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On summability domains

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1. *Introduction.* We denote by ω the space of all complex sequences with the topology given by the semi-norms

$$x \to |\delta^n(x)| \quad (n = 1, 2, \ldots),$$

where $\delta^n(x) = x_n$. An FK-space, E, is a subspace of ω on which there exists a complete metrizable locally convex topology τ, such that the inclusion $(E, \tau) \subset \omega$ is continuous; if τ is given by a single norm then E is a BK-space.

Let $A = (a_{ij}; i = 1, 2, \ldots; j = 1, 2, \ldots)$ be an infinite matrix; then if $x, y \in \omega$ are such that

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j,$$

we write

$$y = Ax.$$

If E is a sequence space (i.e. a subspace of ω) then the summability domain E_A is defined to be the set of all $x \in \omega$ such that Ax (exists and) belongs to E. If E is an FK-space, with a fundamental sequence of semi-norms (p_n), then E_A is an FK-space with the topology given by the semi-norms

$$x \to |\delta^n(x)| \quad (n = 1, 2, \ldots),$$

$$x \to q_i(x) = \sup_k \left| \sum_{j=1}^{k} a_{ij} x_j \right| \quad (i = 1, 2, \ldots),$$

$$x \to p_n(Ax) \quad (n = 1, 2, \ldots).$$

We shall denote by ϕ the space of all sequences eventually equal to zero. For $x \in \omega$, we write

$$P_n x = (x_1, x_2, \ldots, x_n, 0, \ldots);$$

if E is an FK-space containing ϕ we say that

(i) E is a BS-space if $(P_n x; n = 1, 2, \ldots)$ is bounded for each $x \in E$,

(ii) E is an AK-space if $P_n x \to x$ for each $x \in E$.

Let M be a subset of the positive integers N; then the characteristic function of M, χ_M, is given by

$$\chi_M(n) = 1 \quad \text{if} \quad n \in M$$
$$= 0 \quad \text{if} \quad n \notin M.$$

We denote by $\omega(M)$ the set of all complex sequences which are zero outside M and by P_M the natural projection of ω on to $\omega(M)$ given by

$$(P_M x)_n = \chi_M(n)\, x_n.$$

We shall use the notation $e^{(n)} = (0, \ldots, 0, 1, 0, \ldots)$ where the 'one' is in the nth position; at the same time $\delta^{(n)}$ will denote the coordinate map $\delta^n(x) = x_n$.

In this paper we consider several problems concerning the structure of summability domains. In section 2 we obtain some results concerning the circumstances under which a summability domain may be a BK-space. In section 3 we consider the problem posed by Bennett (2) of determining under what circumstances a summability domain may be reflexive or a Montel space. Finally in section 4 we apply functional analysis techniques to the study of simultaneous linear equations in an infinite number of unknowns, deriving some forms of the classical Mittag-Leffler theorem on complex functions.

2. *Structure theorems for summability domains.* The following lemma on subspaces of ω is well known.

LEMMA. *Every closed linear subspace of ω is either finite dimensional or isomorphic to ω, and is complemented in ω.*

Proof. As ω has its weak topology, a closed linear subspace E is weakly complete and metrizable. Hence E is isomorphic to the algebraic dual of a vector space of finite or countably infinite dimension with the weak topology; in the former case E is finite dimensional, in the latter E is isomorphic to ω.

For complementation we may restrict attention to the case where $E \cong \omega$; we then select the natural basis (χ_n) of E with dual functionals $\phi_n \in E'$. By the Hahn–Banach theorem we may extend each ϕ_n to a continuous linear functional ψ_n on ω; then the linear map P given by

$$Px = \sum_{n=1}^{\infty} \psi_n(x)\, \chi_n$$

is a continuous projection of ω onto E by the Banach–Steinhaus Theorem.

The next result is of the type discussed in (6); in the language of (6) we show that the class of space $\mathscr{C}(\omega)$ consists of all locally convex spaces.

