$.

Let $M(t)$ be an increasing convex function on $[0,\infty)$ such that $M(0) = 0$. Denote by L_M the Orlicz space on $[0,1]$ (see e.g. [19], [27]) endowed with the norm

$$\|x\|_{L_M} = \inf\{\lambda : \lambda > 0, \int_0^1 M(|x(t)|/\lambda)dt \le 1\}.$$

Convergence in measure (respectively, in weak topology) of a sequence of measurable functions $\{x_n\}_{n=1}^\infty$ (respectively, from an r.i. space X) to a measurable function x (respectively, from X) is denoted by $x_n \xrightarrow{\mu} x$ (respectively, $x_n \xrightarrow{w} x$).

In this paper, we will consider sequences of martingale differences (with respect to an increasing sequence of σ-subalgebras of the σ-algebra of all Lebesgue measurable subsets of $[0,1]$) and, in particular, we need the following simple result [32].

Proposition 2.1. *If X is a separable r.i. space on $[0,1]$, then every weakly null sequence $\{x_n\}_{n=1}^\infty \subset X$ contains a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that*

$$x_{n_k} = y_k + z_k, \quad k = 1, 2, \ldots,$$

where $\{y_k\}_{k=1}^\infty \subset X$ is a weakly null sequence of martingale differences from X and $\|z_k\| \le 2^{-k}$, $k = 1, 2, \ldots$.

Thus, if the Cesaro means of any weakly null sequence of martingale differences from X converge strongly to zero (that is, $X \in (MDCMP)$), then X has the Banach-Saks property.

3. The disjoint Cesaro mean property (DCMP)

We say that an r.i. space X has the *disjoint Cesaro mean property (DCMP)* if whenever $(x_n)_{n=1}^\infty$ is a disjoint weakly null sequence in X, then

$$\lim_{n\to\infty} \frac{1}{n} \|\sum_{j=1}^n x_j\|_X = 0.$$

Theorem 3.1. *Let X be a separable r.i. space on $[0,1]$. The following conditions on X are equivalent*:

(i) *X has the (DCMP);*

(ii) *Whenever $(x_n)_{n=1}^{\infty}$ is a disjoint weakly null sequence then*

$$\lim_{m\to\infty} \sup \left\{ \frac{1}{m} \left\| \sum_{j\in A} x_j \right\| : \ A \subset \mathbb{N}, \ |A| = m \right\} = 0;$$

(iii) *Whenever W is a relatively weakly compact subset of X, then*

$$\lim_{m\to\infty} \sup \left\{ \frac{1}{m} \| \max_{1\le j\le m} |x_j| \|_X : \ x_1, \dots, x_m \in W \right\} = 0.$$

Proof. (i) $\implies$ (ii). Assume that (ii) is false. Let $M = \sup_n \|x_n\|$. Then there exist $c > 0$ and an increasing sequence $(m_k)_{k=1}^{\infty}$ so that for each k we can find $A_k \subset \mathbb{N}$ with $|A_k| = m_k$ and

$$\left\| \sum_{j\in A_k} x_j \right\|_X \ge c m_k.$$

We can assume that $m_{k+1} > 3Mc^{-1} m_k$ for each k. Let $B_1 = A_1$ and then $B_k = A_k \setminus \cup_{j=1}^{k-1} A_j$ for $k \ge 2$. Then $m_k \le |\cup_{j=1}^{k} B_j| \le \sum_{j=1}^{k} m_j \le 3m_k/2$ and

$$\begin{aligned} \left\| \sum_{j=1}^{k} \sum_{i\in B_j} x_i \right\|_X &\ge \left\| \sum_{i\in B_k} x_i \right\|_X \\ &\ge \left\| \sum_{j\in A_k} x_j \right\|_X - M \sum_{j=1}^{k-1} m_j \\ &\ge c m_k - \frac{3M}{2} m_{k-1} \\ &> \frac{c m_k}{2}. \end{aligned}$$

Let $(r_j)_{j=1}^{\infty}$ be a sequence of integers so that for some increasing sequence $(s_j)_{j=0}^{\infty}$ with $s_0 = 0$ we have $\{r_j\}_{j=s_{k-1}+1}^{s_k} = B_k$ for $k \ge 1$. Then $(x_{r_j})_{j=1}^{\infty}$ contradicts the (DCMP).

(ii) $\implies$ (iii). Let $\sup_{x\in W} \|x\| = M$. For each $m \in \mathbb{N}$ let

$$\varphi(m) = \sup \left\{ \frac{1}{m} \|\chi_E \max_{1\le j\le m} |x_j| \|_X : \ x_1, \dots, x_m \in W, \ m(E) \le m^{-1} \right\}.$$

We argue first that $\lim_{m\to\infty} \varphi(m) = 0$. Indeed if not we may find a sequence $m_k \uparrow \infty$ with $\varphi(m_k) > c > 0$. Hence for each k we may find a finite subset $V_k \subset W$

with $V_k = \{x_{kj}\}_{j=1}^{m_k}$, and measurable subset E_k of $[0,1]$ with $m(E_k) \le m_k^{-1}$ so that

$$\|\chi_{E_k} \max_{j\le m_k} |x_{kj}|\|_X \ge cm_k.$$

Now, by passing to a subsequence, we can further assume the existence of a disjoint sequence of sets $F_k \subset E_k$ so that

$$\|\chi_{E_k\setminus F_k} \max_{j\le m_k} |x_{kj}|\|_X \le cm_k/2.$$

Then we may find disjoint subsets G_{kj} of F_k so that

$$\chi_{F_k} \max_{j\le m_k} |x_{kj}| = \max_{j\le m_k} \chi_{G_{kj}} |x_{kj}|.$$

Now the countable set $\{\chi_{G_{kj}} x_{kj}\}_{j\le m_k, k\ge 1}$ forms a weakly null sequence. However

$$\left\| \sum_{j=1}^{m_k} \chi_{G_{kj}} x_{kj} \right\|_X \ge cm_k/2$$

and this contradicts (ii). Thus $\lim_{m\to\infty} \varphi(m) = 0$.

