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Schauder decompositions in locally convex spaces

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1. *Introduction.* A decomposition of a topological vector space E is a sequence $(E_n)_{n=1}^{\infty}$ of non-trivial subspaces of E such that each x in E can be expressed uniquely in the form $x = \sum_{i=1}^{\infty} y_i$, where $y_i \in E_i$ for each i. It follows at once that a basis $(x_n)_{n=1}^{\infty}$ of E corresponds to the decomposition consisting of the one-dimensional subspaces $E_n = \operatorname{lin}\{x_n\}$; the theory of bases can therefore be regarded as a special case of the general theory of decompositions, and every property of a decomposition may be naturally defined for a basis.

The decomposition $(E_n)_{n=1}^{\infty}$ induces a sequence $(Q_n)_{n=1}^{\infty}$ of projections defined by $Q_n x = y_n$, where $x = \sum_{i=1}^{\infty} y_i$ with $y_i \in E_i$. These projections are obviously orthogonal (i.e. $Q_n Q_m = 0$ if $n \neq m$) and $Q_n(E) = E_n$ for each n; if, in addition, they are continuous, the decomposition is said to be a Schauder decomposition. For a basis $(x_n)_{n=1}^{\infty}$ the projection Q_n takes the form $Q_n(x) = f_n(x)\, x_n$, where f_n is a linear functional and Q_n is continuous if and only if f_n is continuous; it is known that every basis of an F-space or LF-space is a Schauder basis (2). It is, in general, necessary to restrict attention to Schauder decompositions, as non-Schauder decompositions are extremely difficult to deal with.

In recent years, Schauder bases and, to a lesser extent, Schauder decompositions of general locally convex spaces (rather than Banach spaces) have received some attention (see, for example (1), (4), (9), (10)). However, little is as yet known about the structure of bases in this more general setting, since attempts to generalize Banach space arguments usually fail. Nevertheless, it is the purpose of this paper to establish quite strong results about certain types of bases and decompositions familiar in the Banach space theory, in particular shrinking and boundedly complete (or in the terminology to be introduced later, γ-complete) decompositions.

In section 2 certain elementary but necessary facts about Schauder decompositions are established, and the concept of a simple Schauder decomposition is introduced. A necessary and sufficient condition for the semi-reflexivity of space with a Schauder decomposition is obtained in section 3, extending slightly a theorem of Cook (3), who generalized a theorem of James (7) to locally convex spaces.

In section 4 a sequence space technique due to Garling (6) is generalized for use in Schauder decompositions. The duality of boundedly complete and shrinking decompositions is studied in section 5, and in section 6 further study is made of spaces with boundedly complete decompositions.

2. *Basic results.* Let E be a locally convex space, and let $(E_n)_{n=1}^{\infty}$ be a Schauder decomposition for E; then as observed in section 1 $(E_n)_{n=1}^{\infty}$ induces a sequence $(Q_n)_{n=1}^{\infty}$ of continuous projections on E satisfying:

(i) $Q_n Q_m = 0 \quad (m \neq n)$,

(ii) $x = \sum_{n=1}^{\infty} Q_n x \quad$ for each $\quad x \in E$.

Conversely if $(Q_n)_{n=1}^{\infty}$ is a sequence of continuous projections on E satisfying conditions (i) and (ii), it may be readily shown that $(Q_n(E))_{n=1}^{\infty}$ is a Schauder decomposition for E with induced projections $(Q_n)_{n=1}^{\infty}$; it will be convenient to refer to such a sequence of projections as a Schauder decomposition, and the preceding remarks show that this is entirely consistent with the original definition.

Henceforth, if $(Q_n)_{n=1}^{\infty}$ is a Schauder decomposition of E then we shall always write $E_n = Q_n(E)$ and $P_n = \sum_{i=1}^{n} Q_i$.

The map $Q_n: E \to E$ has a dual map Q_n defined on the algebraic dual E^* of E, given by $\langle x, Q_n^* f\rangle = \langle Q_n x, f\rangle$ whenever $x \in E$ and $f \in E^*$. The restriction of Q_n^* to E', the continuous dual of E, will be denoted by Q_n'; it can easily be shown that Q_n' maps E' into E' and is itself continuous for the topology $\sigma(E', E)$. Also $Q_n'(E')$ can be identified naturally with E_n' under the correspondence $Q_n' f \to f|E_n$, where $f|E_n$ denotes the restriction of f to E_n.

If $x \in E$ and $f \in E'$
$$\langle x, f\rangle = \sum_{i=1}^{\infty} \langle Q_i x, f\rangle = \sum_{i=1}^{\infty} \langle x, Q_i' f\rangle$$
so that $f = \sum_{i=1}^{\infty} Q_i' f$ in the topology of $\sigma(E', E)$. Thus $(Q_n')_{n=1}^{\infty}$ is a Schauder decomposition of $\{E', \sigma(E', E)\}$; it will be called the dual decomposition.

The following simple result is of fundamental importance:

PROPOSITION 2·1. *Let E be a locally convex space with a Schauder decomposition $(Q_n)_{n=1}^{\infty}$. Then the weak, Mackey and strong topologies on E induce respectively the weak, Mackey and strong topologies on each E_n.*

Let I_n denote the inclusion map $E_n \to E$ and consider the sequence $E \xrightarrow{Q_n} E_n \xrightarrow{I_n} E$. Each map is continuous for the original topology and hence for, in turn, the weak topologies $\sigma(E, E')$ and $\sigma(E_n, E_n')$, the Mackey topologies $\tau(E, E')$ and $\tau(E_n, E_n')$ and the strong topologies $\beta(E, E')$ and $\beta(E_n, E_n')$. Considering the restriction of the first map to E_n, the required result is obtained.

COROLLARY. *Let E be a locally convex space with a Schauder decomposition $(Q_n)_{n=1}^{\infty}$. Then:*

(i) *If E is barrelled, each E_n is barrelled.*

(ii) *If E is semi-reflexive, each E_n is semi-reflexive.*

Let τ be the topology on E. If E is barrelled $\tau = \tau(E, E') = \beta(E, E')$, and so by the proposition $\tau|E_n = \tau(E_n, E_n') = \beta(E_n, E_n')$. If E is semi-reflexive, the result is obtained by using the dual decomposition (Q_n'); for $\tau(E', E) = \beta(E', E)$ and so $\tau(E_n', E_n) = \beta(E_n', E_n)$, since $Q_n'(E') = E_n'$.

