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Mackey duals and almost shrinking bases

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1. *Introduction.* Suppose (e_n) is a basis of a Banach space E, and that (e'_n) is the dual sequence in E'. Then if (e'_n) is a basis of E' in the norm topology (i.e. (e_n) is shrinking) it follows that E' is norm separable: it is easy to give examples of spaces E for which this is not so. Therefore there are plenty of spaces which cannot have a shrinking basis. This leads one to consider whether it might not be reasonable to replace the norm topology on E' by one which is always separable (provided E is separable). Of course, the weak*-topology $\sigma(E', E)$ is one possibility (Köthe (17), p. 259); then it is trivial that (e'_n) is a weak*-basis of E'. However, if the weak*-topology is separable, then so is the Mackey topology $\tau(E', E)$ on E', and so we may ask whether (e'_n) is a basis of $(E', \tau(E', E))$.

These observations lead us to define an almost shrinking basis (e_n) as one such that (e'_n) is a basis of E' in its Mackey topology. A number of questions naturally arise concerning almost shrinking bases, and it is these questions which we investigate in this paper. As a spin-off of this study, we obtain some interesting examples in the theory of bases in locally convex spaces.

In section 2 we give some results concerning the structure of compact sets in the Mackey dual of a Banach space; in particular, we show that if E' is norm separable, then the Mackey topology and the norm topology define the same compact sets and convergent sequences. In general, the Mackey topology restricted to compact sets is the finest of all separable polar topologies on E'. Thus we obtain that if (e_n) is almost shrinking, then (e'_n) is a basis for E' in any separable polar topology.

In section 3 we switch to problems concerned with bases and give some alternative conditions equivalent to the condition that (e_n) is almost shrinking (Theorem 1). We then give conditions under which an almost shrinking basis is shrinking, and obtain examples of locally convex spaces which are not Banach spaces, but such that every Schauder basis is completely normal. We conclude the section by giving an example of a locally convex space having no basis but having a weak Schauder basis, and an example of a space with no almost shrinking basis.

In section 4 we give a result similar to the classical result of James relating reflexivity to the properties of a given basis (Theorem 3). This leads naturally to the consideration of the problem: in which Banach spaces is every basis almost shrinking? This is solved in Theorem 4, and as a consequence we find that the statement, 'every weak Schauder basis of $(E', \tau(E', E))$ is a basis', can only occur when it reduces to already known results (either $\tau(E', E)$ has the same convergent sequences as its weak topology or $(E', \tau(E', E))$ is a Banach space).

2. *Compact sets.* We suppose that E is a Banach space. The results that follow could be stated in more generality and most will hold for a locally convex space. We shall say a subset K of E' is *limited* if whenever $x_n \to 0$ weakly then

$$\lim_{n\to\infty} \sup_{f\in K} |f(x_n)| = 0,$$

(see Grothendieck(9), Köthe(16) and Webb(24)).

The topology on E' of uniform convergence on weakly null sequences will be denoted by ν.

We will denote by ρ the topology of uniform convergence on subsets of E which are absolutely convex, bounded and weakly metrizable. It is clear that all such sets are weakly precompact, and therefore have the property that every sequence contains a Cauchy subsequence. Let us also point out that by an easy application of Grothendieck's criterion ((7), (17), p. 269) for completeness (E',ν), $(E',\tau(E',E))$ and (E',ρ) are all complete; therefore precompact sets are relatively compact in these topologies.

Proposition 1. *Let K be a subset of E'; the following are equivalent:*

(i) *K is limited.*

(ii) *K is ν-relatively compact.*

(iii) *K is Mackey relatively compact.*

(iv) *K is ρ-relatively compact.*

If E is separable these are equivalent to:

(v) *K is relatively compact in every topology γ on E' such that (E',γ) is separable and γ is $\langle E,E'\rangle$ polar.*

Proof. (i)$\Leftrightarrow$(ii) is an exercise in Grothendieck(9), p. 286 (see also Webb(24)). Clearly (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (ii). Conversely (ii) $\Rightarrow$ (iv) by Grothendieck's result(9), p. 286, since we have observed that if $A \subset E$ is ρ-equicontinuous, then A has the property that every sequence contains a weakly Cauchy sequence. Similarly (ii) $\Rightarrow$ (iii) by an application of Eberlein's theorem ((4), (17), p. 315), which asserts that a weakly compact set in E is weakly sequentially compact and therefore has the same property.