PROPOSITION 1. *Let E be a locally convex space and let $T: E \to \omega$ be a linear map with closed graph. Then T is continuous.*

Proof. Let $T^*: \omega^* \to E^*$ be the (algebraic) adjoint map and let ω' and E' be the subspaces of ω^* and E^* consisting of all continuous linear functionals. By a result of Ptak (see (14), p. 114) $T^{*-1}(E') \cap \omega'$ is $\sigma(\omega', \omega)$ dense in ω'. However, every linear subspace of ω' is $\sigma(\omega', \omega)$ closed, so that we have $\omega' \subset (T^*)^{-1}(E')$, i.e. $T^*(\omega') \subset E'$. Hence T is weakly continuous, and, as ω has its weak topology, it follows that T is continuous.

PROPOSITION 2. *Let E be a locally convex space and let X be a closed linear subspace of $E \oplus \omega$. Let S be the natural projection of $E \oplus \omega$ on to E; then $G = S^{-1}(0) \cap X$ is complemented in X and any complement of G is isomorphic to a linear subspace of E.*

Proof. Clearly $G \subset \omega$, so that G is complemented in ω by the Lemma. If $Q\colon \omega \to G$ is a projection of ω on to G then $P_0 = Q(I-S)$ is a projection of QX on to G. Let P be any continuous projection of X on to G and let $F = P^{-1}(0)$. We shall show that the map $S\colon F \to E$ is an isomorphism of F on to a subspace of E.

First we note that S is continuous. Next we show that S is injective, for if $x \in X$ and $Sx = 0$ then $x \in G$, i.e. $Px = x$; if, furthermore, $x \in F$ then $Px = 0$, i.e. $x = 0$. Hence we may induce a Hausdorff topology ρ on F by the map $S\colon F \to E$, and ρ is weaker than the original topology τ.

Consider the map $T\colon (F,\rho) \to \omega$, where $T = I - S$ is the complementary projection to S in $E \oplus \omega$. Let (x_α) be a net in F such that

$$x_\alpha \to x(\rho)$$

and
$$Tx_\alpha \to y(\tau)$$

Then
$$Sx_\alpha \to Sx(\tau)$$

and so
$$x_\alpha = (T+S)\,x_\alpha \to y + Sx(\tau).$$

As $\tau \geqslant \rho$,
$$x = y + Sx,$$

i.e.
$$y = Tx.$$

Thus T has a closed graph and is continuous, by Proposition 1. As τ is induced by the maps S and T it follows that $\rho = \tau$ and so S is an isomorphism.

THEOREM 1. *Let E be an* FK-*space and A an infinite matrix; then*

$$E_A \cong F \oplus G,$$

where F is isomorphic to a closed subspace of $\prod_{i=1}^{\infty} (c)_i \oplus E$ and G is either finite-dimensional or isomorphic to ω. If E_A is a BK-*space then E_A is isomorphic to a closed subspace of $c \oplus E$.*

Proof. This is a simple extension of Theorem 1 of (2), where it is shown that E_A is isomorphic to a closed subspace of $\omega \oplus \prod_{i=1}^{\infty} (c)_i \oplus E$.

The fact that F is isomorphic to a closed subspace of $\prod_{i=1}^{\infty} (c)_i \oplus E$ follows from the completeness of E_A and hence of F.

For the second part, we observe that only a finite number of the semi-norms $\{p_n, |\delta^n|, q_n\}$ are required to determine the topology on E_A; it is thus clear that G is finite-dimensional and F is isomorphic to a closed subspace of $c \oplus c \oplus \ldots \oplus c \oplus E$. Hence E_A is isomorphic to a subspace of $E \oplus (c \oplus \ldots \oplus c \oplus G) \cong E \oplus c$.

THEOREM 2. *Let E be a* BK-*space and A an infinite matrix such that E_A is a* BS-*space. Then there is a subset M of N such that the projection $P_M\colon E_A \to E_A$ is continuous and*

(i) $P_M(E_A) = \omega(M)$.

