Proc. Camb. Phil. Soc. (1971), **70**, 401
PCPS 70–42
Printed in Great Britain

Some forms of the closed graph theorem

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(*Received* 16 *October* 1970)

In this paper we shall establish some forms of the closed graph theorem for locally convex spaces, using the approach of Pták(17). Our interest is in classifying pairs of locally convex spaces (E,F) which have the property that every closed graph linear mapping $T\colon E \to F$ is continuous; if (E,F) has this property then we shall say that (E,F) is in the class $\mathscr{C}$. If $\mathscr{A}$ is a particular class of locally convex spaces then $\mathscr{C}(\mathscr{A})$ is the class of all E such that $(E,F) \in \mathscr{C}$ for all $F \in \mathscr{A}$.

The first result of this type was obtained by Mahowald(13), (see also Bourbaki(5) Ch. 6, § 1, no. 4, Proposition 11) who showed that if $\mathscr{A}$ is the class of all Banach spaces, then $\mathscr{C}(\mathscr{A})$ is the class of all barrelled spaces. It follows immediately that one can replace $\mathscr{A}$ by the class of all B_r-complete spaces in this statement. In this paper we will determine $\mathscr{C}(\zeta)$ where ζ is the class of all separable B_r-complete spaces.

1. *Metrizable subsets of locally convex spaces.* In this section we list some elementary results concerning metrizable subsets of locally convex spaces. If $\langle E,F\rangle$ is a dual pair then we denote the weak, Mackey and strong topologies on E by $\sigma(E,F)$, $\tau(E,F)$ and $\beta(E,F)$. In particular if (E,τ) is a locally convex space, then the associated weak, Mackey and strong topologies are denoted by $\sigma(E,E')$, $\tau(E,E')$ and $\beta(E,E')$.

LEMMA 1·1. *If E is a separable locally convex space and $U \subset E'$ is equicontinuous then U is $\sigma(E',E)$ metrizable.*

Proof. See Köthe(11) (§ 21.3, p. 259).

The following lemma is a result from general topology, which is difficult to trace in the literature. The only reference we know is Bourbaki(4) Fascicule des Résultats p. 47, where it is given without proof.

LEMMA 1·2. *Let M be a compact metric space and let N be a continuous Hausdorff image of M; then N is compact metrizable.*

Proof. It is clear that N is compact; now consider the Banach space $C(N)$ of all continuous real-valued functions on N with the usual norm and the map $T\colon C(N) \to C(M)$ given by

$$T\phi(m) = \phi(\theta(m))$$

where $\theta\colon M \to N$ is a continuous map of M onto N. Clearly T is an isometric embedding of $C(N)$ into $C(M)$. As $C(M)$ is separable, $C(N)$ is separable and hence the closed unit ball B of $C(N)'$ is metrizable in $\sigma((C(N))', C(N))$ (Lemma 1·1). There is a natural injection of N into B continuous for $\sigma(C(N)', C(N))$ and so, as N is compact, N is homeomorphic to a subset of B; thus N is metrizable.

The following proposition could be derived from Köthe(11) §28.5 (3).

PROPOSITION 1·3. *Let V be an absolutely convex subset of a locally convex space (E, τ): then V is τ-metrizable if and only if the uniformity induced on V by the τ-uniformity on E is metrizable.*

Proof. Suppose V is τ-metrizable; then there is a sequence U_n of τ-closed absolutely convex neighbourhoods of zero in E such that the sequence $(U_n \cap V;\ n = 1, 2, \ldots)$ is a base of τ-neighbourhoods of zero in V.

Let U be any τ-neighbourhood of zero in E, and let $\tilde{U}$ be the vicinity induced by U; i.e. $\tilde{U} \subset E \times E$ and

$$\tilde{U} = \{(x, y);\ x - y \in U\}.$$

We show that there exists n such that

$$\tilde{U}_n \cap (V \times V) \subset \tilde{U} \cap (V \times V).$$

We choose n such that $$U_n \cap V \subset (\tfrac{1}{2}U) \cap V$$

since $(U_n \cap V)$ is a base of neighbourhoods of zero.

