# A REMARK ON QUASI-ISOMETRIES

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ABSTRACT. We show that if $f : B_n \to \mathbb{R}^n$ is an $\epsilon-$quasi-isometry, with $\epsilon < 1$, defined on the unit ball $B_n$ of $\mathbb{R}^n$, then there is an affine isometry $h : B_n \to \mathbb{R}^n$ with $\|f(x) - h(x)\| \le C\epsilon(1 + \log n)$ where $C$ is a universal constant. This result is sharp.

## 1. INTRODUCTION

Suppose $\epsilon > 0$ and $X$ and $Y$ are Banach spaces and let $B_X = \{x : \|x\| \le 1\}$. An $\epsilon$-quasi-isometry $f : B_X \to Y$ is a local homeomorphism such that for any $x \in B_X$ we have

$$(1 + \epsilon)^{-1} \le \liminf_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|} \le \limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|} \le (1 + \epsilon).$$

A map $f : B_X \to Y$ is $\epsilon-$rigid if

$$(1 + \epsilon)^{-1}\|x_1 - x_2\| \le \|f(x_1) - f(x_2)\| \le (1 + \epsilon)\|x_1 - x_2\|, \qquad x_1, x_2 \in B_X.$$

If $\dim X = \dim Y < \infty$, then any $\epsilon-$rigid map is also an $\epsilon-$quasi-isometry (cf. [8]). We refer to [2] for a full discussion.

In this note we will be concerned with the case when $X = Y = \mathbb{R}^n$ with the usual Euclidean norm (we then write $B_n$ for the closed unit ball in $\mathbb{R}^n$). In this context John [6] (discussed in [2]) showed that if $f$ is an $\epsilon$-quasi-isometry one can find an $a \in \mathbb{R}^n$ and an orthogonal linear operator $U$ so that

$$\|f(x) - Ux - a\| \le Cn^{\frac{3}{2}}\epsilon, \qquad x \in B_n.$$

See also a recent paper [1] for related results. Recently the exponent $\frac{3}{2}$ has been improved to $\frac{1}{2}$ by Vestfrid [11]. This raises the question of finding the optimal function $\alpha(n)$ so that given any $\epsilon-$quasi-isometry $f$ one can find an orthogonal $U$ and $a$ such that

$$\|f(x) - Ux - a\| \le \alpha(n)\epsilon, \qquad x \in B_n.$$

In another direction Matouskova [8] has given examples to show that $\alpha(n) \ge c \log n$ for any such function. The aim of this note is to show that this lower estimate is in fact sharp and we can take $\alpha(n) \le C(\log n + 1)$.

In the final section we make a few remarks on the connections between this result and the theory of twisted sums, and make an observation about an old problem of Figiel, Lindenstrauss and Milman.

## 2. MAIN RESULTS

Let us first discuss the example which gives the lower bound (due to Matouskova [8]). Fix $\epsilon > 0$. Assume $n$ is even, say $n = 2m$, and identify $\mathbb{R}^n$ with $\mathbb{C}^m$. For $z = re^{i\theta} \in \mathbb{C}$ let $\varphi_\epsilon(z) = zr^{i\epsilon}$ (if $z = 0$, then $\varphi_\epsilon(z) = 0$). Then $\varphi_\epsilon$ is an $\epsilon$−quasi-isometry of $\mathbb{C}$ (see [6], [2] p. 352) and so it follows easily that if we set

$$f_\epsilon(z_1, \ldots, z_m) = (\varphi_\epsilon(z_1), \ldots, \varphi_\epsilon(z_m)),$$

then $f_\epsilon$ is also an $\epsilon$−quasi-isometry of $B_n$ onto $B_n$.

**Proposition 2.1.** *Suppose* $\epsilon \log m < 2\pi$. *Then for any affine isometry* $h(x) = Ux + a$ *we have*

$$\max_{x \in B_n} \|f_\epsilon(x) - h(x)\| \geq \sin(\tfrac{1}{4}\epsilon \log m).$$

*Remark* 1. Let us remark that this example of Matouskova shows that Theorem 3 of [10] is incorrect, as was pointed out to us by Igor Vestrid.

