ROOTS OF COMPLEX POLYNOMIALS
AND WEYL-HEISENBERG FRAME SETS

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ABSTRACT. A Weyl-Heisenberg frame for $L^2(\mathbb{R})$ is a frame consisting of modulates $E_{mb}g(t) = e^{2\pi i mbt}g(t)$ and translates $T_{na}g(t) = g(t - na)$, $m, n \in \mathbb{Z}$, of a fixed function $g \in L^2(\mathbb{R})$, for $a, b \in \mathbb{R}$. A fundamental question is to explicitly represent the families (g, a, b) so that $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. We will show an interesting connection between this question and a classical problem of Littlewood in complex function theory. In particular, we show that classifying the characteristic functions χ_E for which $(E_m T_n \chi_E)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ is equivalent to classifying the integer sets $\{n_1 < n_2 < \cdots < n_k\}$ so that $f(z) = \sum_{j=1}^{k} z^{n_i}$ does not have any zeroes on the unit circle in the plane.

1. INTRODUCTION

A family of vectors (f_i) in a Hilbert space H is called a **frame** for H if there are constants $A, B > 0$ so that

$$(1.1) \qquad A\|f\|^2 \leq \sum_i |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \text{ for all } f \in H.$$

We call A (resp. B) a **lower** (resp. **upper**) **frame bound** of the frame. The largest A and the smallest B which work in (1.1) above are called the **optimal frame bounds**. If P is an orthogonal projection on H, and (f_i) is a frame for H with frame bounds A, B, then for all $f \in PH$ we have

$$(1.2) \qquad A\|f\|^2 \leq \sum_i |\langle f, f_i \rangle|^2 = \sum_i \langle f, Pf_i \rangle|^2 \leq B\|f\|^2.$$

It follows that (Pf_i) is a frame for PH with the same frame bounds A, B.

An important class of frames used in signal/image processing, data compression etc. are the Weyl-Heisenberg frames. For $f \in L^2(\mathbb{R})$ and a, b real numbers we define **translation by a** (resp. **modulation by b**) by $T_a f(t) = f(t - a)$ (resp. $E_b f(t) = e^{2\pi i bt}f(t)$). For a fixed $g \in L^2(\mathbb{R})$, we say that (g, a, b) generates a **Weyl-Heisenberg frame** (WH-frame for short) if $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$. A fundamental question in this area is to explicitly represent those families (g, a, b) which generate Weyl-Heisenberg frames for $L^2(\mathbb{R})$. Much work has been done on this question by Ron and Shen [13, 14], Janssen [10], Casazza, Christensen

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and Janssen [3], Casazza and Lammers [4], and a host of other authors. The papers [4, 10] concentrate on the question of finding the characteristic functions χ_E so that (χ_E, a, b) generates a Weyl-Heisenberg frame. Janssen [10] has made quite a detailed study of this question, and one consequence of his work is that even this very special case of the general problem is quite a deep question. It is known [7] that $(g, 1, 1)$ generates a WH-frame if and only if $(E_m T_n g)$ is a Riesz basis for $L^2(\mathbb{R})$. In this paper we will show that the question of classifying the characteristic functions χ_E so that $(\chi_E, 1, 1)$ generates a Weyl-Heisenberg frame is equivalent to a classical problem of Littlewood in complex function theory. This shows that this question in WH-frame theory is even more difficult than previously thought, as well as giving important connections between frame theory and complex function theory.

Now we will pass to Littlewood's problem. In 1968 Littlewood [11] studied the class of functions $\mathcal{A}_n$ which consists of polynomials of the form $\sum_{i=1}^n a_i z^i$, with $a_i \in \{0, 1\}$. On page 25 of [11] he writes: "These raise some fascinating questions." One of the main questions raised by Littlewood is:

Problem 1.1 (Littlewood)**.** Classify the integer sets $\{n_1 < n_2 < \cdots < n_k\}$ so that $f(z) = \sum_{j=1}^k z^{n_i}$ does not have any zeroes on the unit circle in the plane.