(ii) $(I - P_M)\,E_A \cong$ *a closed subspace of* $E \oplus c$.

Proof. Let $M = \{j : a_{ij} = 0 \text{ for all } i\}$; then for $x \in \omega(M)$ we have $Ax = 0 \in E$ so that $\omega(M) \subset E_A$. Clearly $P_M\colon E_A \to \omega(M)$ is a continuous projection. Let $L = N - M$; then $(I - P_M)\,E_A = \omega(L) \cap E_A$.

Now the set $P(x) = (P_n x; n = 1, 2, \ldots)$ is bounded in E_A, for $x \in E_A$, and as the map $A: E_A \to E$ is continuous, the set $AP(x)$ is bounded in E. Therefore the maps

$$(AP_n; n = 1, 2, \ldots)$$

are pointwise bounded on E and the semi-norm

$$|||x||| = \sup_n (\|AP_n x\|) + \|x\|$$

(where $\|.\|$ is the norm on E) is continuous on E_A. We shall show that on $\omega(L) \cap E_A$, $|||.|||$ defines the topology of E_A.

For each n
$$\begin{aligned} q_n(x) &= \sup_k \left| \sum_{j=1}^{k} a_{nj} x_j \right| \\ &= \sup_k |\delta^n(AP_k x)| \\ &\leqslant \sup_k (\|\delta^n\| \, \|AP_k x\|) \\ &\leqslant \|\delta^n\| \, |||x|||. \end{aligned}$$

so that each q_n is $|||.|||$ continuous.

For each $n \in L$ there exists a k with $a_{kn} \neq 0$; then we have

$$\begin{aligned} |a_{kn}| \, |x_n| &\leqslant \left| \sum_{j=1}^{n} a_{kj} x_j \right| + \left| \sum_{j=1}^{n-1} a_{kj} x_j \right| \\ &\leqslant \|\delta^k\| \, (\|AP_n x\| + \|AP_{n-1} x\|) \\ &\leqslant 2\|\delta^k\| \, |||x||| \end{aligned}$$

so that
$$|\delta^n(x)| \leqslant \frac{2}{|a_{kn}|} \|\delta^k\| \, |||x|||.$$

Thus on $\omega(L) \cap E_A$, $|||.|||$ defines the topology; we may then apply Theorem 1 to this space which has properties BK and BS. By Theorem 1 $\omega(L) \cap E_A$ is isomorphic to a closed subspace of $c \oplus E$.

COROLLARY. *If c_A is a BS-space then $c_A \cong \omega(M) \oplus X$ where X is isomorphic to a closed subspace of c.*

3. *Montel and reflexive summability domains*

PROPOSITION 3. *Let E be a Montel F-space with defining semi-norms*

$$(p_n; n = 0, 1, 2, \ldots)$$

and suppose that the topology defined by the semi-norms $(p_n; n = 1, 2, \ldots)$ is Hausdorff. Then the semi-norms $(p_n; n = 1, 2, \ldots)$ define the original topology on E.

Proof. Let τ be the original topology on E and let τ_0 be the topology of the semi-norms $(p_n; n = 1, 2, \ldots)$. Then if the identity map $(E, \tau_0) \to (E, \tau)$ is not continuous there exists a sequence x_n in E with

$$x_n \to 0(\tau_0)$$
$$\inf_n p_0(x_n) = \epsilon > 0.$$

Let
$$y_n = \frac{1}{p_0(x_n)} x_n$$
so that we have
$$y_n \to 0(\tau_0)$$
$$p_0(y_n) = 1 \quad (n = 1, 2, \ldots).$$

Then the set (y_n) is τ-bounded and so possesses a τ-convergent subsequence (y_{n_k})
$$y_{n_k} \to y(\tau).$$
Clearly
$$y_{n_k} \to y(\tau_0)$$
so that
$$y = 0$$
as τ_0 is Hausdorff. However,
$$p_0(y) = \lim_{k\to\infty} p_0(y_{n_k})$$
$$= 1$$
and so we have reached a contradiction.