If E is separable, then E' is weak*-separable (Köthe(17), p. 259), and hence Mackey separable. Hence (v) $\Rightarrow$ (iii). Conversely if γ is $\langle E,E'\rangle$-polar and (E',γ) is separable, it is easy to show that any γ-equicontinuous subset of E is weakly metrizable and so $\gamma \leqslant \rho$ so that (iv) $\Rightarrow$ (v).

Corollary *If E' is norm separable, then the Mackey topology $\tau(E',E)$ and the norm topology define the same compact sets and convergent sequences.*

This Corollary has been observed in special cases before; for example, Garling(6), p. 977 shows that $(l_1, \tau(l_1, c_0))$ has the same compact sets as the norm topology on l_1.

We define the topology σ^+ on E as the topology of uniform convergence on Mackey compact subsets of E'. By Proposition 1 and Proposition 1·3 of Webb(24) we have

Proposition 2. *σ^+ is the finest topology on E having the same convergent sequences as the weak topology.*

It is perhaps worth pointing out that although σ^+ has the same convergent sequences as $\sigma(E,E')$, it may define a different topology on the unit ball.

PROPOSITION 3. *σ^+ and the weak topology define the same topology on the unit ball of E if and only if every Mackey compact set in E' is norm compact. In particular this will be the case if E' is norm separable.*

Proof. See Webb(24), Proposition 2·10, $(c) \Leftrightarrow (d)$.

3. *Almost shrinking bases.* Now let $\{e_n\}$ be a Schauder basis of E; we will denote the dual sequence in E' by $\{e'_n\}$ so that for $x \in E$

$$x = \sum_{n=1}^{\infty} e'_n(x)\, e_n.$$

The sequence $\{e'_n\}$ is a Schauder basis of $(E', \sigma(E', E))$; if it is also a basis of E' in the norm topology we say that $\{e_n\}$ is *shrinking*. We shall say that $\{e_n\}$ is *almost shrinking* if $\{e'_n\}$ is a basis for $(E', \tau(E', E))$. This definition is motivated by the fact that E' is always Mackey separable although it may fail to be norm separable; furthermore $\{e'_n\}$ is always a weak basis for $(E', \tau(E', E))$. Therefore in asking whether a basis is almost shrinking we are asking the question of whether a weak Schauder basis of $(E', \tau(E', E))$ is necessarily a Schauder basis.

The operators P_n on E are defined by

$$P_n x = \sum_{i=1}^{n} e'_i(x)\, e_i$$

We recall that $\{e_i\}$ (resp. $\{e'_i\}$) is an equi-Schauder basis of (E, γ) (resp. (E', γ)) if the maps $\{P_n\}$ (resp. $\{P'_n\}$) are equicontinuous.

THEOREM 1. *Let E be a Banach space with a basis $\{e_n\}$. Then the following conditions are equivalent:*

(i) *$\{e_n\}$ is almost shrinking.*
(ii) *If $a_k \to 0$ weakly and $n_k \to \infty$ then $P_{n_k} a_k \to 0$ weakly.*
(iii) *If $a_k \to a$ weakly and $n_k \to \infty$ then $P_{n_k} a_k \to a$, weakly.*
(iv) *$\{e_n\}$ is an equi-Schauder basis of (E, σ^+).*
(v) *$\{e'_n\}$ is an equi-Schauder basis of $(E', \tau(E', E))$.*

Proof. We prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i), and (v) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii) If $a_k \to 0$ weakly then $\{a_k\}$ is $\tau(E', E)$-equicontinuous; hence for $f \in E'$

$$\lim_{k\to\infty} P'_{n_k} f(a_j) = f(a_j)$$

uniformly in j. Therefore

$$\lim_{k\to\infty} P'_{n_k} f(a_k) = f(\lim_{k\to\infty} a_k) = 0,$$

and so $P_{n_k} a_k \to 0$.

