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Positivity

Delta-semidefinite and Delta-convex Quadratic Forms in Banach Spaces

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Abstract. A continuous quadratic form ("quadratic form", in short) on a Banach space X is: (a) *delta-semidefinite* (i.e., representable as a difference of two nonnegative quadratic forms) if and only if the corresponding symmetric linear operator $T\colon X \to X^*$ factors through a Hilbert space; (b) *delta-convex* (i.e., representable as a difference of two continuous convex functions) if and only if T is a UMD-operator. It follows, for instance, that each quadratic form on an infinite-dimensional $L_p(\mu)$ space ($1 \leq p \leq \infty$) is: (a) delta-semidefinite iff $p \geq 2$; (b) delta-convex iff $p > 1$. Some other related results concerning delta-convexity are proved and some open probms are stated.

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Introduction

Let X be a real Banach space. Recall that a function $q\colon X \to \mathbb{R}$ is a *continuous quadratic form* (more precise would be "continuous purely quadratic form") if there exists a continuous bilinear form $b\colon X \times X \to \mathbb{R}$ such that $q(x) = b(x, x)$ for each $x \in X$.

In the present paper, we are interested mainly in the following two isomorphic properties of X.

(D) *Each continuous quadratic form on X is* delta-semidefinite*, i.e., it can be represented as a difference of two nonnegative continuous quadratic forms.*

(dc) *Each continuous quadratic form on X is* delta-convex*, i.e., it can be represented as a difference of two continuous convex functions.*

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Since nonnegative quadratic forms are convex, (D) always implies (dc). The reverse implication is not true, as we shall see in Section 3.

In Section 1, we characterize delta-semidefinite continuous quadratic forms on X as precisely those whose corresponding symmetric linear operator $T\colon X \to X^*$ is factorizable through a Hilbert space. This leads, via known results on factorizability, to sufficient conditions for a Banach space X to satisfy (D). The characterization also implies that the property (D) passes to quotients, and the spaces ℓ_p, $1 \le p < 2$, do not satisfy (D).

In Section 2, we use X-valued Walsh-Paley martingales to prove that a continuous quadratic form on X is delta-convex if and only if the corresponding symmetric linear operator is a UMD-operator. It follows that ℓ_1 not only fails (D) but it also fails (dc).

In Section 3, we discuss relationships between the properties (D), (dc) and the following property.

(Cdc) *Each $C^{1,1}$ function on X is delta-convex.*

It is easy to see that also (Cdc) implies (dc). We show that (dc) and (Cdc) pass to quotients. For each of the properties (D), (dc), (Cdc), we characterize those p's in $[1, \infty]$ for which an infinite-dimensional $L_p(\mu)$ space satisfies the property (Theorem 3.3). It follows that (dc) implies neither (D) nor (Cdc). (The latter should be compared with a result from [10] which says that all Banach space-valued quadratic mappings on X are delta-convex if and only if all Banach space-valued $C^{1,1}$ mappings on X are delta-convex.) We also solve a probm from [20] by proving existence of a function f whose compositions with all "delta-convex curves" (in the sense of [20]) are delta-convex while f is not locally delta-convex. Some of these counterexamples use a result by M. Zelený [23]. Finally, we show that the property (dc) is not stable with respect to direct sums, and we state some open problems.

As usual, B_X and S_X denote the closed unit ball and the unit sphere of the Banach space X, respectively.

1. Delta-semidefinite Quadratic Forms

In what follows, the term "operator" means "bounded linear operator". Recall that an operator $T\colon X \to X^*$ is called *symmetric* if $\langle Tx, y\rangle = \langle Ty, x\rangle$ for all $x, y \in X$ (equivalently: $T^* = T$ on X).

It is easy to see that the formula

$$q(x) = \langle Tx, x\rangle \tag{1}$$

defines a one-to-one correspondence between the continuous quadratic forms q on X and the symmetric operators $T\colon X \to X^*$.
(Indeed, if q is generated by a continuous bilinear form b, it is generated also by the symmetric bilinear form $\frac{b(x,y)+b(y,x)}{2}$. Moreover, there is a unique symmetric bilinear form b that generates q; this follows from the formula

$$2b(x,y) = b(x+y, x+y) - b(x,x) - b(y,y) \tag{2}$$

valid for symmetric b. The rest follows from the well-known one-to-one correspondence, via the formula $b(x,y) = \langle Tx, y\rangle$, between the continuous bilinear forms b on $X \times X$ and the operators $T\colon X \to X^*$.)
If (1) holds for each $x \in X$, we say that T *generates* q.

The formula (2) also implies the following

Fact 1.1. *Each continuous quadratic form q on X is everywhere Fréchet differentiable. Moreover, its Fréchet derivative at x is given by $q'(x) = 2Tx$ where T is the symmetric operator that generates q.*

The following theorem characterizes delta-semidefinite continuous quadratic forms. (Recall that a continuous quadratic form is called *delta-semidefinite* if it is the difference of two nonnegative continuous quadratic forms.) An operator $T\colon X \to Y$ is said to be *factorizable through* Z if there exist operators $A\colon X \to Z$ and $B\colon Z \to Y$ such that $T = BA$.

Theorem 1.2. *Let q be a continuous quadratic form on a Banach space X, and $T\colon X \to X^*$ be the symmetric operator that generates q. Then the following assertions are equivalent:*

(i) *q is delta-semidefinite;*
(ii) *there exists a continuous quadratic form p on X, such that $|q| \le p$;*
(iii) *T is factorizable through a Hilbert space.*

Proof.

(iii) $\Rightarrow$ (i). If $T = BA$ where $A\colon X \to H$ and $B\colon H \to X^*$ are operators, and H is a Hilbert space, then we have

$$q(x) = \langle BAx, x\rangle = \langle Ax, B^*x\rangle_H = \tfrac{1}{4}\|Ax + B^*x\|_H^2 - \tfrac{1}{4}\|Ax - B^*x\|_H^2$$

which shows that q is difference of two nonnegative quadratic forms.

(i) $\Rightarrow$ (ii). If $q = p_1 - p_2$ where p_i $(i = 1, 2)$ are nonnegative continuous quadratic forms, then $|q| \le p_1 + p_2 =: p$.

(ii) $\Rightarrow$ (iii). Let (ii) hold, and let $S\colon X \to X^*$ be the symmetric operator such that $p(x) = \langle Sx, x\rangle$. The function

$$[\cdot,\cdot]\colon X/\mathrm{Ker}(S) \times X/\mathrm{Ker}(S) \to \mathbb{R}, \quad [\xi, \eta] := \langle Sx, y\rangle \text{ where } x \in \xi,\ y \in \eta,$$

is well-defined, bilinear, symmetric, and $[\xi,\xi] \ge 0$ for each $\xi \in X/\mathrm{Ker}(S)$. Moreover, if $p(x) = 0$ for some $x \in X$, then x is a minimizer for p, and hence $0 = p'(x) = 2Sx$ by Fact 1.1. In other words, $[\xi,\xi] = 0$ implies $\xi = 0$. Consequently, $[\cdot,\cdot]$ is an inner product on $X/\mathrm{Ker}(S)$. Let H be the completion of the inner product space $(X/\mathrm{Ker}(S), [\cdot,\cdot])$.

Consider the operator $J = iQ\colon X \to H$ where $Q\colon X \to X/\mathrm{Ker}(S)$ is the quotient map and $i\colon X/\mathrm{Ker}(S) \hookrightarrow H$ is the inclusion map. (J is continuous since $\|Qx\|_H^2 = \langle Sx, x\rangle \le \|S\| \cdot \|x\|^2$ for all $x \in X$.)