Now suppose $m \in \mathbb{N}$ and $N > m^3$. Let $x_1, \ldots, x_N \in W$. Choose disjoint measurable sets G_j so that

$$\max_{j\le N} |x_j| = \sum_{j=1}^{N} \chi_{G_j} |x_j|.$$

Let $y_j = \chi_{G_j} x_j$ and $a_j = m(G_j)$. Let Ω denote the collection of all m-subsets of $\{1, 2, \ldots, N\}$ equipped with normalized counting measure $\mathbb{P}$. Let $\eta_j(\omega) = 1$ if $j \in \omega$ and $\eta_j = 0$ otherwise. Then $\mathbb{E}(\sum_{j=1}^{N} a_j \eta_j) = m/N$ and

$$\mathbb{P}\left(\sum_{j=1}^{N} a_j \eta_j > m^{-1} \right) \le \frac{m^2}{N} < \frac{1}{m}.$$

If $\sum_{j=1}^{N} a_j \eta_j(\omega) \le m^{-1}$ then $E = E(\omega) = \cup_{\eta_j(\omega)=1} G_j$ has measure less than m^{-1} and

$$\left| \sum_{j=1}^{N} \eta_j(\omega) y_j \right| \le \chi_E \max_{\eta_j(\omega)=1} |x_j|.$$

Thus we have for all such ω,

$$\left\| \sum_{j=1}^{N} \eta_j(\omega) y_j \right\|_X \le m\varphi(m).$$

Hence, as the exceptional set has measure less than $1/m$ and the $\|\sum_{j=1}^N \eta_j y_j\|_X$ is bounded by mM

$$\mathbb{E}\left\|\sum_{j=1}^N \eta_j y_j\right\|_X \leq m\varphi(m) + M.$$

This implies that

$$\begin{aligned}\frac{1}{N}\left\|\sum_{j=1}^N y_j\right\| &= \frac{1}{m}\left\|\mathbb{E}\sum_{j=1}^N \eta_j(\omega) y_j\right\| \\ &\leq \varphi(m) + Mm^{-1}.\end{aligned}$$

This clearly implies (iii).

(iii) $\implies$ (i) is trivial. □

Lemma 3.2. *Suppose X is a separable r.i. space containing no copy of ℓ_1. Then every bounded martingale difference sequence is weakly null.*

Proof. Since X contains no copy of ℓ_1, X^* is order-continuous and also separable. Suppose $(d_n)_{n=1}^\infty$ is a bounded martingale difference sequence. It suffices to show that some subsequence is weakly null. By Rosenthal's theorem [28], $(d_n)_{n=1}^\infty$ has a subsequence $(d'_n)_{n=1}^\infty$ which is weakly Cauchy and hence also weakly Cauchy in L_1. Thus $(d'_n)_{n=1}^\infty$ is weakly convergent in L_1 and its limit must be zero since it is a martingale difference sequence. Thus for every $f \in L_\infty$ we have

$$\lim_{n\to\infty} \int f d'_n \, dt = 0.$$

Since L_∞ is dense in X^* this implies that $(d'_n)_{n=1}^\infty$ is weakly null in X. □

Theorem 3.3. *If X is an r.i. space which is p-convex for some $p > 1$, then X has the (DCMP). In the case when X is a separable r.i. space containing no copy of ℓ_1 the opposite assertion holds also.*

Proof. If X is p-convex with constant K where $p > 1$ then we have

$$\|\max(|x_1|, \ldots, |x_n|)\| \leq K n^{1/p} \max(\|x_1\|, \ldots, \|x_n\|), \qquad x_1, \ldots, x_n \in X$$

and so (iii) of Theorem 3.1 is immediate.

Conversely, if X fails to be p-convex for any $p > 1$ then so does $X[2^{-n}, 2^{1-n}]$ for any choice of n. Hence, by [24, Theorem 1.f.12], for any n we can find n disjoint functions $(y_{nj})_{j=1}^n$ with supp $y_{nj} \subset [2^{-n}, 2^{1-n})$, $\|y_{nj}\| = 1$ and $\|\sum_{j=1}^n y_{nj}\|_X > n/2$. But then, by Lemma 3.2, $(y_{nj})_{j\leq n, n\geq 1}$ is a weakly null sequence contradicting (ii) of Theorem 3.1 above. □

Theorem 3.4. *Any Orlicz space L_F on the interval $[0, 1]$ has the (DCMP).*

Proof. Assume $(x_n)_{n=1}^\infty$ is a normalized weakly null disjoint sequence. Let

$$G_n(s) = \int_0^1 F(|sx_n(t)|)dt,$$

so $G_n(1) = 1$ and $G_n(s)/s$ is a non-decreasing function. Let

$$\psi(s) = \lim_{n\to\infty} G_n(s)/s.$$

We claim that $\lim_{s\to 0}\psi(s) = 0$. Indeed if not, we can find $c > 0$, a strictly increasing sequence $(n_k)_{k=1}^\infty$ of integers and a sequence $0 < s_k < 2^{-k}$ such that $G_{n_k}(s_k) > cs_k$. Suppose $\sum a_k x_{n_k}$ converges and $\|\sum a_k x_{n_k}\|_{L_F} < 1$. Then

$$\sum_{k=1}^\infty G_{n_k}(|a_k|) < 1$$

and so

$$\sum_{|a_k|>s_k} |a_k| < c^{-1}.$$

Since $\sum s_k < \infty$ this implies $\sum |a_k| < \infty$ and this contradicts the fact that $(x_n)_{n=1}^\infty$ is weakly null.