The following two definitions are fairly standard:

DEFINITION 2·2. *A Schauder decomposition* (Q_n) *is said to be equi-Schauder if the set of maps* $P_n = \sum_{i=1}^{n} Q_i$ *is equicontinuous.*

A Schauder decomposition (Q_n) *is said to be shrinking if* (Q'_n) *is a Schauder decomposition for* $(E', \beta(E', E))$.

In addition, it will be found necessary to introduce another definition, similar but much weaker than that of a shrinking decomposition.

DEFINITION 2·3. *A Schauder decomposition* (Q_n) *is said to be simple if for each* $f \in E'$, $(P'_n f)_{n=1}^{\infty}$ *is a* $\beta(E', E)$ *bounded set in* E'.

In general, it is true that $(P'_n f)_{n=1}^{\infty}$ is $\sigma(E', E)$-bounded, but not necessarily $\beta(E', E)$-bounded. Suppose λ is defined to be the space of all real-valued sequences (α_n) such that

$$\sup_n \left|\sum_{i=1}^{2n} \alpha_i\right| < \infty \quad \text{and} \quad \sup_n \frac{1}{n}\left|\sum_{i=1}^{2n+1} \alpha_i\right| < \infty,$$

and ϕ is the space of all sequences which are finitely non-zero; then $\langle \lambda, \phi \rangle$ forms a dual pair of sequence spaces and if e_n is the sequence taking the value 1 in the nth place and zero elsewhere, $(e_n)_{n=1}^{\infty}$ is a Schauder basis for $\sigma(\phi, \lambda)$. Let $f \in \lambda$ be the sequence $f(2n-1) = n$ and $f(2n) = -n$, and let $a_n \in \phi$ be defined by

$$a_n = \sum_{i=1}^{n} e_i;$$

then the set $(a_{2n})_{n=1}^{\infty}$ is $\sigma(\phi, \lambda)$ bounded. However, we have $\langle a_{2n}, P_{2n-1} f \rangle = n$, and so $(P'_n f)_{n=1}^{\infty}$ is not $\beta(E', E)$ bounded.

In most cases of interest, however, a Schauder decomposition is simple as is shown by the following proposition.

PROPOSITION 2·4. *If* (Q_n) *is a Schauder decomposition for* E, *then each of the following conditions is sufficient to ensure that* (Q_n) *is simple:*

(i) $\tau(E, E')$ *is sequentially complete,*
(ii) $\tau(E', E)$ *is sequentially complete,*
(iii) (Q_n) *is equi-Schauder,*
(iv) (Q_n) *is shrinking.*

In cases (i) and (ii) the weakly and strongly bounded subsets of E' coincide, and as remarked above $(P'_n f)_{n=1}^{\infty}$ is always weakly bounded. In case (iii) $(P'_n f)_{n=1}^{\infty}$ is equicontinuous and so is strongly bounded; in case (iv) $(P'_n f)_{n=1}^{\infty}$ is strongly convergent and hence certainly bounded.

If H is defined as the subspace of E' given by

$$H = \{f; f = \lim_{n\to\infty} P'_n f \quad \text{in} \quad \beta(E', E)\}$$

then it is clear that (Q'_n) is a Schauder decomposition for H in the topology $\beta(E', E)$, and that if $J = \operatorname{lin} \bigcup_{n=1}^{\infty} Q'_n(E')$, then $J \subset H \subset \bar{J}$ (closure in the topology $\beta(E', E)$). If, however, (Q_n) is simple, a much stronger result can be obtained; the following lemma, of which the simple proof is omitted, is a necessary preliminary.

LEMMA 2·5. *Let $\langle E, F\rangle$ be a dual pair of vector spaces: suppose T is a collection of $\sigma(E,F)$-continuous linear maps from E to E, such that for each $x \in E$, the set $\{t(x); t \in T\}$ is $\beta(E,F)$-bounded. Then T is $\beta(E,F)$ equicontinuous.*

THEOREM 2·6. *Let (Q_n) be a simple Schauder decomposition for E: then $H = \bar{J}$, and (Q'_n) is an equi-Schauder decomposition for $(H, \beta(E', E))$.*

Since (Q_n) is simple, for each $f \in E'$, $(P'_n f)_{n=1}^{\infty}$ is $\beta(E', E)$ bounded and so we can apply Lemma 2·5 to obtain that $(P'_n)_{n=1}^{\infty}$ is $\beta(E', E)$ equicontinuous; hence the set

$$H = \{f; \lim_{n\to\infty} P'_n f = f \quad \text{in} \quad \beta(E', E)\}$$

is closed. Since $J \subset H \subset \bar{J}$, the result follows at once.

COROLLARY. *If (Q_n) is a shrinking Schauder decomposition for E, (Q'_n) is an equi-Schauder decomposition for $(E', \beta(E', E))$.*

THEOREM 2·7. *If E is quasi-barrelled, then if (Q_n) is a simple Schauder decomposition of E, (Q_n) is equi-Schauder.*

If A is an equicontinuous subset of E', the set $P'(A) = \bigcup_{n=1}^{\infty} P'_n(A)$ is strongly bounded in E', since the maps $(P'_n)_{n=1}^{\infty}$ are equicontinuous in the strong topology on E' and A is strongly bounded. Hence $P'(A)$ is equicontinuous and as

$$\bigcap_{n=1}^{\infty} P_n^{-1}(A^0) = [P'(A)]^0 \subset A^0,$$

(Q_n) is equi-Schauder.

In this case the completion of E is barrelled and the simple Schauder decompositions of E are precisely those which are decompositions of the completion of E.

3. *Semi-reflexivity.* It is well known that a Banach space with a basis is reflexive if and only if the basis is shrinking and boundedly complete; recently Cook (3) has shown that this result remains true for decompositions in locally convex spaces provided reflexivity is replaced by semi-reflexivity.

DEFINITION 3·1. *Let (Q_n) be a Schauder decomposition of E; then:*

(i) *(Q_n) is β-complete if, whenever $\left(\sum_{n=1}^{m} x_n\right)_{m=1}^{\infty}$ is a weakly bounded Cauchy sequence with $x_n \in E_n$ then $\sum_{n=1}^{\infty} x_n$ converges (in the original topology on E).*

(ii) *Q_n is γ-complete if, whenever $\left(\sum_{n=1}^{m} x_n\right)_{m=1}^{\infty}$ is a bounded sequence with $x_n \in E_n$ then $\sum_{n=1}^{\infty} x_n$ converges.*

Definition (ii) is exactly that of boundedly complete decompositions (see (4)); however, the terminology is chosen because of the close connexion with concepts of β and γ-duality of sequence spaces studied in (5). Both types of decompositions are 'complete' in the sense of (6).

