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AN ANALOGUE OF THE RADON-NIKODYM PROPERTY FOR NON-LOCALLY CONVEX QUASI-BANACH SPACES

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1. Introduction

In recent years there has been considerable interest in Banach spaces with the Radon-Nikodym Property; see (1) for a summary of the main known results on this class of spaces. We may define this property as follows: a Banach space X has the Radon-Nikodym Property if whenever $T \in \mathscr{L}(L_1, X)$ (where $L_1 = L_1(0, 1)$) then T is differentiable i.e.

$$Tf = \int_0^1 f(x)g(x)\,dx$$

where $g:(0,1) \to X$ is an essentially bounded strongly measurable function.

In this paper we examine analogues of the Radon-Nikodym Property for quasi-Banach spaces. If $0 < p < 1$, there are several possible ways of defining "differentiable" operators on L_p, but they inevitably lead to the conclusion that the only differentiable operator is zero. For example, a differentiable operator on L_1 has the Dunford-Pettis property; operators on L_1 with the Dunford-Pettis property map the unit ball of L_∞ to a compact set (cf (12)). However any operator on L_p ($p < 1$) with this property is zero (4).

Thus we define a quasi-Banach space X to be p-trivial if $\mathscr{L}(L_p, X) = \{0\}$. The concept of p-triviality is then hoped to be an analogue of the Radon-Nikodym property amongst locally p-convex quasi-Banach spaces. It turns out that this hope is fulfilled to some extent. Our main results in Sections 4 and 5 demonstrate an analogue of Edgar's theorem (2) and of the Phelps characterisation of the Radon-Nikodym Property ((1), (9)) to this setting. Precisely we show that a locally p-convex quasi-Banach space is p-trivial if and only if every closed bounded p-convex set is the closed p-convex hull of its "strongly p-extreme points". Our analogue of Edgar's theorem is that if C is a bounded closed p-convex subset of a p-trivial quasi-Banach space then every $x \in C$ may be represented in the form

$$x = \sum_{n=1}^{\infty} a_n u_n$$

where $a_n \geq 0$, $\Sigma\, a_n^p = 1$, and each u_n is a p-extreme point of C. We observe in this connection that a similar Choquet-type theorem for compact p-convex sets was proved in (5).

In our final Section 6 we briefly discuss the associated super-property. Here there is a slight divergence between the Radon-Nikodym Property for Banach spaces and p-triviality for quasi-Banach spaces. A Banach space with the super-Radon-Nikodym property is super-reflexive (**11**); thus there is a space X such that ℓ_1 is not finitely representable in X but which fails the Radon-Nikodym Property (**3**). However if ℓ_p $(0 < p < 1)$ is not finitely representable in a quasi-Banach space then it is p-trivial.

2. Notation

A quasi-norm on a real vector space X is a map $x \mapsto \|x\|$ such that

(1) $\|x\| > 0 \quad$ if $x \neq 0$.
(2) $\|tx\| = |t|\,\|x\| \quad t \in \mathbf{R},\ x \in X$.
(3) $\|x + y\| \leqslant k(\|x\| + \|y\|) \quad x, y \in X$.

where k is the modulus of concavity of the quasi-norm. If $k = 1$, $\|\cdot\|$ is a norm. In general the quasi-norm is r-subadditive $(0 < r \leqslant 1)$ if

(4) $\|x + y\|^r \leqslant \|x\|^r + \|y\|^r \quad x, y \in X$.

The sets $\{x : \|x\| < \alpha\}$ define the base of neighbourhoods for a Hausdorff vector topology on X. If X is complete, we say that X is a *quasi-Banach space*; if the quasi-norm is also r-subadditive then X is an *r-Banach space.*

The Aoki-Rolewicz theorem (**10**, p. 57) asserts that every quasi-norm is equivalent to a quasi-norm which is r-subadditive for some $r > 0$. Here $\|\cdot\|$ and $\|\cdot\|^*$ are *equivalent* if there exists $0 < m \leqslant M < \infty$ such that

$$m\|x\| \leqslant \|x\|^* \leqslant M\|x\| \quad x \in X.$$

A subset C of X is *p-convex* (where $0 < p \leqslant 1$) if given $x, y \in C$ and $0 \leqslant a, b \leqslant 1$ with $a^p + b^p = 1$, then $ax + by \in C$. Observe that if $0 < p < 1$ and C is a closed p-convex set then C contains 0. We say that X is (*locally*) *p-convex* if there is a bounded p-convex neighbourhood of zero; this is equivalent to the existence of an equivalent p-subadditive quasi-norm on X.