*Proof.* Since $f_\epsilon(x) = -f_\epsilon(-x)$ we can clearly replace $h$ by $\frac{1}{2}(h(x)-h(-x))$ and so assume $h(x) = Ux$. Note that $U$ is only assumed real-linear on $\mathbb{C}^m$. Let us suppose $U$ is the best linear approximation to $f_\epsilon$ (not necessarily orthogonal). Let $G$ be the group of unitary operators $V$ on $\mathbb{C}^m$ of the form $V(z_1, \ldots, z_m) = (u_1 z_{\pi(1)}, \ldots, u_m z_{\pi(m)})$ where $|u_j| = 1$ for $1 \leq j \leq m$ and $\pi$ is a permutation of $\{1, 2, \ldots, m\}$. Note that $V^{-1} f_\epsilon(V(z_1, \ldots, z_m)) = f_\epsilon(z_1, \ldots, z_m)$. Hence if $U$ is an optimal approximation we can replace it by

$$U' = \int_G V^{-1} U V \, d\mu(V)$$

where $\mu$ is the Haar measure of $G$, and then $U'$ is also optimal. Now $U'$ commutes with each $V \in G$ and so is complex linear and furthermore must be of the form $re^{i\theta} I$ where $I$ is the identity. Now

$$\|f_\epsilon(1, 0, \ldots, 0) - (re^{i\theta}, 0, \ldots, 0)\| \geq |\sin \theta|$$

and

$$\|f_\epsilon(m^{-\frac{1}{2}}, m^{-\frac{1}{2}}, \ldots, m^{-\frac{1}{2}}) - re^{i\theta}(m^{-\frac{1}{2}}, \ldots, m^{-\frac{1}{2}})\| \geq |\sin(\tfrac{1}{2}\epsilon \log m - \theta)|.$$

Clearly this leads to the result.                                              □

The following result is essentially contained in [3], but we will give some details here for completeness and ease of the reader. As we point out in the final section this result is in some sense derivable from classical results on Banach space theory and the theory of twisted sums, although to do it this way would arguably use more sophisticated notions than we really need.

**Theorem 2.2.** *There is an absolute constant* $C$ *with the following property. Let* $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous map which satisfies the conditions:*
    (1) $\Omega(\lambda x) = \lambda x,$        $\lambda \in \mathbb{R}, \ x \in \mathbb{R}^n,$
    (2) $\|\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2)\| \leq \|x_1\| + \|x_2\|,$        $x_1, x_2 \in \mathbb{R}^n.$
*Then there is a linear map* $A : \mathbb{R}^n \to \mathbb{R}^n$ *with* $\|\Omega(x) - A(x)\| \leq C(\log n + 1)\|x\|$ *for all* $x \in \mathbb{R}^n.$

*Proof.* This theorem is almost proved in [3], but we will sketch the details. We need the following result, which is a fairly simple deduction from the Hahn-Banach Theorem (see Theorem 3.1 of [3]):

**Proposition 2.3.** *Suppose $K$ is a compact convex subset of $\mathbb{R}^m$ and that $f : K \to \mathbb{R}$ is a continuous map satisfying the condition*

$$|f(\tfrac{1}{2}(x_1 + x_2)) - \tfrac{1}{2}(f(x_1) + f(x_2))| \le 1, \qquad x_1, x_2 \in K.$$

*Then there is an affine map $a : K \to \mathbb{R}$ such that*

$$|f(x) - a(x)| \le \log_2 n + \tfrac{5}{2}, \qquad x \in K.$$

Returning to the proof of Theorem 2.2 we will first prove that there is a symmetric linear operator $B$ so that