There is a huge literature on the zeroes of polynomials in $\mathcal{A}_n$ as well as various other related classes (e.g. where the coefficients come from the set $\{-1, 0, 1\}$). For an up to date view of this subject, we refer the reader to [1, 2, 12] and their references.

2. Main results

We need the Zak transform (called the kq-representation by Zak and also called the Weil-Brezin map in the literature) but brought to the level of an "art form" by Janssen (see [5, 8, 9]). We define the **Zak transform** to be the unitary mapping $Z(\cdot)$ from $L^2(\mathbb{R})$ onto $L^2([0, 1]^2)$ that takes the orthonormal basis $\{E_m T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}}$ to the orthonormal basis $\{e^{2\pi i(nx+my)}\}_{n,n \in \mathbb{Z}}$. This is not the usual definition of the Zak transform, but is an equivalent formulation [7]. It is known (see [7, 8, 9]) that

$$(2.1) \qquad Zg(x, y) = \sum_{n \in \mathbb{Z}} g(x + n)e^{2\pi i n y}, \quad \text{for all} \;\; x, y \in [0, 1],$$

and

$$(2.2) \qquad Z(E_m T_n g)(x, y) = e^{2\pi i(nx+my)} Zg(x, y).$$

Now we have the following (somewhat well-known) result.

Proposition 2.1. *Let E be a measurable subset of $[0, 1]$, $F = \bigcup_{n \in \mathbb{Z}} (E + n)$ and $g \in L^2(F)$. The following are equivalent:*
(1) $(E_m T_n g)_{m,n \in \mathbb{Z}}$ is a frame for $L^2(F)$ with optimal frame bounds A, B.
(2) We have

$$0 < A = \operatorname*{ess\,inf}_{(x,y) \in E \times [0,1]} |Zg(x, y)|^2$$

$$\leq \operatorname*{ess\,sup}_{(x,y) \in E \times [0,1]} |Zg(x, y)|^2 = B < \infty.$$

Proof. Since $Z(\cdot)$ is a unitary operator, the frame bounds for $(E_m T_n g)$ are the same as the frame bounds for $(e^{2\pi i(nx+my)} Zg(x,y))$. Also, by equation (2.1) and the definition of E, F, we have

$$(2.3) \qquad Zg(x,y) = 0, \quad \text{for all } x \notin E.$$

Since $(e^{2\pi i(nx+my)})_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1]^2)$, we have for any $F(x,y) \in L^2([0,1]^2)$,

$$(2.4) \qquad \begin{aligned} &\sum_{m,n\in\mathbb{Z}} |\langle F(x,y), e^{2\pi i(nx+my)} Zg(x,y)\rangle|^2 \\ &= \|F(x,y)\overline{Zg(x,y)}\|^2_{L^2(E\times[0,1])}. \end{aligned}$$

It follows that the upper (resp. lower) frame bound for $(e^{2\pi i(nx+my)} Zg(x,y))$ is the square of the norm of the multiplication operator (resp. the inverse of the multiplication operator) $F \to F \cdot Zg$ on $L^2(E \times [0,1])$. A direct calculation shows that these norms are precisely the bounds given in (2) of the proposition. $\square$

We now have

Proposition 2.2. *Let $E_i \subset [0,1]$, $F_i = \bigcup_{n\in\mathbb{Z}}(E_i + n)$, $F = \bigcup_i F_i$, and assume $F_i \cap F_j = \emptyset$, for all $i \neq j$. Let g_i be a function supported on F_i and assume that $g = \sum_i g_i \in L^2(\mathbb{R})$. The following are equivalent:*
(1) $(E_m T_n g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(F)$ with optimal frame bounds A, B.
(2) For each i, $(E_m T_n g_i)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(F_i)$ with optimal frame bounds A_i, B_i, and $0 < A = \inf A_i \leq \sup B_i = B < \infty$.