PROPOSITION 4. *If E is a BK-space and E_A is a Montel space, then E_A is a closed subspace of ω_A.*

Proof. By Proposition 3, the semi-norm
$$x \to \|Ax\|$$
may be discarded from the defining semi-norms so that the topology on E_A is given by the semi-norms $\{|\delta^n|, q_n\}$, i.e. is inherited from ω_A. Then E_A is a closed subspace of ω_A.

THEOREM 3. *Let E be a BK-space and let A be a matrix whose columns are members of E; then E_A is a Montel space if and only if A has only a finite number of non-zero columns (and thus $E_A = \omega$).*

Proof. As the columns of A are members of E we have $\phi \subset E_A$. Now by Proposition 4 E_A is a closed subspace of ω_A, and ω_A is an AK-space; therefore $E_A = \omega_A$.

Thus E_A is an AK-space and therefore a BS-space, and we may apply Theorem 2:
$$E_A = \omega(M) \oplus G,$$
where G is a BK-space. As E_A is a Montel space, G is finite-dimensional, and so $N - M = L$ is finite, i.e. A has only a finite number of non-zero columns. Clearly also $E_A = \omega$.

This partially solves a problem of Bennett (2), who conjectures that the above result is true for any matrix A for $E = c$, l_p or m. He also conjectures that l_A or c_A can only be reflexive under the same conditions. We give a similar result for the case of l_A; however, again we require $\phi \subset l_A$.

PROPOSITION 5. *If l_A is reflexive then l_A is a closed subspace of ω_A.*

Proof. We use an argument similar to Proposition 3. Let τ_0 be the topology inherited by l_A from ω_A and let τ be the original FK-topology. If the identity map $(l_A, \tau_0) \to (l_A, \tau)$ is not continuous, then we may find a sequence $x^{(n)} \in l_A$ such that
$$x^{(n)} \to 0(\tau_0)$$

and $$\|Ax^{(n)}\|_1 = 1 \quad (n = 1, 2, \ldots)$$

(where $\|\,.\,\|_1$ is the usual norm on l).

Then the set $B = \{x^{(n)}\}$ is bounded in l_A and hence is weakly relatively compact; hence on $\bar{B}$ the weak topology on l_A agrees with the topology of coordinatewise convergence. Thus $x^{(n)} \to 0$ weakly in l_A.

Now the map $A: l_A \to l$ is continuous and so $Ax^{(n)} \to 0$ weakly in l. Hence by a well-known property of l (see (1), p. 137) $\|Ax^{(n)}\|_1 \to 0$, contradicting the choice of $\{x^{(n)}\}$. Thus $\tau_0 = \tau$ and the result is proved.

THEOREM 4. *If A is a matrix whose columns are members of l and such that l_A is reflexive, then only finitely many columns of A are non-zero (and $l_A = \omega$).*

Proof. The proof follows from Proposition 5 just as that of Theorem 3 from Proposition 4.

We remark that Theorem 4 holds if we assume that l_A has a separable strong dual, with a very similar proof; we omit the details. Using this result we may show that if A and B are matrices such that $\phi \subset l_A = c_B$ then $l_A = \omega$; this follows from the result that the strong dual of c_B is separable (see (16)).

4. *Solutions of simultaneous linear equations.* In this section we consider the problem of solving an infinite set of simultaneous linear equations

$$\sum_{j=1}^{\infty} a_{ij} x_j = \eta_i \quad (i = 1, 2, \ldots).$$

We obtain a necessary and sufficient condition on the matrix A so that the system has a solution for any sequence (η_i). This problem has been considered by Banach (1), pp. 51–52, Eidelheit (3, 4), Polya (13), Petersen and Baker (9, 10, 11, 12). Banach showed that if the system has a unique solution for any (η_i) then A is row-finite. Polya investigated solutions satisfying

$$\sum_{j=1}^{\infty} |a_{ij}|\,|x_j| < \infty \quad (i = 1, 2, \ldots)$$

and his results were extended by Petersen and Baker. Our results and approach are most closely related to the results of Eidelheit (particularly (4), Satz 1).

