**Reprinted from**

# RECENT PROGRESS IN FUNCTIONAL ANALYSIS

Proceedings of the International Functional Analysis Meeting on the Occasion of the 70th Birthday of Professor Manuel Valdivia, Valencia, Spain, July 3–7, 2000


Edited by

Klaus D. BIERSTEDT
*University of Paderborn*
*Paderborn, Germany*

José BONET
*Technical University of Valencia*
*Valencia, Spain*

Manuel MAESTRE
*University of Valencia*
*Valencia, Spain*

Jean SCHMETS
*University of Liège*
*Liège, Belgium*



2001
ELSEVIER
Amsterdam - London - New York - Oxford - Paris - Shannon - Tokyo

Recent Progress in Functional Analysis
K.D. Bierstedt, J. Bonet, M. Maestre and J. Schmets



# Applications of Banach space theory to sectorial operators

**Nigel Kalton** *

Department of Mathematics
University of Missouri
Columbia
Missouri 65211

To Professor Manuel Valdivia on the occasion of his seventieth birthday.

**Abstract**

*We will discuss recent joint work of the author with Gilles Lancien and Lutz Weis on the theory of sectorial operators. Our presentation is informal and we hope to show how classical Banach space theory finds applications in this area.*
MCS 2000 Primary 47A60; Secondary 47D06

## 1. Introduction

In this survey we will try to show how Banach space methods can be used in the study of sectorial operators. In particular we will show that the *maximal regularity problem* can be considered as a variant of the *complemented subspace problem* solved thirty years ago by Lindenstrauss and Tzafriri [23]. We then show how the ideas developed in resolving this problem lead to a new approach to the theory of operators with an $H^\infty$-calculus initiated by McIntosh [27].

In general, sectorial operators have a very nice and complete theory on Hilbert spaces. The problem is always to try to extend the theory to more general Banach spaces. Even for applications in partial differential equations it is very natural to, at least, consider the classical spaces $L_p$ when $p \neq 2$. It is exactly in considering such problems that Banach space theory has much to offer, because much of the classical theory has the same basic theme of attempting to find what parts of Hilbert space theory can be carried over to Banach spaces.

We will write this survey from the point-of-view of a Banach space specialist and our aim is partly to advertise the interesting results that can be achieved when Banach space methods are applied externally to other (related) fields.

*The author was supported by NSF grant DMS-9870027

## 2. Sectorial operators

Let us introduce some notation. Let $X$ be a complex Banach space, and let $A$ be a closed operator on $X$ with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$. We say that $A$ is *sectorial* if $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense, $A$ is one-one and there exists $0 < \phi < \pi$ and a constant $C$ such that if $\lambda \in \mathbf{C} \setminus \{0\}$ with $|\arg \lambda| > \phi$ then the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ is bounded and $\|\lambda R(\lambda, A)\| \le C$. We can then define the angle of sectoriality $\omega(A)$ as the infimum of all angles $\phi$ for which the above statement holds; clearly $0 \le \omega(A) < \pi$.

Let us observe at this point an important property of sectorial operators. The resolvent $R(\lambda, A)$ is certainly defined for $\lambda$ on the negative real axis. Let $S_t = -tR(-t, A) = t(t + A)^{-1}$ for $t > 0$. Then $S_t$ is uniformly bounded. Next note that if $x = Ay$ then

$$\lim_{t\to 0} \|S_t x\| = \lim_{t\to 0} t\|A(t + A)^{-1} y\| = 0$$

while if $x \in \mathcal{D}(A)$

$$\lim_{t\to\infty} \|x - S_t x\| = \lim_{t\to\infty} \|(t + A)^{-1} Ax\| = 0.$$

Since both the domain and range are dense this means that $\lim_{t\to 0} S_t x = 0$ and $\lim_{t\to\infty} S_t x = x$ for all $x \in X$.

Suppose $A$ is sectorial with angle $\omega = \omega(A)$. Then we can define a functional calculus for $A$ as follows. Suppose $f$ is bounded and analytic on a sector $\Sigma_\phi := \{z : |\arg z| < \phi\}$ where $0 < \omega < \phi$. Suppose $\omega < \nu < \phi$ and consider the contour $\Gamma_\nu = \{t \exp(i\nu \operatorname{sgn} t) : -\infty < t < \infty\}$. Then we can formally "define"

$$f(A)x = -\frac{1}{2\pi i} \int_{\Gamma_\nu} f(\zeta) R(\zeta, A) x \, d\zeta. \tag{2.1}$$

Of course (2.1) is not well-defined without some restrictions on $f$. The most natural is that

$$\sup_{\zeta\in\Sigma_\phi} \int_0^\infty |f(t\zeta)| \frac{dt}{t} < \infty \tag{2.2}$$

as this implies that the integral in (2.1) is convergent as a Bochner integral for all $x \in X$, and that $f(A)$ is bounded with

$$\|f(A)\| \le C \sup_{\zeta\in\Sigma_\phi} \int_0^\infty |f(t\zeta)| \frac{dt}{t} \tag{2.3}$$

where $C$ depends only on $A$ and $\phi$.