If $x \in \mathrm{Ker}(S)$, then $p(x) = q(x) = 0$. Since $p + q$ is a nonnegative quadratic form generated by the symmetric operator $T + S$, the same argument as above shows that $Tx + Sx = 0$. This proves that $\mathrm{Ker}(S) \subset \mathrm{Ker}(T)$. Consequently, the operator

$$T_0\colon X/\mathrm{Ker}(S) \to X^*, \quad T_0\xi := Tx \text{ where } x \in \xi,$$

is well-defined. We claim that T_0 is continuous also in the norm generated by the inner product $[\cdot,\cdot]$. To prove this, consider $\xi \in X/\mathrm{Ker}(S)$ and $y \in X$ such that $\|\xi\|_H \le 1$ and $\|y\| \le 1$. Fix $x \in \xi$, and denote $\eta = Qy$. Then $\|\eta\|_H^2 = \langle Sy, y\rangle \le \|S\|$, and $|\langle T_0\xi, y\rangle| \le |\langle Tx, y\rangle| = \frac{1}{2}\,|q(x+y) - q(x) - q(y)| \le \frac{1}{2}\,[p(x+y) + p(x) + p(y)]$ $= \frac{1}{2}\left[\|\xi+\eta\|_H^2 + \|\xi\|_H^2 + \|\eta\|_H^2\right] \le \frac{1}{2}\left[(1 + \|S\|^{1/2})^2 + 1 + \|S\|\right]$.

Thus T_0 has a unique extension to an operator from H into X^*; let us denote it by T_0 again. Then $T = T_0 J$ is the desired factorization through H. □

Corollary 1.3. (a) *A Banach space X has the property (D) (see Introduction) if and only if each symmetric operator $T\colon X \to X^*$ is factorizable through a Hilbert space.*

(b) *The property (D) passes to quotients, and hence also to complemented subspaces.*

Proof. (a) follows immediately from Theorem 1.2. Let us show (b). Let X satisfy (D), and let L be a closed subspace of X. Let $T\colon X/L \to (X/L)^* = L^\perp$ be a symmetric operator. Consider the operator $S = iTQ\colon X \to X^*$ where $Q\colon X \to X/L$ is the quotient map, and $i\colon L^\perp \to X^*$ is the inclusion isometry. Since $Q^* = i$, we have $\langle Sx, y\rangle = \langle T(Qx), (Qy)\rangle$ $(x, y \in X)$, which shows that S is symmetric. By (a), S is factorizable through a Hilbert space. Now, Proposition 7.3 in [9] implies that T factors through a Hilbert space, too. □

Operators that are factorizable through a Hilbert space were intensively studied (the main reference is [16], see also [9]), and there exist many sufficient conditions for factorizability of all operators between two given spaces. Thus, by Theorem 1.2, we obtain various sufficient conditions for validity of the property (D) (defined in Introduction); we collect them in the following theorem.

For the classical notion of modulus of smoothness, see e.g. [13]. For the notion of type and cotype, see e.g. [13] or [9]. We shall need the following notion of second order differentiability, studied in [2].

Definition 1.4. (a) *Let f be a continuous convex function on a Banach space X. We say that f is* second order differentiable *at a point $x_0 \in X$ if there exist $x_0^* \in X^*$ and a continuous quadratic form q on X such that, for each $v \in X$,*

$$f(x_0 + tv) = f(x_0) + x_0^*(v)t + q(v)t^2 + o(t^2) \qquad \text{as } t \to 0.$$

(b) *A* second order differentiable norm *is a norm which is second order differentiable at each nonzero point.*

Remark 1.5. *It follows from results in [2] that a norm $\|\cdot\|$ on X is second order differentiable iff it is Fréchet (equivalently: Gâteaux) smooth and its derivative $\|\cdot\|'\colon X \setminus \{0\} \to X^*$ is* weak**-Gâteaux differentiable.*

Theorem 1.6. *Let X be a Banach space. Each continuous quadratic form on X is delta-semidefinite (and hence delta-convex), provided* at least one *of the following conditions is satisfied.*

(a) *X has type 2.*
(b) *X^* has cotype 2, and X has the approximation property.*
(c) *X^* has cotype 2, and X does not contain $\ell_1(n)$'s uniformly.*
(d) *X^* has cotype 2, and X is a Banach lattice.*
(e) *$X = C(K)$ for some compact space K.*
(f) *$X = L_p(\mu)$ for $2 \le p \le \infty$ and some positive measure μ.*
(g) *$X = c_0(I)$ for some set I.*
(h) *X admits a uniformly smooth renorming with modulus of smoothness of power type 2 (i.e., $\varrho_X(\tau) \le a\tau^2$ for some $a > 0$).*
(i) *X has the Radon–Nikodým property and admits an equivalent second order differentiable norm.*

Proof. (a) follows from Corollary 3.6 and Proposition 3.2 in [16].
(b) see Theorem 4.1 in [16].
(c) follows from (a) by [15, Corollary 2.5].
(d) follows from Theorems 8.17 and 8.11 in [16].
(e) follows e.g. from (d) since $C(K)^*$ has cotype 2 (see [16, p.34]).
(f) the case $p < \infty$ follows from (a) (see [13, p.73]); the case $p = \infty$ follows from (e) (see [13, Theorem 1.b.6]).
(g) follows from (e) by Corollary 1.3(b), since $c_0(\Gamma)$ is a closed hyperplane in $c(\Gamma) = C(K)$ where K is the one-point compactification of the discrete set Γ.
(h) follows from (a) (see Theorem 1.e.16 in [13]).
(i) follows from (h) by the following reasoning. If the norm of X is second order differentiable, then this norm is Lipschitz-smooth at each point of S_X by [2]; this implies (by [11, Lemma 2.4]) that the gradient of the norm is pointwise Lipschitz at each point of S_X. By [7, Corollary III.2], if X has also the RNP, then it satisfies (h). $\square$

Results in [19, Section 5] imply that, for each $1 < p < 2$, there exists an operator $U : \ell_p \to \ell_{p^*}$ (where $\frac{1}{p} + \frac{1}{p^*} = 1$) such that U is not factorizable through a Hilbert space. Since U, constructed in [19], is also symmetric, we obtain one more corollary to Theorem 1.2.

Corollary 1.7. *The space ℓ_p fails the property (D) whenever* $1 \le p < 2$.
(The case of $p = 1$ follows from Corollary 1.3(b) and from the well-known fact that each separable Banach space is isometric to a quotient of ℓ_1.)

2. Delta-convex Quadratic Forms

Let X be a Banach space. Recall that a continuous function $\varphi : X \to \mathbb{R}$ is *delta-convex* if it is the difference of two continuous convex functions on X. It is easy to see that φ is delta-convex if and only if there exists a (necessarily convex) continuous function ψ on X such that both $\pm\varphi + \psi$ are convex. Every such function ψ is called a *control function* for φ. Denoting

$$\Delta^2\varphi(x, y) := \varphi(x + y) + \varphi(x - y) - 2\varphi(x)\,, \qquad x, y \in X,$$

it is easy to see that ψ is a control function for φ if and only if $|\Delta^2\varphi(x, y)| \leq \Delta^2\psi(x, y)$ for all $x, y \in X$.

Since every nonnegative quadratic form is convex, each delta-semidefinite quadratic form is delta-convex. As we shall see in Section 3, the converse is not true in general.

In this section, we use X-valued Walsh-Paley martingales to study delta-convexity of quadratic forms. We recall all needed definitions and properties to make our exposition self-contained.

