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HOMOMORPHISMS OF GROUP ALGEBRAS WITH NORM LESS THAN $\sqrt{2}$

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We show that two locally compact abelian groups G_1 and G_2 are isomorphic if there exists an algebra isomorphism T of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| < \sqrt{2}$. This constant is best possible. The same result is proved for locally compact connected groups, but for the general locally compact group, the result is proved under the hypothesis $\|T\| < 1.246$. Similar results are given for the algebras $C(G)$ and $L^\infty(G)$ when G is compact. In the abelian case, we give a representation theorem for isomorphisms satisfying $\|T\| < \sqrt{2}$.

1. Introduction. In **[13]**, Wendel proved that, for locally compact groups G_1 and G_2, if T is an algebra isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| \leqq 1$, then G_1 and G_2 are isomorphic. Similar results for $M(G)$, $C_c(G)$ and $L^p(G)$ have been proved in **[3]**, **[5]**, **[6]**, **[7]**, **[11]**, **[12]**, **[14]** and **[15]**. For abelian groups, better results are known. In **[8]**, it is shown that two locally compact abelian groups G_1 and G_2 are isomorphic if there exists an algebra isomorphism T of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| < \frac{1}{2}\sqrt{5}$, and in view of a result of Saeki in **[9]**, this can be improved to the condition $\|T\| < \frac{1}{2}(1+\sqrt{2})$, (see **[8]** 4.6.3 (c)). Saeki's later paper **[10]** makes it possible to extend the result even further to the condition $\|T\| < \frac{1}{4}(1+\sqrt{17})$.

We prove the result for abelian groups with the condition $\|T\| < \sqrt{2}$ and this is the best possible constant, as Example 1 shows. In fact we characterize all algebra isomorphisms T of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| < \sqrt{2}$ as follows:

Case 1. If $0 < \|T\| < \frac{1}{2}(1+\sqrt{3})$, then T has the form

$$(Tf)(x) = \psi(x) f(s(x))$$

where $s\colon G_2 \to G_1$ is a topological isomorphism and $\psi \in \hat{G}_2$. In this case T is an isometry.

Case 2. If $\frac{1}{2}(1+\sqrt{3}) \leqq \|T\| < \sqrt{2}$, then T has the form

$$(Tf)(x) = \psi(x)\{\tfrac{1}{2}(1+\varphi(x)) f(s(x)) + \tfrac{1}{2}(1-\varphi(x)) f(s(x)u)\}$$

where $s: G_2 \to G_1$ is a topological isomorphism, $\psi, \varphi \in \hat{G}_2$ with φ of odd order n, and $u \in G_1$ is an element of order 2. In this case $\|T\| = \sqrt{2}\cos \pi/4n$.

Again $\frac{1}{2}(1+\sqrt{3})$ is the best possible constant as Example 2 shows. It is interesting to note that the isomorphisms given in case (2) above correspond to the idempotents characterized by Saeki in **[10]**. We give similar results for homomorphisms. In **[8]** Chapter 4, algebra homomorphisms of $L^1(G_1)$ into $M(G_2)$ are represented by means of piecewise affine maps from a subset of $\hat{G}_2$ to $\hat{G}_1$, (see **[8]** p. 78 for definitions). Our proofs do not however use this representation. We prove the result first for discrete groups by a computational argument, and then use this to prove the general result. An advantage of our method is that it generalises to give corresponding results in the nonabelian case. We show that if G_1 and G_2 are locally compact connected groups and T is an algebra isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| < \sqrt{2}$, then G_1 and G_2 are isomorphic. Without connectedness, we can prove the result under the condition $\|T\| < \lambda_0$ where λ_0 is the root of a cubic equation ($\lambda_0 \sim 1{\cdot}247$). In these cases we cannot describe the form of the isomorphism, nor do we know whether these constants are best possible. There are corresponding results in each case for isomorphisms of the convolution algebras $M(G)$, $L^\infty(G)$ and $C(G)$ (in the latter cases we assume G compact).

The idea for the paper came from the generalization of the Banach–Stone theorem due to Cambern **[2]** and Amir **[1]** which states that, for compact Hausdorff spaces X and Y, if $T: C(X) \to C(Y)$ is a linear isomorphism with $\|T\|\|T^{-1}\| < 2$, then X and Y are homeomorphic. In the event, we were able to bypass this result because the extra condition we have when X and Y are groups, that T is a convolution algebra isomorphism enables us to find the map from X to Y directly and under the weaker hypothesis of $\|T\| < \sqrt{2}$.

NOTATION. As usual, $C(G)$, $L^1(G)$, $L^\infty(G)$ and $M(G)$ will denote the continuous functions, the integrable functions, the bounded measurable functions, and the bounded measures respectively on a locally compact group G. To avoid confusion we will write $l_1(G)$ in place of $L^1(G)$ when G is discrete. We will use f, g, etc. to denote elements of $l_1(G)$ or $L^1(G)$ and x, y, z, u, etc. to denote elements of the group G or the corresponding elements of $l_1(G)$. $\hat{G}$ will denote the dual group of G if G is abelian, and $\hat{f}$ the Fourier transform of $f \in L^1(G)$. If s is a homomorphism of G_1 into G_2, then $\hat{s}$ will denote the induced homomorphism of $\hat{G}_2$ into $\hat{G}_1$.

We begin with some examples from finite groups. Let $\mathbf{Z}_n$ denote the cyclic group of order n.

EXAMPLE 1. The group algebras of $\mathbf{Z}_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ are isomorphic under the map defined by

$$Tx = \tfrac{1}{2}(1+i)y + \tfrac{1}{2}(1-i)z$$

where x is a generator of $\mathbf{Z}_4$ and y, z are generators of $\mathbf{Z}_2 \times \mathbf{Z}_2$. As a map of $l_1(\mathbf{Z}_4)$ $(= L^1(\mathbf{Z}_4))$ into $l_1(\mathbf{Z}_2 \times \mathbf{Z}_2)$ (or $C(\mathbf{Z}_4)$ into $C(\mathbf{Z}_2 \times \mathbf{Z}_2)$) $\|T\| = \sqrt{2}$.

EXAMPLE 2. There is an isomorphism of $l_1(\mathbf{Z}_6)$ (or $C(\mathbf{Z}_6)$) into itself which is not induced directly by a group isomorphism. If x is a generator of $\mathbf{Z}_6$ define T by

$$Tx = -\tfrac{1}{2}x + \frac{i\sqrt{3}}{2}x^4.$$

Then $\|T\| = \frac{1}{2}(1+\sqrt{3})$ which is less than $\sqrt{2}$.

EXAMPLE 3. There is an algebra embedding of $l_1(\mathbf{Z}_2)$ in $l_1(\mathbf{Z}_4)$ of small norm which is not induced by a group map. If x generates $\mathbf{Z}_2$ and y generates $\mathbf{Z}_4$, define T by

$$Tx = \tfrac{1}{4}(1-i)y + \tfrac{1}{2}y^2 + \tfrac{1}{4}(1+i)y^3.$$

Then $Tx^2 = \frac{1}{2}e + \frac{1}{4}(1+i)y + \frac{1}{4}(1-i)y^3$ which is an idempotent in $l_1(\mathbf{Z}_4)$. Also $\|T\| = \frac{1}{2}(1+\sqrt{2})$.

EXAMPLE 4. There exists an algebra epimorphism of $l_1(\mathbf{Z}_6)$ onto $l_1(\mathbf{Z}_2 \times \mathbf{Z}_2)$ which is induced by a group homomorphism which is not an epimorphism. Let x generate $\mathbf{Z}_6$, y and z generate $\mathbf{Z}_2 \times \mathbf{Z}_2$, and define T by

$$Tx = \tfrac{1}{2}y - \frac{i\sqrt{3}}{2}z.$$

Then $Tx^2 = -\frac{1}{2}e - (i\sqrt{3}/2)yz$ and $Tx^3 = -y$, and T is clearly an epimorphism, but there is no epimorphism of $\mathbf{Z}_6$ onto $\mathbf{Z}_2 \times \mathbf{Z}_2$. In this case $\|T\| = \frac{1}{2}(1+\sqrt{3})$.