We shall need an idea due to Garling (5). Let bv denote the space of sequences of bounded variation, i.e. such that

$$\sum_{i=1}^{\infty} |x_i - x_{i+1}| < \infty.$$

Then bv is a BK-space under the norm

$$\|x\|_{bv} = \sum_{i=1}^{\infty} |x_i - x_{i+1}| + \lim_{n\to\infty} |x_n|.$$

If X is any subset of ω then $B^*(X)$ is the linear span of the set $\{bx = (b_i x_i); b \in bv, x \in X\}$.

Let A be an infinite matrix; we shall say that A is *essential* if (1) the rows $(a^{(n)})$ of A are linearly independent, (2) for each n,

$$B^*(a^{(1)} \ldots a^{(n)}; e^{(1)} \ldots e^{(n)}) \cap \operatorname{lin}(a^{(1)}, a^{(2)}, \ldots)$$

is finite-dimensional.

THEOREM 5. *The system of equations*

$$\sum_{j=1}^{\infty} a_{ij} x_j = \eta_i$$

is soluble for any sequence (η_i) *if and only if* A *is essential.*

Proof. We first show that (1) and (2) are necessary for the system to possess a solution for all η_i. It is easily seen that (1) is necessary; now suppose that (2) fails. Then there exists N and a sequence $f^{(n)} \in \phi$, not contained in a finite-dimensiona subset, such that

$$\sum_{k=1}^{\infty} f_k^{(n)} a^{(k)} \in B^*(a^{(1)} \ldots a^{(N)}, e^{(1)} \ldots e^{(N)}).$$

Therefore
$$\sum_{k=1}^{\infty} f_k^{(n)} a^{(k)} = \sum_{i=1}^{N} b^{(i,n)} a^{(i)} + \sum_{i=1}^{N} \gamma_i^{(n)} e^{(i)}.$$

Let
$$\lambda_n = \max_i \|b^{(i,n)}\|_{bv} + \max_i |\gamma_i^{(n)}|.$$

Then for $x \in \omega_A$

$$\left| \sum_{i=1}^{\infty} f_i^{(n)} \sum_{j=1}^{\infty} a_j^{(i)} x_j \right| \leqslant \sum_{i=1}^{N} \|b^{(i,n)}\|_{bv} \left(\sup_m \left| \sum_{j=1}^{m} a_j^{(i)} x_j \right| \right) + \sum_{i=1}^{N} |\gamma_i^{(n)}| \cdot |x_j|$$
$$\leqslant \lambda_n \left(\sum_{i=1^m}^{N} \left| \sum_{j=1}^{m} a_j^{(i)} x_j \right| + \sum_{i=1}^{N} |x_j| \right),$$

so that if $\sum_{i=1}^{\infty} a_{ij} x_j = \eta_i$ is soluble, then

$$\sup_n \left| \lambda_n^{-1} \sum_{i=1}^{\infty} f_i^{(n)} \eta_i \right| < \infty.$$

This clearly contradicts the assumption that the system is always soluble, since the sequence $f^{(n)}$ is not contained in any finite-dimensional subset of ϕ.