Let $n \geq 1$ be an integer, $\Gamma = \{-1, 1\}$, $f\colon \Gamma^n \to X$. Then the *expectation* of f is defined as $\mathbb{E}f = 2^{-n}\sum_{\eta\in\Gamma^n} f(\eta) = \int_{\Gamma^n} f\,d\mathbb{P}$, where $\mathbb{P} = \mathbb{P}_n$ is the uniformly distributed probability measure on Γ^n.

For $0 \leq k \leq n$, consider the σ-algebra $\Sigma_k = \{A \times \Gamma^{n-k} : A \subset \Gamma^k\}$. Obviously, a function $f\colon \Gamma^n \to X$ is Σ_k-measurable if and only if f depends only on the first k coordinates (in particular, all Σ_0-measurable functions are constant). For this reason, we sometimes view Σ_k-measurable functions on Γ^n as functions on Γ^k.

For $f\colon \Gamma^n \to X$ and $0 \leq k \leq n$, the *conditional expectation of F w.r.t. Σ_k* is the Σ_k-measurable function $\mathbb{E}(f|\Sigma_k)\colon \Gamma^n \to X$ which has the same integral (w.r.t. $\mathbb{P}$) as f over each element of Σ_k. It is easy to see that it is given by

$$\mathbb{E}(f|\Sigma_k)(\omega) = \textstyle\int_{\Gamma^{n-k}} f(\omega, \cdot)\,d\mathbb{P}_{n-k}\,, \qquad \omega \in \Gamma^k.$$

Note that $\mathbb{E}(f|\Sigma_0) \equiv \mathbb{E}f$, $\mathbb{E}(f|\Sigma_n) = f$, and $\mathbb{E}(\mathbb{E}(f|\Sigma_k)) = \mathbb{E}f$.

In this paper, we consider only Walsh-Paley martingales of finite length.

Definition 2.1. *Let X be a Banach space. An X-valued* Walsh-Paley martingale *is any finite sequence $(f_0, \ldots, f_n)$ of X-valued functions on Γ^n such that $f_k = \mathbb{E}(f_n|\Sigma_k)$ for each $0 \leq k \leq n$; or equivalently, each f_k is Σ_k-measurable, and*

$$f_k(\omega) = \tfrac{1}{2} f_{k+1}(\omega, -1) + \tfrac{1}{2} f_{k+1}(\omega, 1) \quad \textit{whenever } 0 \leq k < n \textit{ and } \omega \in \Gamma^k.$$

Given a Walsh-Paley martingale $(f_0, \ldots, f_n)$, the corresponding martingale differences *are the functions $df_k = f_k - f_{k-1}$ $(1 \leq k \leq n)$.*

Remark 2.2. *Let $(f_0, \ldots, f_n)$ be an X-valued Walsh-Paley martingale. The above definition easily implies the following properties.*

(a) $\mathbb{E}(df_k|\Sigma_j) = 0$ *whenever* $0 \leq j < k \leq n$.
(b) $(f_0, \ldots, f_n, f_n, \ldots, f_n)$ *is a Walsh-Paley martingale no matter how many times f_n is repeated.*
(c) $(Tf_0, \ldots, Tf_n)$ *is a Y-valued Walsh-Paley martingale whenever $T\colon X \to Y$ is linear.*
(d) *If $0 \leq m < n$ and $\overline{\omega} \in \Gamma^m$, then the finite sequence $(g_0, \ldots, g_{n-m})$, where $g_k := f_{m+k}(\overline{\omega}, \cdot)\colon \Gamma^{n-m} \to X$, is a Walsh-Paley martingale.*

(e) $df_k(\omega) = \frac{1}{2}\omega_k\,[f_k(\omega_1,\dots,\omega_{k-1},1) - f_k(\omega_1,\dots,\omega_{k-1},-1)]$ *whenever* $1 \le k \le n$ *and* $\omega \in \Gamma^k$.
(f) $f_{k-1}(\omega) \pm df_k(\omega) = f_k(\omega_1,\dots,\omega_{k-1},\pm\omega_k)$ *whenever* $1 \le k \le n$ *and* $\omega \in \Gamma^k$.

A finite sequence $(\varepsilon_1,\dots,\varepsilon_n)$ of functions on Γ^n is said to be *predictable* if ε_k is Σ_{k-1}-measurable for each $1 \le k \le n$.

Lemma 2.3. *Let $(f_0,\dots,f_n)$ be an X-valued Walsh-Paley martingale.*

(a) *If $\varphi\colon X \to \mathbb{R}$ is a function, then $\mathbb{E}\sum_{k=1}^n \Delta^2\varphi(f_{k-1},df_k) = 2\mathbb{E}\varphi(f_n) - 2\varphi(f_0)$.*
(b) *If $1 \le k \le n$ and $w\colon \Gamma^n \to \mathbb{R}$ is Σ_{k-1}-measurable, then $\mathbb{E}(w\,df_k) = 0$.*
(c) *If $(g_0,\dots,g_n)$ is an X^*-valued Walsh-Paley martingale and $(\varepsilon_1,\dots,\varepsilon_n)$ is a predictable sequence of real-valued functions, then $\mathbb{E}\langle df_k, \varepsilon_j dg_j\rangle = 0$ whenever $k \ne j$ and $k,j \in \{1,\dots,n\}$.*
(d) $(\mathbb{E}\max\limits_{0\le k\le n}\|f_k\|^p)^{1/p} \le \frac{p}{p-1}(\mathbb{E}\|f_n\|^p)^{1/p}$ *for each real $p > 1$.*

Proof. (a) Recall that f_0 is constant. By Remark 2.2(f), the left-hand side equals

$$\sum_{k=1}^n \int_{\Gamma^k} [\varphi(f_k(\omega)) + \varphi(f_k(\omega_1,\dots,\omega_{k-1},-\omega_k)) - 2\varphi(f_{k-1}(\omega))]\,d\mathbb{P}_k(\omega)$$
$$= 2\sum_{k=1}^n (\mathbb{E}\varphi(f_k) - \mathbb{E}\varphi(f_{k-1})) = 2\mathbb{E}\varphi(f_n) - 2\mathbb{E}\varphi(f_0).$$

follows easily from Remark 2.2(a).

Let, e.g., $k < j$. Obviously, $\langle df_k, \varepsilon_j dg_j\rangle$ is Σ_j-measurable. For each fixed $\omega \in \Gamma^{j-1}$, we have

$$\mathbb{E}(\langle df_k, \varepsilon_j dg_j\rangle|\Sigma_{j-1})(\omega) = \mathbb{E}\langle df_k(\omega), \varepsilon_j(\omega)dg_j(\omega,\cdot)\rangle = \langle df_k(\omega), \varepsilon_j(\omega)\mathbb{E}dg_j(\omega,\cdot)\rangle = 0$$

by Remark 2.2(d),(a). Thus $\mathbb{E}\langle df_k, \varepsilon_j dg_j\rangle = \mathbb{E}\left(\mathbb{E}(\langle df_k, \varepsilon_j dg_j\rangle|\Sigma_k)\right) = 0$. The case $k > j$ is similar.
(d) For real-valued martingales, this is the well-known Doob's L_p-inequality (see e.g. [22, Theorem 14.11]). In the general case, consider the Walsh-Paley martingale $(g_0,\dots,g_n)$ given by $g_k = \mathbb{E}(\|f_n\|\,|\Sigma_k)$. Then $\|f_k\| = \|\mathbb{E}(f_n|\Sigma_k)\| \le g_k$ and $\|f_n\| = g_n$. Hence, using the scalar case, we get $(\mathbb{E}\max\limits_{0\le k\le n}\|f_k\|^p)^{1/p} \le (\mathbb{E}\max\limits_{0\le k\le n} g_k^p)^{1/p} \le \frac{p}{p-1}(\mathbb{E}g_n^p)^{1/p} = \frac{p}{p-1}(\mathbb{E}\|f_n\|^p)^{1/p}$. □

Lemma 2.4. *Let φ be a continuous real function on a Banach space X. In order that φ is delta-convex it is necessary and sufficient that there is a continuous function $\varrho\colon X \to [0,\infty)$ such that if $(f_0,\dots,f_n)$ is an X-valued Walsh-Paley martingale then*

$$\mathbb{E}\sum_{k=1}^n |\Delta^2\varphi(f_{k-1},df_k)| \le \mathbb{E}\varrho(f_n). \tag{3}$$

Moreover, in this case, ϱ can always be taken convex.