The dual map T^* is a monomorphism of $C(\mathbf{Z}_2 \times \mathbf{Z}_2)$ into $C(\mathbf{Z}_6)$ which is not induced by an epimorphism, (see Theorem 3.5).

2. Discrete abelian groups. In this section we shall study algebra homomorphisms $T: l_1(G_1) \to l_1(G_2)$ which preserve the

identity. Although we shall restrict attention to the abelian case, our first result is true for any pair of groups G_1 and G_2, and we shall use it later in §4. We shall denote the identities of G_1 and G_2 by e_1 and e_2 respectively.

PROPOSITION 2.1. *Let G_1 and G_2 be any two groups and suppose $T: l_1(G_1) \to l_1(G_2)$ is a bounded algebra homomorphism such that $Te_1 = e_2$. Then if $x \in G_1$ and $Tx = \sum_{i=1}^{\infty} \alpha_i y_i$ where $y_i \in G_2$ are distinct, then there exists j, such that $|\alpha_j| \geq \|T\|^{-1}$. If $\|T\| < \sqrt{2}$, then j is unique.*

Proof. Let $Tx^{-1} = \sum_{i=1}^{\infty} \beta_i y_i^{-1} + \sum_{i=1}^{\infty} \gamma_i z_i$ where the (z_i) are distinct and disjoint from (y_i^{-1}). Then $e_2 = Tx * Tx^{-1}$ and so identifying the coefficient of e_2 we obtain $\sum_{i=1}^{\infty} \alpha_i \beta_i = 1$. Hence

$$\begin{aligned} 1 &\leqq \sup |\alpha_i| (\Sigma |\beta_i|) \\ &\leqq (\sup |\alpha_i|) \|Tx^{-1}\| \end{aligned}$$

so that $\sup |\alpha_i| \geqq \|T\|^{-1}$. Since $\alpha_i \to 0$, there exists j such that $|\alpha_j| \geqq \|T\|^{-1}$.

Suppose $\|T\| < \sqrt{2}$, and k is another index such that $|\alpha_k| \geqq \|T\|^{-1}$. Then $\|Tx\| \geqq |\alpha_j| + |\alpha_k| \geqq 2 \cdot \|T\|^{-1} \geqq \sqrt{2}$, which is a contradiction.

On $l_1(G)$, there is an involution defined by $(\Sigma \alpha_i x_i)^* = \Sigma \bar{\alpha}_i x_i^{-1}$. We shall also define the l_2-norm by $\|\Sigma \alpha_i x_i\|_2 = \sqrt{(\Sigma |\alpha_i|^2)}$ (where the (x_i) are distinct).

PROPOSITION 2.2. *Suppose G_1 and G_2 are abelian groups and $T: l_1(G_1) \to l_1(G_2)$ is a bounded algebra homomorphism such that $Te_1 = e_2$. Then T is a *-map and for $x \in G_1$, $\|Tx\|_2 = 1$.*

Proof. Let χ be any character on G_2. Then χ induces a multiplicative linear functional $\tilde{\chi}: l_1(G_2) \to C$. $\tilde{\chi} \circ T$ is multiplicative on $l_1(G_1)$, and hence for $f \in l_1(G_1)$

$$\tilde{\chi}(Tf^*) = \overline{\tilde{\chi}(Tf)}$$

This is true for any $\chi \in \hat{G}_2$ and hence $Tf^* = (Tf)^*$ (since $l_1(G_2)$ is semi-simple).

Now $e_2 = Tx * Tx^{-1} = Tx * (Tx)^*$. Hence if $Tx = \sum_{i=1}^{\infty} \alpha_i y_i$, we have, identifying the coefficient of e_2

$$1 = \Sigma |\alpha_i|^2 = \|Tx\|_2^2.$$

For the next three lemmas, we assume that G_1 and G_2 are abelian and $T: l_1(G_1) \to l_1(G_2)$ is an algebra homomorphism such that $Te_1 = e_2$ and $\|T\| < \sqrt{2}$. We define, for $x \in G_1$, $t(x) \in G_2$ by

$$Tx = \alpha t(x) + \sum_{i=1}^{\infty} \beta_i y_i$$

where $|\alpha| \geqq \|T\|^{-1}$ and $t(x) \neq y_i$ for any i. t is then a well-defined map by Proposition 2.1. It follows from the fact that T is a *-map that $t(x^{-1}) = (t(x))^{-1}$. However, in general, t is not a homomorphism. The next two lemmas investigate the consequences of $t(x^2)$ being equal or not equal to $(t(x))^2$.

LEMMA 2.3. *Let* $x \in G_1$ *and* $t(x^2) = (t(x))^2$. *Suppose* $Tx = \alpha t(x) + f$ *and* $Tx^2 = \beta t(x^2) + g$ *where* f *and* g *contain no terms in* $t(x)$ *and* $t(x^2)$ *respectively. Then* $\|g\| \geqq (2/\|T\|^2)\|f\|$.

Proof. By multiplication

$$Tx^2 = \alpha^2 t(x)^2 + 2\alpha t(x) * f + f * f.$$

We consider two cases:

(a) If $|\alpha| \geqq \|T\|^{-\frac{1}{2}}$

then
$$\begin{aligned} \|g\| &\geqq 2|\alpha|\|f\| - \|f\|^2 \\ &= (3|\alpha| - \|Tx\|)\|f\| \\ &\geqq (3\|T\|^{-\frac{1}{2}} - \|T\|)\|f\|. \end{aligned}$$

For $1 \leqq \theta \leqq 2^{\frac{3}{4}}$, let $\varphi(\theta) = 3\theta - \theta^2$, then since $\varphi''(\theta) = -2$, φ is concave and so

$$\varphi(\theta) \geqq \min(\varphi(1), \varphi(2^{\frac{3}{4}})) \quad \text{for} \quad 1 \leqq \theta \leqq 2^{\frac{3}{4}}.$$

Hence
$$3\theta - \theta^2 \geqq 2$$

(since $\varphi(1) = 2$, and $\varphi(2^{\frac{3}{4}}) = 2(3.2^{-\frac{1}{4}} - \sqrt{2}) > 2$). Thus since $1 \leqq \|T\| \leqq \sqrt{2}$, and letting $\theta = \|T\|^2$

$$\|g\| \geqq \frac{2}{\|T\|^2}\|f\|.$$

(b) Next suppose $|\alpha| < \|T\|^{-\frac{1}{2}}$ (note that this implies that $\|T\| > 1$). For convenience we shall let $k = \|T\|^{-\frac{1}{2}}$. Then $f * f$ will contain a term in $t(x)^2$ of magnitude at least $|\beta| - |\alpha|^2 \geqq k^2 - |\alpha|^2 > 0$. Hence

$$\|g\| \geqq 2|\alpha|\|f\| - \|f\|^2 + k^2 - |\alpha|^2$$
$$= k^2 - (|\alpha| - \|f\|)^2$$

and, as $\|f\| \neq 0$, $\|g\|/\|f\| \geqq 1/\|f\|\ (k^2 - (|\alpha| - \|f\|)^2)$. If we define

$$\varphi(\theta) = \theta^{-1}(k^2 - (|\alpha| - \theta)^2),$$

then

$$\varphi'(\theta) = \theta^{-2}(|\alpha|^2 - k^2) - 1 \leqq 0 \quad \text{for} \quad \theta > 0.$$

Hence as $\|f\| \leqq \|T\| - |\alpha|$

$$\frac{\|g\|}{\|f\|} \geqq \frac{1}{k^{-2} - |\alpha|}\,(k^2 - (2|\alpha| - k^{-2})^2),$$

where $k^2 < |\alpha| < k$. If we write $\gamma = k^{-2} - |\alpha|$

$$\frac{\|g\|}{\|f\|} \geqq \frac{1}{\gamma}\,(k^2 - (k^{-2} - 2\gamma)^2)$$

and $k^{-2} - k < \gamma < k^{-2} - k^2$.