THEOREM 3·2. *Let E be a locally convex space with a Schauder decomposition (Q_n); the following are equivalent:*

(i) *E is semi-reflexive.*

(ii) *(Q_n) is γ-complete and shrinking, and each E_n is semi-reflexive.*

(iii) *(Q_n) is β-complete and shrinking, and each E_n is semi-reflexive.*

The equivalence of (i) and (ii) has been shown by Cook (3); as (ii) obviously implies (iii), we show that (iii) implies (i). Suppose that χ is a strongly continuous linear functional on E'; as (Q_n) is shrinking

$$f = \sum_{i=1}^{\infty} Q_i' f \quad \text{in} \quad \beta(E', E)$$

and

$$\langle \chi, f \rangle = \sum_{i=1}^{\infty} \langle \chi, Q_i' f \rangle.$$

Define $\Gamma_n \colon E_n' \to E'$ by
$$\langle x, \Gamma_n f \rangle = \langle Q_n x, f \rangle.$$

Then Γ_n is weakly continuous and hence strongly continuous; thus $f \to \langle \chi, \Gamma_n f \rangle$ is a continuous linear functional on $\{E_n', \beta(E_n', E_n)\}$. As E_n' is semi-reflexive, there exists $x_n \in E_n$ with

$$\langle x_n, f \rangle = \langle \chi, \Gamma_n f \rangle.$$

Now let $f \in E'$ and let $f_n \in E_n'$ be its restriction to E_n; then:

$$\begin{aligned} \langle \chi, Q_n' f \rangle &= \langle \chi, \Gamma_n f_n \rangle \\ &= \langle x_n, f_n \rangle \\ &= \langle x_n, f \rangle \end{aligned}$$

and so

$$\langle \chi, f \rangle = \sum_{n=1}^{\infty} \langle x_n, f \rangle.$$

Hence $\left(\sum_{n=1}^{m} x_n \right)_{m=1}^{\infty}$ is weakly Cauchy, and so there exists x with

$$x = \sum_{n=1}^{\infty} x_n,$$

hence

$$\begin{aligned} \langle x, f \rangle &= \sum_{n=1}^{\infty} \langle x_n, f \rangle \\ &= \langle \chi, f \rangle \end{aligned}$$

and so $\chi \in E$; thus E is semi-reflexive.

Corollary. *Let E be a Mackey space (i.e. the topology on E is the Mackey topology $\tau(E, E')$) with a Schauder decomposition (Q_n). Then E is barrelled, if and only if:*

(i) *Each E_n is barrelled.*

(ii) *(Q_n) is a Schauder decomposition for E in the strong topology $\beta(E, E')$.*

(iii) *(Q_n') is β-complete (or γ-complete) for $(E', \sigma(E', E))$.*

4. *The $\sigma\gamma$-topology and B-invariance.* In order to study γ-complete decomposition in detail in section 5, it is now necessary to introduce several concepts first studied by Garling (5, 6) in sequence spaces; as many of the results of this section are simple generalizations of results of Garling, their proofs may be omitted or condensed.

We recall that the sequence space *cs* of all sequences (α_i), such that $\sum_{i=1}^{\infty} \alpha_i$ converges, is a Banach space under the norm

$$\|\alpha\| = \sup_n \left| \sum_{i=1}^{n} \alpha_i \right|,$$

and that its dual may be identified with the sequence space bv of all sequences (α_i) of 'bounded variation', such that

$$\sum_{i=1}^{\infty} |\alpha_i - \alpha_{i+1}| < \infty;$$

the dual norm on bv is given by

$$\|\alpha\| = \lim_{n\to\infty} |\alpha_n| + \sum_{i=1}^{\infty} |\alpha_i - \alpha_{i+1}|.$$

We shall let B be the closed unit ball of bv, and B_0 be the subset of B consisting of all α such that $\lim_{n\to\infty} \alpha_n = 0$.

Suppose now that E is a locally convex space with a Schauder decomposition (Q_n); we define E^β to be the subspace of the algebraic dual E^* consisting of all $f \in E^*$ such that (i) $Q_n^* f \in Q_n'(E')$ for each n, (ii) $\langle x,f\rangle = \sum_{i=1}^{\infty} \langle Q_i x, f\rangle$ whenever $x \in E$.

It is an immediate consequence of the definition that $E' \subset E^\beta \subset E^*$, and (Q_n) is a Schauder decomposition for E in the topology $\sigma(E, E^\beta)$. It can also be seen that (Q_n') is a β-complete decomposition for $(E', \sigma(E', E))$ if and only if $E' = E^\beta$.

If $f \in E$ and $x \in E$ then the sequence $(\langle Q_n x, f\rangle)_{n=1}^{\infty}$ belongs to cs; and so if $b \in B$, $\sum_{i=1}^{\infty} b_i \langle Q_i x, f\rangle$ converges whenever $f \in E$ and $x \in E$. Thus we define an action of B on E by $f \to b\,.f$, where

$$\langle x, b\,.f\rangle = \sum_{i=1}^{\infty} b_i \langle Q_i x, f\rangle.$$

If G is a subspace of E, we define

$$B(G) = (b\,.g;\ b \in B \text{ and } g \in G) \quad \text{and} \quad B_0(G) = (b\,.g;\ b \in B_0 \text{ and } g \in G);$$

and let $$B^*(G) = \operatorname{lin} B(G) \quad \text{and} \quad B_0^*(G) = \operatorname{lin} B_0(G).$$

A subspace G of E is said to be B-invariant if $G = B(G)$ (and hence $G = B^*(G)$); since $B^*(G) = \operatorname{lin}(B_0(G) \cup G)$ always, this is equivalent to $B_0(G) \subset G$. It is obvious at once that E^β is B-invariant.

In Lemmas 4·1–4·3 the sequence (y_n) in E will be of the form

$$y_n = \sum_{i=1}^{n} x_i \quad \text{with} \quad x_i \in E_i.$$

LEMMA 4·1. *Let (Q_n) be a Schauder decomposition for E, and suppose $J \subset G \subset E^\beta$; then (Q_n) is a Schauder decomposition for $\{E, \sigma(E, G)\}$ and (y_n) is convergent in the topology $\sigma(E, G)$ if and only if it is convergent in the original topology.*

The simple proof is omitted.