If C is a p-convex subset of X then a point x of C is *p-extreme* if $x = a_1x_1 + a_2x_2$ with $x_1, x_2 \in X$ and $0 < a_1, a_2 < 1$, $a_1^p + a_2^p = 1$ implies that $x = x_1 = x_2$.

A point $x \in C$ is *strongly p-extreme* if whenever $y_n, z_n \in C$, $0 \leqslant a_n, b_n \leqslant 1$, $a_n^p + b_n^p = 1$ and $a_ny_n + b_nz_n \to x$ then $\max(a_n, b_n) \to 1$. According to our definition 0 is never strongly p-extreme, although it may well be p-extreme. We regard strongly p-extreme points as an analogue of denting points.

The set of p-extreme points of C is denoted ∂_pC. If A is any set its p-convex hull is denoted by co_pA and its closed p-convex hull by $\overline{\mathrm{co}}_pA$.

3. p-trivial spaces

We define a quasi-Banach space X to be *p-trivial* $(0 < p < 1)$ if $\mathscr{L}(L_p, X) = \{0\}$, where $L_p = L_p(0, 1)$. As we observed in the introduction, this is the appropriate generalisation, to the case $p < 1$, of the Radon-Nikodym property for Banach spaces. In this section, we observe some examples of p-trivial quasi-Banach spaces.

Theorem 3.1. *Suppose X satisfies either of the following conditions:*

(a) *For any closed infinite-dimensional subspace Y of X there exists $q > p$ and a q-convex quasi-Banach space Z such that $\mathscr{L}(Y, Z) \neq \{0\}$.*

(b) *For any closed infinite-dimensional subspace Y of X there exists an F-space and a non-zero compact linear operator $T : Y \to Z$.*

Then X is p-trivial.

Proof. We prove only (b). Suppose $S \in \mathscr{L}(L_p, X)$ and $S \neq 0$. Then since $L_p^* = \{0\}$, $Y = \overline{S(L_p)}$ is infinite-dimensional. Let $T: Y \to Z$ be a non-zero compact operator on Y. Then TS is a non-zero compact operator on L_p, contradicting the results of (4).

A quasi-Banach space X is *pseudo-dual* if there exists a Hausdorff vector topology τ on X such that the unit ball of x is relatively compact (cf (8)).

Theorem 3.2. *Let X be a p-trivial quasi-Banach space and let Y be a closed subspace of X which is either q-convex for some $q > p$ or isomorphic to a pseudo-dual space. Then X/Y is p-trivial.*

Proof. In either case a linear operator $S: L_p \to X/Y$ may be lifted to a linear operator $\tilde{S}: L_p \to X$ (see (8)).

Theorem 3.3. *Let X be a quasi-Banach space, and let Y be a closed p-trivial subspace of X such that X/Y is p-trivial.*

Then X is p-trivial.

Proof. Immediate.

Theorem 3.4. *Let X be a quasi-Banach space which possesses no infinite-dimensional subspace isomorphic to a Hilbert space. Then X is p-trivial.*

Proof. This is immediate from (4) Theorem 3.4.

The author has recently constructed a non-p-trivial space which is p-convex, but contains no copy of L_p; details will appear elsewhere.

Theorem 3.5. *Suppose X is a subspace of L_p. Then X is p-trivial if and only if X has no subspace isomorphic to L_p.*

Proof. By the results of (6), if $T \in \mathscr{L}(L_p, L_p)$ and $T \neq 0$, there is a subspace Y of L_p, such that $Y \cong L_p$ and $T|Y$ is an isomorphism.

4. Edgar's theorem for p-trivial spaces

Our first main result generalises Edgar's theorem (2) on Banach spaces with the Radon-Nikodym property.