If $\psi(\theta) = \theta^{-1}(k^2 - (k^{-2} - 2\theta)^2)$

then $\psi'(\theta) = \theta^{-2}(k^{-4} - k^2) - 4$

and so $\psi''(\theta) \leqq 0$ for $\theta \geqq 0$. Therefore ψ is concave and so for $k^{-2} - k < \gamma < k^{-2} - k^2$

$$\psi(\gamma) \geqq \min(\psi(k^{-2} - k),\ \psi(k^{-2} - k^2)).$$

Now

$$\psi(k^{-2} - k) = \frac{1}{k^{-2} - k}\,(k^2 - (2k - k^{-2})^2)$$
$$= \frac{1}{k^2(1 - k^3)}\,(k^6 - (2k^3 - 1)^2)$$
$$= \frac{1}{k^2}\,(3k^3 - 1)$$
$$= 2k^4 + \frac{1}{k^2}\,(2k^3 - 1)(1 - k^3).$$
$$\geqq 2k^4$$

since $2^{-1/4} < k < 1$.

Similarly

$$\begin{aligned}\psi(k^{-2}-k^2) &= \frac{1}{k^{-2}-k^2}\,(k^2-(2k^2-k^{-2})^2)\\ &= \frac{1}{k^2(1-k^4)}\,(k^6-(2k^4-1)^2)\\ &= 2k^4+\frac{1}{k^2(1-k^4)}\,(k^6-(2k^4-1)^2-2k^6(1-k^4))\\ &= 2k^4+\frac{1}{k^2(1-k^4)}\,(2k^4-1)(k^6-2k^4+1)\\ &\geqq 2k^4\end{aligned}$$

since $2k^4-1>0$ and $k^6-2k^4+1=(1-k^2)(1+k^2-k^4)>0$.

Thus

$$\frac{\|g\|}{\|f\|}\geqq 2k^4=\frac{2}{\|T\|^2}\,.$$

LEMMA 2.4. *Suppose* $t(x)^2\neq t(x^2)$. *Then*

$$Tx=\alpha t(x)+\beta t(x)u+f$$

where $u=t(x^2)t(x)^{-2}\in G_2$ *has order* 2, $f\in l_1(G_2)$ *does not contain terms in* $t(x)$ *or* $t(x)u$, *and*

(i) $$|\alpha|+|\beta|>1.29$$

(ii) $$\|f\|<0.13$$

(iii) $$|\alpha\bar{\beta}+\bar{\alpha}\beta|<0.017$$

(iv) $$\left|\frac{\bar{\alpha}\beta}{|\alpha|}+\frac{\alpha\bar{\beta}}{|\beta|}\right|<0.60.$$

Proof. Writing $u=t(x^2)t(x)^{-2}$ we can expand Tx in the form $\alpha t(x)+\beta t(x)u+f$. We have immediately

(1) $$|\alpha|^2+|\beta|^2+\|f\|_2^2=1$$

(2) $$|\alpha|+|\beta|+\|f\|\leqq\|T\|$$

using Proposition 2.2. In addition the coefficient of $t(x^2)=t(x)^2u$ in $(Tx)^2$ is at least $\|T\|^{-1}$ and so

$$2|\alpha\beta| + \|f\|_2^2 \geqq \|T\|^{-1} \tag{3}$$

since if $f = \Sigma_{y\in G_2}\gamma(y)y$, then the coefficient of $t(x)^2u$ is

$$2\alpha\beta + \sum_{y\in G_2}\gamma(y)\gamma(t(x)^2uy^{-1})$$

and $|\Sigma_{y\in G_2}\gamma(y)\gamma(t(x)^2uy^{-1})| \leqq \|f\|_2^2$ by Cauchy's inequality.

First we note that from (1) and (3)

$$(|\alpha| - |\beta|)^2 \leqq 1 - \|T\|^{-1} < 0.30$$

so that

$$|\alpha| - |\beta| < 0.55 \tag{4}$$

Now let $A = |\alpha| + |\beta|$. Then from (1) and (3)

$$A^2 + 2\|f\|_2^2 \geqq 1 + \|T\|^{-1} > 1 + \frac{1}{\sqrt{2}}$$

and as

$$\|f\|_2 \leqq \|Tx\| - A < \sqrt{2} - A,$$

we have

$$A^2 + 2(\sqrt{2} - A)^2 > 1 + \frac{1}{\sqrt{2}},$$

i.e.

$$3A^2 - 4\sqrt{2}A + 4 > 1 + \frac{1}{\sqrt{2}}$$

or

$$\left(A - \frac{2\sqrt{2}}{3}\right)^2 > \frac{1}{3\sqrt{2}} - \frac{1}{9}.$$

Therefore either

$$A > \frac{2\sqrt{2}}{3} + \frac{1}{3}\sqrt{\left(\frac{3}{\sqrt{2}} - 1\right)}$$

$$> \frac{2}{3}(1{\cdot}41) + \frac{1}{3}(1{\cdot}05) = 1{\cdot}29$$

or

$$A < \frac{2\sqrt{2}}{3} - \frac{1}{3}\sqrt{\left(\frac{3}{\sqrt{2}} - 1\right)}$$

$$< \frac{2}{3}(1{\cdot}42) - \frac{1}{3}(1{\cdot}04) = {\cdot}60.$$

Since $A > |\alpha| > 0{\cdot}70$, we have $A > 1{\cdot}29$. Thus we have established (i).

For (ii) note that $\|f\| < \|T\| - 1{\cdot}29 < 1{\cdot}42 - 1{\cdot}29 = 0{\cdot}13$ from (2).

Next, we note that since T is a *-map

$$Tx^{-1} = \bar{\alpha}t(x)^{-1} + \bar{\beta}t(x)^{-1}u^{-1} + f^*.$$

Multiplying by Tx, we consider the coefficient of u^{-1}, which must be zero. If $u^2 \neq e_2$, we obtain that

$$|\alpha\bar{\beta}| \leqq |\beta|\|f^*\| + \|f\| \cdot \|f^*\|$$

$$0{\cdot}70|\beta| \leqq 0{\cdot}13|\beta| + 0{\cdot}02$$

from which we conclude $|\beta| \leqq 0{\cdot}04$.

However $|\alpha| + |\beta| > 1{\cdot}29$ and $|\alpha| - |\beta| < 0{\cdot}55$ so that $|\beta| > 0{\cdot}37$ and so we have a contradiction. Hence $u^2 = e_2$. Again identifying coefficients of u

$$|\alpha\bar{\beta} + \beta\bar{\alpha}| \leqq \|f\| \cdot \|f^*\| < 0{\cdot}017.$$

Finally

$$\left|\frac{\bar{\alpha}\beta}{|\alpha|} + \frac{\alpha\bar{\beta}}{|\beta|}\right| \leqq \left|\frac{\bar{\alpha}\beta}{|\alpha|} - \frac{\bar{\alpha}\beta}{|\beta|}\right| + \left|\frac{\bar{\alpha}\beta}{|\beta|} + \frac{\alpha\bar{\beta}}{|\beta|}\right|$$

$$\leqq |\alpha| - |\beta| + |\beta|^{-1}|\bar{\alpha}\beta + \alpha\bar{\beta}|$$

$$\leqq 0{\cdot}55 + \frac{0{\cdot}017}{0{\cdot}37} < 0{\cdot}60.$$

LEMMA 2.5. *The set* $\{t(x^2)t(x)^{-2}: x \in G_1\}$ *contains at most one element* $u \neq e_2$.