Conversely suppose A is essential; we shall show that the map $A: \omega_A \to \omega$ maps ω_A on to a dense barrelled subspace of ω. It is easy to see that (1) implies that $A(\omega_A)$ is dense in ω. Now suppose (1) and (2) hold and that $C \subset \phi = \omega'$ is pointwise bounded on $A(\omega_A)$, but not finite-dimensional. Then there exists a sequence $f^{(n)} \in C$, not contained in a finite-dimensional subspace, such that for $x \in \omega_A$

$$\sup_n |f^{(n)}(Ax)| < \infty.$$

Thus $\{f^{(n} A\}$ is equicontinuous on ω_A and so there exists $r > 0$ and $\alpha_1 \ldots \alpha_r$, $\beta_1 \ldots \beta_r$ such that

$$|f^{(n}(Ax)| \leqslant \sum_{i=1}^{r} \alpha_i q_i(x) + \sum_{i=1}^{r} \beta_i |\delta^i|(x)|.$$

Therefore
$$f^{(n)}(Ax) = \sum_{i=1}^{r} \alpha_i g_i^{(i,n)}(x) + \sum_{i=1}^{r} \beta_i \gamma_i^{(n)} x_i,$$

where $|\gamma_i^{(n)}| \leqslant 1$, and
$$|g^{(i,n)}(x)| \leqslant q_i(x) \quad (x \in \omega_A).$$

Identifying cs' with bv we see that $g^{(i,n)}$ takes the form

$$g^{(i,n)}(x) = \sum_{j=1}^{\infty} a_j^{(i)} b_j^{(i,n)} x_j,$$

where $b^{(i,n)} \in bv$ and $\|b^{(i,n)}\| \leqslant 1$.

Let $y_k^{(n)} = f^{(n)}(Ae^{(k)})$; then

$$\begin{aligned} y^{(n)} &= \sum_{i=1}^{\infty} f_i^{(n)} a^{(i)} \\ &= \sum_{i=1}^{\infty} \alpha_i b^{(i,n)}.a^{(i)} + \sum_{i=1}^{r} \gamma_i^{(n)} \beta_i e^{(i)} \end{aligned}$$

and so $\Sigma f_i^{(n)} a^{(i)} \in B^*(a^{(1)} \ldots a^{(r)}, e^{(1)} \ldots e^{(r)})$. Since the $a^{(i)}$ are linearly independent, it follows from (2) that the sequence $\{f^{(n)}\}$ is contained in a finite-dimensional subset of ϕ, contradicting the assumption.

It follows $A(\omega_A)$ is barrelled in ω, and so by the open mapping theorem (see (14), p. 116) the map $A: \omega_A \to A(\omega_A)$ is open. In particular it follows that $A(\omega_A)$ is complete in ω, and hence $A(\omega_A) = \omega$, as required.

Suppose now that A is a matrix satisfying $a_{nk} \neq 0$ for all n and k. Then Polya (13) gives the following sufficient condition for A to be essential

$$\lim_{k\to\infty} \frac{a_{mk}}{a_{nk}} = 0 \quad \text{whenever} \quad n \leqslant m.$$

We recall that a sequence (x_k) is Césaro $(C, 1)$ convergent to x if

$$\frac{x_1 + \ldots + x_k}{k} \to x$$

and this we will write $$\lim_c x_k = x.$$

LEMMA. *If* $\lim_c x_k = 0$ *and* $b = (b_k) \in bv$ *then*

$$\lim_c b_k x_k = 0.$$

Proof. $$b_1 x_1 + \ldots + b_k x_k = \sum_{i=1}^{k} (b_i - b_{i+1}) \sum_{j=1}^{i} x_j + b_{k+1}\left(\sum_{j=1}^{k} x_j\right)$$

so that $$\begin{aligned} |b_1 x_1 + \ldots + b_k x_k| &\leqslant \sup_{m\leqslant k} \left|\sum_{i=1}^{m} x_i\right| \left(\sum_{i=1}^{k} |b_i - b_{i+1}| + |b_{k+1}|\right) \\ &\leqslant \|b\|_{bv} \sup_{m\leqslant k} \left|\sum_{i=1}^{m} x_i\right|. \end{aligned}$$