Now fix $0 < s < 1$, $a > 1$ and suppose $n > as^{-1}$. Then

$$\int F\left(\frac{a}{n}(|x_1(t) + \cdots + x_n(t)|)\right) dt = \sum_{k=1}^n G_k\left(\frac{a}{n}\right) \le \frac{a}{n}\sum_{k=1}^n \frac{G_k(s)}{s}.$$

Hence

$$\lim_{n\to\infty}\int F\left(\frac{a}{n}(|x_1(t) + \cdots + x_n(t)|)\right) dt \le a\psi(s)$$

and since $\lim_{s\to 0}\psi(s) = 0$ we have

$$\lim_{n\to\infty}\int F\left(\frac{a}{n}(|x_1(t) + \cdots + x_n(t)|)\right) dt = 0 \qquad (a > 1),$$

i.e.

$$\lim_{n\to\infty}\frac{1}{n}\|x_1 + \cdots + x_n\|_{L_F} = 0.$$

□

4. The martingale difference Cesaro mean property (MDCMP)

Let us recall that an r.i. space X has the martingale difference Cesaro mean property (MDCMP) if for every weakly null martingale difference sequence $(d_n)_{n=1}^\infty$ in X we have

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{k=1}^n d_k\right\|_X = 0.$$

Lemma 4.1. *If X has the (MDCMP) then X has the (DCMP).*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a disjoint weakly null sequence. Consider the sequence $y_k = x_k \otimes r_k$ in $X[0,1]^2$. Here $(r_k)_{k=1}^{\infty}$ denote the standard Rademachers. Then

$$|y_k| \leq |x_k| \otimes \chi_{[0,1]} \qquad k = 1, 2, \dots$$

so that $(y_k)_{k=1}^{\infty}$ is relatively weakly compact (see also [11]). It is also a martingale difference sequence and hence weakly null. The Lemma follows trivially. □

Lemma 4.2. *If X has the (MDCMP) then for every weakly compact set W we have*

$$\lim_{s \to 0} \sup_{x \in W} \|\sigma_s x\|_X = 0.$$

Proof. Let W be a weakly compact set for which criterion fails. Then we may find a sequence (x_n) in W so that $\|\sigma_{2^{-2^n}} x_n\|_X \geq c > 0$ for each n. Consider the sequence in $X([0,1]^2)$ defined by $y_1 = y_2 = 0$ and then

$$y_k = x_n \otimes r_k \qquad 2^n + 1 \leq k \leq 2^{n+1}, \ n \geq 1.$$

Then $|y_k| \leq |x_n| \otimes \chi_{[0,1]}$ for $2^n + 1 \leq k \leq 2^{n+1}$. As in Lemma 4.1 $(y_k)_{k=1}^{\infty}$ is weakly null. On the other hand,

$$\left| \sum_{k=2^n+1}^{2^{n+1}} y_k \right| \geq 2^n |x_n| \otimes \chi_{E_n}$$

where $E_n = \{t : r_k(t) = 1, \ 2^n + 1 \leq k \leq 2^{n+1}\}$ has measure $2^{2^{-n}}$. Therefore,

$$\left\| \sum_{k=2^n+1}^{2^{n+1}} y_k \right\|_X \geq 2^n \|\sigma_{2^{-2^n}} x_n\| \geq c 2^n,$$

whence

$$\frac{1}{2^{n+1}} \left\| \sum_{k=1}^{2^{n+1}} y_k \right\|_X \geq c/2 \qquad n = 1, 2, \dots$$

which contradicts (MDCMP). □

Theorem 4.3. *Let X be a separable r.i. space with $\alpha_X > 0$ and the (DCMP). Then X has the (MDCMP).*

Proof. Let $(d_k)_{k=1}^{\infty}$ be a weakly null martingale difference sequence. Let $M = \sup_n \|d_n\|$. By Theorem 3.1 we have an estimate

$$\| \max_{1 \leq j \leq n} |d_j| \|_X \leq n \varphi(n)$$

where $\lim_{n\to\infty}\varphi(n)=0$. By a result of Johnson and Schechtman [14] we have

$$\begin{aligned}\left\|\sum_{k=1}^n d_k\right\|_X &\le \left\|\sup_{m\le n}\left|\sum_{k=1}^m d_k\right|\right\|_X\\ &\le C\left\|\left(\sum_{k=1}^n |d_k|^2\right)^{1/2}\right\|_X\\ &\le C\left\|\max_{j\le n}|d_j|\right\|_X^{1/2}\left\|\sum_{j=1}^n |d_j|\right\|_X^{1/2}\\ &\le CM^{1/2}\varphi(n)^{1/2}n,\end{aligned}$$

so that

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{k=1}^n d_k\right\|_X=0.$$

□

Theorem 4.4. *Let X be a separable r.i. space on $[0,1]$ containing no subspace isomorphic to ℓ_1 (e.g. suppose X is reflexive). Then the following conditions are equivalent:*

(i) *X has the (MDCMP);*

(ii) *X is p-convex for some $p>1$ and $\alpha_X>0$;*

(iii) *If $(d_n)_{n=1}^\infty$ is a bounded martingale difference sequence then $\lim_{n\to\infty}\frac{1}{n}\|d_1+\cdots+d_n\|_X=0$.*

Proof. (ii) $\Longrightarrow$ (i). Apply Theorem 3.3 and Theorem 4.3.