(ii) $\Rightarrow$ (iii) If $a_k \to a$ weakly then $a_k - a \to 0$ and so $P_{n_k}(a_k - a) \to 0$. However, $P_{n_k} a \to a$ and therefore $P_{n_k} a_k \to a$.

(iii) $\Rightarrow$ (v) Let W be a weakly compact subset of E; we will show that

$$P(W) = \bigcup_{n=1}^{\infty} P_n(W)$$

is relatively weakly compact. By Eberlein's Theorem, it is enough to show that $P(W)$ is relatively weakly sequentially compact. Let $f_k \in P(W)$ be a sequence, so that $f_k = P_{n_k} w_k$, $w_k \in W$; by selection of a subsequence we may suppose that $w_k \to w$ weakly.

If n_k contains a bounded subsequence, then we may select a subsequence of $\{f_k\}$ lying in the relatively compact (because finite-dimensional) set $\bigcup_{n=1}^{N} P_n(W)$. Hence we may assume $n_k \to \infty$ and therefore $P_{n_k} w_k \to w$, so that $P(W)$ is relatively weakly compact.

This shows that the maps P'_n are Mackey-equicontinuous and hence

$$P'_n f \to f \quad T(E', E)$$

on a Mackey closed subspace of E'. However, $P'_n e'_k \to e'_k$ $\tau(E', E)$ and the $\{e'_k\}$ are fundamental in $(E', \tau(E', E))$; hence $\{e'_k\}$ is an equi-Schauder basis of $(E', \tau(E', E))$.

(v) $\Rightarrow$ (i) Immediate.

(v) $\Rightarrow$ (iv) Let K be Mackey compact; then as $\{P'_n\}$ is equicontinuous it follows that $P'_n f \to f$ uniformly in the Mackey topology on K. Now we quote Theorem 1 of (19) to deduce that $P'(K) = \bigcup_{n=1}^{\infty} P'_n(K)$ is Mackey precompact and hence relatively compact. Therefore the maps P_n are σ^+-equicontinuous, and $\{e_n\}$ is an equi-Schauder basis of (E, σ^+).

(iv) $\Rightarrow$ (ii) If $a_k \to 0$ weakly then $a_k \to 0$ (σ^+), and as $\{P_{n_k}\}$ is σ^+-equicontinuous $P_{n_k} a_k \to 0$ (σ^+).

McArthur and Retherford (19) have shown that no basis of a Banach can be weakly equi-Schauder (cf. (2)); Theorem 1 shows that almost shrinking bases are 'sequentially' equi-Schauder in the weak topology (cf. conditions (ii) and (v)). At the same time, every Schauder basis of $(E', \tau(E', E))$ is equi-Schauder ((i) implies (v)).

It is also worth pointing out that if $\{e_n\}$ is almost shrinking then $\{e'_n\}$ is a Schauder basis of E' in any (E, E') polar topology γ such that (E', γ) is separable (see Proposition 1).

We next consider the distinction between almost shrinking and shrinking bases. We recall ((14)) that a sequence $\{e_n\}$ in a locally convex space is *normal* if there exists a sequence $\{\alpha_n\}$ of scalars and a neighbourhood V of O such that $\alpha_n x_n \notin V$ for all n, and $\{\alpha_n x_n\}$ is bounded ; a Schauder basis in which every block basic sequence is normal is called *completely normal*. It is shown in (14) that a Fréchet space with a completely normal basis is a Banach space.