(2.1) $$|\langle x, \Omega(x) - Bx \rangle| \le 5(\log_2 n + 1)\|x\|^2.$$

We define $\varphi(x) = \langle x, \Omega(x) \rangle$. Let us observe that $\varphi(\lambda x) = \lambda^2 \varphi(x)$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Also, for $x_1, x_2 \in \mathbb{R}^n$,

$$\varphi(x_1 + x_2) + \varphi(x_1 - x_2) = \langle x_1, \Omega(x_1 + x_2) + \Omega(x_1 - x_2) \rangle$$
$$+ \langle x_2, \Omega(x_1 + x_2) - \Omega(x_1 - x_2) \rangle$$

and hence

$$\left| \tfrac{1}{2}(\varphi(x_1 + x_2) + \varphi(x_1 - x_2)) - \varphi(x_1) - \varphi(x_2) \right| \le (\|x_1\| + \|x_2\|)^2$$
$$\le 2(\|x_1\|^2 + \|x_2\|^2).$$

Then if $P$ is a positive-definite operator we define $\nu_P$ to be the probability measure on $\mathbb{R}^n$ with Fourier transform

$$\int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} d\nu_P(x) = e^{-\frac{1}{2}\langle P\xi, \xi \rangle}.$$

For each positive definite matrix $P$ we define

$$f(P) = \int_{\mathbb{R}^n} \varphi(x) d\nu_P(x).$$

Observe that $f(\lambda P) = \lambda f(P)$ if $\lambda \ge 0$. Since $\nu_{P+Q} = \nu_P * \nu_Q$

$$f(P + Q) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(x_1 + x_2) d\nu_P(x_1) d\nu_Q(x_2).$$

By symmetry

$$f(P + Q) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(x_1 + x_2) d\nu_P(x_1) d\nu_Q(x_2)$$
$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tfrac{1}{2}\left(\varphi(x_1 + x_2) + \varphi(x_1 - x_2)\right) d\nu_P(x_1) d\nu_Q(x_2)$$
$$= \rho + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\varphi(x_1) + \varphi(x_2)\right) d\nu_P(x_1) d\nu_Q(x_2)$$

where

$$|\rho| \le 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\|x_1\|^2 + \|x_2\|^2) d\nu_P(x_1) d\nu_Q(x_2) \le 2 \operatorname{tr}(P + Q).$$

Now let $\Gamma_0 = \{P : \operatorname{tr} P = 1\}$. Then for $P, Q \in \Gamma_0$ we have

$$|f(\tfrac{1}{2}(P + Q)) - \tfrac{1}{2}(f(P) + f(Q))| \le 2.$$

Now $\Gamma_0$ is a compact convex subset in the $\frac{1}{2}n(n+1)$–dimensional space $\Sigma_n$ of symmetric matrices and so by Proposition 2.3 we have an affine map $a : \Gamma_0 \to \mathbb{R}$ such that

$$|f(P) - a(P)| \le 5(\log_2 n + 1), \qquad P \in \Gamma_0.$$

Extending $a$ by homogeneity we obtain a linear map $h$ on the space $\Sigma_n$ so that

$$|f(P) - h(P)| \le 5(\log_2 n + 1)\, \mathrm{tr}\, P$$

if $P \ge 0$. For $x \in \mathbb{R}^n$ let $x \otimes x$ be the rank-one operator $y \to \langle y, x \rangle x$. Then $q(x) = h(x \otimes x)$ is a quadratic form and so there is a symmetric operator $S$ so that $q(x) = \langle x, Sx \rangle$ for $x \in \mathbb{R}^n$. Now $f(x \otimes x) = \varphi(x)$ and so we conclude that (2.1) holds.

To complete the proof we consider $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and define $\Omega'(\mathbf{x}) = (0, \Omega(x_1))$ if $\mathbf{x} = (x_1, x_2)$. By (2.1) we find a symmetric $2n \times 2n$ matrix $B$ so that

$$|\langle \mathbf{x}, \Omega'(\mathbf{x}) - B(\mathbf{x}) \rangle| \le 5(\log_2 n + 2)\|\mathbf{x}\|^2.$$