Theorem 4.1. *Suppose $0 < p < 1$ and that X is a p-trivial quasi-Banach space.*

Suppose C is a closed bounded p-convex subset of X and that $x \in C$. Then there exists a sequence $u_n \in \partial_p C$ and $a_n \geq 0$ such that $\Sigma\, a_n^p \leq 1$ and

$$x = \sum_{n=1}^{\infty} a_n u_n.$$

Proof. We shall assume the contrary and produce a contradiction. Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0, 1]$. For a sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}$ let $L_p(\mathcal{A})$ denote the closed subspace of $L_p[0, 1] = L_p(\mathcal{B})$ of all $\mathcal{A}$-measurable functions. Let Ω denote the first uncountable ordinal. We shall construct, by transfinite induction, an increasing transfinite sequence of σ-algebras $\mathcal{B}_\alpha$ $(1 \leq \alpha < \Omega)$ and of linear operators $T_\alpha : L_p(\mathcal{B}_\alpha) \to X$ such that

(1) $\mathcal{B}_1 = \{[0, 1], \phi\}$ and $T_1(c.1) = cx$ where 1 denotes the characteristic function of $[0, 1]$.

(2) If $\alpha < \beta$ and $f \in L_p(\mathcal{B}_\alpha)$ then $T_\beta f = T_\alpha f$.

(3) If $f \in L_p(\mathcal{B}_\alpha)$, $f \geq 0$ and $\|f\|_p \leq 1$ then $Tf \in C$.

(4) If $\epsilon_\alpha = \inf\{\Sigma_{n=1}^{\infty} \lambda(B_n)^{1/p} : B_n \in \mathcal{B}_\alpha;\ \bigcup_{n=1}^{\infty} B_n = [0, 1]\}$ then $\{\epsilon_\alpha : 1 \leq \alpha < \Omega\}$ is strictly decreasing.

Of course if we can satisfy (1), (2), (3), (4) then we have an immediate contradiction since any well-ordered subset of $\mathbf{R}$ is countable.

Define $\mathcal{B}_1$, T_1 as above. Now suppose $1 < \alpha < \Omega$ and that $\mathcal{B}_\beta$, T_β have been defined for $\beta < \alpha$. If α is a limit ordinal, let $\mathcal{B}_\alpha$ be the σ-algebra generated by $\bigcup(\mathcal{B}_\beta : \beta < \alpha)$. Since

$$\|T_\beta\| \leq 2k \sup_{y \in C} \|y\| \qquad \beta < \alpha$$

we can define T_α to be the unique extension of each T_β to $L_p(\mathcal{B}_\alpha)$. Then conditions (2), (3), (4) are immediate.

Next suppose $\alpha = \gamma + 1$. Let $(B_j^\gamma : j \in J)$ be a maximal family of disjoint atoms of β_γ i.e., $\lambda(B_j^\gamma) > 0$ and $B \in \mathcal{B}_\gamma$, $B \subset B_j^\gamma$ implies either $\gamma(B) = \lambda(B_j^\gamma)$ or $\lambda(B) = 0$. J is at most countable; let $B^* = [0, 1] \backslash \bigcup_J B_j^\gamma$. Since X is p-trivial we have $T_\gamma | L_p(B^*, \mathcal{B}_\gamma) = 0$.

Let $a_j = \lambda(B_j^\gamma)^{1/p}$ and $v_j = a_j^{-1} T 1_{B_j^\gamma}$ $(j \in J)$. Then

$$x = T_\gamma(1) = \sum_{j \in J} a_j v_j.$$

Hence by assumption there exists i such that $v_i \notin \partial_p C$ i.e.,

$$v_i = su + tw$$

where $u, w \in C$, $s, t > 0$ and $s^p + t^p = 1$.

Choose $A \in \mathcal{B}$ such that $A \subset B_i^\gamma$, and $\lambda(A) = (sa_i)^p$.

Let $\mathcal{B}_\alpha$ be the σ-algebra generated by adjoining A to $\mathcal{B}_\gamma$. Extend T_γ by defining

$$T_\alpha 1_A = sa_i u.$$

Then

$$T_\alpha 1_{B_i^\gamma \backslash A} = ta_i w$$

and conditions (2), (3) follow easily. For (4), observe that

$$\epsilon_\gamma = \sum_{j \in J} a_j$$

while

$$\epsilon_\alpha = \sum_{j \neq i} a_j + (s + t)a_i < \epsilon_\gamma.$$

This completes the proof.