THEOREM 2. *Let $\{e_n\}$ be an almost shrinking basis of a Banach space E. Then the following are equivalent*:

(i) *E' is norm-separable*;

(ii) *$\{e_n\}$ is shrinking*;

(iii) *$\{e'_n\}$ is a completely normal basis of $(E', \tau(E', E))$.*

Proof. (i) $\Leftrightarrow$ (ii) By Proposition 1 and Corollary.

(ii) $\Rightarrow$ (iii) Let (f_n) be a block basic sequence with respect to e'_n. Thus

$$f_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e'_i$$

where $p_0 = 0 < p_1 < p_2 \ldots$, and $f_n \neq 0$ for every n. We may choose α_n such that $\|\alpha_n f_n\| = 1$ for all n; then the sequence $(\alpha_n f_n)$ is $\tau(E', E)$-bounded.

Let $E'_n = \lim\{e'_{p_{n-1}+1}, \ldots, e'_{p_n}\}$ in E', and define a linear functional χ_n such that $\chi_n(\alpha_n f_n) = 1 = \|\chi_n\|$. Then

$$\chi_n(f) = \sum_{i=p_{n-1}+1}^{p_n} c_i f(e_i) \quad (f \in E'_n).$$

We define θ_n on E' by

$$\theta_n(f) = \sum_{i=p_{n-1}+1}^{p_n} c_i f(e_i) \quad (f \in E').$$

Thus

$$\theta_n(f) = f(x_n),$$

where

$$x_n = \sum_{i=p_{n-1}+1}^{p_n} c_i e_i$$

in E.

Now since $\{e_n\}$ is a basis of E

$$\sup \|P_n\| = K < \infty$$

and

$$\sup \|P'_n\| = K.$$

Thus

$$|\theta_n(f)| = \left|\chi_n\left(\sum_{i=p_{n-1}+1}^{p_n}\right) f(e_i)\, e'_i\right|$$

$$\leqslant 2K\|f\|.$$

Hence

$$\|x_n\| = \|\theta_n\| \leqslant 2K$$

and so $\{x_n\}$ is a bounded block basic sequence. We now quote Theorem 5·4 of (**14**) to deduce that $x_n \to 0$ weakly, and hence that $\{x_n\}$ is $\tau(E', E)$-equicontinuous. Let $V = \{f : |f(x_n)| \leqslant \frac{1}{2},\ n = 1, 2, \ldots\}$; then V is a $\tau(E', E)$-neighbourhood of zero, and $(\alpha_n f_n) \notin V$ for all n. Therefore (f_n) is normal, as required.

(iii) $\Rightarrow$ (ii). Suppose $\{e_n\}$ is not shrinking and suppose $P'_n f \nrightarrow f$ in norm in E'. Then there is a sequence $0 = p_0 < p_1 < p_2 \ldots$, such that if

$$g_n = \sum_{i=p_{n-1}+1}^{p_n} f(e_i)\, e'_i$$

then

$$\inf \|g_n\| = \epsilon > 0.$$

However, $\{g_n\}$ is $\tau(E', E)$-normal and so there is a sequence $\{\alpha_n\}$ of scalars such that $\{\alpha_n g_n\}$ is bounded and there is an absolutely convex neighbourhood V of zero with $\alpha_n g_n \notin V$ for all n. Then

$$\sup_n \|\alpha_n g_n\| < \infty$$

so that

$$\sup_n |\alpha_n| = a < \infty.$$

Thus $g_n \notin a^{-1} V$. However, $\{e'_n\}$ is a basis of $(E', \tau(E', E))$ and so Σg_n converges in $\tau(E', E)$. Hence we have reached a contradiction.

Theorem 2 enables us to give examples of locally convex spaces in which every Schauder basis is completely normal, but which are not Banach spaces. For example the space $(l_1, \tau(l_1, c_0))$ has this property.