Proof. Suppose $t(x^2)t(x)^{-2} = u \neq e_2$ and $t(y^2)t(y)^{-2} = v \neq e_2$ with $v \neq u$. Then write

$$Tx = \alpha_1 t(x) + \beta_1 t(x)u + \gamma_1 t(x)v + \delta_1 t(x)uv + f$$

$$Ty = \alpha_2 t(y) + \beta_2 t(y)v + \gamma_2 t(y)u + \delta_2 t(y)uv + g$$

where the terms are disjoint ($\gamma_1, \gamma_2, \delta_1$ and δ_2 may be zero). By our assumptions, $t(x)t(y)$, $t(x)t(y)u$, $t(x)t(y)v$, $t(x)t(y)uv$ are distinct (note that u and v have order 2 by the preceding lemma). The sum of all terms in the product

$$(\gamma_1 t(x)v + \delta_1 t(x)uv + f) * (\gamma_2 t(y)u + \delta_2 t(y)uv + g)$$

may be estimated by $(0{\cdot}13)^2 < 0{\cdot}017$. Hence by considering each of the four elements above, we obtain

$$\begin{aligned}
\|Txy\| &\geqq |\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2| + |\alpha_1\gamma_2 + \beta_1\alpha_2 + \delta_1\beta_2| + |\alpha_1\beta_2 + \beta_1\delta_2 + \gamma_1\alpha_2| \\
&\quad + |\alpha_1\delta_2 + \beta_1\beta_2 + \delta_1\alpha_2| - 0{\cdot}017 \\
&\geqq |\alpha_1\alpha_2|\left(1 + \operatorname{Re}\left(\frac{\beta_1\gamma_2}{\alpha_1\alpha_2} + \frac{\gamma_1\beta_2}{\alpha_1\alpha_2}\right)\right) + |\beta_1\alpha_2|\left(1 + \operatorname{Re}\left(\frac{\alpha_1\gamma_2}{\beta_1\alpha_2} + \frac{\delta_1\beta_2}{\beta_1\alpha_2}\right)\right) \\
&\quad + |\alpha_1\beta_2|\left(1 + \operatorname{Re}\left(\frac{\beta_1\delta_2}{\alpha_1\beta_2} + \frac{\gamma_1\alpha_2}{\alpha_1\beta_2}\right)\right) + |\beta_1\beta_2|\left(1 + \operatorname{Re}\left(\frac{\alpha_1\delta_2}{\beta_1\beta_2} + \frac{\delta_1\alpha_2}{\beta_1\beta_2}\right)\right) \\
&\qquad\qquad - 0{\cdot}017 \\
&\geqq (|\alpha_1| + |\beta_1|)(|\alpha_2| + |\beta_2|) + \operatorname{Re}\gamma_1\left(\frac{|\alpha_1\alpha_2|}{\alpha_1\alpha_2}\beta_2 + \frac{|\alpha_1\beta_2|}{\alpha_1\beta_2}\alpha_2\right) \\
&\quad + \operatorname{Re}\gamma_2\left(\frac{|\beta_1\alpha_2|}{\beta_1\alpha_2}\alpha_1 + \frac{|\alpha_1\alpha_2|}{\alpha_1\alpha_2}\beta_1\right) + \operatorname{Re}\delta_1\left(\frac{|\beta_1\alpha_2|}{\beta_1\alpha_2}\beta_2 + \frac{|\beta_1\beta_2|}{\beta_1\beta_2}\alpha_2\right) \\
&\quad + \operatorname{Re}\delta_2\left(\frac{|\alpha_1\beta_2|}{\alpha_1\beta_2}\beta_1 + \frac{|\beta_1\beta_2|}{\beta_1\beta_2}\alpha_1\right) - 0{\cdot}017.
\end{aligned}$$

Now

$$\left|\frac{|\alpha_1\alpha_2|}{\alpha_1\alpha_2}\beta_2 + \frac{|\alpha_1\beta_2|}{\alpha_1\beta_2}\alpha_2\right| = \left|\frac{\bar{\alpha}_2\beta_2}{|\alpha_2|} + \frac{\bar{\beta}_2\alpha_2}{|\beta_2|}\right| < 0.60$$

by Lemma 2.4, and we can estimate each term similarly.

Hence

$$\begin{aligned}
\|Txy\| &\geqq (|\alpha_1| + |\beta_1|)(|\alpha_2| + |\beta_2|) - 0{\cdot}60(|\gamma_1| + |\gamma_2| + |\delta_1| + |\delta_2|) \\
&\qquad - 0{\cdot}017 \\
&\geqq (1{\cdot}29)^2 - 0{\cdot}60(0{\cdot}26) - 0{\cdot}017 \\
&> 1{\cdot}66 - 0{\cdot}16 - 0{\cdot}02 = 1{\cdot}48
\end{aligned}$$

which is a contradiction, since $\|Txy\| < \sqrt{2}$.

We now come to the basic theorem.

THEOREM 2.6. *Let G_1 and G_2 be abelian groups with identities e_1 and e_2, and let $T: l_1(G_1) \to l_1(G_2)$ be an algebra homomorphism such that $Te_1 = e_2$. Suppose $\|T\| < \sqrt{2}$; then if*

(a) $\|T\| < \frac{1}{2}(1+\sqrt{3})$, *$T$ takes the form*

$$Tx = \psi(x)s(x)$$

where $\psi \in \hat{G}_1$ and $s: G_1 \to G_2$ is a homomorphism, and then T is an isometry;

or (b) $\|T\| \geqq \frac{1}{2}(1+\sqrt{3})$, *$T$ takes the form*

$$Tx = \psi(x)(\tfrac{1}{2}(1+\varphi(x))s(x) + \tfrac{1}{2}(1-\varphi(x))s(x)u)$$

where $\varphi, \psi \in \hat{G}_1$ and φ has odd order, $s: G_1 \to G_2$ is a homomorphism and $u \in G_2$ is an element of order 2. In this case $\|T\| = \sqrt{2}\cos \pi/4n$, where n is the order of φ. In both cases, if T is a monomorphism, then s is a monomorphism, and if T is an isomorphism then so is s.

COROLLARY 2.7. *If there is an algebra isomorphism $T: l_1(G_1) \to l_1(G_2)$ with $\|T\| < \sqrt{2}$, then G_1 and G_2 are isomorphic.*

REMARK. Example 1 shows that $\sqrt{2}$ is the best possible constant in the Theorem and its Corollary. Example 2 shows that $\frac{1}{2}(1+\sqrt{3})$ is best possible for (a). Example 4 shows that in case (b) T can be an epimorphism while s is *not* an epimorphism.

Proof of Theorem 2.6. Let ϵ_2 denote the identity character on G_2. Then $\epsilon_2 \circ T$ is a nonzero character ψ on G_1 (since $Te_1 = e_2$). If we define $S: l_1(G_1) \to l_1(G_1)$ by $Sx = \psi^{-1}(x)x$ then S is algebra automorphism, and so by considering TS in place of T we may reduce the problem to supposing $\epsilon_2 \circ T = \epsilon_1$.

Using the notation of the preceding lemmas, suppose first that $t(x)^2 = t(x^2)$ for all $x \in G_1$. Then by Lemma 2.3 if

$$Tx = \alpha t(x) + f$$

then

$$Tx^{2^n} = \beta t(x)^{2^n} + g$$

where $\|g\| \geqq (2/\|T\|^2)^n \|f\|$. Since $2 > \|T\|^2$, we conclude that $f = 0$ and hence $Tx = \alpha t(x)$ where $\alpha = \alpha(x)$. Since $\epsilon_2 \circ T = \epsilon_1$, $\alpha \equiv 1$, i.e. $Tx = t(x)$ and t is a homomorphism. We thus obtain the result of (a).