Let

$$B(\mathbf{x}) = (B_{11}x_1 + B_{12}x_2, B_{21}x_1 + B_{22}x_2)$$

where $B_{jk}$ are $n \times n$ matrices for $1 \le j, k \le 2$. Then if $\epsilon_1, \epsilon_2 = \pm 1$ we obtain

$$\left| \langle x_2, \Omega(x_1) \rangle - \epsilon_1 \epsilon_2 \sum_{j,k \le 2} \epsilon_j \epsilon_k \langle x_j, B_{jk} x_k \rangle \right| \le 5(\log_2 n + 2)(\|x_1\|^2 + \|x_2\|^2).$$

Averaging over choices of sign and noting that $B_{12} = B_{21}^t$ we obtain (if $A = 2B_{21}$)

$$|\langle x_2, \Omega(x_1) - Ax_1 \rangle| \le 5(\log_2 n + 2)(\|x_1\|^2 + \|x_2\|^2), \qquad x_1, x_2 \in \mathbb{R}^n.$$

Taking $\|x_1\| = \|x_2\| = 1$ and using homogeneity, this leads to the inequality

$$\|\Omega(x) - Ax\| \le 10(\log_2 n + 2)\|x\|, \qquad x \in \mathbb{R}^n.$$

This completes the proof.                                                          $\square$

The following lemma follows from Lemma 2.5 of Matouskova [8]; a more general lemma for arbitrary Banach spaces has been proved by Vestfrid [11].

**Lemma 2.4.** *Suppose $f : B_n \to \mathbb{R}^n$ is an $\epsilon$-quasi-isometry where $\epsilon < 10^{-2}$. Then if $\|x_1\|, \|x_2\| \le \frac{1}{2}$,*

$$\|f(\tfrac{1}{2}(x_1 + x_2)) - \tfrac{1}{2}(f(x_1) + f(x_2))\| \le 65\epsilon\|x_1 - x_2\|.$$

**Lemma 2.5.** *Let $g : [0,1] \to \mathbb{R}$ be a continuous function which satisfies*

$$(2.2) \qquad |g(t) - \tfrac{1}{2}(g(t+s) + g(t-s))| \le s, \qquad 0 < s \le \min(t, \tfrac{1}{2}(1-t)).$$

*Then*

$$\max_{0 \le t \le 1} |g(t)| \le 14 + 12 \max_{0 \le t \le \frac{1}{2}} |g(t)|.$$

*Proof.* Let $M = \max_{0 \le x \le \frac{1}{2}} |g(x)|$. It will suffice to show that $g(t) \le 12M + 14$ since then one can obtain the same estimate for $-g$.

Let $h(t) = (1-t)\log(1-t)$ (with $h(1) = 0$). Note $h$ is continuous and $-\frac{1}{2} \le h(t) \le 0$ for $0 \le t \le 1$. We also observe that

$$\frac{3}{2}t \log \frac{3}{2}t + \frac{1}{2}t \log \frac{1}{2}t = 2t \log t + t \log \frac{3\sqrt{3}}{4}.$$

Then if $\frac{1}{3} \leq t \leq 1$ we have

(2.3) $$\tfrac{1}{2}(h(\tfrac{1}{2}(3t-1)) + h(\tfrac{1}{2}(1+t))) \geq h(t) + \tfrac{1}{8}(1-t)$$

since $\log \frac{3\sqrt{3}}{4} > \frac{1}{4}$. Note that if $g$ satisfies (2.2),

(2.4) $$\tfrac{1}{2}(g(\tfrac{1}{2}(3t-1)) + g(\tfrac{1}{2}(1+t))) \geq g(t) - \tfrac{1}{2}(1-t).$$

It follows from (2.3) and (2.4) that if $g$ satisfies (2.2), then $g + 4h$ cannot have a maximum for $\frac{1}{3} \leq t < 1$. This implies that

(2.5) $$g(t) \leq \max(\max_{0 \leq x \leq \frac{1}{3}} g(x), g(1)) + 2, \qquad 0 \leq t \leq 1.$$