(i) $\Longrightarrow$ (ii). By Theorem 3.3 and Lemma 4.1 X is p-convex for some $p>1$. Now assume $\alpha_X=0$. Then there is a sequence $(x_n)_{n=1}^\infty$ in X with $\|x_n\|_X=1$ and $\|\sigma_{4^{-n}}x_n\|_X\ge\frac{1}{2}$. Let $(y_n)_{n=1}^\infty$ be a sequence so that $\operatorname{supp} y_n\subset(2^{-n},2^{1-n})$ and y_n has the same distribution as $\sigma_{2^{-n}}x_n$. By Lemma 3.2, (y_n) is weakly null. However, $\|\sigma_{2^{-n}}y_n\|_X\ge\frac{1}{2}$ and this contradicts Lemma 4.2.

The equivalence of (i) and (iii) follows from Lemma 3.2. □

Remark 4.5. *Note that the implication (ii) $\Longrightarrow$ (iii) holds for every r.i. space. Indeed, since $\alpha_X>0$, we have due to [14],*

$$\left\|\sum_{k=1}^n d_k\right\|_X\le C\left\|\left(\sum_{k=1}^n d_k^2\right)^{1/2}\right\|_X.$$

By the assumption, we have $\|d_k\|_X \le K$, $k \ge 1$ *for some* $K > 0$. *Thus, the preceding estimate implies for* $r \in (1, \min\{2, p\}]$ *that*

$$\left\|\sum_{k=1}^{n} d_k\right\|_X \le C \left\|\left(\sum_{k=1}^{n} |d_k|^r\right)^{1/r}\right\|_X \le CKn^{1/r},$$

whence the assertion follows.

The following corollary immediately follows from Remark 4.5 and Proposition 2.1.

Corollary 4.6. *If a separable r.i. space* X *satisfies conditions of* (ii) *of Theorem 4.4, then* X *has the Banach-Saks property.*

Recall, that Freniche [12] proved that L_p $(1 \le p < \infty)$ has the martingale difference Cesaro mean property. The following theorem gives precise characterization of the class Orlicz spaces with the (MDCMP).

Theorem 4.7. *An Orlicz space* L_F *on* $[0, 1]$ *has the (MDCMP) if and only if* $\alpha_{L_F} > 0$.

Proof. If $\alpha_{L_F} = 0$ then by [9, Theorem 5.5 (ii)], we know that the separable part $(L_F)_0$ of the Orlicz space L_F fails the Banach-Saks property. Hence, due to Proposition 2.1, $(L_F)_0$ (and moreover L_F) does not possess the (MDCMP).

If $\alpha_{L_F} > 0$, then L_F is separable and we may use Theorem 4.3 and Theorem 3.4. □

Theorem 4.8. *Let* $M(\varphi)$ *be a Marcinkiewicz space on* $[0, 1]$. *The following conditions are equivalent:*

(i) $\varphi(+0) > 0$ *or* $0 < \gamma_\varphi \le \delta_\varphi < 1$;

(ii) *for every bounded martingale difference sequence* $\{d_k\}_{k=1}^{\infty} \subset M(\varphi)$, *we have*

$$\frac{1}{n}\left\|\sum_{k=1}^{n} d_k\right\|_{M(\varphi)} \to 0;$$

(iii) $M(\varphi)$ *has the (MDCMP).*

Proof. If $\varphi(+0) > 0$, then $M(\varphi) = L_1$. Therefore, to prove $(i) \implies (ii)$, we need only to note that the space $M(\varphi)$ is p-convex as soon as $p > \frac{1}{\gamma_\varphi}$ and $\alpha_{M(\varphi)} = 1 - \delta_\varphi > 0$ (see [25], [20]) and use Remark 4.5. The implication $(ii) \implies (iii)$ is trivial. To prove that $(iii) \implies (i)$, assume that (i) does not hold. Thanks to [4], we then have that the separable part $M_0(\varphi)$ of the Marcinkiewicz space $M(\varphi)$ does not have the Banach-Saks property. This immediately implies that one can locate in the space $M_0(\varphi)$ a weakly null sequence of martingale differences whose Cesaro means do not converge strongly. □

5. Rearrangement invariant spaces satisfying Dunford-Pettis criterion of relative weak compactness

Here, we introduce (see also [21]) a class of r.i. spaces which do not satisfy the conditions of Theorem 4.4 but still possess the (MDCMP).

Definition 5.1. *An r.i. space X on $[0,1]$ is said to have (Wm)-property ($X \in (Wm)$), if from $\{x_n\}_{n=1}^{\infty} \subset X$, $x_n \xrightarrow{w} 0$ and $x_n \xrightarrow{\mu} 0$ it follows that $\|x_n\|_X \to 0$.*

Proposition 5.2. *If an r.i. space X is p-convex for some $p > 1$, then $X \notin (Wm)$.*

Proof. Let $\{f_n\}_{n=1}^{\infty} \subset X$ be pairwise disjoint functions and let $\|f_n\|_X = 1$, $n = 1, 2, \ldots$. For any $n \in \mathbf{N}$ and any $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, we have

$$\|\sum_{i=1}^{n} \alpha_i f_i\|_X \le M(\sum_{i=1}^{n} |\alpha_i|^p)^{1/p}, \tag{5.1}$$

where $M > 0$ is some constant. We define a linear operator $T : l_p \mapsto X$ by

$$T(e_i) = f_i, \quad i = 1, 2, \ldots.$$

where $\{e_i\}_{i=1}^{\infty}$ is the standard basis in l_p. By (5.1), T is a continuous mapping onto $[f_n]_{n=1}^{\infty}$. Since $f_n \xrightarrow{w} 0$ in X, $f_n \xrightarrow{\mu} 0$, and $\|f_n\|_X = 1$, $n \in \mathbf{N}$, we infer that $X \notin (Wm)$. □