Remark. It is easy, given Theorem 4.1, to modify the representation of x so that $\Sigma\, a_n^p = 1$. This follows from the fact that $0 \in C$ (see (5) for the details).

5. Geometric characterisations of p-trivial spaces

Suppose that C is a bounded p-convex set with 0 as an interior point (this implies that X is p-convex). Denote by C_0 the interior of C. Then if $x \in C$ and $0 \leqslant t < 1$, $tx \in C_0$. Let us define a function $\varphi : C_0 \to \mathbf{R}$ by

$$\varphi(x) = \inf \sum_{n=1}^{\infty} a_n$$

where the infimum is taken over all non-negative series $\Sigma\, a_n$ such that $\Sigma\, a_n^p = 1$ and there exist $u_n \in C_0$ with

$$x = \sum_{n=1}^{\infty} a_n u_n$$

Let us observe that the infimum may be taken instead over all non-negative series $\Sigma\, a_n$ such that $\Sigma\, a_n^p \leqslant 1$ and

$$x = \sum_{n=1}^{\infty} a_n u_n.$$

For if

$$x = \sum_{n=1}^{\infty} a_n u_n$$

where $a_n \geqslant 0$ and $\Sigma\, a_n^p \leqslant 1$, then for any N, we may write

$$x = \sum_{n=1}^{\infty} a_n u_n + \alpha(0 + 0 + \cdots + 0)$$

where $\alpha^p = N^{-1}(1 - \Sigma\, a_n^p)$, and there are N zero terms. Thus

$$\varphi(x) \leqslant \sum a_n + N\alpha$$

$$= \sum a_n + N^{1-1/p}\left(1 - \sum a_n^p\right)^{1/p}.$$

Letting $N \to \infty$ we see that

$$\varphi(x) \leqslant \sum a_n$$

For $x \in C$, we define

$$\varphi_*(x) = \liminf_{y \to x} \varphi(y).$$

$$\varphi^*(x) = \limsup_{y \to x} \varphi(y).$$

Thus φ_* is lower-semi-continuous and φ^* is upper-semi-continuous on C and $\varphi_* \leq \varphi^*$. Let

$$V = \{x \in C : \varphi^*(x) = 1\}$$
$$W = \{x \in C : \varphi_*(x) = 1\}.$$

Then V is closed and W is a G_δ-set; also $W \subset V$. Clearly any member of W is strongly p-extreme for C.

The following lemmas prepare our main theorem. We assume that X is p-trivial.

Lemma 5.1. *If $x \in C_0$, there exist $v_m \in V$ and $a_m \geq 0$ such that $\Sigma\, a_m^p \leq 1$ and $\Sigma\, a_m v_m = x$.*

Proof. (cf. Theorem 4.1). Suppose $x \in C_0$. Let $\mathcal{B}_1 = \{(0,1), \phi\}$ and define $T_1 : L_p(\mathcal{B}_1) \to X$ by $T_1(c.1) = cx$ By induction we construct an increasing sequence of atomic sub-σ-algebras $\mathcal{B}_n$ of $\mathcal{B}$ and a sequence of linear operators $T_n : L_p(\mathcal{B}_n) \to X$ such that

(1) $T_{n+1}|L_p(\mathcal{B}_n) = T_n$. $n \geq 2$,

(2) $T_n\{f : f \in L_p(\mathcal{B}_n);\ \ f \geq 0, \|f\|_p \leq 1\} \subset C_0$.

Indeed suppose $\mathcal{B}_n$ has atoms $(B_j^n : j \in J)$ where J is at most countable. Let $b_j = \lambda(B_j^n)^{1/p}$, $j \in J$ and

$$u_j = b_j^{-1} T_n 1_{B_j^n}.$$

Then $u_j \in C_0$. Then write

$$u_j = \sum_{i=1}^{\infty} a_{ij} w_{ij}$$

where $w_{ij} \in C_0$, $\Sigma\, a_{ij}^p = 1$

$$\sum_{i=1}^{\infty} a_{ij} \leq \tfrac{1}{2}(1 + \varphi(u_j))$$

(the sum may, of course, be finitely non-zero). Split each B_j^n into atoms $\{B_{ij}^n : i = 1, 2, \ldots\}$ where $\lambda(B_{ij}^n) = a_{ij}^p b_j^p$. Let $\mathcal{B}_{n+1} = \sigma\{B_{ij}^n : j \in J, i = 1, 2, \ldots\}$ and define T_{n+1} on $L_p(\mathcal{B}_{n+1})$ so that

$$T_{n+1} 1_{B_{ij}^n} = b_j^{-1} a_{ij}^{-1} w_{ij}.$$

It is easy to verify the conditions.