If $g(1) \leq 0$, then we immediately obtain $g(t) \leq M + 2$ for $0 \leq t \leq 1$ and the proof is complete. If $g(1) > 0$, we define $f(t) = tg(1) - g(t)$ and note that $f$ also satisfies (2.2). For $0 \leq t \leq \frac{1}{3}$ we have $f(t) \leq M + \frac{1}{3}g(1)$ and we also have $f(1) = 0$. By (2.5)

$$f(t) \leq \tfrac{1}{3}g(1) + M + 2.$$

Letting $t = \frac{1}{2}$ we have

$$\tfrac{1}{6}g(1) \leq g(\tfrac{1}{2}) + M + 2 \leq 2M + 2.$$

Hence $g(1) \leq 12M + 12$ and we now use (2.5) to complete the proof. $\qquad \square$

**Theorem 2.6.** *There is a universal constant $C$ with the following property. Let $f : B_n \to \mathbb{R}^n$ be an $\epsilon-$quasi-isometry where $0 < \epsilon < 1$. Then there exists $a \in \mathbb{R}^n$ and an orthogonal linear map $U$ with*

$$\|f(x) - Ux - a\| \leq C\epsilon(\log n + 1)$$

*for $x \in B_n$.*

*Proof.* It suffices to consider the case $\epsilon < 10^{-2}$. We may assume $f(0) = 0$. Let us then define

$$\Omega(x) = \|x\| \left( f\left(\frac{x}{2\|x\|}\right) - f\left(-\frac{x}{2\|x\|}\right) \right), \qquad x \in \mathbb{R}^n.$$

$\Omega$ is clearly homogeneous. Next suppose $\|x\| = 1$ and $\|y\| = 1$. Define $h(t) = \langle f(tx) - \Omega(tx), y \rangle$ for $-1 \leq t \leq 1$. Note that by Lemma 2.4, applied to the ball $tx + (1 - |t|)B_X$, we have

$$|h(t) - \tfrac{1}{2}(h(t+s) + h(t-s))| \leq 200\epsilon s, \qquad 0 < s \leq \min(|t|, \tfrac{1}{2}(1-|t|)).$$

In particular if we apply the Whitney Lemma ([12]) there is an affine function $a$ so that $|h(t) - a(t)| \leq 100\epsilon$ for $-\frac{1}{2} \leq t \leq \frac{1}{2}$. Now

$$f(\pm\tfrac{1}{2}x) - \Omega(\pm\tfrac{1}{2}x) = \tfrac{1}{2}(f(\tfrac{1}{2}x) + f(-\tfrac{1}{2}x)).$$

Then $h(\frac{1}{2}) = h(-\frac{1}{2})$ and by Lemma 2.4, $|h(\frac{1}{2})| \leq 65\epsilon$. Thus

$$|a(t)| \leq |h(\tfrac{1}{2})| + 100\epsilon \leq 200\epsilon$$

for $0 \leq t \leq \frac{1}{2}$. Hence

$$|h(t)| \leq 300\epsilon, \qquad |t| \leq \tfrac{1}{2}.$$

Now by Lemma 2.5 (used on both $[0,1]$ and $[-1,0]$) we obtain

$$|h(t)| \leq C_0\epsilon, \qquad |t| \leq 1,$$

where $C_0 < 10^4$. We conclude that

$$\|f(x) - \Omega(x)\| \le C_0 \epsilon, \qquad \|x\| \le 1.$$

Now if $x_1, x_2 \in \mathbb{R}^n$ we define $b = 2(\|x_1\| + \|x_2\|)$ and then note that

$$\|\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2)\|$$
$$= b\|\Omega(b^{-1}(x_1 + x_2)) - \Omega(b^{-1}x_1) - \Omega(b^{-1}x_2)\|$$
$$\le C_0 \epsilon b + b\|2f(\tfrac{1}{2}(b^{-1}(x_1 + x_2))) - f(b^{-1}x_1) - f(b^{-1}x_2)\|$$
$$\le C_0 \epsilon b + 200\epsilon\|x_1 - x_2\| \le C_1 \epsilon(\|x_1\| + \|x_2\|)$$

where $C_1 = 2C_0 + 200$.