Proof. Let φ be delta-convex with a control function ψ. By adding a suitable affine function, we can (and do) suppose that $\psi \geq 0$. Now Lemma 2.3(a) implies

$$\begin{aligned}\mathbb{E}\sum_{k=1}^{n}|\Delta^2\varphi(f_k, df_k)| &\leq \mathbb{E}\sum_{k=1}^{n}\Delta^2\psi(f_k, df_k)\\ &= 2\mathbb{E}\psi(f_n) - 2\psi(f_0)\\ &\leq 2\mathbb{E}\psi(f_n).\end{aligned}$$

Thus (3) holds with $\varrho = 2\psi$ (which is convex).

Conversely, if (3) holds, we may define

$$\psi(x) = \tfrac{1}{2}\inf\left\{\mathbb{E}\varrho(f_n) - \mathbb{E}\sum_{k=1}^{n}|\Delta^2\varphi(f_{k-1}, df_k)|\right\} \tag{4}$$

where the infimum is taken over all X-valued Walsh-Paley martingales with $f_0 = x$.

Suppose $x, u \in X$, $y = x + u, z = x - u$ and $\varepsilon > 0$. Using Remark 2.2(b) and the definition of ψ, pick X-valued Walsh-Paley martingales $(f_0, \ldots, f_n)$ and $(g_0, \ldots, g_n)$ such that $f_0 = y$, $g_0 = z$ and

$$\begin{aligned}\psi(y) &> \frac{1}{2}\,\mathbb{E}\varrho(f_n) - \frac{1}{2}\,\mathbb{E}\sum_{k=1}^{n}|\Delta^2\varphi(f_{k-1}, df_k)| - \varepsilon\\ \psi(z) &> \frac{1}{2}\,\mathbb{E}\varrho(g_n) - \frac{1}{2}\,\mathbb{E}\sum_{k=1}^{n}|\Delta^2\varphi(g_{k-1}, dg_k)| - \varepsilon.\end{aligned}$$

Form a new Walsh-Paley martingale $(h_0, \ldots, h_{n+1})$ by setting $h_0 = x$ and, for $1 \leq k \leq n+1$,

$$h_k(\eta_1, \ldots, \eta_{n+1}) = \begin{cases} f_{k-1}(\eta_2, \ldots, \eta_{n+1}) & \text{if } \eta_1 = 1\\ g_{k-1}(\eta_2, \ldots, \eta_{n+1}) & \text{if } \eta_1 = -1.\end{cases}$$

Note that, for example,

$$\mathbb{E}\varrho(h_{n+1}) = \int\limits_{\{\eta_1=1\}} \varrho(h_{n+1})\,d\mathbb{P}_{n+1} + \int\limits_{\{\eta_1=-1\}} \varrho(h_{n+1})\,d\mathbb{P}_{n+1} = \frac{1}{2}\mathbb{E}\varrho(f_n) + \frac{1}{2}\mathbb{E}\varrho(g_n).$$

Thus

$$\begin{aligned}2\psi(x) &\leq \mathbb{E}\varrho(h_{n+1}) - \mathbb{E}\sum_{k=1}^{n+1}|\Delta^2\varphi(h_{k-1}, dh_k)|\\ &= \tfrac{1}{2}\,\mathbb{E}\varrho(f_n) + \tfrac{1}{2}\,\mathbb{E}\varrho(g_n)\\ &\quad -|\Delta^2\varphi(x,u)| - \frac{1}{2}\,\mathbb{E}\sum_{k=1}^{n}|\Delta^2\varphi(f_{k-1}, df_k)| - \frac{1}{2}\,\mathbb{E}\sum_{k=1}^{n}|\Delta^2\varphi(g_{k-1}, dg_k)|\\ &\leq \psi(x) + \psi(y) + 2\varepsilon - |\Delta^2\varphi(x,u)|.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that $|\Delta^2\varphi(x,u)| \leq \Delta^2\psi(x,u)$ whenever $x, u \in X$. Thus ψ is a midconvex (or Jensen convex) function which is locally bounded since $0 \le \psi \le \varrho/2$. Consequently (see [18, p.215]), ψ is a continuous convex function. Thus ψ is a control function for φ. $\square$

Observation 2.5. *For each $g\colon \Gamma^n \to [0,\infty)$ and $p>0$, we have*

$$\sum_{j=1}^{\infty} 2^{jp}\,\mathbb{P}(g>2^j) \le \frac{2^p}{2^p-1}\,\mathbb{E}g^p.$$

Indeed, $\int g^p\,d\mathbb{P} \ \ge\ \sum\limits_{j=1}^{\infty} \int\limits_{\{2^{j-1}<g\le 2^j\}} g^p\,d\mathbb{P} \ \ge\ \sum\limits_{j=1}^{\infty} 2^{(j-1)p}\,\big[\mathbb{P}(g>2^{j-1})-\mathbb{P}(g>2^j)\big]$

$$= \sum_{j=0}^{\infty} 2^{jp}\,\mathbb{P}(g>2^j) - \sum_{j=1}^{\infty} 2^{(j-1)p}\,\mathbb{P}(g>2^j) \ge (1-2^{-p})\sum_{j=1}^{\infty} 2^{jp}\,\mathbb{P}(g>2^j).$$

Let $p>0$. Recall that a function $\varphi\colon X\to\mathbb{R}$ is called *positively p-homogeneous* if $\varphi(tx)=t^p\varphi(x)$ whenever $t\ge 0$, $x\in X$.

Lemma 2.6. *Suppose $p>1$. A continuous positively p-homogeneous function $\varphi\colon X\to\mathbb{R}$ is delta-convex if and only if there is a constant C such that for all X-valued Walsh-Paley martingales $(f_0,\dots,f_n)$ we have*

$$\mathbb{E}\sum_{k=1}^{n} |\Delta^2\varphi(f_{k-1},df_k)| \le C\mathbb{E}\|f_n\|^p. \tag{5}$$

Proof. Assume φ is delta-convex. Let ϱ be the corresponding continuous function from Lemma 2.4. Choose $r>0$ so that $C_0 := \sup\{\varrho(x) : \|x\|\le r\}<\infty$. Then

$$\mathbb{E}\sum_{k=1}^{n} |\Delta^2\varphi(f_{k-1},df_k)| \le \mathbb{E}\varrho(f_n) \le C_0 \qquad \text{whenever } \|f_n\|_\infty \le r,$$

where $\|g\|_\infty = \max_{\eta\in\Gamma^n}|g(\eta)|$ as usual. Hence, for an arbitrary Walsh-Paley martingale $(f_0,\dots,f_n)$, p-homogeneity implies that

$$\mathbb{E}\sum_{k=1}^{n} |\Delta^2\varphi(f_{k-1},df_k)| \le C_1\|f_n\|_\infty^p \tag{6}$$

where $C_1 = C_0/r^p$.