Now let $\mathcal{B}_\infty = \sigma(\bigcup_{n=1}^{\infty} \mathcal{B}_n)$ and let T be the unique continuous extension of each T_n to $L_p(\mathcal{B}_\infty)$. Then $\mathcal{B}_\infty$ has atoms $\{B_j : j \in J_\infty\}$ and since X is $L_{p\text{-trivial}}$

$$x = \sum_{j \in J_\infty} a_j v_j$$

where $a_j = \lambda(B_j)^{1/p}$ and $v_j = a_j^{-1} T 1_{B_j}$. Clearly $v_j \in C$ and $\Sigma\, a_j^p \leq 1$. It remains to show that $v_j \in V$.

For each n, let A_j^n be the atom of $\mathcal{B}_n$ including B_j. Then $\bigcap_n A_j^n = B_j$; let

$$z_j^n = \lambda(A_j^n)^{-1/p} T 1_{A_j^n}.$$

Then $z_j^n \in C_0$ and $z_j^n \to v_j$. Now for each n,

$$\tfrac{1}{2}(1 + \varphi(z_j^n)) \geq (\lambda(B_j)/\lambda(A_j^n))^{1/p}$$

and hence $\varphi(z_j^n) \to 1$. Thus $v_j \in V$.

Since X is necessarily p-convex we can assume that the norm on X is p-subadditive. We also choose $\delta > 0$ such that $\{x : \|x\| \leq \delta\}$ is contained in C.

Lemma 5.2. *Suppose $x \in C_0$ and $0 \leq t < 1$. Then*

$$\varphi(tx) \leq t\varphi_*(x).$$

Proof. Suppose $\epsilon > 0$, and

$$\epsilon \leq \delta t^{-1}(1 - t^p)^{1/p}.$$

Then there exists u, $\|u\| < \epsilon$ such that $x - u \in C_0$ and

$$\varphi(x - u) < \varphi_*(x) + \epsilon.$$

Hence

$$x - u = \sum_{n=1}^{\infty} a_n x_n$$

where $x_n \in C_0$, $a_n \geq 0$, $\Sigma\, a_n^p = 1$ and

$$\Sigma\, a_n \leq \varphi_*(x) + \epsilon.$$

Thus

$$tx = \sum_{n=1}^{\infty} t a_n x_n + \frac{t\epsilon}{\delta}\left(\frac{\delta u}{\epsilon}\right)$$

and

$$\sum t^p a_n^p + t^p \epsilon^p \delta^{-p} \leq 1.$$

By the remark at the beginning of the section,

$$\begin{aligned}\varphi(tx) &\leq t\left(\sum a_n\right) + t\epsilon\delta^{-1} \\ &\leq t(\varphi_*(x) + \epsilon) + t\epsilon\delta^{-1}.\end{aligned}$$

Now let $\epsilon \to 0$,

$$\varphi(tx) \leq t\varphi_*(x).$$

Lemma 5.3. *Suppose $x_n \in C$, $a_n \geq 0$ and $\Sigma a_n^p \leq 1$. Then*

$$\varphi_*\left(\sum_{n=1}^{\infty} a_n x_n\right) \leq \sum_{n=1}^{\infty} a_n \varphi_*(x_n).$$

Proof. For $\epsilon > 0$, there exist $u_n \in C_0$ such that

$$\|x_n - u_n\| \leq \epsilon.$$

and

$$u_n = \sum_{k=1}^{\infty} c_{nk} v_{nk}$$

where $v_{nk} \in C_0$, $c_{nk} \geq 0$, $\Sigma c_{nk}^p \leq 1$ and

$$\sum_k c_{nk} \leq \varphi_*(x_n) + \epsilon.$$

Thus

$$\left\|\sum_{n=1}^{\infty} a_n x_n - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_n c_{nk} v_{nk}\right\| \leq \epsilon.$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_n c_{nk} &\leq \sum_{n=1}^{\infty} a_n(\varphi_*(x_n) + \epsilon) \\ &\leq \sum_{n=1}^{\infty} a_n \varphi_*(x_n) + \epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ we obtain the result.