Now we can apply Theorem 2.2 to deduce the existence of a linear map $A$ so that

$$\|\Omega(x) - Ax\| \le C_2(1 + \log n)\epsilon\|x\|, \qquad x \in \mathbb{R}^n,$$

for some absolute constant $C_2$. If $\|x\| \le 1$ we have

$$\|f(x) - Ax\| \le (C_2(1 + \log n) + C_0)\epsilon$$

and hence if $\|x\| = 1$ we have

$$1 - C_3(\log n + 1)\epsilon \le \|Ax\| \le 1 + C_3(\log n + 1)$$

where $C_3$ is a universal constant. This implies there is an orthogonal transformation $U$ with $\|A - U\| \le C_3(\log n + 1)$. Then $U$ satisfies the conclusion of the theorem. $\qquad \square$

## 3. REMARKS

In fact Theorem 2.2 can be reached from known results in Banach space theory using the theory of twisted sums (and more specifically twisted Hilbert spaces). We sketch the ideas. One can use the map $\Omega$ to define a $(2n)$-dimensional Banach space $X$ with a Euclidean subspace $E$ with $\dim E = n$ so that $X/E$ is also Euclidean. It then follows from results of Figiel, Lindenstrauss and Milman [5] (see also [4]) that $X$ has type 2 constant $T_2(X) \le C(1 + \log n)$. But then Maurey's extension theorem [9] produces a projection $P : X \to E$ with $\|P\| \le T_2(X)$. This projection induces the linear approximation $A$ in the standard way.

We would like to take this opportunity to resolve a question raised in [5] concerning such twisted sums. In [5], Figiel, Lindenstrauss and Milman ask for an estimate on the Banach-Mazur distance $d_X = d(X, \ell_2^{2n})$ when $X$ is as above (i.e. $\dim X = 2n$ and $X$ has a subspace $E$ of dimension $n$ with $E$ and $X/E$ isometrically Euclidean). They note that in [4] an example is given with $d_X \ge c(\log n)^{\frac{1}{2}}$ and they obtain an upper estimate of the type $C(\log n + 1)^2$. Later in [7] an example was produced with $d_X \ge c \log n$. We now point out that this is sharp, as the upper estimate can be improved to $C(\log n + 1)$. Curiously this requires nothing other than the results of [5]. The bound established in [5] uses Kwapien's theorem and a logarithmic estimate on both type and cotype; by using Maurey's theorem one eliminates a logarithmic factor.

**Theorem 3.1.** *There is a universal constant $C$ such that if $X$ is a Banach space of dimension $2n$ with a subspace $E$ of dimension $n$ with $E, X/E$ isometrically Euclidean, then $d(X, \ell_2^{2n}) \le C(1 + \log n)$.*

*Proof.* As remarked above $T_2(X) \leq C(1 + \log n)$ for some universal constant $C$ (see Theorem 6.5 of [5]). By Maurey's extension theorem [9] there is a projection $P : X \to E$ with $\|P\| \leq T_2(X)$. Let $Q$ be the quotient map $Q : X \to X/E$. Define a Euclidean norm on $X$ by

$$\|x\|_H^2 = \tfrac{1}{2}(\|P\|^{-2}\|Px\|^2 + \|Qx\|^2).$$

Then $\|x\|_E \leq \|x\|$. Conversely assume $\|x\| \geq 8\|P\|\|x\|_H$. Then $\|Qx\| \leq \tfrac{1}{4}\|P\|^{-1}\|x\|$. Pick $e \in E$ so that $\|x - e\| = \|Qx\|$. Then

$$\|Px\| \geq \|e\| - \|P(x - e)\| \geq \|x\| - \|x - e\|(1 + \|P\|).$$

Hence

$$\|Px\| \geq \frac{3\|P\| - 1}{4\|P\|}\|x\| \geq \frac{1}{2}\|x\|$$

and

$$\|x\|_H \geq \frac{1}{2\|P\|}\|x\|,$$

which is a contradiction. Thus

$$\|x\|_H \leq \|x\| \leq 8T_2(X)\|x\|$$

and the proof is complete.                                                              $\square$

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