LEMMA 4·2. *Under the same hypotheses as Lemma 4·1, the following statements are equivalent:*

(i) *(y_n) is $\sigma(E,G)$ bounded.*

(ii) *(y_n) is $\sigma(E,B_0^*(G))$ bounded.*

(iii) *(y_n) is $\sigma(E,B_0^*(G))$ Cauchy.*

That (iii) implies (ii) is clear; the proof is completed by showing that (ii) implies (i) and (i) implies (iii). Suppose (y_n) is $\sigma(E,G)$ unbounded; then a subsequence (y_{n_j}) may be chosen such that for some $g\in G$

$$|\langle y_{n_j},g\rangle| \geqslant 4^j+2^j\sum_{i=1}^{n_{j-1}}|\langle x_i,g\rangle|+|\langle y_{n_{j-1}},g\rangle|.$$

Letting $b_i = 2^{-j}$ when $n_{j-1}+1 \leqslant i \leqslant n_j$, $(b_i)\in B_0$ and it is easily seen that

$$|\langle y_{n_j}, b.g\rangle| \geqslant 2^j,$$

so that (y_n) is $\sigma(E,B_n^*(G))$ unbounded.

Now suppose that (y_n) is $\sigma(E,G)$ bounded, and let $g\in G$ and $b\in B_0$. If $n \geqslant m$

$$\begin{aligned}|\langle y_n-y_m, b.g\rangle| &= \left|\sum_{i=m+1}^{n} b_i\langle y_i-y_{i-1},g\rangle\right|\\ &\leqslant \left|\sum_{i=m+1}^{n}(b_i-b_{i+1})\,\langle y_i,g\rangle+b_{n+1}\langle y_n,g\rangle+b_{m+1}\langle y_m,f\rangle\right|\\ &\leqslant \sup_j|\langle y_j,g\rangle|\left(\sum_{i=m+1}^{n}|b_i-b_{i+1}|+|b_{n+1}|+|b_{m+1}|\right)\\ &\to 0 \quad\text{as}\quad n,m\to\infty.\end{aligned}$$

This completes the proof of Lemma 4·2. An immediate consequence of this and Lemma 4·1 is that (Q_n) is a γ-complete decomposition for $\sigma(E,B_0^*(G))$ if and only if it is a γ-complete decomposition for $\sigma(E,G)$. However, in this paper we shall only use the corresponding results for $B^*(G)$, which are consequences of the following lemma.

LEMMA 4·3. *Under the same hypotheses as Lemma* 4·1 *(y_n) is convergent (resp. Cauchy: resp. bounded) for the topology $\sigma(E,B^*(G))$ if and only if (y_n) is convergent (resp. Cauchy: resp. bounded) for the topology $\sigma(E,G)$.*

The equivalence of convergence follows from Lemma 4·1, while the other two cases are immediate consequences of Lemma 4·2 and the fact that $B^*(G) = \operatorname{lin}(G \cup B_0^*(G))$. From this we obtain at once the next proposition which will be very useful in section 5.

PROPOSITION 4·4. *If (Q_n) is a Schauder decomposition for E and $J\subset G\subset E^\beta$ then (Q_n) is β-complete (resp. γ-complete) for $\sigma(E,G)$ if and only if it is β-complete (resp. γ-complete) for $\sigma(E,B^*(G))$.*

Since $f = \lim_{n\to\infty} P'_n f$ in the topology $\sigma(E',E)$ the set $P(f) = (P'_n f)_{n=1}^\infty$ is weakly bounded for each $f\in E'$, and so we can define an $\langle E,E'\rangle$ polar topology $\sigma\gamma(E,E')$ of uniform convergence on the sets $(P(f), f\in E')$; the notation is derived from (5). This topology for E may also be considered as given by the collection of semi-norms $(p_f; f\in E')$, where $p_f(x) = \sup|\langle P_n x,f\rangle|$. Although $\sigma(E,E')$ is not a 'natural' topology on E, for it depends on the decomposition (Q_n), its properties are extremely useful; the remainder of this section is devoted to its study.

PROPOSITION 4·5. *$\sigma\gamma(E,E')$ is the minimal $\langle E,E'\rangle$ polar topology for which (Q_n) is an equi-Schauder decomposition.*

Suppose $x \in E$, and $f \in E'$; then

$$p_f(x - P_m x) = \sup_{n>m} |\langle P_n x - P_m x, f\rangle|$$

and so

$$\lim_{m\to\infty} P_m x = x$$

in $\sigma\gamma(E,E')$. Also $p_f(P_m x) \leqslant p_f(x)$ for each $m, x \in E$ and $f \in E'$; and so (Q_n) is an equi-Schauder decomposition for $\sigma\gamma(E,E')$.

Conversely suppose τ is an $\langle E,E'\rangle$ polar topology for which (Q_n) is an equi-Schauder decomposition. Then if $f \in E'$, f is τ-continuous and so $p_f(x) = \sup_n |\langle P_n x, f\rangle|$ is τ-continuous, i.e. $\tau \geqslant \sigma\gamma(E,E')$.

COROLLARY. *$\sigma\gamma(E,E')$ is complete (resp. sequentially complete: resp. quasi-complete) if and only if:*

(i) *(Q_n) if γ-complete for the original topology.*

(ii) *Each E_n is weakly complete (resp. weakly sequentially complete: resp. weakly quasi-complete or semi-reflexive).*

This is a direct application of the main theorem of (8) and the fact that if $x_i \in E_i$ the sequence $\left(\sum_{i=1}^{n} x_i\right)_{n=1}^{\infty}$ is $\sigma(E,E')$ Cauchy if and only if it is $\sigma\gamma(E,E')$ Cauchy.

PROPOSITION 4·6. *The dual of $(E, \sigma\gamma(E,E'))$ is $B^*(E')$.*

If $f \in E'$, define $T_f: E \to cs$ by $(T_f(x))_n = \langle Q_n x, f\rangle$. Then $\|T_f x\|_{cs} = p_f(x)$, and so T_f is $\sigma\gamma(E,E')$ continuous. Hence if $b \in B$ the linear functional $x \to \langle T_f(x), b\rangle = \langle x, b.f\rangle$ is continuous, and so $B^*(E')$ is contained in the dual of $(E, \sigma\gamma(E,E'))$.

Conversely let $\phi \in (E, \sigma\gamma(E,E'))'$ be such that $|\phi(x)| \leqslant p_f(x)$ for all $x \in E$. Then if $T_f(x) = 0$, $\langle x, \phi\rangle = 0$ and so we may define a linear functional ψ on $T_f(E)$ by

$$\langle T_f(x), \psi\rangle = \langle x, \phi\rangle$$

and

$$|\langle T_f(x), \psi\rangle| \leqslant \|T_f(x)\|_{cs}.$$

Using the Hahn–Banach theorem ψ may be extended to a linear functional b on cs such that $\|b\|_{bv} \leqslant 1$, i.e. $b \in B$.