Then if $(x, y) \in \tilde{U}_n \cap (V \times V)$, we have $x \in V$ and $y \in V$, so that, as V is absolutely convex, $\frac{1}{2}(x-y) \in V$. However, $\frac{1}{2}(x-y) \in \frac{1}{2}U_n \subset U_n$ so that $\frac{1}{2}(x-y) \in U_n \cap V$; hence $\frac{1}{2}(x-y) \in \frac{1}{2}U$ and so $x - y \in U$, i.e. $(x, y) \in \tilde{U} \cap (V \times V)$.

Thus when considering absolutely convex subsets of a locally convex space, it is unnecessary to distinguish between uniform metrizability and topological metrizability. We note that a uniform space is metrizable if and only if its completion is metrizable; this fact is used in Theorem 1·4.

THEOREM 1·4. *Let (E, τ) be a locally convex space and let V be a precompact τ-uniformly metrizable subset of E; then the closed absolutely convex cover of V, $\bar{\Delta}(V)$, is τ-metrizable.*

Proof. Let $(\tilde{E}, \tilde{\tau})$ be the completion of (E, τ), and consider $\overline{V} \subset \tilde{E}$; then $\overline{V}$ is compact and $\tilde{\tau}$-(uniformly) metrizable. Consider the map $T: E' (= \tilde{E}') \to C(\overline{V})$ where $C(\overline{V})$ is the separable Banach space of all continuous functions on $\overline{V}$; let T^* denote the adjoint map $[C(\overline{V})]' \to (E')^*$. Then T^* is $\sigma([C(\overline{V})]', C(\overline{V})) \to \sigma((E')^*, E')$ continuous. Let B be the closed unit ball of $[C(\overline{V})]'$; then by (1·1), B is $\sigma([C(\overline{V})]', C(\overline{V}))$-compact and metrizable, and by (1·2), $T^*(B)$ is $\sigma((E')^*, E')$-compact and metrizable. Identifying $\tilde{E}$ as a subspace of $(E')^*$, $T^*(B) \cap \tilde{E}$ is $\sigma(\tilde{E}, E')$-closed, absolutely convex and metrizable; clearly if $x \in V$, then $\delta_x \in B$ where $\delta_x(\phi) = \phi(x)$, and $T^*\delta_x = x$.

Let W be the closed absolutely convex cover of V in $\tilde{E}$; then W is $\tilde{\tau}$-compact, and by the remarks above $W \subset T^*(B) \cap \tilde{E}$. On W, $\tilde{\tau} = \sigma(\tilde{E}, E')$ (as W is compact) and so W is $\tilde{\tau}$-metrizable; hence $W \cap E = \bar{\Delta}(V)$ is τ-metrizable.

COROLLARY. *If (x_n) is a τ-Cauchy sequence, then the closed absolutely convex cover of $\{x_n\}_{n=1}^{\infty}$ is τ-metrizable.*

Proof. We may assume $\{x_n\}$ is a sequence of distinct elements; then the completion V of $\{x_n\}$ in the τ-uniformity is isomorphic topologically with the one-point compactification of the integers. As V is compact there is a unique uniformity (see Kelley(10), p. 197–8) on V inducing this topology, and this is metrizable; it follows that $\{x_n\}$ is τ-uniformly metrizable.

2. *The closed graph theorem in separable spaces.* We recall that a locally convex space E is B_r-complete (see (17)), if, given a dense subspace V of $(E', \sigma(E', E))$ such that $V \cap A$ is closed in A for every equicontinuous set A in E', then $V = E'$.

Let ζ denote the collection of all separable B_r-complete locally convex spaces and let ζ_B be the collection of separable Banach spaces; in this section we identify $\mathscr{C}(\zeta)$ and $\mathscr{C}(\zeta_B)$ with a certain class of locally convex spaces.