Next suppose $\{t(x)^{-2}t(x^2): x \in G\} = \{e_2, u\}$ where u is an element of order 2. Let $H = \{e_2, u\}$, and let $\pi: G_2 \to G_2/H$ be the quotient

map. Denote by $P: l_1(G_2) \to l_1(G_2/H)$ the induced algebra epimorphism, and consider $PT: l_1(G_1) \to l_1(G_2/H)$. As $\|PT\| \leqq \|T\| < \sqrt{2}$, we can appeal to the preceding lemmas. Let $\tilde{t}: G_1 \to G_2/H$ be the map defined before Lemma 2.3. The mass of Tx concentrated in the coset $\tilde{t}(x)$ is at least $\|T\|^{-1}$, and hence as $2\|T\|^{-1} > \|T\|$, it follows that $\pi t(x) = \tilde{t}(x)$. Thus $\tilde{t}(x^2) = \pi t(x^2) = \tilde{t}(x)^2$ since $u \in H$, and so by the preceding paragraph

$$PTx = \tilde{t}(x)$$

and $\tilde{t}: G_1 \to G_2/H$ is a homomorphism.

Thus for $x \in G_1$

$$Tx = \frac{1+\theta(x)}{2}\, t(x) + \frac{1-\theta(x)}{2}\, t(x)u + f$$

where $f = (e_2 - u) * \Sigma_1^{\infty} \gamma_i y_i$, and there is precisely one y_i in each coset of H. Then $\|f\| = 2\Sigma|\gamma_i| < \|T\| - 1$ and hence $\|f\|_2^2 = 2\Sigma|\gamma_i|^2 < \frac{1}{2}(\|T\| - 1)^2$. However

$$\tfrac{1}{2} + \tfrac{1}{2}|\theta(x)|^2 + \|f\|_2^2 = 1$$

so that

$$|\theta(x)|^2 \geqq 1 - (\|T\| - 1)^2 = 2\|T\| - \|T\|^2.$$

Now suppose

$$Tx^2 = \frac{1+\theta(x^2)}{2}\, t(x^2) + \frac{1-\theta(x^2)}{2}\, t(x^2)u + g.$$

Then

$$\|g\| \geqq 2\left\| f * \left(\frac{1+\theta(x)}{2}\, t(x) + \frac{1-\theta(x)}{2}\, t(x)u \right) \right\| - \|f\|^2$$

and since $f = \frac{1}{2}(e_2 - u) * f$

$$\begin{aligned} \|g\| &\geqq 2\|\theta(x)t(x) * f\| - \|f\|^2 \\ &= 2|\theta(x)|\|f\| - \|f\|^2 \\ &\geqq \|f\|(2\sqrt{(2\|T\| - \|T\|^2)} - \|T\| + 1) \\ &= K\|f\| \end{aligned}$$

where $K = 1 + 2\sqrt{(2\|T\| - \|T\|^2)} - \|T\| > 1$ since $\|T\| < 8/5$.

Thus it follows that $\| Tx^{2^n} \| \geqq K^n \| f \|$, and hence that $f = 0$, i.e. T has the form

$$Tx = \tfrac{1}{2}(1 + \theta(x))t(x) + \tfrac{1}{2}(1 - \theta(x))t(x)u.$$

By multiplying out, it is easy to see that

$$\theta(xy) = \pm \theta(x)\theta(y)$$

and hence that θ^2 is a character on G_1. Since $\| T \| < \sqrt{2}$, $| \theta(x) \pm i | \geqq \epsilon > 0$ for all $x \in G_1$ and some $\epsilon > 0$. Hence θ^2 has odd order, n, say. Define $\varphi = (\theta^2)^{(n+1)/2}$; then φ is also a character of order n, since $(n, (n+1)/2) = 1$. Now $\varphi(x) = \theta^{n+1}(x) = \pm \theta(x)$, and hence by defining $s(x) = t(x)$ or $t(x)u$ we may express T in the form

$$Tx = \tfrac{1}{2}(1 + \varphi(x))s(x) + \tfrac{1}{2}(1 - \varphi(x))s(x)u.$$

Since φ is a character it follows easily that s is a homomorphism. The norm of T is then

$$\begin{aligned} &\sup_{x \in G_1} \tfrac{1}{2}(|1 + \varphi(x)| + |1 - \varphi(x)|) \\ &\quad = \max_{1 \leqq k \leqq n} \left(\left| \cos \frac{k\pi}{n} \right| + \left| \sin \frac{k\pi}{n} \right| \right) \\ &\quad = \max_{1 \leqq k \leqq n} \sqrt{2} \sin \left(\frac{k\pi}{n} + \frac{\pi}{4} \right) \\ &\quad = \sqrt{2} \cos \frac{\pi}{4n} . \end{aligned}$$

If $n = 3$

$$\sqrt{2} \cos \frac{\pi}{4n} = \frac{1 + \sqrt{3}}{2} .$$

If T is a monomorphism in case (b) (case (a) is trivial), let $M \subset G_2$ be the set such that $s(M) = e_2$. Then $Tm \in \operatorname{lin}(e_2, u)$ for $m \in M$, and so M has at most two members. If $M \neq \{e_1\}$, there exists $v \in M$ and $v^2 = e_1$. Hence $\varphi(v) = +1$ since φ is of odd order and so $Tv = \psi(v)e_2$, contradicting the fact that T is a monomorphism.

If T is an isomorphism, let $N = s(G_1)$. If $u \notin N$ then there must exist $x \in G_1$ such that $s(x)u = u$ and $1 - \varphi(x) \neq 0$. But then $s(x) = e_1$ and by the result for monomorphisms, $x = e_1$, $\varphi(x) = 1$. Thus $u \in N$ and so $Tx \in l_1(s(G_1))$ for $x \in G_1$. Thus $s(G_1) = G_2$ since T is onto.

3. General case. We now use the results for discrete groups to establish the corresponding theorems for locally compact abelian groups.

First consider homomorphisms from $L^1(G_1)$ into $M(G_2)$. If I is such a homomorphism, then T possesses a unique extension $\bar{T}: M(G_1) \to M(G_2)$ which is continuous on bounded sets from the strong operator topology (of $M(G_1)$ operating on $L^1(G_1)$) to the weak*-topology (of $M(G_2) = C_0(G_2)^*$) (see **[4]** Theorem 4.1.1 and Remark afterwards). We also observe that in the strong-operator topology, the unit ball of $M(G_1)$ is the closed absolutely convex cover of the Dirac measures $\{\delta_x : x \in G_1\}$ (see **[4]** Lemma 1.1.3). Hence $\bar{T}$ is uniquely determined by its behaviour on the atomic measures, which we identify with $l_1(G_1)$.

THEOREM 3.1. *Let G_1 and G_2 be locally compact abelian groups and $T: L^1(G_1) \to M(G_2)$ be an algebra homomorphism such that $\|T\| < \sqrt{2}$ and $\bar{T}\delta_{e_1} = \delta_{e_2}$. Then*

(i) *if $\|T\| < \frac{1}{2}(1+\sqrt{3})$, then $\|T\| = 1$ and T takes the form*

$$\widehat{Tf}(\chi) = \hat{f}(\hat{s}(\chi)\psi) \qquad (\chi \in \hat{G}_2)$$

where $\psi \in \hat{G}_1$ and $s: G_1 \to G_2$ is a continuous homomorphism;

(ii) *if $\|T\| \geqq \frac{1}{2}(1+\sqrt{3})$, then T takes the form*

$$\begin{aligned} \widehat{Tf}(\chi) &= \hat{f}(\hat{s}(\chi)\psi) \quad \text{if} \quad \chi(u) = 1 \\ &= \hat{f}(\hat{s}(\chi)\psi\varphi) \quad \text{if} \quad \chi(u) = -1 \end{aligned}$$

where φ, $\psi \in \hat{G}_1$, φ is of odd order n, $s: G_1 \to G_2$ is a continuous homomorphism, and $u \in G_2$ is an element of order 2. Then $\|T\| = \sqrt{2}\cos \pi/4n$.