Now, fix any X-valued Walsh-Paley martingale $(f_0,\dots,f_n)$ with $\mathbb{E}\|f_n\|^p=1$. (By p-homogeneity, it suffices to prove (5) for such martingales.) Let $\eta\in\Gamma^n$. We define $m_0(\eta)=0$ and, for any integer $r\ge 1$,

$$M_r(\eta) = \{0\le k<n : \max\{\|f_k(\eta)\pm df_{k+1}(\eta)\|\} > 2^r\}$$

and

$$m_r(\eta) = \begin{cases} \min M_r(\eta) & \text{if } M_r(\eta)\ne\emptyset,\\ n & \text{if } M_r(\eta)=\emptyset.\end{cases}$$

For each $m\in\{0,\dots,n\}$, the set $\{m_r=m\}$ belongs to Σ_m by Remark 2.2(f). (Thus the functions m_r are so-called "stopping times".) Hence it can be written in the form

$$\{m_r=m\} = A_{r,m}\times\Gamma^{n-m}, \qquad \text{where } A_{r,m}\subset\Gamma^m.$$

We have

$$\begin{aligned}
&\mathbb{E} \sum_{m_{r-1}<k\le m_r} |\Delta^2\varphi(f_{k-1}, df_k)| \\
&\quad = \sum_{m=0}^{n-1} \int_{\{m_{r-1}=m\}} \sum_{m<k\le m_r(\eta)} |\Delta^2\varphi(f_{k-1}(\eta), df_k(\eta))|\, d\mathbb{P}_n(\eta) \\
&\quad = \sum_{m=0}^{n-1} \int_{A_{r-1,m}} \left(\int_{\Gamma^{n-m}} \sum_{m<k\le m_r(\eta)} |\Delta^2\varphi(f_{k-1}(\omega,\xi), df_k(\omega,\xi))|\, d\mathbb{P}_{n-m}(\xi) \right) d\mathbb{P}_m(\omega)\,.
\end{aligned}$$

The expression in parentheses can be seen as

$$\mathbb{E} \sum_{m<k\le n} |\Delta^2\varphi(g_{k-1}(\omega,\cdot), dg_k(\omega,\cdot))|\,, \tag{7}$$

where

$$g_k(\omega,\xi) = \begin{cases} f_k(\omega,\xi) & \text{if } m<k\le m_r(\omega,\xi), \\ f_{m_r(\omega,\xi)}(\omega,\xi) & \text{if } m_r(\omega,\xi)<k\le n. \end{cases}$$

Since $(g_k(\omega,\cdot))_{k=m}^n$ is a Walsh-Paley martingale by Remark 2.2(d), and the definition of m_r implies $\|f_{m_r(\eta)}(\eta)\| = \|f_{m_r(\eta)-1}(\eta) + df_{m_r(\eta)}(\eta)\| \le 2^r$, we can majorize the expression (7) (using (6)) by

$$C_1\|g_n(\omega,\cdot)\|_\infty^p = C_1\|f_{m_r(\omega,\cdot)}(\omega,\cdot)\|_\infty^p \le C_1 2^{rp}.$$

Thus

$$\begin{aligned}
\mathbb{E}\sum_{k=1}^n |\Delta^2\varphi(f_{k-1}, df_k)| &= \sum_{r=1}^\infty \mathbb{E} \sum_{m_{r-1}<k\le m_r} |\Delta^2\varphi(f_{k-1}, df_k)| \\
&\le C_1 \sum_{r=1}^\infty 2^{rp} \sum_{m=0}^{n-1} \mathbb{P}_m(A_{r-1,m}) = C_1 \sum_{r=1}^\infty 2^{rp}\, \mathbb{P}(m_{r-1}<n).
\end{aligned}$$

Now, for $r>1$, Remark 2.2(f) implies that

$$\mathbb{P}(m_{r-1}<n) \le \mathbb{P}\left(\max_{1\le k<n} \max\{\|f_k \pm df_{k+1}\|\} > 2^{r-1} \right) \le 2\mathbb{P}\left(\max_{0\le k\le n} \|f_k\| > 2^{r-1} \right).$$

This gives (via Observation 2.5 and Lemma 2.3(d))

$$\begin{aligned}
\mathbb{E}\sum_{k=1}^n |\Delta^2\varphi(f_{k-1}, df_k)| &\le C_1 2^p + C_1 2 \sum_{r=2}^\infty 2^{rp}\, \mathbb{P}(\max_{0\le k\le n} \|f_k\| > 2^{r-1}) \\
&= C_1 2^p + C_1 2^{p+1} \sum_{j=1}^\infty 2^{jp}\, \mathbb{P}(\max_{0\le k\le n} \|f_k\| > 2^j) \\
&\le C_1 2^p + C_1 \frac{2^{2p+1}}{2^p-1}\, \mathbb{E} \max_{0\le k\le n} \|f_k\|^p \\
&\le C_2\,(1+\mathbb{E}\|f_n\|^p) = 2C_2\,,
\end{aligned}$$

where C_2 is a suitable constant depending only on p. Thus (5) holds.

The converse follows trivially from Lemma 2.4 by putting $\varrho(x) = C\|x\|^p$. $\square$

Corollary 2.7. *Suppose $p \geq 1$. Then every positively p-homogeneous delta-convex function $\varphi\colon X \to \mathbb{R}$ has a control function which is positively p-homogeneous.*

Proof. The case $p = 1$ was proved in [20, Lemma 1.21]. Assume $p > 1$. By Lemma 2.6, (3) holds with $\varrho(x) = C\|x\|^p$. By the proof of Lemma 2.4, the formula (4) defines a positively p-homogeneous control function for φ. $\square$

Remark 2.8. *Let us remark that natural analogues of Lemma 2.4, Lemma 2.6 and Corollary 2.7 hold also for mappings $\Phi\colon X \to Y$ (instead of functions $\varphi\colon X \to \mathbb{R}$), where "delta-convex function" is replaced by "delta-convex mapping" (as defined in [20]) and, in the terms involving Φ, the absolute value is replaced by the norm of Y. This follows from [20, Proposition 1.13].*

Definition 2.9. *Let X and Y be Banach spaces. We say that a linear operator $T\colon X \to Y$ is a* UMD-operator *if there exists a constant $C > 0$ such that*

$$\mathbb{E}\|\textstyle\sum_{k=1}^n \varepsilon_k\, T df_k\|^2 \leq C\, \mathbb{E}\|f_n\|^2 \tag{8}$$

whenever $(f_0, \ldots, f_n)$ is an X-valued Walsh-Paley martingale and $\varepsilon_1, \ldots, \varepsilon_n$ are numbers in $\{-1, 1\}$. We say that X is a UMD-space *if the identity $I\colon X \to X$ is a UMD-operator.*

Remark 2.10. (a) *It is easy to see that a composition of two bounded linear operators is a UMD-operator whenever at least one of them is. In particular, if at least one of X, Y is a UMD-space, then each bounded linear operator $T\colon X \to Y$ is a UMD-operator.*

(b) *Suppose $p > 1$ is a real number. Then X is a UMD-space if and only if is a constant $c_p > 0$ such that*

$$\mathbb{E}\|\sum_{k=1}^n \varepsilon_k\, df_k\|^p \leq c_p\, \mathbb{E}\|f_n\|^p$$

whenever $\varepsilon_k = \pm 1$ ($1 \leq k \leq n$) and $(f_0, \ldots, f_n)$ is a Walsh-Paley martingale. Moreover, in this case the above inequality holds also for general (i.e. not necessarily Walsh-Paley) martingales. (See p.67 and Lemma 7.1 in [6].)

(c) *Every UMD-space is superreflexive. (See e.g. [17, p.222] or [1, Proposition 2].)*

For us the following result, which was proved in [21], will be important. Let us remark that a similar result for general martingales in UMD-spaces was proved by Burkholder [5] and, as remarked in [4, p.502], his proof can be easily modified to prove the same for UMD-operators defined using general martingales.