Thus

$$\langle x, \phi\rangle = \langle T_f x, b\rangle = \sum_{i=1}^{\infty} b_i \langle Q_i x, f\rangle,$$

i.e. $\phi \in B(E')$, and the result follows.

COROLLARY. *The following are equivalent:*

(i) *E' is B-invariant.*

(ii) *$B_0(E') \subset E'$.*

(iii) *$\sigma\gamma(E,E')$ is an $\langle E,E'\rangle$ dual topology.*

(iv) *There exists an $\langle E,E'\rangle$ dual topology τ such that (Q_n) is an equi-Schauder decomposition for (E,τ).*

(v) *For each f, $P(f)$ is equicontinuous for the Mackey topology $\tau(E,E')$.*

That (i) and (ii) are equivalent has already been observed, and Proposition 4·6 shows that (i) and (iii) are equivalent; (v) and (iii) are obviously equivalent. Condition (iii) implies (iv) by letting $\tau = \sigma\gamma(E, E')$; and (iv) implies (iii) since $\sigma\gamma(E, E') \leqslant \tau$ by Proposition 4·5.

We recall that (Q_n) is simple if $P(f)$ is strongly bounded for each f; it follows that, if E' is B-invariant then certainly (Q_n) is simple; and the converse holds if E is quasi-barrelled or E' is strongly sequentially complete (this last follows by using condition (ii) of the corollary).

DEFINITION 4·7. *(Q_n) is B-simple if E' is B-invariant.*

To conclude this section, we consider the bounded sets of E in the topology $\sigma\gamma(E, E')$; in this respect we find an interesting criterion for (Q_n) to be simple (Proposition 4·8). Theorem 4·9 is essentially theorem 10 of (5) translated into a new setting.

PROPOSITION 4·8. *(Q_n) is simple if and only if $\sigma(E, E')$ and $\sigma\gamma(E, E')$ define the same bounded sets.*

$P(f)$ is strongly bounded if and only if for every $\sigma(E, E')$ bounded set A

$$\sup_{a \in A} \sup_n |\langle a, P'_n f\rangle| < \infty,$$

i.e.

$$\sup_{a \in A} p_f(a) < \infty,$$

and the result follows.

THEOREM 4·9. *Suppose that (Q_n) is a β-complete Schauder decomposition for E such that each E_n is weakly sequentially complete. Let $f_i \in Q'_i(E')$ and $g_n = \sum_{i=1}^{n} f_i$; suppose $(g_n)_{n=1}^{\infty}$ is $\sigma(E', E)$-bounded; for every $\sigma\gamma(E, E')$-bounded set A:*

$$\sup_n \sup_{a \in A} |\langle a, g_n\rangle| < \infty.$$

The conditions that (Q_n) is β-complete and each E_n is weakly sequentially complete imply, using the Corollary to Proposition 4·5, that $\sigma\gamma(E, E)$ is sequentially complete. The method is then precisely that of (5) theorem 10 and is omitted.

COROLLARY. *Under the hypotheses of Theorem 4·8, $\sigma\gamma(E, E')$ and $\sigma(E, E^\beta)$ define the same bounded sets.*

If $f \in E^\beta$ then $(P^*_n f)_{n=1}$ is a $\sigma(E', E)$ Cauchy sequence, and is therefore $\sigma(E', E)$-bounded. Hence if A is $\sigma\gamma(E, E')$-bounded using Theorem 4·8

$$\sup_n \sup_{a \in A} |\langle a, P^*_n f\rangle| < \infty$$

and so

$$\sup_{a \in A} |\langle a, f\rangle| < \infty \quad \text{since} \quad f \in E^\beta,$$

i.e. A is $\sigma(E, E^\beta)$-bounded.

5. *Shrinking and γ-complete decompositions.* Let (x_n) be a Schauder basis for E, and (f_n) the dual sequence in E'; as before, let H be the set of all $f \in E'$ such that $P'_n f \to f$ in the topology $\beta(E', E)$, with the relativized topology. Then if E is a Banach space, each of the following statements is true (see (7) and (11)):

(I) If (x_n) is γ-complete, then H' is naturally isomorphic to E (loosely, $H' = E$).

(II) If (x_n) is γ-complete, (f_n) is a shrinking basis for H.

(III) If (f_n) is a shrinking basis for H, (x_n) is γ-complete.

(IV) If (x_n) is shrinking, (f_n) is a γ-complete basis for E'.

(V) If (f_n) is a γ-complete basis for H, then (x_n) is shrinking (i.e. $H = E'$).

The problem therefore naturally arises: under what circumstances can we generalize these results to locally convex spaces? This problem has been considered by Dubinsky and Retherford (4), who produce counter-examples to show that, in general, (II), (III) and (IV) fail to hold; and it may be noted that each of their examples is a simple basis. We shall show, however, that (I) and (V) hold for simple bases, and (III) holds if E is sequentially complete; we shall also derive a necessary and sufficient condition for (II) to hold when E is sequentially complete.

Since this work generalizes without difficulty to Schauder decompositions, all the results are obtained in this form; usually it is necessary to assume that each E_n is semi-reflexive.

LEMMA 5·1. *If (Q_n) is a simple Schauder decomposition for E and $f \in B_0(E')$ then $(P_n^* f)_{n-1}^{\infty}$ is a $\beta(E', E)$ Cauchy sequence.*

Since $f \in B_0(E')$, let $f = b\,.\,g$ where $b \in B_0$ and $g \in E'$; let p be any $\beta(E', E)$ continuous semi-norm on E'. Suppose $m \geqslant n > 1$.

$$\begin{aligned} p\left(\sum_{i=n+1}^{m} Q_i^* f\right) &= p\left(\sum_{i=n+1}^{m} b_i Q_i' g\right) \\ &= p\left(\sum_{i=n+1}^{m} (b_i - b_{i+1}) P_i' g + b_{m+1} P_m' g - b_{n+1} P_n' g\right) \\ &\leqslant \left(\sum_{i=n+1}^{m} |b_i - b_{i+1}| + |b_{m+1}| + |b_{n+1}|\right) \sup_k p(P_k' g) \\ &\to 0 \quad \text{as} \quad m, n \to \infty, \end{aligned}$$

since (Q_n) is simple; and so $P_n^* f$ is a Cauchy sequence as required.

COROLLARY. *If (Q_n) is B-simple then H is B-invariant.*

If $f \in B_0(E')$ then by the lemma $P_n' f \to f$ in the strong topology $\beta(E', E)$. Hence $B_0(E') \subset H$, and so $B_0(H) \subset H$; thus H is B-invariant.