Let $T: E \to F$ be a linear map; then we shall denote by L_T the subspace of F' of all f such that $f \circ T$ is a continuous linear map on E. Thus if $T^*: F^* \to E^*$ is the algebraic adjoint of T, then $L_T = (T^*)^{-1}(E') \cap F'$. Lemma 2·1 is due to Pták(17).

LEMMA 2·1 *T has closed graph if and only if L_T is $\sigma(F', F)$-*dense *in F'.*

LEMMA 2·2. *If $(E', \sigma(E', E))$ is sequentially complete then L_T is $\sigma(F', F)$-sequentially closed.*

Proof. Suppose $f_n \in L_T$ and $f_n \to f\sigma(F', F)$; then $T^*f_n \to T^*f\sigma(E^*, E)$, and so T^*f_n is a $\sigma(E', E)$-Cauchy sequence. Hence T^*f_n converges in E', and $T^*f \in E'$, i.e. $f \in L_T$.

We pause in the logical development of this section to give an application of Lemma 2·2 to derive an extension of a recent theorem of McWilliams (15) (see also Civin and Yood(6)).

THEOREM 2·3. *Let E be a locally convex space which is sequentially dense in $E''(= (E', \beta(E', E))')$ in the topology $\sigma(E'', E')$; suppose F is weakly sequentially complete and that $T: E \to F$ is continuous. Then T maps bounded sets into weakly relatively compact sets.*

Proof. Consider T': $(F', \sigma(F', F)) \to (E', \sigma(E', E''))$; then by Lemma 2·2 $L_{T'} \subset E''$ is $\sigma(E'', E')$-sequentially closed. However $E \subset L_{T'}$ (in the natural embedding of E in E'') and so $L_{T'} = E''$, i.e. T' is continuous. Let $\tilde{T}$ denote the adjoint map $\tilde{T}: E'' \to F$; then $\tilde{T}$ is $\sigma(E'', E') \to \sigma(F, F')$ continuous and its restriction to E is T. Now suppose $B \subset E$ is bounded; then B is $\beta(E', E)$-equicontinuous and so $\bar{B} \subset (E'', \sigma(E'', E'))$ is $\sigma(E'', E')$-compact; hence $\tilde{T}(\bar{B})$ is $\sigma(F, F')$-compact, i.e. $T(B)$ is $\sigma(F, F')$-relatively compact.

COROLLARY. *If E satisfies the conditions of the theorem every continuous map $T: E \to l^1$ maps bounded sets into relatively compact sets.*

We observe that E satisfies the conditions of the theorem if $(E', \beta(E', E))$ is separable. The next theorem characterises Mackey spaces which belong to $\mathscr{C}(\zeta)$.

THEOREM 2·4. *Let E be a Mackey space; then the following are equivalent:*

(i) $E \in \mathscr{C}(\zeta)$,

(ii) $E \in \mathscr{C}(c_0)$,

(iii) *$(E', \sigma(E', E))$ is sequentially complete.*

Proof. (i) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (iii). Let $f_n \in E'$ be $\sigma(E', E)$-Cauchy and define the map $T: E \to c$ (where c is the space of convergent sequences) by $Tx = (f_n(x))_{n=1}^{\infty}$. Then T has closed graph and as $c \cong c_0$, it follows that T is continuous; hence $\lim_{n\to\infty} f_n(x)$ is a continuous linear functional on E, so that $\lim_{n\to\infty} f_n = f \in E'$.

(iii) $\Rightarrow$ (i). Let F be separable and B_r-complete and let $T: E \to F$ have a closed graph; then by (2·1), $L_T \subset F'$ is $\sigma(F', F)$-dense, while by (2·2), L_T is $\sigma(F', F)$-sequentially closed. Let U be a $\sigma(F', F)$-closed equicontinuous subset of F'; then U is $\sigma(F', F)$-metrizable and $U \cap L_T$ is sequentially closed in U. Thus $U \cap L_T$ is closed in U for all such U, and thus $L_T = F'$, as F is B_r-complete. In particular T is weakly continuous and therefore Mackey continuous.