Proof. (1) $\Rightarrow$ (2): Let P_i be the orthogonal projection of $L^2(F)$ onto $L^2(F_i)$ given by $P_i f = f|_{F_i}$. Now, $P_i g = g_i$, and, as we observed in the introduction, $(g_i, 1, 1)$ generates a WH-frame for $L^2(F_i)$ with frame bounds A, B. Hence, $A \leq A_i \leq B_i \leq B$.

(2) $\Rightarrow$ (1): By Proposition 2.1 and our assumptions in (2), for each i we have

$$A \leq A_i \leq \operatorname*{ess\,inf}_{(x,y)\in F_i\times[0,1]} |Zg_i(x,y)|^2$$

$$\leq \operatorname*{ess\,sup}_{(x,y)\in F_i\times[0,1]} |Zg_i(x,y)|^2 \leq B_i \leq B.$$

Now, $Zg = \sum_i Zg_i$ and by our assumption that $F_i \cap F_j = \emptyset$, for all $i \neq j$, we have

$$\text{support } Zg_i \cap \text{support } Zg_j = \emptyset, \quad \text{for all } i \neq j.$$

It follows that

$$\operatorname*{ess\,sup}_{(x,y)\in F\times[0,1]} |Zg(x,y)|^2 = \sup_i \operatorname*{ess\,sup}_{(x,y)\in F_i\times[0,1]} |Zg_i(x,y)|^2 = \sup_i B_i = B.$$

Similarly,

$$A = \inf_i A_i = \operatorname*{ess\,inf}_{(x,y)\in F\times[0,1]} |Zg(x,y)|^2.$$

It follows that $(g, 1, 1)$ generates a WH-frame for $L^2(F)$ with frame bounds A, B by Proposition 2.1 $\square$

Now we have a constructive characterization of certain Weyl-Heisenberg frame sets for $a = b = 1$. To simplify the notation, we call a measurable subset $F \subset \mathbb{R}$ a **Weyl-Heisenberg frame set for** (a, b) if (χ_F, a, b) generates a Weyl-Heisenberg frame for $L^2(\mathbb{R})$.

Theorem 2.3. *Fix integers $n_1 < n_2 < \cdots < n_k$. The following are equivalent:*

(1) The set $F = \bigcup_{j=1}^{k}([0,1) + n_j)$ is a Weyl-Heisenberg frame set for $(1,1)$ with frame bounds A, B.

(2) We have $A \leq |\sum_{j=1}^{k} z^{n_j}|^2 \leq B$, for all $|z| = 1$.

(3) For every measurable set $E \subset [0,1]$ of positive measure, and for the set $F_0 = \bigcup_{j=1}^{k}(E + n_j)$, $(E_m T_n \chi_{F_0})_{m,n \in \mathbb{Z}}$ is a frame for $L^2(F_0)$ with frame bounds A, B.

Proof. In Proposition 2.1, if $g = \chi_F$, then

$$Zg(x,y) = \chi_{[0,1]}(x) \sum_{j=1}^{k} e^{2\pi i n_j y} = \sum_{j=1}^{k} e^{2\pi i n_j y}, \quad \text{for all} \ \ x, y \in [0,1].$$

Hence,

$$A \leq \operatorname*{ess\,inf}_{(x,y) \in [0,1]^2} |Zg(x,y)|^2 \leq \operatorname*{ess\,sup}_{(x,y) \in [0,1]^2} |Zg(x,y)|^2 \leq B,$$

if and only if for all $|z| = 1$,

$$A \leq |\sum_{j=1}^{k} z^{n_j}|^2 \leq B.$$

This proves the equivalence of (1) and (2), which is clearly implied by (3).