PROPOSITION 4. *An unconditional basis is always almost shrinking.*

Proof. For $f \in E'$

$$f = \Sigma f(e_i)\, e_i', \quad \sigma(E', E)$$

and series converges subseries. Now by the Orlicz–Pettis Theorem (18).

$$f = \Sigma f(e_i)\, e_i' \tau, \quad (E', E).$$

We conclude this section by giving an example, a space with no almost shrinking basis. Let C be the space of all continuous real functions on $[0, 1]$; then C has a basis (the original result of Schauder (20)). However, if $f_n \in C'$ and $f_n \to f \tau(C', C)$ then it follows from results of Grothendieck ((8), Theoréme 2 combined with Proposition 1 above, or Edwards (5), p. 283 and p. 621), that $f_n \to f$ weakly in C'. Therefore any basis of $(C', \tau(C', C))$ would be a weak basis and therefore a basis of C' in the norm topology; however, C' is inseparable. The space $(C', \tau(C', C))$ is therefore a locally convex space with a weak Schauder basis but which has no basis; we believe that this is the first such example. Previous examples have been given of separable locally convex spaces without bases ((21), (22) and (12)) or of weak Schauder bases which fail to be bases ((1), (3)). It should be noted that the space above is not 'ω-separable' or 'sequentially separable' (see (12) and (23) p. 210).

4. *Properties of almost shrinking bases.* A basis (e_n) of a Banach space E is called *γ-complete* or *boundedly complete* if whenever

$$\sup_N \left\| \sum_{i=1}^N a_i e_i \right\| < \infty,$$

then $\Sigma a_i e_i$ converges. It is called *β-complete* if whenever

$$\left\{ \sum_{i=1}^N a_i e_i ; N = 1, 2, \ldots \right\}$$

is weakly Cauchy then $\Sigma a_i e_i$ converges. The classical theorem of James (10) states E is reflexive if and only if (e_n) is both γ-complete and shrinking; a modification of this result (13) states that E is reflexive if and only if (e_n) is both β-complete and shrinking. We now give a partial result of a similar type concerning almost shrinking bases.

THEOREM 3. *Let (e_n) be a β-complete almost shrinking basis of a Banach space E. Then E is weakly sequentially complete.*

Proof. By Theorem 1, (e_n) is an equi-Schauder basis of (E, σ^+). As (e_n) is β-complete, it follows that (e_n) is complete for (E, σ^+) in the sense of (11); hence (E, σ^+) is complete and therefore E is weakly sequentially complete.

It will be shown later that the converse of this result is false (i.e. there is a basis of a weakly sequentially complete Banach space which is not almost shrinking). A question naturally arises from Theorem 3: if E is weakly sequentially complete, is (E, σ^+) complete? A counterexample would yield a weakly sequentially complete Banach space with no almost shrinking basis.

The results of James have led naturally to the study of spaces E in which every basis is shrinking, or every basis is γ-complete (25), or every basis is β-complete (15). In each case a neat characterization of such spaces is obtained (in the first two cases E is reflexive, in the final case E is weakly sequentially complete). It is natural therefore to attempt to classify spaces in which every basis is almost shrinking.

THEOREM 4. *Let E be a Banach space with a basis. Then the following are equivalent*:

(i) *Either E is reflexive, or the weak topology and the norm topology define the same convergent sequences.*

(ii) *Every basis of E is almost shrinking.*

(iii) *Every basis of E is almost shrinking and β-complete.*

Proof. (i) $\Rightarrow$ (iii) In either case E is weakly sequentially complete (since in the latter case, a weakly Cauchy sequence must be norm convergent). Hence any basis of E must be β-complete. If E is reflexive, any basis is shrinking and therefore almost shrinking. On the other hand, if weak sequential convergence coincides with norm sequential convergence, then σ^+ is the norm topology and hence (iv) of Theorem 1 is satisfied for any basis.

(iii) $\Rightarrow$ (ii) Immediate.