Corollary. *If $\mathscr{M}$ is the class of Mackey spaces then $\mathscr{M} \cap \mathscr{C}(\zeta) \subset \mathscr{C}(\mathscr{R})$ where $\mathscr{R}$ is the class of reflexive Banach spaces.*

Proof. This follows from Theorem 2·4 and Theorem 1 (iii) of McIntosh(14).

We now consider the class $\mathscr{C}(\zeta)$ in general; we shall require the following lemma. Suppose E is a locally convex space and $V \subset E'$ is bounded and absolutely convex; then we may form a Banach space E_V by completing the quotient space $E/V^{\perp}$ with the norm induced by V.

Lemma 2·5. (i) *The natural quotient map $T: E \to E_V$ has closed graph.*

(ii) *If V is $\sigma(E', E)$ metrizable then E_V is separable.*

Proof. (i) This is proved in (13).

(ii) Consider the adjoint map $T^*: E_V^* \to E^*$ which is $\sigma(E_V^*, E_V) \to \sigma(E^*, E)$ continuous; let B be the closed unit ball of E_V', so that $B \subset E_V' \subset E_V^*$. Then $T^*(B) \subset V^{00}$ (polars taken with respect to the duality $\langle E, E^*\rangle$); however, V^{00} is the $\sigma(E^*, E)$-closure of V and is complete. By the remarks preceding Theorem 1·4, V^{00} is $\sigma(E^*, E)$-metrizable. Now T^* is injective and B is $\sigma(E_V^*, E_V)$-compact, so that $T^*: B \to T^*(B)$ is a homeomorphism for the topologies $\sigma(E_V^*, E_V)$ and $\sigma(E^*, E)$. Hence B is $\sigma(E_V', E_V)$-metrizable and therefore E_V is separable (see Dunford and Schwartz(7), p. 426).

Theorem 2·6. *If E is a locally convex space then the following conditions on E are equivalent:*

(i) $E \in \mathscr{C}(\zeta)$

(ii) $E \in \mathscr{C}(C[0,1])$

(iii) $E \in \mathscr{C}(\zeta_B)$

(iv) *Every $\sigma(E', E)$ bounded metrizable absolutely convex set in E' is equicontinuous.*

Proof. (i) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (iii). By the Banach–Mazur Theorem every separable Banach space may be embedded isometrically in $C[0, 1]$ (see (2)).

(iii) $\Rightarrow$ (iv). If V is $\sigma(E', E)$ bounded metrizable and absolutely convex in E', the map $T: E \to E_V$ has closed graph (Lemma 2·5) and is continuous. Hence

$$\|Tx\| = \sup_{f \in V} |f(x)|$$

is a continuous semi-norm on E, i.e. V is equicontinuous.

(iv) $\Rightarrow$ (i). By the Corollary to Theorem 1·4, every $\sigma(E', E)$ Cauchy sequence is equicontinuous, and hence convergent in E'. By Theorem 2·4, $(E, \tau(E, E')) \in \mathscr{C}(\zeta)$.

Let $T: E \to F$ be a closed graph linear map where F is separable and B_r-complete; then T is $(E, \tau(E, E')) \to F$ continuous, and so has an adjoint $T': F' \to E'$. Let $V \subset F'$ be $\sigma(F', F)$-closed absolutely convex and equicontinuous; then V is $\sigma(F', F)$-metrizable (1·1) and $\sigma(F', F)$-compact and so by Lemma 1·2, $T'(V)$ is $\sigma(E', E)$-bounded absolutely convex and metrizable. Hence $T'(V)$ is equicontinuous and so T' maps equicontinuous sets into equicontinuous sets; thus T is continuous.