Proposition 5.3. *If an r.i. space $X \in (Wm)$, then X is separable.*

Proof. If X is non-separable, then there exists a sequence of pairwise disjoint normalized elements $\{x_n\}_{n=1}^{\infty}$ such that the space $[x_n]_{n=1}^{\infty}$ spanned by $\{x_n\}_{n=1}^{\infty}$ in X is a lattice isomorphic to the space l_∞ [23, 1.a.7]. To see that $X \notin (Wm)$ it suffices to verify that $x_n \xrightarrow{w} 0$ in X and $x_n \xrightarrow{\mu} 0$ as $n \to \infty$. The latter convergence is obvious. To see the former, observe that since $l_\infty^* = \pi(l_1) \oplus \pi(l_1)^\perp$, where $\pi(l_1)$ is the subspace of all order continuous functionals on l_∞ and $\pi(l_1)^\perp$ is the subspace of all singular functionals [18], the sequence of standard unit vectors $\{e_n\}_{n=1}^{\infty}$ converges weakly in l_∞. □

Proposition 5.4. *If an r.i. space $X \in (Wm)$, then X has the (MDCMP).*

Proof. If $(d_n)_{n=1}^{\infty}$ is a weakly null martingale difference sequence in X then

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} d_k\|_{L_1} = 0$$

since L_1 has the (MDCMP) (e.g. by [12] or Theorem 4.7). Since this sequence is weakly null in X we have

$$\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} d_k\|_X = 0.$$

□

Now we show that the following equivalent characterization of (Wm)-spaces, mimicking the classical Dunford-Pettis characterization of relatively weakly compact subsets in L_1-spaces, holds.

Theorem 5.5. *An r.i. space X has the property (Wm) if and only if each relatively weakly compact set $K \subset X$ satisfies*

$$\lim_{E\subseteq[0,1],\ m(E)\to 0} \sup_{x\in K} \|x\chi_E\|_X = 0. \tag{5.2}$$

Proof. Assume that the condition (5.2) holds, but there exists a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $x_n \xrightarrow{w} 0$, $x_n \xrightarrow{\mu} 0$ and, however, for some $\varepsilon > 0$ and some subsequence $\{\bar{x}_n\}_{n=1}^\infty \subset \{x_n\}_{n=1}^\infty$, we have

$$\|\bar{x}_n\|_X \geq \varepsilon, \quad n = 1, 2, \dots . \tag{5.3}$$

We set $E_n := \{t \in [0,1] \ : \ |\bar{x}_n(t)| > \frac{\varepsilon}{3}\}$, $n = 1, 2, \dots$. Since $\bar{x}_n \xrightarrow{\mu} 0$, it follows that $m(E_n) \to 0$. Since the set $\{\bar{x}_n\}_{n=1}^\infty$ is relatively weakly compact, it follows from the assumption that there exists $\delta > 0$ such that for any $E \subset [0,1]$, $m(E) < \delta$

$$\|\bar{x}_n\chi_E\|_X \leq \frac{\varepsilon}{3}, \quad n = 1, 2, \dots .$$

Thus, if $N \geq 1$ is such that $m(E_n) < \delta$, $n \geq N$, we obtain $\|\bar{x}_n\chi_{E_n}\|_X \leq \frac{\varepsilon}{3}$, $n \geq N$. On the other hand, by the definition of E_n, $\|\bar{x}_n\chi_{[0,1]\setminus E_n}\|_X \leq \frac{\varepsilon}{3}$, $n = 1, 2, \dots$. Therefore, $\|\bar{x}_n\|_X \leq \frac{2\varepsilon}{3}$ $(n \geq N)$, which contradicts (5.3).

To prove the converse assertion, assume that $X \in (Wm)$ and that there exists a relatively weakly compact set $K \subset X$, for which (5.2) does not hold. Due to Proposition 5.3 the space X is separable, and so, using standard arguments, we obtain that there exist $\eta > 0$, $\{\bar{x}_n\}_{n=1}^\infty \subset K$ and a sequence of pairwise disjoint sets $\{F_n\}_{n=1}^\infty \subset [0,1]$, $n = 1, 2, \dots$, such that

$$\|\bar{x}_n\chi_{F_n}\|_X \geq \eta, \quad n = 1, 2, \dots . \tag{5.4}$$

Setting $y_n := \bar{x}_n\chi_{F_n}$, we obviously have $y_n \xrightarrow{\mu} 0$. To obtain a contradiction with the assumption $X \in (Wm)$, it suffices to show that the sequence $\{y_n\}_{n=1}^\infty$ is weakly null in X. It follows from the weak compactness criteria [11] that the set $\{y_n\}_{n=1}^\infty$ is relatively weakly compact and therefore, without loss of generality, we may assume (passing to a subsequence, if necessary) that $\{y_n\}_{n=1}^\infty$ converges weakly to some $z \in X$. Now, the fact that $z = 0$ follows from [7, Lemma 5.3]. □

Next, we characterize Lorentz and Orlicz spaces having the (Wm)-property.

Proposition 5.6. *If $\varphi(t)$ is an increasing concave function on $[0,1]$ such that $\varphi(+0) = 0$, then the Lorentz space $\Lambda(\varphi)$ has the (Wm)-property.*

Proof. By Theorem 5.5, it suffices to show that for any relatively weakly compact set $K \subset \Lambda(\varphi)$ condition (5.2) holds. The latter follows immediately from [30, Theorem 1]. □

Remark 5.7. *The assumption* $\varphi(+0) = 0$ *is equivalent to the assumption* $\Lambda(\varphi) \neq L_\infty$.