With the aid of Lemma 5·1, (I) may now be investigated; Theorem 5·2 gives necessary and sufficient conditions for (I) to hold.

THEOREM 5·2. *Let (Q_n) be a Schauder decomposition for E. Then $H' = E$ if and only if (Q_n) is γ-complete and simple, and each E_n is semi-reflexive.*

First we suppose $H' = E$; let χ be a continuous linear functional on $(E_n', \beta(E_n', E_n))$.

We define a linear functional η on E' by $\langle \eta, f \rangle = \langle \chi, \bar{f} \rangle$ where $\bar{f}$ is the restriction of f to E_n; there exists a bounded subset A of E_n such that whenever $\phi \in E_n'$

$$|\langle \chi, \phi \rangle| \leqslant \sup_{a \in A} |\langle a, \phi \rangle|,$$

and since A is a bounded subset of E with

$$|\langle \eta, f \rangle| \leqslant \sup_{a \in A} |\langle a, f \rangle|$$

whenever $f \in E'$, η is a $\beta(E', E)$ continuous linear functional on E'. By assumption there exists $y \in E$ such that whenever $f \in H$, $\langle y, f\rangle = \langle \eta, f\rangle$. Suppose $\phi \in E'_n$; define $\Gamma_n \phi \in E'$ by $\langle x, \Gamma_n \phi\rangle = \langle Q_n x, \phi\rangle$ whenever $x \in E$. Thus we have

$$\begin{aligned}\langle Q_n y, \phi\rangle &= \langle y, \Gamma_n \phi\rangle \\ &= \langle \eta, \Gamma_n \phi\rangle \\ &= \langle \chi, \phi\rangle \quad \text{for all} \quad \phi \in E'_n.\end{aligned}$$

Thus E_n is semi-reflexive.

Now let $f \in E'$, and let $P(f) = (P'_n f)_{n=1}^{\infty}$; we wish to show that $P(f)$ is $\beta(E', E)$-bounded. We have $P(f) \subset H$, and $H' = E$ is the topology $\beta(E', E)$; but $P(f)$ is certainly $\sigma(H, E)$-bounded, and so it is $\beta(E', E)$-bounded.

Finally let $x_n \in E_n$ be any sequence such that

$$\left(\sum_{i=1}^{n} x_i\right)_{n=1}^{\infty}$$

is bounded. Then the set $\left(\sum_{i=1}^{n} x_i\right)$ is an equicontinuous collection of linear functionals on $(E', \beta(E', E))$; and so the set of $f \in E'$ such that $\sum_{n=1}^{\infty} \langle x_n, f\rangle$ converges, is a closed subspace, and includes each $Q'_n(E')$. Thus for each $f \in H$, $\sum_{n=1}^{\infty} \langle x_n, f\rangle$ converges and is a continuous functional on H; by assumption there exists $x \in E$ such that

$$\langle x, f\rangle = \sum_{n=1}^{\infty} \langle x_n, f\rangle$$

for each $f \in H$. If $f \in E'$, $Q'_n f \in H$ and so $\langle x, Q'_n f\rangle = \langle x_n, f\rangle$; thus $x_n = Q_n x$ and

$$\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} Q_i x = x$$

converges.

For the converse, we use Lemma 5·1; suppose that (Q_n) is γ-complete and simple with each E_n semi-reflexive. Certainly $E \subset H'$; now suppose $\chi \in H'$. The map $\Gamma_n \colon E'_n \to E'$ used in the earlier part of the proof is weakly and hence also strongly continuous and maps E' into $Q'_n(E')$; and so if $\phi \in E'_n$, $\langle \chi, \Gamma_n \phi\rangle$ is strongly continuous linear functional on E'_n. Thus there exists $x_n \in E_n$ such that $\langle x_n, \phi\rangle = \langle \chi, \Gamma_n \phi\rangle$ for all $\phi \in E'_n$. If $f \in E'$ and f_n is its restriction to E_n,

$$\langle x_n, f\rangle = \langle x_n, f_n\rangle = \langle \chi, \Gamma_n f_n\rangle,$$

where $$\langle x, \Gamma_n f_n\rangle = \langle Q_n x, f\rangle = \langle x, Q'_n f\rangle$$

for each $x \in E$; thus $\langle x_n, f\rangle = \langle \chi, Q'_n f\rangle$.

Now let $b \in B_0$; then $\sum_{i=1}^{n} b_i Q'_i f$ is a Cauchy sequence, and so $\sum_{i=1}^{\infty} b_i \langle \chi, Q'_i f\rangle$ converges for each $f \in E'$. Hence $\sum_{i=1}^{n} \langle \chi, Q'_i f\rangle$ is a bounded sequence for each $f \in E'$, and so $\sum_{i=1}^{n} \langle x_i, f\rangle$ is a bounded sequence. Therefore $\left(\sum_{i=1}^{n} x_i\right)$ is a bounded sequence, and since (Q_n) is γ-complete, there exists $x \in E$ such that $x = \sum_{i=1}^{\infty} x_i$.

Then $$\langle x,f\rangle = \sum_{i=1}^{\infty} \langle x_i,f\rangle = \sum_{i=1}^{\infty} \langle \chi, Q_i' f\rangle = \langle \chi,f\rangle,$$

and so $H' = E$.

We now consider statements (II) and (III); it is clearly necessary to consider the strong dual of H. We observe first that $Q_n''(H')$ may be identified with E_n, when E_n is semi-reflexive; for let $\chi \in H'$ be such that $Q_n''\chi = \chi$, then for $\phi \in E_n'$ the map $\phi \to \langle \chi, \Gamma_n \phi\rangle$ is strongly continuous, where Γ_n is the map used above. Thus since E_n is semi-reflexive, there exists $x \in E_n$ such that whenever $\phi \in E_n'$, $\langle \chi, \Gamma_n \phi\rangle = \langle x, \phi\rangle$; hence if $f \in H$, $\langle \chi, f\rangle = \langle Q_n''\chi, f\rangle = \langle \chi, Q_n' f\rangle = \langle \chi, \Gamma_n f_n\rangle = \langle x, f\rangle$, where f_n is the restriction of f to E_n. Thus $Q_n''(H') = E_n$ is a natural sense.