Corollary. *If $E \in \mathscr{C}(\zeta)$, and V is a $\sigma(E', E)$-compact metrizable set, then the closed absolutely convex cover of V is $\sigma(E', E)$-compact.*

Proof. As there is precisely one uniformity inducing a given topology on a compact set (see (10), p. 197) we may conclude that any metric on V inducing the topology $\sigma(E', E)$, also induces the uniformity; thus V is uniformly metrizable. By Theorem 1·4 $\bar{\Delta}(V)$ is uniformly metrizable and hence equicontinuous; it is therefore weakly compact.

The reader may compare this corollary with the well-known result that if E is barrelled, then the closed absolutely convex cover of any $\sigma(E', E)$-compact set is $\sigma(E', E)$-compact.

We now give an example of a Mackey space in $\mathscr{C}(\zeta)$ which is not barrelled. Let l^∞ be the space of all bounded sequences, and l^1 the space of all absolutely convergent series, under the usual duality; then the space $(l^\infty, \tau(l^\infty, l^1))$ is not barrelled (as the unit ball of the Banach space l^1 is not weakly compact). However $\sigma(l^1, l^\infty)$ is sequentially complete (see Dunford and Schwartz(7), p. 290) and therefore

$$(l^\infty, \tau(l^\infty, l^1)) \in \mathscr{C}(\zeta).$$

Now consider the set $\{e^{(n)}\}$ in l^1 where $e^{(n)} = (0, \ldots, 0, 1, 0, \ldots)$ with the one in the nth position. Then the set $\{e^{(n)}\}$ is discrete in the topology $\sigma(l^1, l^\infty)$, and therefore is bounded and metrizable. However, the semi-norm

$$\sup_n |x_n| = \sup_n |\langle x, e^{(n)} \rangle|$$

is not continuous on $(l^\infty, \tau(l^\infty, l^1))$, so that $\{e^{(n)}\}$ is not equicontinuous.

3. *On $\mathscr{C}(c_0)$ and $\mathscr{C}(\zeta)$.*

Theorem 3·1. *$E \in \mathscr{C}(c_0)$ if and only if every $\sigma(E', E)$ Cauchy sequence is equicontinuous.*

Proof. If $E \in \mathscr{C}(c_0)$ then for a $\sigma(E', E)$-Cauchy sequence (f_n) in E' define $T: E \to c$ by

$$Tx = (f_n(x))_{n=1}^\infty$$

T has closed graph and as $c \cong c_0$, T is continuous; thus $(f_n)_{n=1}^\infty$ is $\sigma(E', E)$ equicontinuous.

Conversely if every $\sigma(E', E)$ Cauchy sequence is equicontinuous, it follows that $(E', \sigma(E', E))$ is sequentially complete, i.e. $(E, \tau(E, E')) \in \mathscr{C}(c_0)$ by Theorem 2·4; hence if $T: E \to c_0$ has closed graph then T is weakly continuous. Thus if $\phi_n(x) = (Tx)_n$ then $\phi_n \in E'$ and $\phi_n \to 0$ $\sigma(E', E)$. Hence $\|Tx\| = \sup_n |\phi_n(x)|$ is a continuous semi-norm on E.

We remark that $\mathscr{C}(c_0) \neq \mathscr{C}(\zeta)$; for let E be a separable Banach space and let τ_0 be the topology on E of uniform convergence on $\sigma(E', E)$-null sequences. It is clear that

$(E,\tau_0)\in\mathscr{C}(c_0)$; however if $(E,\tau_0)\in\mathscr{C}(\zeta)$ then τ_0 must be the norm topology; then there is a $\sigma(E',E)$ null sequence (f_n) in E' with

$$\|x\| \leqslant \sup_n |f_n(x)| \leqslant K\|x\|$$

for all $x\in E$, i.e. E is isomorphic to a subspace of c_0.