Therefore $$\left|\frac{1}{k}(b_1 x_1 + \ldots + b_k x_k)\right| \leqslant \sup_{m\leqslant k} \left|\frac{m}{k} s_m\right|$$

where $$s_m = \frac{1}{m} \sum_{i=1}^{m} x_i \to 0.$$

Let $$S = \sup_n |s_n| < \infty,$$

and for given $\epsilon > 0$, choose $k_0(\epsilon)$ such that, for $k \geqslant k_0$,

$$|s_k| \leqslant \epsilon.$$

Then let $k_1 \geqslant Sk_0\epsilon^{-1}$ and suppose $k \geqslant k_1$ and $m \leqslant k$; if $m \geqslant k_0$

$$\left|\frac{m}{k}s_m\right| \leqslant \frac{m}{k}\epsilon,$$

while if $m < k_0$

$$\left|\frac{m}{k}s_m\right| \leqslant S\left(\frac{k_0}{k_1}\right) \leqslant \epsilon.$$

Hence

$$\lim_{k\to\infty}\sup_{m\leqslant k}\frac{ms_m}{k} = 0$$

and the lemma is proved.

This lemma is equivalent to Beispiel 4·2 of Zeller (15).

PROPOSITION 6. *Suppose A is a matrix with $a_{nk} \neq 0$ for $1 \leqslant k < \infty$, $1 \leqslant n < \infty$, and suppose that for $m < n$*

$$\lim_{k\to\infty}{}_c \frac{a_{mk}}{a_{nk}} = 0.$$

Then A is essential.

Proof. We first verify (2). Suppose

$$\sum_{i=1}^{N} c_i a^{(i)} \in B^*(a^{(1)}\ldots a^{(n)}, e^{(1)}\ldots e^{(n)}),$$

i.e.

$$\sum_{i=1}^{N} c_i a^{(i)} = \sum_{i=1}^{n} b^{(i)}a^{(i)} + \sum_{i=1}^{n} d_i e^{(i)},$$

where $b^{(i)} \in bv$.

Thus

$$\sum_{i=1}^{N} c_i a_{ik} = \sum_{i=1}^{n} b_k^{(i)} a_{ik} + d_k \quad (k = 1, 2, \ldots).$$

Suppose $N > n$

$$\sum_{i=1}^{N} c_i \frac{a_{ik}}{a_{Nk}} = \sum_{i=1}^{n} b_k^{(i)} \frac{a_{ik}}{a_{Nk}} + \frac{d_k}{a_{Nk}}$$

and therefore taking $(C, 1)$ limits as $k \to \infty$, we obtain

$$\sum_{i=1}^{N} c_i \lim{}_c \frac{a_{ik}}{a_{Nk}} = \sum_{i=1}^{n} \lim{}_c b_k^{(i)} \frac{a_{ik}}{a_{Nk}} + \lim{}_c \frac{d_k}{a_{Nk}}.$$

It now follows by the assumptions and the lemma that

$$c_N = 0.$$

In this way we may show that $\sum_{i=1}^{N} c_i a^{(i)} \in \operatorname{lin}(a^{(1)}\ldots a^{(n)})$, and hence (2) is proved. Now if

$$\sum_{i=1}^{N} c_i a^{(i)} = 0$$

then it follows that $\sum_{i=1}^{N} c_i a^{(i)} \in \operatorname{lin}(a^{(1)})$ and hence that $c_1 = c_2 = \ldots c_N = 0$; thus (1) is also proved.

One version of the Mittag–Leffler Theorem (see (6) or (5), p. 299) states that if (z_n) is a sequence of distinct points in $\mathbf{C}$ with no accumulation points and η_n is any sequence of complex numbers there is an entire function f with

$$f(z_n) = \eta_n \quad (n = 1, 2, \ldots).$$

Polya (13) studies this theorem by means of simultaneous linear equations and proves also that if (m_n) is any sequence of integers there is an entire function satisfying

$$f^{(m_n)}(z_n) = \eta_n \quad (n = 1, 2, \ldots).$$

We observe that Proposition 6 yields these results very quickly. As an example of the application of Proposition 6 we give the following theorems.