Proposition 5.8. *Let* F *be an increasing convex function on* $(0, \infty)$ *such that* $F(0)$=0. *The Orlicz space* $L_F \in (Wm)$ *if and only if either*

(i) $L_F = L_1$, *or*
(ii) *the complementary function* G *to* F *satisfies*

$$\lim_{x\to+\infty} \frac{G(Cx)}{G(x)} = \infty, \tag{5.5}$$

for some $C > 0$.

Proof. The fact that $L_1 \in (Wm)$ follows immediately from the classical Dunford-Pettis description of relatively weakly subsets in L_1 (see also Proposition 5.6). Let us assume that (5.5) holds. Using [2, Theorem 2.8], we observe that any relatively weakly compact set $K \subset L_F$ satisfies condition (5.2). Note that the assumption $F \in \Delta_2$, which was, in addition, requested in [2, Theorem 2.8] is a consequence of (5.5) (see [19, Theorem 6.5]). Theorem 5.5 now implies that $L_F \in (Wm)$.

For the converse implication, it suffices to show that the conditions $L_F{\in}(Wm)$ and $L_F \neq L_1$ imply (5.5). Since $L_F \neq L_1$, we may assume that F is an N-function. In other words,

$$\lim_{x\to0+} \frac{F(x)}{x} = \lim_{x\to+\infty} \frac{x}{F(x)} = 0.$$

This allows us to employ the Ando's criterion of relative weak compactness in Orlicz spaces [27, p. 144] which we may use, since L_F is separable by Proposition 5.3. By that criterion, the set $\{x_n\}_{n=1}^\infty \subset L_F$ is relatively weakly compact if and only if

$$\lim_{u\to0} \sup_{n\in\mathbf{N}} \frac{1}{u} \int_0^1 F(u|x_n(t)|)\, dt = 0. \tag{5.6}$$

The plan of the proof is as follows. Assuming that (5.5) does not hold, we shall show that there exists a sequence $\{x_n\}_{n=1}^\infty \subset L_F$, $\|x_n\|_{L_F} = 1$, $x_n \xrightarrow{\mu} 0$, such that (5.6) holds. The latter would immediately imply that $L_F \notin (Wm)$, which is sufficient to establish the converse implication.

Let us now assume that (5.5) does not hold, or equivalently, for any $C > 0$ there exists $M = M(C) > 0$ such that

$$G(Ct_k) \le MG(t_k), \quad k = 1, 2, \dots, \tag{5.7}$$

for some $t_k \to \infty$. Passing to the inverse function, we obtain

$$CG^{-1}(\tau_k) \le G^{-1}(M\tau_k), \quad k = 1, 2, \dots, \tag{5.8}$$

where $\tau_k = G(t_k) \to \infty$. Recall that the fundamental function φ_{L_F} of the Orlicz space L_F is given by $\varphi_{L_F}(t) = \|\chi_{(0,t)}\|_{L_F} = \frac{1}{F^{-1}(1/t)}$, $t > 0$ [19, Ch. 2, § 9, p. 72]. Moreover, from [20, Ch. 2, § 4, p. 144], we know that $\varphi_X(t)\cdot\varphi_{X^\times}(t) = t$, $0 \le t \le 1$.

Since $(L_F)^\times = L_G$ [19, Ch. 2, § 14], we obtain $G^{-1}(\tau)\cdot F^{-1}(\tau) = \tau$ ($\tau \geq 1$). Therefore, using (5.8), we see that the assumption that (5.5) fails is equivalent to the assumption that for any $C > 0$ there exists $M = M(C) > 0$ such that

$$F(v_k) \geq MF\left(\frac{C}{M}v_k\right), \quad k = 1,2,\dots,$$

for some $v_k \uparrow \infty$ as $k \to \infty$. The latter guarantees that for every $n \geq 1$ there exists $M_n > 0$ and $v_k^n \uparrow \infty$ as $k \to \infty$ such that

$$F(v_k^n) \geq M_n F\left(\frac{nv_k^n}{M_n}\right), \quad n,k = 1,2,\dots. \tag{5.9}$$

We can assume that $M_n \uparrow \infty$ (see (5.7)) and $v_k^n \uparrow \infty$ as $k \to \infty$ so that for $\alpha_k^n := \frac{1}{F(v_k^n)}$, we have

$$\sum_{k=1}^{\infty} \alpha_k^n \leq 2^{-n}, \quad n = 1,2\dots. \tag{5.10}$$

Since the function $F(t)/t$ is increasing as $t \to \infty$, we infer from (5.9) that for all $u \in (0, n/M_n)$

$$\frac{\alpha_k^n}{u}F(uF^{-1}(1/\alpha_k^n)) \leq \frac{\alpha_k^n}{n}M_nF\left(\frac{n}{M_n}v_k^n\right) \leq \frac{1}{n}, \quad n,k = 1,2,\dots.$$

Thus, for the diagonal sequence $\beta_n := \alpha_n^n$, $n = 1,2,\dots$, we have

$$\frac{\beta_k}{u}F(uF^{-1}(1/\beta_k)) \leq \frac{1}{n}, \quad 0 < u < \frac{n}{M_n}, \quad k \geq n. \tag{5.11}$$

By (5.10), we have $\sum_{n=1}^\infty \beta_n < 1$, and so there exist pairwise disjoint sets $E_n \subset [0,1]$, such that $m(E_n) = \beta_n$, $n \in \mathbf{N}$.