(ii) $\Rightarrow$ (i) Let (e_n) be a basis of E, and suppose that E contains a sequence $x_n \to 0$ weakly such that $\|x_n\| \geqslant 1$. Then by a standard 'gliding hump' technique we may find a subsequence (y_n) of (x_n) and a block basic sequence (z_n) with $\|y_n - z_n\| \to 0$ and

$$z_n = \sum_{i=p_{n-1}+1}^{p_n} e_i'(y_n)\, e_i$$

where $\|z_n\| \geqslant \frac{1}{2}$ and $p_0 = 0 < p_1 < p_2 \ldots$. Then by Theorem 1, (ii), $z_n \to 0$ weakly.

Now suppose (e_n) is not shrinking: then there is a bounded block basic sequence (w_n) and $f_0 \in E'$ such that $f_0(w_n) = 1$ for all n (see (14) Theorem 5·4). Thus we may select a bounded block basic sequence (v_n) of the form

$$v_n = \sum_{i=q_{n-1}+1}^{q_n} c_i e_i$$

where $q_0 = 0 < q_1 < q_2 \ldots$, and such that (v_{2n}) is a subsequence of (w_n) and (v_{2n+1}) is a subsequence of (z_n).

Next we define
$$u_{2n} = v_{2n}$$
$$u_{2n+1} = v_{2n+1} - v_{2n},$$

so that $u_{2n}, u_{2n+1} \in \operatorname{lin}(e_{q_{2n-1}+1}, \ldots e_{q_{2n+1}})$ for $n = 1, 2, \ldots$. We shall show that there is a basis (t_n) of E such that $t_{q_{2n+1}} = u_{2n+1}$ and $t_{q_{2n+1}-1} = u_{2n}$. To do this, we quote Proposition 2 of (15), which is a modification of Zippin's Lemma (25). It must be shown that

$$\inf_c \inf_n \|u_{2n+1} + c u_{2n}\| = \lambda > 0.$$

Suppose on the contrary that $\lambda = 0$; then since u_{2n} and u_{2n+1} are linearly independent for each n, we may deduce the existence of sequences $n_k \to \infty$ and (c_k) such that

$$\|u_{2n_k+1} + c_k u_{2n_k}\| \to 0.$$

Now
$$\lim_{k\to\infty} (f_0(u_{2n_k+1}) + c_k f_0(u_{2n_k})) = 0,$$

and therefore
$$\lim_{k\to\infty} (c_k - 1) = 0.$$

Hence
$$\lim_{k\to\infty} \|u_{2n_k+1} + u_{2n_k}\| = 0.$$

i.e.
$$\lim_{k\to\infty}\|v_{2n_k+1}\| = 0,$$
contrary to construction. Hence there is a basis (t_n) as required.

Now
$$t_{q_{2n+1}}+t_{q_{2n+1}-1} = v_{2n+1} \to 0 \text{ weakly.}$$
Hence if (t_n) is almost shrinking
$$t_{q_{2n+1}} \to 0 \text{ weakly.}$$
However,
$$t_{q_{2n+1}} = v_{2n}$$
and
$$f_0(t_{q_{2n+1}}) = 1 \quad \text{for all } n.$$
Again we have a contradiction, and we deduce that (e_n) is shrinking. Hence every basis of E is shrinking and hence (25) E is reflexive.

We can now give the counter-example to the converse of Theorem 3. The space $L(0, 1)$ is weakly sequentially complete, but by Theorem 4 must possess a basis which is not almost shrinking. Another example is provided by the space $l_1 \oplus l_2$ in which each component satisfies the conditions of Theorem 4, but the whole space does not; this example has a basis which is unconditional and therefore β-complete and almost shrinking, but not every basis can be of this type. Another interpretation of Theorem 4 may be given as follows: it is shown that if every weak Schauder basis of $(E', \tau(E', E))$ is a Schauder basis, then either $(E', \tau(E', E))$ is a Banach space (E is reflexive) or, $\tau(E', E)$ defines the same convergent sequences as its weak topology $\sigma(E', E)$ (it may be easily shown that this is equivalent to the statement that σ^+ is the norm topology). Thus $(E', \tau(E', E))$ can only have the property that every weak Schauder basis is a basis if this is a consequence of already known results; there are no other spaces with this property.

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