Proof. Let $P: M(G_2) \to l_1(G_2)$ be the natural projection; P is an algebra homomorphism. Consider $P\bar{T}: l_1(G_1) \to l_1(G_2)$. Clearly $\|\bar{T}\| = \|T\|$ and $\|P\| = 1$, and so $\|P\bar{T}\| < \sqrt{2}$. By Theorem 2.6, either

(i) $P\bar{T}\delta_x = \psi(x)\delta_{s(x)}$

or (ii) $P\bar{T}\delta_x = \psi(x)[\frac{1}{2}(1+\varphi(x))\delta_{s(x)} + \frac{1}{2}(1-\varphi(x))\delta_{s(x)u}]$.

Consider case (i). Then

$$\bar{T}\delta_x = \psi(x)\delta_{s(x)} + \mu$$

where μ is nonatomic, and $\|\mu\| < \sqrt{2} - 1$.

$$\bar{T}\delta_{x^2} = \psi(x^2)\delta_{s(x^2)} + \nu$$

where ν is nonatomic and

$$\begin{aligned}\|\nu\| &\geqq \|2\psi(x)\delta_{s(x)} * \mu\| - \|\mu * \mu\| \\ &\geqq (2 - \|\mu\|)\|\mu\| \\ &\geqq (3 - \sqrt{2})\|\mu\|.\end{aligned}$$

By iteration we obtain a contradiction (cf. 2.6) unless $\mu = 0$, since $3 - \sqrt{2} > 1$.

Thus

$$\bar{T}\delta_x = \psi(x)\delta_{s(x)}.$$

On the Dirac measures, both the strong-operator and weak*-topologies agree with the group topologies, so that s and ψ are continuous. The form of T now follows from the remarks preceding the Theorem, and replacing ψ by ψ^{-1}.

For case (ii) we compose with the quotient $Q: M(G_2) \to M(G_2/H)$ where $H = \{e_2, u\}$ is a subgroup of G_2. Then

$$P'Q\bar{T}\delta_x = \psi(x)\delta_{\pi s(x)}$$

where P' is the natural map $P': M(G_2/H) \to l_1(G_2/H)$, and π is the quotient map $\pi: G_2 \to G_2/H$. As above ψ and πs are continuous and

$$Q\bar{T}\delta_x = \psi(x)\delta_{\pi s(x)}.$$

Thus

$$\bar{T}\delta_x = P\bar{T}\delta_x + \mu$$

where μ is nonatomic and satisfies $\mu = \frac{1}{2}(\delta_{e_2} - \delta_u) * \mu$. Then

$$\bar{T}\delta_{x^2} = P\bar{T}\delta_{x^2} + \nu$$

where ν is nonatomic and satisfies

$$\begin{aligned}\nu &= \psi(x)[(1 + \varphi(x))\mu * \delta_{s(x)} + (1 - \varphi(x))\mu * \delta_{s(x)u}] + \mu * \mu \\ &= 2\psi(x)\varphi(x)\mu * \delta_{s(x)} + \mu * \mu.\end{aligned}$$

Thus

$$\begin{aligned}\|\nu\| &\geqq 2\|\mu\| - \|\mu\|^2 \\ &\geqq (3 - \sqrt{2})\|\mu\|\end{aligned}$$

and arguing as before $\mu = 0$, i.e.

$$\bar{T}\delta_x = P\bar{T}\delta_x.$$

Now

$$\bar{T}\delta_x = \psi(x)[\tfrac{1}{2}(1+\varphi(x))\delta_{s(x)} + \tfrac{1}{2}(1-\varphi(x))\delta_{s(x)u}]$$

where ψ is continuous. To prove φ continuous, since φ has finite order, it is sufficient to establish that $\{x: \varphi(x) = 1\}$ is closed. Suppose $x_\alpha \to x$ and $\varphi(x_\alpha) = 1$. Then

$$\bar{T}\delta_{x_\alpha} = \psi(x_\alpha)\delta_{s(x_\alpha)}$$

so that

$$\bar{T}\delta_x = \psi(x)\delta_z$$

where $s(x_\alpha) \to z$. Hence $\varphi(x) = \pm 1$; but φ has odd order so that $\varphi(x) = 1$ and φ is continuous. It follows that s is also continuous. The general form of T follows as above, again replacing ψ by ψ^{-1}.

The proof above clearly shows that a homomorphism T of $M(G_1)$ into $M(G_2)$ with $\|T\| < \sqrt{2}$ and $T\delta_{e_1} = \delta_{e_2}$ has the form on atomic measures given by either

(1) $T\delta_x = \psi(x)\delta_{s(x)}$, or

(2) $T\delta_x = \psi(x)[\tfrac{1}{2}(1+\varphi(x))\delta_{s(x)} + \tfrac{1}{2}(1-\varphi(x))\delta_{s(x)u}]$

with ψ, φ, s and u as before.

However we cannot deduce that s is continuous or the form of T on the nonatomic measures. For example, (see **[8]**, 3.4.1) let G be a compact infinite group and let G_d be the same group with the discrete topology. Then the algebra homomorphism $P: M(G) \to l_1(G) = M(G_d)$ as in the proof of the theorem has the above form, but the identity map is not continuous.

COROLLARY 3.2. *Suppose T is an algebra isomorphism of $L^1(G_1)$ onto $L^1(G_2)$* [resp: *$M(G_1)$ onto $M(G_2)$*] *with $\|T\| < \sqrt{2}$. Then G_1 and G_2 are isomorphic.*

Proof. By **[8]** 4.6.4 it is sufficient to prove the result in the L^1-case. In this case $\bar{T}$ is also an isomorphism and satisfies $\bar{T}\delta_{e_1} = \delta_{e_2}$. By Theorem 3.1 and Theorem 2.6 there is an isomorphism $s: G_1 \to G_2$ and s is continuous.

Since T^{-1} is onto $L^1(G_1)$, T^{-1} has a unique extension to $M(G_2)$ (without any continuity requirements). Thus $\bar{T}^{-1}$ is the unique extension and is hence continuous for the strong operator and weak*-topologies on bounded sets.

The form of $\bar{T}$ is either

(1) $\bar{T}\delta_x = \psi(x)\delta_{s(x)}$

or (2) $\bar{T}\delta_x = \psi(x)[\tfrac{1}{2}(1+\varphi(x))\delta_{s(x)} + \tfrac{1}{2}(1-\varphi(x))\delta_{s(x)u}]$.

Thus either

(1) $\bar{T}^{-1}\delta_x = \psi^{-1}(s^{-1}(x))\delta_{s^{-1}(x)}$

or (2) $\bar{T}^{-1}\delta_x = \psi^{-1}(s^{-1}(x))$
$\times [\frac{1}{2}(1+\varphi^{-1}(s^{-1}(x))\delta_{s^{-1}(x)} + \frac{1}{2}(1-\varphi^{-1}(s^{-1}(x))\delta_{s^{-1}(xu)}]$.

[Note that $\varphi(s^{-1}(u)) = 1$ since $u^2 = e_2$ and φ is of odd order.]

The continuity of s^{-1} follows as in Theorem 3.1.

Theorem 3.1 is false if $\bar{T}\delta_{e_1} \neq \delta_{e_2}$ (see Example 3). However with a stronger condition on the norm, we have the following result, essentially due to Saeki **[9]** (see **[8]** 4.6.3.).