PROPOSITION 5·3. *Let (Q_n) be a simple Schauder decomposition of E. Then on E, $\beta(H', H)$ coincides with the topology of uniform convergence on the strongly bounded subsets of E'.*

Clearly $\beta(H', H)$ is weaker than the topology of uniform convergence on the strongly bounded subsets of E'. Now suppose A is strongly bounded; then $P'(A) = \bigcup_{n=1}^{\infty} P_n'(A)$ is also strongly bounded, as the maps (P_n') are equicontinuous. Thus $P'(A) \subset H$ is $\beta(H', H)$-equicontinuous, and $[P'(A)]^0 \subset A^0$, so that A^0 is a $\beta(H', H)$-neighbourhood of zero.

COROLLARY 1. *If E is sequentially complete and (Q_n) is a simple Schauder decomposition with each E_n semi-reflexive such that (Q_n'') is shrinking for H, then $H' = E$ and (Q_n) is a strong decomposition of E.*

By Proposition 5·3, the topology $\beta(H', H)$ coincides with the topology $\beta(E, E')$ on E, as E is sequentially complete. As stated before Proposition 5·3, (Q_n'') is a decomposition of H' into subspaces which can be identified with $(E_n)_{n=1}^{\infty}$. For $\chi \in H'$, $(P_n''\chi)$ is a $\beta(H', H)$-Cauchy sequence in E; hence $P_n''\chi$ converges in E and $\chi \in E$. Thus $H' = E$ and the result follows.

COROLLARY 2. *Under the hypotheses of Corollary* 1, *(Q_n) is a γ-complete decomposition of E (see* (III)).

Proof by Theorem 5·2 and Corollary 1.

It will be seen that Corollary 2 provides an affirmative answer for (III) if E is sequentially complete and (Q_n) is simple, while Corollary 1 shows that it is unlikely that under the same conditions we can obtain (II) unless E is barrelled. If E is barrelled then we do not need sequential completeness, for we have the following result, proved for bases in (4):

COROLLARY 3. *If E is barrelled and (Q_n) is a γ-complete Schauder decomposition of E with each E_n semi-reflexive (and so reflexive), then (Q_n') is shrinking for H.*

Any Schauder decomposition of E is equi-Schauder and hence simple (Proposition 2·4); hence by Theorem 5·2 $H' = E$, and by Proposition 5·3 $\beta(H', H)$ is the original topology on E.

To conclude this section we consider statements (IV) and (V); Proposition 5·4 is simply a generalization of a result of Dubinsky and Retherford (4), and so the proof is omitted.

Proposition 5·4. *If E is a locally convex space such that every strongly bounded sequence in E' is equicontinuous and (Q_n) is a shrinking Schauder decomposition, then (Q'_n) is a γ-complete Schauder decomposition for E' in its strong topology.*

Finally we prove (V) for simple decompositions.

Proposition 5·5. *If (Q_n) is a simple Schauder decomposition of E and (Q'_n) is a γ-complete decomposition of H, then (Q_n) is shrinking (i.e. $H = E'$).*

If $f \in E'$ then $\left(\sum_{i=1}^{n} Q'_i f\right)_{n=1}^{\infty}$ is a strongly bounded sequence in H, and so $\sum_{i=1}^{\infty} Q'_i f$ converges; hence $f \in H$.

6. *The structure of H.* In this section we obtain various results concerning H particularly when (Q_n) is a γ-complete decomposition of E; in particular sufficient conditions for H to be barrelled are derived. We shall also relate, in general, the structure of E', to the structure of the space $B^*(E')$ defined in section 4.

Theorem 6·1. *If (Q_n) is simple with each E_n semi-reflexive then H is quasi-barrelled.*

In this case (Q''_n) is a decomposition of H' into the sub-spaces E_n. Suppose $A \subset H'$ is $\beta(H', H)$-bounded; then as (Q'_n) is an equi-Schauder decomposition of H, and is thus simple, (Q''_n) is equi-Schauder on $(H', \beta(H', H))$ and so $P''(A) = \bigcup_{n=1}^{\infty} P''_n(A)$ is $\beta(H', H)$-bounded in E. Therefore, $P''(A)$ is bounded in E by Proposition 5·3, and

$$[P''(A)]^0 \cap H \subset A^0,$$

and the result follows.

Lemma 6·2. *If (Q_n) is a B-simple Schauder decomposition of E with each E_n semi-reflexive, then on H, $\beta(E', E) = \beta(H, E) = \beta(H, H')$, and so H is barrelled.*

Let $L = B_0^*(E') \subset E'$; then (Q'_n) is a Schauder decomposition of $(L, \sigma(L, E))$ and we may identify the space L^β (see section 4) with respect to this decomposition, and E may be considered as a subspace of L^β. The dual decomposition (Q'_n) of $(L^\beta, \sigma(L^\beta, L))$ is a β-complete Schauder decomposition of L^β into the subspaces E_n.

Suppose A is $\sigma(L^\beta, L)$-bounded in L^β and $f \in E'$; then if $b \in B_0$,

$$\sup_{a \in A} |\langle a, b.f\rangle| < \infty,$$

and so, using the Principle of Uniform Boundedness on the space bv_0,

$$\sup_n |\langle a, P'_n f\rangle| < \infty$$

so that

$$\sup_n |\langle P''_n a, f\rangle| < \infty$$

and $P''(A) = \bigcup_{n=1}^{\infty} P''_n(A)$ is a $\sigma(E, E')$-bounded subset of E. Clearly

$$[P''(A)]^0 \cap L \subset A^0 \cap L,$$

and so A is $\beta(E, E')$-equicontinuous on L. Hence on L, $\beta(E, E') \geqslant \beta(L, L^\beta) \geqslant \beta(L, L')$, where L' is the dual of L in the topology $\beta(E, E')$. Hence L is barrelled, and as $H = \bar{L}$, H is barrelled and on H $\beta(E', E) = \beta(H, H') \geqslant \beta(H, E) \geqslant \beta(E', E)$.

THEOREM 6·3. *If E is barrelled and (Q_n) is a Schauder decomposition of E into reflexive subspaces then H is barrelled.*

(Q_n) is equi-Schauder and hence B-simple by Proposition 4·6 and Corollary.

In the general case when (Q_n) is not B-simple, it is convenient to consider $B^*(E')$ instead of E'; this is explained by the next lemma.