Fact 2.11. *Let $T\colon X \to Y$ be a UMD-operator between Banach spaces X, Y. Then there exists a constant $C > 0$ such that (8) holds whenever $(f_0, \ldots, f_n)$ is an X-valued Walsh-Paley martingale and $(\varepsilon_1, \ldots, \varepsilon_n)$ is a predictable sequence of $\{\pm 1\}$-valued functions (i.e., each ε_k is Σ_{k-1}-measurable).* (See [21].)

Theorem 2.12. *Let q be a continuous quadratic form on X and $T\colon X \to X^*$ the symmetric operator that generates q. Then q is delta-convex if and only if T is a UMD-operator.*

Proof. Let T be a UMD-operator, let $(f_0, \dots, f_n)$ be an X-valued Walsh-Paley martingale. Using the identity

$$\Delta^2 q(x, y) = 2q(y), \tag{9}$$

we can write, for $\eta \in \Gamma^n$ and $1 \le k \le n$,

$$|\Delta^2 q(f_{k-1}(\eta), df_k(\eta))| = 2|q(df_k(\eta))| = 2\varepsilon_k(\eta) q(df_k(\eta)) = 2\varepsilon_k(\eta)\langle df_k(\eta), T df_k(\eta)\rangle,$$

where $\varepsilon_k(\eta) = 1$ if $q(df_k(\eta)) \ge 0$, $\varepsilon_k(\eta) = -1$ if $q(df_k(\eta)) < 0$. Observe that ε_k is Σ_{k-1}-measurable by Remark 2.2(e) since q is an even function; in other words, the sequence $(\varepsilon_1, \dots, \varepsilon_n)$ is predictable. Using Lemma 2.3, we can write

$$\begin{aligned}
\mathbb{E}\sum_{k=1}^{n} |\Delta^2 q(f_{k-1}, df_k)| &= 2\mathbb{E}\sum_{k=1}^{n}\langle df_k, \varepsilon_k T df_k\rangle = 2\mathbb{E}\left\langle \sum_{j=1}^{n} df_j\,, \sum_{k=1}^{n}\varepsilon_k T df_k \right\rangle \\
&= 2\mathbb{E}\left\langle f_n - f_0, \sum_{k=1}^{n}\varepsilon_k T df_k \right\rangle = 2\mathbb{E}\left\langle f_n, \sum_{k=1}^{n}\varepsilon_k T df_k \right\rangle \\
&\le 2(\mathbb{E}\|f_n\|^2)^{1/2}\left(\mathbb{E}\|\sum_{k=1}^{n}\varepsilon_k T df_k\|^2\right)^{1/2} \le 2\sqrt{C}\ \mathbb{E}\|f_n\|^2\,,
\end{aligned}$$

where C is the constant from Fact 2.11.

For the converse, suppose that q is delta-convex. Consider any X-valued Walsh-Paley martingale $(f_0, \dots, f_n)$ with $\mathbb{E}\|f_n\|^2 = 1$, and numbers $\varepsilon_k \in \{-1, 1\}$ $(1 \le k \le n)$. It is easy to see that there exists $h_n\colon \Gamma^n \to X$ such that $\mathbb{E}\|h_n\|^2 = 1$ and

$$\mathbb{E}\left(\|\sum_{k=1}^{n}\varepsilon_k T df_k\|^2\right)^{1/2} \le 2\mathbb{E}\left\langle h_n, \sum_{k=1}^{n}\varepsilon_k T df_k \right\rangle\,;$$

indeed, denoting $g = \sum_{k=1}^{n}\varepsilon_k T df_k$, one can put $h(\eta) := t(\eta)v(\eta)$, where $t\colon \Gamma^n \to [0, \infty)$ satisfies $\mathbb{E}t^2 = 1$ and $(\mathbb{E}\|g\|^2)^{1/2} = \mathbb{E}(t \cdot \|g\|)$, and $v(\eta) \in S_X$ is such that $\|g(\eta)\| \le 2\langle v(\eta), g(\eta)\rangle$. Let $(h_0, \dots, h_n)$ be the Walsh-Paley martingale given by h_n (i.e., $h_k = \mathbb{E}(h_n|\Sigma_k)$, $1 \le k \le n$). Then Lemma 2.6 and the identities

$$\langle x, Ty\rangle = (1/4)\left(q(x+y) - q(x-y)\right) \quad \text{and} \quad (9)$$

imply

$$\begin{aligned}
\mathbb{E}\left(\|\sum_{k=1}^{n}\varepsilon_k T df_k\|^2\right)^{1/2} &\leq 2\mathbb{E}\left\langle h_n, \sum_{k=1}^{n}\varepsilon_k T df_k\right\rangle = \qquad \text{(as above)}\\
&= 2\mathbb{E}\sum_{k=1}^{n}\langle dh_k, \varepsilon_k T df_k\rangle \leq 2\mathbb{E}\sum_{k=1}^{n}|\langle dh_k, T df_k\rangle|\\
&\leq \frac{1}{2}\mathbb{E}\sum_{k=1}^{n}|q(d(f_k+h_k))| + \frac{1}{2}\mathbb{E}\sum_{k=1}^{n}|q(d(f_k-h_k))|\\
&\leq \frac{1}{4}C\,\mathbb{E}\|f_n+h_n\|^2 + \frac{1}{4}C\,\mathbb{E}\|f_n-h_n\|^2\\
&\leq \frac{1}{2}C\mathbb{E}\left(\|f_n\|+\|h_n\|\right)^2 \leq C\left(\mathbb{E}\|f_n\|^2+\mathbb{E}\|h_n\|^2\right) \leq 2C.
\end{aligned}$$

Thus T is a UMD-operator. □

The following theorem is an immediate consequence of Theorem 2.12 and Remark 2.10(a).

Theorem 2.13. *Let X be a Banach space. Then every continuous quadratic form on X is delta-convex if and only if every symmetric operator $T : X \to X^*$ is a UMD-operator. In particular, if X is a UMD-space, then every continuous quadratic form on X is delta-convex.*

Theorem 2.14. *There exists a continuous quadratic form on ℓ_1 which is not delta-convex.*

Proof. Let $J\colon \ell_1 \to \ell_\infty$ be an isometric embedding (recall that every separable Banach space isometrically embeds into ℓ_∞). Consider the continuous quadratic form on $\ell_1 = \ell_1 \oplus_1 \ell_1$, given by

$$q(x,y) = \langle y, Jx\rangle + \langle x, Jy\rangle\,, \qquad x,y \in \ell_1\,,$$

which is generated by the symmetric operator $T(x,y) = (Jy, Jx)$. If q were delta-convex, T would be a UMD-operator. But then J would be a UMD-operator; consequently, ℓ_1 would be a UMD-space. But this is false by Remark 2.10(c). □

Let us conclude with a simple but useful proposition.

Proposition 2.15. *Let $p > 0$. Let $\varphi\colon X \to \mathbb{R}$ be a p-homogeneous function on a Banach space X. Then φ is delta-convex if and only if φ is delta-convex on a convex neighborhood of the origin.*

Proof. Let φ be delta-convex on a convex neighborhood U of the origin, and let $\psi\colon U \to \mathbb{R}$ be a corresponding control function. There exists $\delta > 0$ such that ψ is bounded on δB_X. A simple homogeneity argument shows that φ is delta-convex on each rB_X ($r > 0$) with a bounded control function of the form $\varrho(x) = c_1\psi(c_2 x)$. Then φ is delta-convex on X by [12, Theorem 16]. □

3. Further Results and Open Problems

We shall consider the following three properties of a Banach space X, defined already in Introduction.