THEOREM 6. *Let (z_n) be a sequence of distinct complex numbers with no accumulation point in C; let M be any subset of N such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_M(k) = 1.$$

Then for a sequence (η_n) of complex numbers there is an entire function f of the form

$$f(z) = t_0 + \sum_{k\in M} t_k z^k$$

and such that

$$f(z_n) = \eta_n.$$

Proof. Suppose, without loss of generality, that $0 \in \{z_n\}$; re-order (z_n) so that $z_0 = 0$ and

$$|z_n| \geqslant |z_{n-1}| \quad (n = 1, 2, \ldots).$$

Then we require to solve the equations

$$\sum_{k\in M} x_k z_n^k = \eta_n - \eta_0 \quad (n = 1, 2, \ldots)$$

for the variables $(x_k;\ k \in M)$. If $(t_k;\ k\in M)$ is a solution, then the function

$$f(z) = t_0 + \sum_{k\in M} t_k z^k,$$

where $t_0 = \eta_0$ solves the original problem; for since $\sum_{k\in M} t_k z^k$ converges for each z_n and $|z_n| \to \infty$, it is clear the series converges for all $z \in \mathbf{C}$.

We write $M = (m_1, m_2, \ldots)$, where $m_1 \leqslant m_2 \leqslant m_3, \ldots$, and consider the matrix A given by $a_{nk} = z_n^{m_k}$, for $k \geqslant 1$, $n \geqslant 1$.

For $1 \leqslant p < n$

$$\frac{a_{pk}}{a_{nk}} = \left(\frac{z_p}{z_n}\right)^{m_k} = \zeta^{m_k},$$

where $|\zeta| \leqslant 1$ and $\zeta \neq 1$. By the conditions of the theorem

$$\lim_{q\to\infty} \frac{m_q}{q} = 1.$$

Now

$$\left|\sum_{k=1}^{q} \zeta^{m_k} - \sum_{k=1}^{m_q} \zeta^k\right| \leqslant m_q - q$$

and hence
$$\left|\sum_{k=1}^{q} \zeta^{m_k}\right| \leqslant (m_q - q) + \left|\frac{1-\zeta^{m_q+1}}{1-\zeta}\right|$$
$$\leqslant (m_q - q) + \frac{2}{|1-\zeta|}.$$

Therefore
$$\left|\frac{1}{q}\sum_{k=1}^{q} \zeta^{m_k}\right| \leqslant \frac{m_q}{q} - 1 + \frac{2}{q|1-\zeta|} \to 0,$$

i.e.
$$\lim\nolimits_c \frac{a_{pk}}{a_{nk}} = 0.$$

Hence A is essential and, by Theorem 5, the result follows.

If $|z_n| > |z_{n-1}|$ for all n then the above proof is simplified since
$$\lim_{k\to\infty} \frac{a_{pk}}{a_{nk}} = 0$$
whenever $p < n$; we then need no restrictions on the density of M.

COROLLARY. *If $|z_n| > |z_{n-1}|$ for all n, then Theorem 5 is true for any infinite subset M of N.*

However, in general, the theorem fails quite simply without some restrictions on M. Thus if M consists of the even integers only then we can find no entire function
$$f(z) = t_0 + \sum_{k=1}^{\infty} t_k z^{2k}$$
with
$$f(n) = n \quad (n = 0, \pm 1, \pm 2, \ldots).$$

In this case we only have
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_M(k) = \tfrac{1}{2}.$$

Finally let us observe that the same theorems and proofs apply when the open unit disc D replaces **C**.

THEOREM 7. *Let (z_n) be any sequence in D having no accumulation point in D; if M is a subset of N such that*
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_M(k) = 1$$
and if (η_k) is any sequence of complex numbers, then there is a function f, regular on D, of the form
$$f(z) = t_0 + \sum_{k\in M} t_k z^t$$
and such that $f(z_k) = \eta_k$, for $k = 1, 2, \ldots$.

If $|z_n| > |z_{n-1}|$ for all n, then we may take for M any infinite subset of N.

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