LEMMA 6·4. *Let (Q_n) be a Schauder decomposition for E.*

(i) *On E', $\tau(B^*(E'), E) = \tau(E', E)$.*

(ii) *If (Q_n) is simple, $\beta(B^*(E'), E) = \beta(E', E)$ on E'.*

If $f \in E'$, $b \in B_0$ and p is a $\tau(E', E)$ continuous semi-norm on E', then, by a similar method to that in Lemma 4·2,

$$p\left(\sum_{i=m+1}^{n} b_i Q_i' f\right) \leqslant \sup_i p(P_i' f)\left(\sum_{i=m+1}^{n} |b_i - b_{i+1}| + |b_{n+1}| + |b_{m+1}|\right),$$

and so if F is a completion of $(E', \tau(E', E))$, then we may identify $B^*(E')$ as a subspace of F. Thus we have

$$E' \subset B^*(E') \subset F.$$

The inclusion maps are continuous for the weak topologies $\sigma(E', E)$, $\sigma(B^*(E'), E)$ and $\sigma(F, E)$, and hence also for the corresponding Mackey topologies. Thus on E'

$$\tau(E', E) \geqslant \tau(B^*(E'), E) \geqslant \tau(F, E).$$

However, (F, τ) is a completion of $\{E', \tau(E', E)\}$ for some locally convex τ; hence $(F, \tau)' = (E', \tau(E', E))' = E$, and so $\tau \leqslant \tau(F, E)$. Hence on E':

$$\tau = \tau(F, E) = \tau(B^*(E'), E) = \tau(E', E).$$

If (Q_n) is simple, by Proposition 4·8 $\sigma(E, E')$ and $\sigma\gamma(E, E')$ define the same bounded sets; and by Proposition 4·6 $\sigma(E, E')$ and $\sigma(E, B^*(E'))$ define the same bounded sets. Hence on E', $\beta(B^*(E'), E) = \beta(E', E)$.

THEOREM 6·5. *Let (Q_n) be a Schauder decomposition for E, such that each E_n is weakly sequentially complete. Then the following are equivalent:*

(i) *$B^*(E')$ is $\tau(B^*(E'), E)$ barrelled.*

(ii) *$\tau(E', E)$ is quasi-barrelled and (Q_n) is β-complete.*

(iii) *$\tau(E', E)$ is quasi-barrelled and (Q_n) is γ-complete.*

It is obvious that (iii) implies (ii); suppose then (ii) holds and A is $\sigma(E, B^*(E'))$-bounded. Then A is $\sigma\gamma(E, E')$-bounded; by the Corollary to Proposition 4·5, $\sigma\gamma(E, E')$ is sequentially complete, and so defines the same bounded sets as $\beta(E, E')$. Thus A is $\beta(E, E')$-bounded, and by assumption $\tau(E', E)$ equicontinuous; hence by Lemma 6·4 A is $\tau(B^*(E'), E)$ equicontinuous, and so $B^*(E')$ is barrelled.

Now suppose (i) holds, and that A is $\beta(E, E')$-bounded. Then A is $\sigma(E, E')$-bounded, hence $\sigma(E, B^*(E'))$-bounded, and so $\tau(B^*(E'), E)$ equicontinuous. Thus by Lemma 6·4 A is $\tau(E', E)$ equicontinuous, and so E' is $\tau(E', E)$ quasi-barrelled. Also (Q_n) is γ-complete for $(E, \tau(E, B^*(E'))$ by Theorem 3·1, and by Proposition 4·4 γ-complete for E in the original topology.

This theorem gives another characterization of semi-reflexivity.

THEOREM 6·6. *E is semi-reflexive if and only if:*

(i) *(Q_n) is simple and β-complete.*

(ii) *E' is $\tau(E',E)$ quasi-barrelled.*

(iii) *Each E_n is weakly sequentially complete.*

If (i), (ii) and (iii) hold, then by Theorem 6·5 $B^*(E')$ is $\tau(B^*(E'),E)$-barrelled, and so by Lemma 6·4, on E'

$$\tau(E',E) = \tau(B^*(E'),E) = \beta(B^*(E'),E) = \beta(E',E).$$

In the final theorem we obtain two necessary and sufficient conditions for H to be barrelled, when (Q_n) is γ-complete and simple.

THEOREM 6·7. *Let (Q_n) be a Schauder decomposition for E, with each E_n semi-reflexive. The following are equivalent:*

(i) *(Q_n) is γ-complete simple and H is barrelled.*

(ii) *(Q_n) is β-complete simple for $(E,\sigma(E,H))$.*

(iii) *$\sigma(E,H)$ is sequentially complete.*

If (i) holds, then by Theorem 5·2 $H' = E$, and so $(E,\sigma(E,H))$ is semi-reflexive. Thus $\sigma(E,H)$ is quasi-complete and so (i) implies (iii).

If $\sigma(E,H)$ is sequentially complete, then (Q_n) is β-complete for $(E,\sigma(E,H))$ and, since $\sigma(H,E)$ and $\sigma\gamma(H,E)$ then define the same bounded sets, also simple. Hence (iii) implies (ii).

Suppose (ii) holds. Let $x_n \in E_n$ be such that $\left(\sum_{i=1}^{n} x_i\right)_{n=1}^{\infty}$ is $\sigma(E, E')$-bounded; then for $f \in H, f = \lim_{n\to\infty} P'_n f$ in $\sigma(E',E)$ and so

$$\lim_{m\to\infty} \sup_{n} \left|\left\langle \sum_{i=1}^{n} x_i, f - P'_m f \right\rangle\right| = 0$$

hence

$$\lim_{m\to\infty} \sup_{n>m} \left| \sum_{i=m+1}^{n} \langle x_i, f\rangle \right| = 0$$

and so $\left(\sum_{i=1}^{n} x_i\right)_{n=1}^{\infty}$ is $\sigma(E,H)$-Cauchy; thus there exists $x \in E$, with $Q_i x = x_i$ and $x = \sum_{i=1}^{\infty} x_i$. (Q_n) is therefore γ-complete. Since (Q_n) is simple for $\sigma(E,H)$, $\sigma(E,H)$ and $\sigma\gamma(E,H)$ define the same bounded sets; hence by Theorem 4·9 so do $\sigma(E,H)$ and $\sigma(E,E^{\beta})$. Hence, $\sigma(E,E')$ and $\sigma\gamma(E,E')$ define the same bounded sets. By Proposition 4·8 (Q_n) is simple for the original topology on E.

We have therefore (Q_n) γ-complete and simple, and so by Theorem 6·1 H is quasi-barrelled and $H' = E$; thus H is $\tau(H,E)$ quasi-barrelled. Hence we can apply Theorem 6·6 to $(E,\sigma(E,H))$ to obtain that H is barrelled. Thus (ii) implies (i).

This result is quite surprising as the condition that $\sigma(E,H)$ is sequentially complete is apparently mild. It shows that in theorem 1·8, page 273 of (4), the condition '(f^n) is shrinking' is irrelevant, and so the theorem does not show the intended duality between shrinking and boundedly complete bases.

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