(D) *Each continuous quadratic form on X is delta-semidefinite.*
(dc) *Each continuous quadratic form on X is delta-convex.*
(Cdc) *Each $C^{1,1}$ function $f\colon X \to \mathbb{R}$ is delta-convex.*

Recall that a function (or mapping) f is $C^{1,1}$ if the Fréchet derivative $f'(x)$ exits for each x and the mapping f' is Lipschitz.

We have seen that (D) passes to quotients (Corollary 1.3). Let us observe the same result for properties (dc) and (Cdc).

Lemma 3.1. *If X is a Banach space with property (dc) (respectively, (Cdc)), then for any closed subspace E of X, the quotient X/E has property (dc) (respectively, (Cdc)).*

Proof. Let $Q\colon X \to X/E$ be the quotient map and let $f\colon X/E \to \mathbb{R}$ be a continuous function such that $f \circ Q$ is delta-convex. We show that f is delta-convex (which proves both assertions). Let $\psi\colon X \to \mathbb{R}$ be a continuous convex function such that $\psi \pm (f \circ Q)$ is convex; we can assume $\psi \geq 0$. Define $\hat{\psi}\colon X/E \to \mathbb{R}$ by $\hat{\psi}(y) = \inf\{\psi(x) : Qx = y\}$. Then it is easy to prove that $\hat{\psi} \pm f$ is convex. Moreover, the (convex) function $\hat{\psi}$ is continuous since it is easily seen to be bounded on a neighborhood of the origin. □

Since each continuous quadratic form is $C^{1,1}$ (by Fact 1.1), we always have the implications

$$\text{(D)} \implies \text{(dc)} \impliedby \text{(Cdc)}\,.$$

As we shall see in the next theorem, no two of the above three properties are equivalent.

Let us start with the following corollary of [10, Theorem 11]. A norm on X is said to have modulus of convexity of power type 2 if, for some $c > 0$, $\delta_X(\varepsilon) \geq c \cdot \varepsilon^2$ whenever $\varepsilon \in (0, 2]$ (where δ_X is the usual modulus of convexity; see e.g. [13]).

Fact 3.2. *Let X be a Banach space that admits a uniformly convex renorming with modulus of convexity of power type 2. Then X satisfies (Cdc) and hence also (dc).*

By an $L_p(\mu)$ space we mean an infinite-dimensional space $L_p(\Omega, \Sigma, \mu)$ where (Ω, Σ, μ) is a positive measure space. This class includes the spaces $L_p(0, 1)$ and ℓ_p. For such spaces, we have the following theorem which summarizes results of [10], [23] and of the present paper.

Theorem 3.3. *Let X be an infinite-dimensional $L_p(\mu)$ space with $1 \leq p \leq \infty$.*

(a) *X satisfies* (D) *if and only if $p \geq 2$.*
(b) *X satisfies* (Cdc) *if and only if $1 < p \leq 2$.*
(c) *X satisfies* (dc) *if and only if $p > 1$.*

Proof. (a) If $p \geq 2$ then $L_p(\mu)$ satisfies (D) by Theorem 1.6(f). If $p < 2$ then $L_p(\mu)$ fails (D) since it contains a complemented copy of ℓ_p which fails (D) by Corollary 1.7.

(b) If $1 < p \leq 2$, then the standard norm on $X = L_p(\mu)$ has modulus of convexity of power type 2 (see [8, Corollary V.1.2]). By Fact 3.2, each such space satisfies (Cdc). $L_1(\mu)$ fails (Cdc) since it fails (dc) (see (c) below). Now, let $2 < p < \infty$. For such p, M. Zelený [23] proved that ℓ_p fails (Cdc); thus $L_p(\mu)$ fails (Cdc), too. Finally, to see that also $L_\infty(\mu)$ fails (Cdc), it suffices to show that $L_\infty(0,1)$ fails (Cdc); indeed, the spaces $L_\infty(0,1)$ and ℓ_∞ are isomorphic by [14], and $L_\infty(\mu)$ contains a complemeted copy of ℓ_∞. By [13, Corollary 2.f.5], $(\ell_4)^* = \ell_{4/3}$ isometrically embed in $L_1(0,1)$; consequently, ℓ_4 is isomorphic to a quotient of $L_\infty(0,1)$. Hence $L_\infty(0,1)$ fails (Cdc) by Lemma 3.1, since we already know that ℓ_4 fails (Cdc).

(c) $L_1(\mu)$ fails (dc) since it contains a complemented copy of ℓ_1 which fails (dc) by Theorem 2.14. For $p > 1$, the space $L_p(\mu)$ satisfies (dc) since, by (a) and (b), it satisfies (D) (if $p \geq 2$) or (Cdc) (if $p \leq 2$). Alternately one may observe that $L_p(\mu)$ is a UMD-space if $1 < p < \infty$ using Remark 2.10(b) and then apply Theorem 2.13. ☐

Remark 3.4. *By Theorem 3.3,* (dc) $\not\Rightarrow$ (D) *and also* (dc) $\not\Rightarrow$ (Cdc). *It is interesting to compare the second non-implication with the following result from [10] about vector-valued mappings:* every Banach space-valued continuous quadratic mapping on X is delta-convex if and only if every Banach space-valued $C^{1,1}$ mapping on X is delta-convex.

Delta-convex functions via delta-convex curves

There is another corollary to the above results. It is connected with Problem 6 in [20]. In that paper, delta-convex mappings between Banach spaces (a generalization of delta-convex functions) were defined and widely studied. We do not state the definition here; it can be found also in [10] together with a survey of principal results. We confine ourselves to stating an equivalent definition (see [20, Theorem 2.3]) of a delta-convex mapping of one real variable.

Definition 3.5. *Let $I \subset \mathbb{R}$ be an open interval, X be a Banach space. A mapping $\varphi\colon I \to X$ is* delta-convex *on I if the right derivative $\varphi'_+(t)$ exists at each $t \in I$ and the mapping φ'_+ has bounded variation on each compact subinterval of I.*

For real-valued functions, Problem 6 in [20] asks: *suppose that X is a Banach space and $f\colon X \to \mathbb{R}$ is a function such that $f \circ \varphi$ is a delta-convex function on* $(0,1)$ *whenever $\varphi\colon (0,1) \to X$ is a delta-convex mapping; is then f locally delta-convex?* The following example answers in negative this problem. (Let us remark that the vector-valued case was solved in negative already in [10].)

Example 3.6. *Let X be an infinite-dimensional $L_p(\mu)$ space where either $p = 1$ or $2 < p < \infty$. Then there exists a continuous function $f\colon X \to \mathbb{R}$ such that f is delta-convex on no neighborhood of* 0*, and $f \circ \varphi$ is delta-convex on* $(0,1)$ *for each delta-convex mapping $\varphi\colon (0,1) \to X$.*

Proof. The case $p = 1$. By Theorem 3.3(b), there exists a continuous quadratic form q on X such that q is not delta-convex. By Proposition 2.15, q is delta-convex on no neighborhood of 0. By Proposition 14 in [10], $q \circ \varphi$ is delta-convex for each delta-convex "curve" φ. Thus we can put $f = q$.