In (18), Webb introduces the notion of sequentially barrelled spaces; E is sequentially barrelled, if, whenever $f_n \to 0$ $\sigma(E',E)$ then $(f_n)_{n=1}^{\infty}$ is equicontinuous. We remark that $(l^1,\tau(l^1,c_0))$ is sequentially barrelled (for it can be easily shown that the absolutely convex cover of weakly null sequence in c_0 is relatively weakly compact); however $(l^1,\tau(l^1,c_0))$ does not belong to $\mathscr{C}(c_0)$ or equivalently $\mathscr{C}(\zeta)$ (by Theorem 2·4), since the identity map $(l^1,\tau(l^1,c_0)) \to (l^1,\beta(l^1,c_0))$ has closed graph but is not continuous.

A locally convex space E is a Mazur space (see Wilansky(19), p. 50) if every sequentially continuous linear functional on E is continuous; it is clear that any semi-bornological space (i.e. a space in which every bounded linear functional is continuous, (19), p. 51) is a Mazur space, but the converse is false (the space $(l^1,\sigma(l^1,c_0))$ is a Mazur space but not semi-bornological). Further results concerning Mazur spaces are given in (9) and (16).

THEOREM 3·2. *If E is a Mazur space then $(E,\tau(E,E'))$ is sequentially barrelled if and only if every weakly bounded set in E is strongly bounded.*

Proof. If $(E,\tau(E,E'))$ is sequentially barrelled then by (18) Proposition 4·1, every weakly bounded set in E is strongly bounded. Conversely let $f_n \to 0$ $\sigma(E',E)$; then define $T: E\to c_0$ by $Tx = (f_n(x))_{n=1}^{\infty}$.

If $x_k \to 0$ in E, then as every weakly bounded set in E is strongly bounded,

$$\sup_n \sup_k |f_n(x_k)| < \infty,$$

i.e.

$$\sup_k \|Tx_k\| < \infty.$$

Considering c_0 as a subspace of l^∞, (Tx_k) is bounded in l^∞ and $(Tx_k)_n \to 0$ each n; as the closed unit ball of l^∞ is $\sigma(l^\infty,l^1)$-compact, it follows that

$$\lim_{k\to\infty}\left|\sum_{n=1}^{\infty} a_n f_n(x_k)\right| = 0,$$

whenever $\sum_{n=1}^{\infty} |a_n| \leqslant 1$.

Thus, as E is a Mazur space $\sum_{n=1}^{\infty} a_n f_n \in E'$, and so T is continuous

$$(E,\sigma(E,E')) \to (c_0,\sigma(c_0,l^1)).$$

Thus T is Mackey-continuous and so $\sup_n |f_n(x)|$ is a continuous semi-norm on $(E,\tau(E,E'))$, i.e. E is $\tau(E,E')$ sequentially barrelled.

THEOREM 3·3. *If E is a Mackey Mazur space then $E\in\mathscr{C}(\zeta)$ (i.e. $\sigma(E',E)$ is sequentially complete) if and only if every weakly bounded set in E is strongly bounded.*

Proof. If $(E',\sigma(E',E))$ is sequentially complete then every weakly bounded set in E is strongly bounded (see Köthe(11), §20.11 (8) or Mackey(12), p. 194, Lemma). The converse follows from Corollary 4·5 of (18) and Theorem 3·2.

A class of Mackey Mazur spaces is examined in (3). It follows from the next theorem that not every Mackey space in $\mathscr{C}(\zeta)$ is a Mazur space.

THEOREM 3·4. *Every sequentially complete DF-space belongs to* $\mathscr{C}(\zeta)$.

Proof. If E is a DF-space and $V \subset E'$ is $\sigma(E',E)$ bounded absolutely convex and $\sigma(E',E)$ metrizable, then V is $\sigma(E',E)$ separable (for V is $\sigma(E',E)$-precompact). Hence V is equicontinuous, for it is strongly bounded and contained in the closure of a countable set.