Lemma 5.4. *W is dense in V.*

Proof. Let

$$M = \sup_{x \in C} \|x\|.$$

Fix $n \in \mathbf{N}$ and let

$$W_n = \{x \in C : \varphi_*(x) > 1 - 1/n\}.$$

Then W_n is relatively open in C. We shall show that $W_n \cap V$ is dense in V. Fix $x \in V$ and $\epsilon > 0$.
Choose $\upsilon > 0$ such that

$$(1 - \upsilon)^{p/1-p} - \upsilon > 1 - 1/n.$$

and

$$\upsilon^p + M^p[\upsilon^p + (1 - (1 - \upsilon)^{1/1-p})^p + (1 - (1 - \upsilon)^{p/1-p})] < \epsilon^p.$$

Since $x \in V$ there exists $y \in C_0$ with

$$\varphi(y) > 1 - \upsilon$$

and

$$\|x - y\| < \upsilon.$$

Since $y \in C_0$, there exists τ, $1 < \tau < 1 + \upsilon$ such that $\tau y \in C_0$ and then we have

$$\varphi_*(\tau y) > 1 - \upsilon$$

by Lemma 5.2.

Now by Lemma 5.1

$$\tau y = \sum_{m=1}^{\infty} a_m v_m$$

where $v_m \in V$, $a_m \geq 0$ and $\Sigma\, a_m^p \leq 1$. Then by Lemma 5.3

$$\sum_{m=1}^{\infty} a_m \varphi_*(v_m) > 1 - \upsilon$$

and in particular

$$\sum a_m > 1 - \upsilon.$$

Suppose $a_1 \geq a_2 \geq \dots$; then

$$a_1 > (1 - \upsilon)^{1/(1-p)}$$

and hence

$$\sum_{m=2}^{\infty} a_m^p < 1 - (1 - \upsilon)^{p/(1-p)}.$$

Thus

$$\begin{aligned} a_1 \varphi_*(v_1) &> 1 - \upsilon - \sum_{m=2}^{\infty} a_m \\ &> (1 - \upsilon)^{p/(1-p)} - \upsilon. \end{aligned}$$

In particular

$$\varphi_*(v_1) > (1 - \upsilon)^{p/(1-p)} - \nu$$

so that $v_1 \in W_n$; also

$$\begin{aligned} \|x - v_1\|^p &\leq \|x - y\|^p + \|\tau y - y\|^p + \|\tau y - v_1\|^p \\ &\leq \upsilon^p + \upsilon^p M^p + \left((1 - a_1)^p + \sum_{m=2}^{\infty} a_m^p\right) M^p. \\ &= \upsilon^p + M^p[\upsilon^p + [1 - (1 - \upsilon)^{1/(1-p)}]^p + 1 - (1 - \upsilon)^{p/(1-p)}] \\ &< \epsilon^p. \end{aligned}$$

Thus it follows that $W_n \cap V$ is dense in V. Since V is closed in X and $W_n \cap V$ is relatively open in V, we may deduce from the Baire Category Theorem that $(\bigcap_{n=1}^{\infty} W_n) \cap V$ is dense in V i.e., W is dense in V.

Lemma 5.5. $C = \overline{\mathrm{co}_p}\, W$.

Proof. $\overline{\mathrm{co}_p}\, W = \overline{\mathrm{co}_p}\, V \supset C_0$ by Lemma 5.1. Since $\overline{C_0} = C$, we have the result.

The next theorem is our main result of the section, and may be regarded as a p-convex analogue of the characterisation of the Radon-Nikodym property for Banach spaces given by Phelps (9 Theorem 5).