THEOREM 3.3. *Let G_1 and G_2 be locally compact abelian groups and T an algebra homomorphism of $L^1(G_1)$ into $M(G_2)$ with $\|T\| < \frac{1}{2}(1+\sqrt{2})$. Then $\|T\| = 1$ and T has the factorization*

$$L^1(G_1) \xrightarrow{S} M(G_2/H_2) \xrightarrow{\pi^*} M(G_2)$$

where H_2 is a compact subgroup of G_2, s is a continuous homomorphism of G_1 into G_2/H_2, $\psi \in \hat{G}_1$, $\rho \in \hat{H}_2$,

$$\widehat{Sf}(\chi) = \hat{f}(\hat{s}(\chi)\psi) \qquad (\chi \in \widehat{G_2/H_2})$$

and $\pi\colon C_0(G_2) \to C_0(G_2/H_2)$ is defined by

$$(\pi f)(\xi) = \int_{H_2} f(xy)\rho(y^{-1})dm_{H_2}(y^{-1}) \qquad (\xi = H_2x).$$

Proof. Let $\bar{T}$ be the extension of T to $M(G_1)$ as in Theorem 3.1. Then $\bar{T}\delta_{e_1}$ is an idempotent in $M(G_2)$ with $\|\bar{T}\delta_{e_1}\| < \frac{1}{2}(1+\sqrt{2})$. By Saeki **[9]**, $\|\bar{T}\delta e_1\| = 1$ and $\bar{T}\delta_{e_1} = \rho m_{H_2}$ where H_2 is a compact subgroup of G_2 and $\rho \in \hat{H}_2$, (see **[4]** Theorem 2.1.4). Since ρm_{H_2} is the identity for the image of $\bar{T}$, T must factor through $M(G_2/H_2)$.

Let $L^1(G) \xrightarrow{S} M(G_2/H_2) \xrightarrow{\pi^*} M(G_2)$ be the factorization with π defined as in the statement of the theorem. Then $S\delta_{e_1} = \delta_{\xi_1}$ where ξ_1 is the identity of G_2/H_2, and since $\|\pi^*\| = 1$, $\|S\| < \frac{1}{2}(1+\sqrt{2})$. The result now follows from Theorem 3.1.

Finally in this section, we give the corresponding results for $C(G)$ and $L^\infty(G)$ with G a compact group.

THEOREM 3.4. *Let G_1 and G_2 be compact abelian groups and let T be an algebra homomorphism of $C(G_1)$ into $C(G_2)$ with T one-to-one and $\|T\| < \sqrt{2}$. Then there exists a group homomorphism s of G_2 into G_1 and $\psi \in \hat{G}_2$ such that either*

(1) *$\|T\| = 1$ and T has the form*

$$(Tf)(x) = \psi(x) f(s(x))$$

or (2) $\|T\| = \sqrt{2}\cos \pi/4n$ *for an odd number* $n > 1$ *and* T *has the form*

$$(Tf)(x) = \psi(x)[\tfrac{1}{2}(1+\varphi(x))f(s(x)) + \tfrac{1}{2}(1-\varphi(x))f(s(x)u)]$$

for some $\varphi \in \hat{G}_2$ *of order* n *and* $u \in G_1$, *of order* 2.

Proof. The characters on a compact abelian group are the only idempotents of norm less than 2. Thus T maps characters into characters. It follows that $(Tf)(e_2) = f(e_1)$ for all $f \in C(G_1)$, and as in **[16]** Theorem 2, T^* is a homomorphism of $M(G_2)$ into $M(G_1)$. Since $T^*\delta_{e_2} = \delta_{e_1}$ and $\|T^*\| < \sqrt{2}$, the result follows from Theorem 3.1.

Note that s need not be an epimorphism (Example 4).

COROLLARY 3.5. *Suppose* T *is an isomorphism of* $C(G_1)$ *onto* $C(G_2)$ *with* $\|T\| < \sqrt{2}$. *Then* G_1 *and* G_2 *are isomorphic.*

Proof. This follows from Corollary 3.2.

Note that $\sqrt{2}$ is again the best possible constant (Example 1).

THEOREM 3.6. *Let* G_1 *and* G_2 *be compact abelian groups and* $T: L^\infty(G_1) \to L^\infty(G_2)$ *be a one-one algebra homomorphism. Then*

(i) *if* $\|T\| < \frac{1}{2}(1+\sqrt{3})$

$$Tf(x) = \psi(x) f(s(x))$$

where $s: G_2 \to G_1$ *is a continuous homomorphism and* $\psi \in \hat{G}_2$

(ii) *if* T *is an isomorphism onto* $L^\infty(G_2)$ *and* $\|T\| < \sqrt{2}$ *then* G_1 *and* G_2 *are isomorphic.*

Proof. As T maps characters to characters, T maps $C(G_1)$ to $C(G_2)$. If T is an isomorphism then $T: C(G_1) \to C(G_2)$ is an isomorphism and case (ii) follows from 3.5. For case (i) we observe by 3.4 that

$$Tf(x) = \psi(x) f(s(x)) \qquad f \in C(G_1)$$

where $\psi \in \hat{G}_2$ and $s: G_2 \to G_1$ is a continuous homomorphism. Let $H = \ker s$ and consider $P: L^\infty(G_2) \to L^\infty(G_2/H)$ the natural projection

$$P\varphi[\pi(x)] = \int_H \varphi(xh)\, dm_H(h)$$

where m_H is the Haar measure on H, and π the quotient map.

Then $PT: L^\infty(G_1) \to L^\infty(G_2/H)$ is an algebra homomorphism and $PT(C(G_1)) = C(G_2/H)$. Hence for $g \in C(G_2/H)$, $f \in L^\infty(G_1)$,

$$PTf * g = PT(f * (PT)^{-1}g) = P\tilde{T}f * g$$

where

$$\tilde{T}f(x) = \psi(x) f(s(x)).$$

It follows that

$$PTf = P\tilde{T}f$$

and hence that $Tf = \tilde{T}f + k$ where

$$\int_H k(xy)dm_H(y) = 0 \qquad (x \in G_2).$$

Now $T(f * f) = \tilde{T}(f * f)$ and so

$$2\tilde{T}f * k + k * k = 0.$$

$\tilde{T}f$ is constant on cosets of H and so $\tilde{T}f * k = 0$. Thus $k * k = 0$ and $k = 0$ since G_2 is abelian.

4. Non-abelian groups. In the non-abelian case, we cannot expect results about the form of near isometries. If G is compact, but not abelian, there exist many isomorphisms of $C(G)$ [or $L^1(G)$] onto itself with norms arbitrarily close to one. An isometry can be perturbed in different ways by automorphisms of the minimal ideals. However we can still ask whether isomorphisms of the algebras determine isomorphisms of the groups. Again we begin with the discrete case. Let G_1 and G_2 be arbitrary groups and $T: l_1(G_1) \to l_1(G_2)$ be an algebra homomorphism with $\|T\| < \sqrt{2}$ and $Te_1 = e_2$. By Proposition 2.1, the map $t: G_1 \to G_2$ is well-defined. But Proposition 2 is false in the non-abelian case since T need not be a *-map. If we impose a stronger condition on the norm of T, we get that t is a homomorphism directly.

LEMMA 4.1. *If $\|T\| < \lambda_0$ where λ_0 is the largest root of the equation $\lambda^3 + \lambda^2 - 2\lambda - 1 = 0$, then t is a homomorphism.*

Proof. Let $Tx = \alpha t(x) + f$ and $Ty = \beta t(y) + g$ where $|\alpha| \geqq 1/\|T\|$, $|\beta| \geqq 1/\|T\|$, with f and g disjoint from $t(x)$ and $t(y)$ respectively. Then the modulus of the coefficient of $t(x)t(y)$ in Txy is greater than $|\alpha||\beta| - \|f\|\|g\|$. Now if $t(x)t(y) \neq t(xy)$, this must be less than $\|T\| - 1/\|T\|$. Thus we must have

$$\begin{aligned}\|T\| - \frac{1}{\|T\|} &\geqq |\alpha||\beta| - \|f\|\|g\| \\ &\geqq |\alpha||\beta| - (\|T\| - |\alpha|)(\|T\| - |\beta|) \\ &= (|\alpha| + |\beta|)\|T\| - \|T\|^2 \\ &\geqq 2 - \|T\|^2.\end{aligned}$$

But this is impossible since $\|T\| < \lambda_0$ and λ_0 is the largest root of $\lambda - 1/\lambda = 2 - \lambda^2$.