Webb (18), p. 361 gives an example of a DF-space which is not a Mazur space (following an example of Grothendieck (8)).

We recall that an example of a non-barrelled space in $\mathscr{C}(\zeta)$ is $(l^\infty, \tau(l^\infty, l^1))$; in this case the strong topology $\beta(l^\infty, l^1)$ is inseparable. The next Proposition shows that this is not coincidental.

PROPOSITION 3·5. *If* $E \in \mathscr{C}(\zeta)$ *and* $(E, \beta(E, E'))$ *is separable then* E *is barrelled.*

Proof. If $V \subset E'$ is $\sigma(E',E)$ bounded and absolutely convex, then V is $\beta(E,E')$ equicontinuous, and hence by Lemma 1·1 is $\sigma(E',E)$ metrizable. Therefore by Theorem 2·6, V is equicontinuous.

From this we may derive the following result of Webb (19) (which may also be derived directly).

COROLLARY. *If* E *is a Mackey space with* $\sigma(E',E)$ *sequentially complete, then if* $(E, \beta(E,E'))$ *is separable,* E *is barrelled.*

Further applications of Theorems 2·4 and 2·6 are given in (3); in particular we observe that Theorem 1 of Bachelis and Rosenthal (1), may be deduced quickly from Theorem 2·4.

The author is extremely grateful to the referee for many interesting and very valuable comments.

REFERENCES

(1) BACHELIS, G. F. and ROSENTHAL, H. P. On unconditionally converging series and biorthogonal systems in a Banach space (to appear).
(2) BANACH, S. *Theorie des Operations Lineares* (Warsaw, 1932).
(3) BENNETT, G. and KALTON, N. J. On FK-spaces containing c_0, (to appear).
(4) BOURBAKI, N. *Elements de Mathématique*, Livre 3, Topologie Generale (Paris, 1959).
(5) BOURBAKI, N. *Elements de Mathématique*, Livre 6, Integration (Paris, 1959).
(6) CIVIN, P. and YOOD, B. Quasi-reflexive spaces. *Proc. Amer. Math. Soc.* 8 (1957), 906–911.
(7) DUNFORD, N. and SCHWARTZ, J. T. *Linear Operators*, Vol. 1 (New York, 1955).
(8) GROTHENDIECK, A. Sur les espaces (F) et (DF). *Summa Brasil. Math.* 3 (1954), 57–123.
(9) ISBELL, J. R. Mazur's theorem. *General Topology and its Relations to Morden Analysis and Algebra* (*Prague Symposium*) (Prague, 1962).
(10) KELLEY, J. L. *General Topology* (New York, 1955).
(11) KÖTHE, G. *Topological Vector Spaces* (Berlin, 1969).
(12) MACKEY, G. W. On infinite dimensional linear spaces. *Trans. Amer. Math. Soc.* 57 (1945), 155–207.
(13) MAHOWALD, M. Barrelled spaces and the closed graph theorem. *J. London Math. Soc.* 36 (1961), 108–110.
(14) MCINTOSH, A. G. On the closed graph theorem. *Proc. Amer. Math. Soc.* 20 (1969), 397–404.

(15) McWILLIAMS, R. D. On certain Banach spaces which are w^*-sequentially dense in their duals. *Duke Math. J.* **37** (1970), 121–126.
(16) MROWKA, S. On the form of pointwise continuous positive functionals and isomorphisms of function spaces. *Studia Math.* **21** (1961), 1–14.
(17) PTÁK, V. Completeness and the open mapping theorem. *Bull. Soc. Math. France* **86** (1958), 41–74.
(18) WEBB, J. H. Sequential convergence in locally convex spaces. *Proc. Cambridge Philos. Soc.* **64** (1968), 341–364.
(19) WILANSKY, A. *Topics in Functional Analysis* (Berlin, 1967).