We now show that (1) implies (3). By (1), $(\chi_F, 1, 1)$ generates a WH-frame for $L^2(F)$ with frame bounds A, B. Let P be the orthogonal projection of $L^2(F)$ onto $L^2(F_0)$ given by $Pf = f|_{F_0}$. Then $P\chi_F = \chi_{F_0}$, and, as we have observed in the introduction, it follows that $(\chi_{F_0}, 1, 1)$ generates a WH-frame for $L^2(F_0)$ with frame bounds A, B. $\qquad\qquad\square$

We call a measurable set $F \subset \mathbb{R}$ an **elementary A-Weyl-Heisenberg frame set of length k** if $F = \bigcup_{j=1}^{k}(E + n_j)$ for some (n_j) and some measurable subset E in $[0,1)$ of positive measure and we have

$$A \leq \inf_{|z|=1} |\sum_{j=1}^{k} z^{n_j}|^2.$$

We end with our classification of all WH-frame sets for $a = b = 1$. We first note that for $F \subset \mathbb{R}$, functions in the span of $(E_m T_n g)_{m,n \in \mathbb{Z}}$ have their support in $\bigcup_{n \in \mathbb{Z}}(F + n)$. Hence, a necessary condition for F to be a WH-frame set for $(1,1)$ is that $|\mathbb{R} - \bigcup_n (F + n)| = 0$.

Theorem 2.4. *Let F be a subset of $\mathbb{R}$ for which $|\mathbb{R} - \bigcup_{n \in \mathbb{Z}}(F + n)| = 0$ a.e. The following are equivalent:*

(1) The set F is a Weyl-Heisenberg frame set for (1,1).

(2) There are constants $k, A > 0$ so that $F = \bigcup_{i \in I} F_i$ (I is finite or infinite), where each F_i is an elementary A-Weyl-Heisenberg frame set of length $\leq k$ and $(F_i + n) \cap (F_j + m) = \emptyset$ for all $i \neq j$ and all $m, n \in \mathbb{Z}$.

Proof. (1) $\Rightarrow$ (2): Let F be a WH-frame set. For each $j \in \mathbb{N}$ let $F_j = \{x \in [0,1) : \#(x+\mathbb{N}) \cap F\# = j\}$, where $\#$ denotes cardinality. A consequence of the WH-frame identity (see [7], Theorem 4.1.2, p. 648) is that $\sum_n |\chi_F(x+n)|^2 \leq B$ a.e. Hence,

there is a $k \in \mathbb{N}$ so that $|F_j| = 0$, for all $j > k$. Now for $j \leq k$ and any finite set of integers $\zeta = \{n_1 < n_2 < \cdots < n_j\}$ let $F_{j,\zeta} = \{x \in F_j : x + n_\ell \in F, \ 1 \leq \ell \leq j\}$. The distinct non-empty $(F_{j,\zeta})$ form a countable family of sets which are pairwise disjoint under translation by $n \in \mathbb{Z}$ and which by Proposition 2.2 satisfies (2).

(2) $\Rightarrow$ (1): Fix i and consider $(\chi_{F_i}, 1, 1)$. Since F_i is an elementary A-WH-frame set of length k, there are a $m \leq k$ and a set $E_i \subset [0,1]$ with $F_i = \bigcup_{j=1}^{m}(E_i + n_j)$ for some $n_1 < n_2 < \cdots < n_m$ so that for all $|z| = 1$ we have

$$A \leq |\bigcup_{j=1}^{m} z^{n_j}|^2 \leq m \leq k.$$

By Theorem 2.3, $(\chi_{F_i}, 1, 1)$ generates a WH-frame with frame bounds A, k. Since the (F_i) are disjoint, we have $\chi_F = \sum_i \chi_{F_i}$. Also, $(F_i + n) \cap (F_j + m) = \emptyset$, for all $i \neq j$, $m, n \in \mathbb{Z}$, implies that $E_i \cap E_j = \emptyset$ for all $i \neq j$. Hence,

$$|F| = \sum_i |F_i| \leq k \sum_i |E_i| \leq k,$$

and it follows that $\chi_F \in L^2(\mathbb{R})$. Now, by Proposition 2.2, $(\chi_F, 1, 1)$ is a frame for $L^2(F)$ with frame bounds A, k. $\qquad \square$

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