Thus $t(xy) = t(x)t(y)$ for all $x, y \in G_1$ and so t is a homomorphism.

Note. $\lambda_0 \sim 1{\cdot}247$.

With the condition $\|T\| < \sqrt{2}$, we can show that t is not too far from a homomorphism in the following sense.

LEMMA 4.2. *For $x \in G_1$, the set $\{t(y)t(z)\colon yz = x\}$ is finite.*

Proof. If $Ty = \alpha t(y) + f$ and $Tz = \beta t(z) + g$ with $|\alpha| \geqq 1/\|T\|$, $|\beta| \geqq 1/\|T\|$ and f and g disjoint from $t(y)$ and $t(z)$ respectively, then the modulus of the coefficient of $t(y)t(z)$ in $Tyz = Tx$ is greater than

$$|\alpha||\beta| - \|f\|\|g\| \geqq 2 - \|T\|^2 \quad \text{as before.}$$

Since this is positive and $\|Tx\| < \sqrt{2}$, it follows that the set $\{t(y)t(z)\colon yz = x\}$ is finite.

These two lemmas give corresponding results for locally compact groups. We give only the results for isomorphisms, though clearly there are slightly more general results.

THEOEREM 4.3. *Let G_1 and G_2 be locally compact groups and T an algebra isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| < \lambda_0$ where λ_0 is the largest root of the equation $\lambda^3 + \lambda^2 - 2\lambda - 1 = 0$, then G_1 and G_2 are isomorphic.*

Proof. By **[4]** §4, there is a unique extension $\tilde{T}$ of T from $M(G_1)$ onto $M(G_2)$ which will also be an isomorphism with $\|\tilde{T}\| < \lambda_0$, and which is continuous on bounded sets as a map from the strong operator topology into the weak* topology.

Now restricting to the atomic measures on G_1 and using Lemma 4.1, we have an isomorphism $t\colon G_1 \to G_2$. t is continuous by the continuity of $\tilde{T}$ as in Theorem 3.1, so it remains only to prove that t^{-1} is continuous. Suppose not. Then there exists a compact neighborhood V of e_1 in G_1 such that $t(V)$ is not a neighborhood of e_2. By taking U

such that $UU^{-1} \subset V$ if necessary, we can assume that the measure of $t(V)$ is zero. Let χ_V denote the characteristic function of V and m_1 the Haar measure on G_1. Then $\chi_V \in L^1(G_1)$ and by [4] Lemma 1.1.2, $(1/m_1(V))\chi_V$ can be approximated in the strong operator topology by elements in the convex hull of $\{\delta_x : x \in V\}$. Suppose $\Sigma_{i=1}^n \lambda_i \delta_{x_i}$ is such an element. Then for each i, $\tilde{T}\delta_{x_i} = \alpha(x_i)\delta_{t(x_i)} + \nu_i$ where $|\alpha(x_i)| > 1/\|T\|$, $\nu_i(\{t(x_i)\}) = 0$ and $\|\nu_i\| \leqq \|T\| - 1/\|T\|$.

Then

$$\tilde{T}\left(\sum_1^n \lambda_i \delta_{x_i}\right) = \sum_1^n \lambda_i \alpha(x_i)\delta_{t(x_i)} + \sum_1^n \lambda_i \nu_i.$$

Now $(1/m_1(V))T\chi_V$ is the w^* limit of such elements. Thus $(1/m_1(V))T\chi_V = \mu + \nu$ where $\operatorname{supp}(\mu) \subset t(V)$ and $\|\nu\| \leqq \|T\| - 1/\|T\|$.

Since $t(V)$ has measure zero, μ is a singular measure. But $T\chi_V \in L^1(G_2)$ and so $\|(1/m_1(V))T\chi_V\| \leqq \|\nu\| \leqq \|T\| - 1/\|T\|$. But there exists a net of such V such that $(1/m_1(V))\chi_V$ tends to δ_{e_1} in the strong operator topology. Therefore $(1/m_1(V))T\chi_V \to \delta_{e_2}$ in the w^*-topology, which is a contradiction, since $\|(1/m_1(V))T\chi_V\| \leqq \|T\| - 1/\|T\| < 1$. Hence t^{-1} is continuous and the result is proved.

COROLLARY 4.4. *Let G_1 and G_2 be compact groups and T an algebra isomorphism of $C(G_1)$ $[L^\infty(G_1)]$ onto $C(G_2)$ $[L^\infty(G_2)]$ with $\|T\| < \lambda_0$. Then G_1 and G_2 are isomorphic.*

Proof. By [12] p. 861, it is sufficient to prove the result for T mapping $C(G_1)$ onto $C(G_2)$. The adjoint map T^* is an algebra isomorphism of $M(G_2)$ onto $M(G_1)$ which maps $L^1(G_2)$ onto $L^1(G_1)$ ([4] Theorem 1). The result now follows from 4.3.

THEOREM 4.5. *Let G_1 and G_2 be locally compact connected groups and T an algebra isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ with $\|T\| < \sqrt{2}$. Then G_1 and G_2 are isomorphic.*

Proof. As in the proof of 4.3, t is a continuous one-to-one map from G_1 onto G_2. By Lemma 4.2, for $x \in G_1$, $\{t(y)t(z) : yz = x\}$ is a finite set. But it is the image of G_1 under the continuous map $y \mapsto t(y)t(y^{-1}x)$. Since G_1 is connected, it is a one point set, and since $t(e_1) = e_2$, this point is $t(x)$. Thus t is an isomorphism. The continuity of t^{-1} follows as in 4.3 since $\|T\| - 1/\|T\|$ is still less than one.

COROLLARY 4.6. *Let G_1 and G_2 be connected compact groups and T an algebra isomorphism of $C(G_1)$ $[L^\infty(G_1)]$ onto $C(G_2)$ $[L^\infty(G_2)]$ with $\|T\| < \sqrt{2}$. Then G_1 and G_2 are isomorphic.*

REFERENCES

1. D. Amir, *On isomorphisms of continuous function spaces*, Israel J. Math., **3** (1965), 205–210.
2. M. Cambern, *A generalised Banach–Stone theorem*, Proc. Amer. Math. Soc., **17** (1966), 396–400.
3. R. E. Edwards, *Bipositive and isometric isomorphisms of some convolution algebras*, Canad. J. Math., **17** (1965), 839–846.
4. F. P. Greenleaf, *Norm decreasing homomorphisms of group algebras*, Pacific J. Math., **15** (1965), 1187–1219.
5. B. E. Johnson, *Isometric isomorphisms of measure algebras*, Proc. Amer. Math. Soc., **15** (1964), 186–188.
6. S. K. Parrott, *Isometric multipliers*, Pacific J. Math., **25** (1968), 159–166.
7. R. Rigelhof, *Norm-decreasing homomorphisms of group algebras*, Trans. Amer. Math. Soc., **136** (1969), 361–372.
8. W. Rudin, *Fourier Analysis on Groups*, Interscience (New York) 1960.
9. S. Saeki, *On norms of idempotent measures*, Proc. Amer. Math. Soc., **19** (1968), 600–602.
10. ———, *On norms of idempotent measures II*, Proc. Amer. Math. Soc., **19** (1968), 367–371.
11. R. S. Strichartz, *Isometric isomorphisms of measure algebras*, Pacific J. Math., **15** (1965), 315–317.
12. ———, *Isomorphisms of group algebras*, Proc. Amer. Math. Soc., **17** (1966), 858–862.
13. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math., **2** (1952), 251–261.
14. G. V. Wood, *A note on isomorphisms of group algebras*, Proc. Amer. Math. Soc., **25** (1970), 771–775.
15. ———, *Isomorphisms of L^p group algebras*, J. London Math. Soc., **4** (1972), 425–428.
16. ———, *Homomorphisms of group algebras*, Duke Math. J., **41** (1974), 255–261.

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