Set $x_n := F^{-1}(1/\beta_n)\chi_{E_n}$, $n = 1,2,\dots$. We have $\|x_n\|_{L_F} = 1$ and $x_n \xrightarrow{\mu} 0$. We observe that condition (5.6) may be rewritten for the set $K := \{x_n\}_{n=1}^\infty \subset L_F$ as

$$\lim_{u\to 0} \sup_{n=1,2,\dots} \frac{\beta_n}{u}F(uF^{-1}(\beta_n^{-1})) = 0. \tag{5.12}$$

For any $\varepsilon > 0$, we choose $n \in \mathbf{N}$ such that $n > \frac{1}{\varepsilon}$. Since $\frac{F(t)}{t} \to 0$ as $t \to 0$, it follows that there exists $\delta_1 > 0$ such that for any $u \in (0,\delta_1)$, we have

$$\frac{\beta_k}{u}F(uF^{-1}(\beta_k^{-1})) < \varepsilon, \quad k = 1,2,\dots,n-1.$$

Due to (5.11) and the choice n, we see that the inequality above also holds for $k \geq n$, whenever $0 < u < \frac{n}{M_n}$. This shows that (5.12) holds and so (5.6) holds for the set $\{x_n\}_{n=1}^\infty \subset L_F$ defined above. This completes the proof. □

Remark 5.9. *If an Orlicz space L_F coincides with some Lorentz space Λ_ψ, then, by Proposition 5.6, we have $L_F \in (Wm)$ and so the complementary function G to the function F satisfies (5.5). The converse implication is not correct. In other words, there exist Orlicz spaces L_F, different from any Lorentz space, for which the function G satisfies (5.5).*

Example 5.10. *Consider the function G given for sufficiently large $t > 0$ by*

$$G(t) = t^{\ln\ln\ln t}.$$

An immediate computation shows that $G'(t) \geq 0$ and $G''(t) \geq 0$, if t is large enough. Consequently, we may assume that $G(t)$ is equivalent to an increasing convex function on $(0, \infty)$. It is easy to check that this function satisfies (5.5) and therefore, by Proposition 5.8, the Orlicz space $L_F \in (Wm)$, where F is the complementary function to G.

At the same time, L_F does not coincide with any Lorentz space. In fact, observing that $L_F^\times = L_G$ and $\Lambda(\varphi)^\times = M(\varphi)$ (see e.g. [20]), we note that the equality $L_F = \Lambda(\varphi)$ implies $L_G = M(\varphi)$ (where $M(\varphi)$ is the Marcinkiewicz space). The latest equality holds if and only if the function $G^{-1}(1/t)(0 < t \leq 1)$ (the "maximal" function of the space $M(\varphi)$ with the fundamental function $\varphi_{M(\varphi)}(t) = \frac{1}{G^{-1}(1/t)}$) belongs to the Orlicz space L_G [29]. But this is not the case, since straightforward calculations show that for every $\lambda > 0$.

$$\int_0^1 G\left(\frac{G^{-1}(1/t)}{\lambda}\right) dt = \infty.$$

6. The (MDCMP) and the Banach-Saks property in general r.i. spaces

We have established in the preceding sections that in classical r.i. spaces (Orlicz, Lorentz and Marcinkiewicz spaces) satisfying the Banach-Saks property Cesaro means of an arbitrary weakly null martingale difference sequence converge strongly. In this section, we shall demonstrate that this is no longer true for general r.i. spaces.

Proposition 6.1. *Let F be a translation-invariant Banach sequence space with the Fatou property so that the canonical basis $(e_n)_{n=1}^\infty$ is a normalized unconditional basis of F. For $\xi = (\xi_j)_{j=1}^\infty \in E$ let $L\xi = (\xi_2, \xi_3, \ldots)$ and $R\xi = (0, \xi_1, \xi_2, \ldots)$. Suppose*

$$\left(\lim_{n\to\infty} \|L^n\|^{1/n}\right)\left(\lim_{n\to\infty} \|R^n\|^{1/n}\right) < 2. \tag{6.1}$$

Then there is a separable r.i. space X with the Fatou property so that:

(i) $0 < \alpha_X < \beta_X < 1$;

(ii) *If $v_n = \chi_{[2^{-n}, 2^{-n+1})}$ then $(v_n/\|v_n\|_X)_{n=1}^\infty$ is equivalent to the canonical basis of F;*

(iii) *Every normalized disjoint sequence $(x_n)_{n=1}^\infty$ has a subsequence equivalent to a block basic sequence of $(e_n)_{n=1}^\infty$ in F.*

Proof. Fix $a < 1$ so that

$$1 \leq \lim_{n\to\infty} \|L^n\|^{1/n} < 2^{1-a}$$

and

$$1 \le \lim_{n\to\infty} \|R^n\|^{1/n} < 2^a.$$

Let $\mathbb{J} = \{-1, -2, \ldots\}$. We define a sequence space E modelled on $\mathbb{J}$ by $\xi \in F$ if and only if $(2^{-an}\xi_{-n})_{n=1}^{\infty} \in E$ with the norm

$$\|\xi\|_E = \|\sum_{j=1}^{\infty} 2^{-aj}\xi_{-j}e_j\|_F.$$

Now in the terminology of [16]

$$\kappa_+(E) = 2^a \lim_{n\to\infty} \|L^n\|^{1/n} < 2 \ \text{ and } \ \kappa_-(E) = 2^{-a} \lim_{n\to\infty} \|R^n\|^{1/n} < 1.$$

Hence by Proposition 5.1 of [16] there is an r.i. space X on $[0,1]$ such that

$$\|f\|_X \approx \|\sum_{n=1}^{\infty} 2^{-an} f^*(2^{-n})e_n\|_F$$

and furthermore

$$2^{\alpha_X} = (\kappa_-(E))^{-1} \ \text{ and } \ 2^{\beta_X} = \kappa_+(E)$$

so that (i) is proved. Moreover, we see that X is separable (since the simple functions are dense there) and X has the Fatou property (since F does). For (ii) we note that this follows from (ii) of Proposition 5.1 of [16].