Theorem 5.6. *Let X be a p-convex quasi-Banach space. Then X is p-trivial if and only if every closed bounded p-convex subset of X is the closed p-convex cover of its strongly p-extreme points.*

Proof. Suppose X is not p-trivial and that $T : L_p \to X$ is a bounded linear operator. Let U be the unit ball of L_p and consider $\overline{T(U)}$. Suppose x is strongly p-extreme for $\overline{T(U)}$. Then there exists $f_n \in U$ with $Tf_n \to x$. However for each f_n we may write (by splitting the interval)

$$f_n = (\tfrac{1}{2})^{1/p} g_n + (\tfrac{1}{2})^{1/p} h_n$$

where $g_n, h_n \in U$. Thus $2^{-1/p} Tg_n + 2^{-1/p} Th_n \to x$ and so we have a contradiction. Hence $\overline{T(U)}$ has no strongly p-extreme points.

Conversely suppose X is p-trivial and D is a closed bounded p-convex subset of X. Let S be the set of strongly p-extreme points for D.

Let B be the closed unit ball of X and for $\delta > 0$ let $C = C_\delta = \overline{\mathrm{co}_p}\,(D \cup \delta B)$. Using the notation of the preceding lemmas, $C = \overline{\mathrm{co}_p}\, W$. However W is contained in the set T_δ of strongly p-extreme points for C.

Suppose $x \in T_\delta$ and $\|x\| > \delta$. Then there exist $y_n \in D$ and, $w_n \in \delta B$, $0 \leq a_n \leq 1$, such that

$$a_n y_n + (1 - a_n^p)^{1/p} w_n \to x.$$

Hence $\max(a_n, (1 - a_n^p)^{1/p}) \to 1$. It is easy to see that since $\|x\| > \delta$ we have $a_n \to 1$ and hence $x \in D$. This implies that $x \in S$.

Now suppose $z \in D$ and $z \notin \overline{\mathrm{co}_p}\, S$. Let

$$\delta = \tfrac{1}{2}\, d(z, \overline{\mathrm{co}_p}\, S) = \tfrac{1}{2} \inf(\|z - v\| : v \in \overline{\mathrm{co}_p}\, S).$$

Then since $\lambda\, \overline{\mathrm{co}_p}\, S \subset \overline{\mathrm{co}_p}\, S$ for $0 \leq \lambda \leq 1$, we have $z \notin \overline{\mathrm{co}_p}\,(S \cup \delta B)$.

However $S \cup \delta B \supset T_\delta$ and hence $z \notin \overline{\mathrm{co}_p}\, T_\delta$, and we have a contradiction.

Corollary 5.7. *X is p-trivial if and only if every closed bounded p-convex set has a strongly p-extreme point.*

6. Remarks on super-properties

For the purposes of this section we shall restrict our comments to quasi-Banach spaces X which have a quasi-norm which is r-subadditive for some $r > 0$. We say that a quasi-Banach Y is finitely representable in a quasi-Banach space X if given any

$\epsilon > 0$ and any finite-dimensional subspace L of Y there is a subspace M of X with $\dim M = \dim L$ such that there is an isomorphism $T: L \to M$ with $\|T\| \|T^{-1}\| < 1 + \epsilon$.

If (P) is a property of quasi-Banach spaces, then we say that X has the property *super-(P)* if any space finitely representable in X has property (P).

Theorem 6.1. *If $0 < p < 1$, the following conditions on X are equivalent:*
(1) *X is super-p-trivial.*
(2) *ℓ_p is not finitely representable in X.*
(3) *X is q-convex for some $q > p$.*

Proof. (2)$\Leftrightarrow$(3) is proved in (7). (3)$\Rightarrow$(1) is obvious. For (1)$\Rightarrow$(2) observe that if ℓ_p is finitely representable in X then so is L_p.

The interest in the above theorem is that the analogy with the Radon-Nikodym Property breaks down at this point. Pisier (11) has shown that X has the super-Radon-Nikodym property if and only if X is super-reflexive. An example of James (3) shows that this is not the same as "ℓ_1 is not finitely representable in X" (i.e., X is B-convex).

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From our remarks in the introduction, the class of p-trivial spaces may also be regarded as a generalisation to quasi-Banach spaces of the class of Banach spaces X such that every $T \in \mathscr{L}(L_1, X)$ has the Dunford-Pettis property. This class is strictly larger than the class of spaces with the Radon-Nikodym Property.

The referee also calls our attention to a paper of W. Fischer and U. Scholer (13) who study a (different) generalisation of the Radon-Nikodym Property in quasi-Banach spaces. It is not clear at present how their work relates to the content of this paper.

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