The case $2 < p < \infty$ follows in a similar way using [23] instead of Theorem 3.3. Indeed, by [23], there exists a $C^{1,1}$ function $g\colon \ell_p \to \mathbb{R}$ that is not delta-convex. A careful look at the proof in [23] shows that the function constructed therein is d.c. on no neighborhood of 0. Consider ℓ_p as a complemented subspace of $X = L_p(\mu)$ and extend g to a $C^{1,1}$ function on the whole X by $f = g \circ P$ where P is a bounded linear projection of X onto ℓ_p. Then f has the desired property by [10, Proposition 14] again. $\square$

It is natural to ask the following

Problem 3.7. *Does there exist a function f as in Example 3.6 for each at least two-dimensional Banach space X? Or, at least, for each infinite-dimensional Banach space X?*

Stability with respect to direct sums

Consider two Banach spaces X_1 and X_2. Since $(X_1 \oplus X_2)^* = X_1^* \oplus X_2^*$, each bounded linear operator operator $T\colon X_1 \oplus X_2 \to (X_1 \oplus X_2)^*$ can be represented as an operator-valued matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ where $T_{ij}\colon X_j \to X_i^*$. It is an easy exercise to verify that:

(I) T is factorizable through a Hilbert space if and only if each T_{ij} is;
(II) T is a UMD-operator if and only if each T_{ij} is;
(III) T is symmetric if and only if $T_{ij}^* = T_{ji}$ on X_i whenever $i, j \in \{1, 2\}$ (equivalently, T_{11} and T_{22} are symmetric and $T_{12}^* = T_{21}$ on X_1).

Hence we have the following consequence of Corollary 1.3 and Theorem 2.12. (Given Banach spaces X and Y, we denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X into Y.)

Corollary 3.8. *Let X_1 and X_2 be Banach spaces. Then:*

(a) *$X_1 \oplus X_2$ has the property (D) if and only if X_1 and X_2 have (D) and every element of $\mathcal{L}(X_1, X_2^*)$ is factorizable through a Hilbert space;*
(b) *$X_1 \oplus X_2$ has the property (dc) if and only if X_1 and X_2 have (dc) and every element of $\mathcal{L}(X_1, X_2^*)$ is a UMD-operator.*

In particular, if X is isomorphic to X^2 then

(a') *X has the property (D) if and only if every element of $\mathcal{L}(X, X^*)$ is factorizable through a Hilbert space;*
(b') *X has the property (dc) if and only if every element of $\mathcal{L}(X, X^*)$ is a UMD-operator.*

The following Corollary is immediate using Remark 2.10(a) for part (b).

Corollary 3.9. *Let X be a Banach space.*

(a) *If X satisfies (D) and H is a Hilbert space, then $X \oplus H$ satisfies (D).*
(b) *If X satisfies (dc) and U is a UMD-space, then $X \oplus U$ satisfies (dc).*

Observation 3.10. *The adjoint T^* is a UMD-operator if and only if T is a UMD-operator. In particular, X is a UMD-space if and only if X^* is.*
To see this, note that $T\colon X \to Y$ is a UMD-operator if and only if the operators

$$T_{n,\varepsilon} := \sum_{k=1}^{n} \varepsilon_k T(E_{k,X} - E_{k-1,X})\colon L_2(\Gamma^n, X) \to L_2(\Gamma^n, Y), \quad n \in \mathbb{N},\ \varepsilon \in \{-1,1\}^n,$$

are equi-bounded, where $E_{k,X} := \mathbb{E}(\cdot|\Sigma_k)\colon L_2(\Gamma^n, X) \to L_2(\Gamma^n, X)$. But then also the corresponding adjoints are equi-bounded. Moreover, it is easy to see that

$$T^*_{n,\varepsilon} = \sum_{k=1}^{n} \varepsilon_k (E_{k,X^*} - E_{k-1,X^*})T^* = \sum_{k=1}^{n} \varepsilon_k T^*(E_{k,Y^*} - E_{k-1,Y^*})$$

which means that T^* is a UMD-operator. The reverse implication follows easily from this one.

Proposition 3.11. *Let X be a Banach space. Let Q be the continuous quadratic form on $X \oplus X^*$, given by $Q(x, x^*) = x^*(x)$. Then the following three assertions are equivalent:*

(i) *$X \oplus X^*$ satisfies (D);*
(ii) *Q is delta-semidefinite;*
(iii) *X is isomorphic to a Hilbert space.*

And also the following three assertions are equivalent:

(i') *$X \oplus X^*$ satisfies (dc);*
(ii') *Q is delta-convex;*
(iii') *X is a UMD-space.*

Proof. The implications (iii)⇒(i)⇒(ii) are obvious. Assume (ii). Since Q is generated by the symmetric operator $T\colon X \oplus X^* \to X^* \oplus X^{**}$, $T(x, x^*) = \frac{1}{2}(x^*, x)$, it follows from Theorem 1.2 that the identity $I\colon X \to X$ factors through a Hilbert space. It is easy to see that this implies (iii). (Indeed, if $I = BA$ is a factorization through a Hilbert space H, then AB is a bounded linear projection onto a closed subspace $H_0 = A(X)$ of H. Then A is a linear isomorphism between X and H_0.)

The implication (i')⇒(ii') is obvious. If (ii') holds, then (as above, via Theorem 2.12) the identity $I\colon X \to X$ is a UMD-operator, which gives (iii'). Finally, if (iii') holds, then $X \oplus X^*$ is a UMD-space (indeed, it suffices to apply (II) before Corollary 3.8 to the identity operator of $X \oplus X^*$, taking into account Observation 3.10). Hence (i') holds by Theorem 2.13. □

As far as we know, the following question is open.

Problem 3.12. *Is the property (D) stable with respect to making direct sums of two spaces? Equivalently, if Banach spaces X_1 and X_2 have property (D), does it imply that each $S \in \mathcal{L}(X_1, X_2^*)$ is factorizable through a Hilbert space?*

We conjecture that the answer is negative, but we do not know any counterexample. However, the following observation shows that a possible counterexample cannot be found by using only spaces provided by Theorem 1.6.

Observation 3.13. *Let each of given two Banach spaces X_1 and X_2 satisfy at least one of the conditions (a)–(i) in Theorem 1.6. Then $X_1 \oplus X_2$ has (D).*

Proof. By Corollary 3.8, it suffices to show that every operator $S \in \mathcal{L}(X_1, X_2^*)$ factors through a Hilbert space. By the proof of Theorem 1.6, each of the spaces X_i $(i = 1, 2)$ has at least one of the following three properties:

(α) X_i has type 2;
(β) X_i^* has cotype 2, and X_i has the approximation property;
(γ) X_i^* has cotype 2, and X_i is a Banach lattice.

First observe that this implies that X_2^* has cotype 2 (see e.g. [16, Proposition 3.2]). Now, if X_1 satisfies (α), apply [16, Corollary 3.6]; if X_1 satisfies (β), apply [16, Theorem 4.1]; if X_1 satisfies (γ), apply [16, Theorems 8.17 and 8.11]. □

Note that Proposition 3.11 implies that Problem 3.12 will have a negative answer once the following probm is solved in negative.

Problem 3.14. *Let X and X^* satisfy (D). Does it imply that X is isomorphic to a Hilbert space?*

For the property (dc) we have the following theorem.

Theorem 3.15. *The property (dc) is not stable under making direct sums.*

Proof. By [3], there exists a Banach lattice X such that X is not a UMD-space and X satisfies an upper-3 estimate and a lower-4 estimate (see [13] for definitions). By [13, Theorem 1.f.7], X is 2-convex and 5-concave. Then X admits a uniformly smooth renorming with modulus of smoothness of power type 2 by [13, Theorem 1.f.1], which implies that X has (D) (see Theorem 1.6(h)), and hence (dc). By duality (see [13, p.63]), X^* admits a uniformly convex renorming with modulus of convexity of power type 2; consequently, X^* has (dc) by Fact 3.2. By Proposition 3.11, $X \oplus X^*$ fails (dc) since X is not a UMD-space. □

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