At last, we show (iii). For each $n = 1, 2, \ldots$ we define

$$z_n(t) = \sum_{k=1}^{\infty} x_n^*(2^{-k})\chi_{[2^{-k-1},2^{-k})}.$$

It is clear that $x_n^*(2t) \le z_n(t) \le x_n^*(t)$ and that $\|z_n\|_X \ge 1/2$. Hence if $(y_n)_{n=1}^{\infty}$ is any disjoint sequence equimeasurable with $(z_n)_{n=1}^{\infty}$ then $(y_n)_{n=1}^{\infty}$ is equivalent to $(x_n)_{n=1}^{\infty}$.

Now since $m(\text{supp } z_n) \le m(\text{supp } x_n)$ we may find increasing sequences of integers $(n_k)_{k=1}^{\infty}$ and $(r_k)_{k=0}^{\infty}$ with $r_0 = 0$ so that

$$\|z_{n_k} - z_{n_k}\chi_{[2^{-r_k},2^{-r_{k-1}})}\|_X < 4^{-k}.$$

Let

$$w_k = z_{n_k}\chi_{[2^{-r_k},2^{-r_{k-1}})}.$$

Then $(w_k)_{k=1}^{\infty}$ is a disjoint sequence in X with $\|w_k\|_X \ge 1/4$. Furthermore we can find y_k equimeasurable with z_{n_k} with disjoint supports so that $w_k = y_k\chi_{[2^{-r_k},2^{-r_{k-1}})}$. Thus $\|y_k - w_k\|_X < 4^{-k}$ and so $(w_k)_{k=1}^{\infty}$ is equivalent to $(y_k)_{k=1}^{\infty}$ and hence to $(x_{n_k})_{k=1}^{\infty}$.

However $(w_k)_{k=1}^{\infty}$ is block basic with respect to $(v_n)_{n=1}^{\infty}$ and hence equivalent to a block basic sequence of $(e_n)_{n=1}^{\infty}$ in F. $\square$

Theorem 6.2. *There is an r.i. space X on $[0,1]$ which has the Banach-Saks property, but which does not have the (MDCMP).*

Proof. We take for F in Proposition 6.1 a translation-invariant Banach sequence space with the Fatou property satisfying condition (6.1) and such that the canonical basis $\{e_k\}_{k=1}^{\infty}$ is a normalized 1-unconditional basis of F satisfying

(a) $e_k \xrightarrow{w} 0$ in F;
(b) there exists $C > 0$ such that $\|\sum_{k=1}^{n} e_k\|_F \geq Cn$, for infinitely many $n \in \mathbf{N}$;
(c) for any block basis $\{f_k\}_{k=1}^{\infty}$ of $\{e_k\}_{k=1}^{\infty}$ such that $f_k \xrightarrow{w} 0$, there exists a subsequence $\{f_{k_i}\}_{i=1}^{\infty} \subseteq \{f_k\}_{k=1}^{\infty}$, such that

$$\frac{1}{m}\|\sum_{i=1}^{m} f_{k_i}\|_F \to 0.$$

Let $v_n = \chi_{[2^{-n},2^{-n+1})}$ and $w_n = v_n/\|v_n\|_X (n = 1, 2, \dots)$. Then, by Proposition 6.1 and assumptions (a) and (b), we have $w_k \xrightarrow{w} 0$ in X and $\|\sum_{k=1}^{n} w_k\|_X \geq C_1 n$ for infinitely many $n \in \mathbf{N}$ for some $C_1 > 0$. Therefore, X does not have the (MDCMP).

We shall show that X has the Banach-Saks property. Since X is separable and has the Fatou property, then by [9, Th. 4.5], it suffices to check only that each weakly null disjointly supported sequence from X contains a subsequence whose Cesaro means converge strongly to zero. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be an arbitrary sequence of pairwise disjoint functions such that $x_n \xrightarrow{w} 0$ and $\|x_n\|_X = 1$. By Proposition 6.1, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$, which is equivalent to some block basis $\{f_k\}_{k=1}^{\infty}$ of $\{e_n\}_{n=1}^{\infty}$. Clearly, $f_k \xrightarrow{w} 0$, and condition (c) implies that there exists a subsequence $\{f_{k_i}\}_{i=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ such that

$$\frac{1}{m}\|\sum_{i=1}^{m} f_{k_i}\|_F \to 0.$$

Therefore, $\frac{1}{m}\|\sum_{i=1}^{m} x_{n_{k_i}}\|_F \to 0$ and thus X has the Banach-Saks property. □

Remark 6.3. *It can be easily verified that the conditions of Theorem 6.2 hold for $F = (\sum \oplus_n \ell_1^{2^n})_{\ell_p}$ $(1 < p < \infty)$ with the norm*

$$\|\xi\|_F = \left(\sum_{r=1}^{\infty}\left(\sum_{k=2^{r-1}}^{2^r-1} |\xi_k|\right)^p\right)^{1/p}.$$

In particular, it is not hard to check that for every $n = 0, \pm 1, \pm 2, \dots$

$$\max(\|L^n\|, \|R^n\|) \leq \log_2 |n| + 2.$$

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