It is often easier to require that $f$ satisfies an estimate of the form $|f(z)| \le C|z|^\delta (1 + |z|^2)^{-\delta}$ for $z \in \Sigma_\phi$ where $\delta > 0$. The set of such functions is denoted $H_0^\infty(\Sigma_\phi)$. It is easily established that the map $f \to f(A)$ is an algebra homomorphism on $H_0^\infty(\Sigma_\phi)$. Now if we set

$$\varphi_n(z) = \frac{(n^2 - 1)z}{(1 + nz)(n + z)}$$

then $\varphi_n(A) = S_n - S_{1/n}$. It is then possible to show that if $f \in H^\infty(\Sigma_\phi)$ that $\lim_{n\to\infty} (\varphi_n f)(A)x$ exists for all $x \in X$ if and only if $\sup_n \|(\varphi_n f)(A)\| < \infty$. It is then possible to define $f(A)x$ unambiguously as $\lim_{n\to\infty} (\varphi_n f)(A)x$.

To summarize let us define $\mathcal{H}(A)$ to be the space of all (germs of) functions $f$ which are analytic and bounded on some sector $\Sigma_\phi$ where $\phi > \omega(A)$ and such that $\sup_n \|(\varphi_n f)(A)\| < \infty$. Then $\mathcal{H}(A)$ is an algebra and we have constructed an algebra homomorphism $f \to f(A)$ from $\mathcal{H}(A) \to \mathcal{L}(X)$. It may also be shown that if $(f_n)$ is a sequence in $H^\infty(\Sigma_\phi)$, where $\phi > \omega$, such that $\sup_n \|f_n\|_{H^\infty}(\Sigma_\phi) < \infty$ and $f_n(z) \to f(z)$ for all $z \in \Sigma_\phi$ then the conditions $f_n \in \mathcal{H}(A)$ and $\sup_n \|f_n(A)\| < \infty$ imply that $f \in \mathcal{H}(A)$ and $f_n(A)x \to f(A)x$ for all $x \in X$.

Notice that the space $\mathcal{H}(A)$ contains all functions $(\lambda - z)^{-1}$ for $|\arg \lambda| > \omega$ and also $H_0^\infty(\Sigma_\phi)$ for $\phi > \omega$. In particular if $\omega < \pi/2$ then we can see that $e^{-wz} \in \mathcal{H}(A)$ for $|\arg w| + \omega < \pi/2$ since $e^{-wz} - wz(1+wz)^{-1} \in H_0^\infty(\Sigma_\phi)$ whenever $|\arg w| + \phi < \pi/2$. In fact if $\psi + \omega < \pi/2$ the set of operators $\{e^{-wA} : w \in \Sigma_\psi\}$ is uniformly bounded and hence we have:

**Proposition 2.1** *If $A$ is sectorial with $\omega(A) < \frac{\pi}{2}$ then $-A$ is the generator of a bounded analytic semigroup.*

Conversely suppose $-A$ is the generator of a bounded strongly continuous semigroup. Then the equation

$$R(\lambda, A) = \int_0^\infty e^{\lambda t} e^{-tA} dt$$

shows that $A$ is sectorial with $\omega(A) \leq \frac{\pi}{2}$.

We say that $A$ has an $H^\infty$*−calculus* or is $H^\infty$*−sectorial* if there exists $\phi < \pi$ so that $H^\infty(\Sigma_\phi) \subset \mathcal{H}(A)$. Then we set $\omega_H(A) = \inf\{\phi : H^\infty(\Sigma_\phi) \subset \mathcal{H}(A)\}$. This notion was originally introduced by McIntosh [27] for operator son Hilbert spaces, and later an extensive study was undertaken in [9].

Note that $A$ has an $H^\infty$−calculus then in particular it has *bounded imaginary powers (BIP)* i.e. $A^{it}$ is a bounded operator for all real $t$. Indeed it is further true that if $\phi > \omega_H(A)$ then $A$ satisfies the estimate $\|A^{it}\| \leq Ce^{\phi t}$. If $X$ is a Hilbert space then (BIP) is actually equivalent to an $H^\infty$-calculus, but this is false for the spaces $L_p$ when $p \neq 2$ [9].

*Example 1.* The primary motivating example to consider here is the case of the differentiation operator $Af = f'$ on the space $L_p(\mathbf{R})$ where $1 \leq p < \infty$. Here $\mathcal{D}(A) = \{f \in L_p : f' \in L_p\}$. It is easy to see that $-A$ is the generator of the translation semigroup $(e^{-tA}f)(s) = f(s-t)$. Thus $A$ is sectorial and $\omega(A) = \frac{\pi}{2}$. Now if $f \in H^\infty(\Sigma_\phi)$ where $\phi > \frac{\pi}{2}$ then the boundedness of $f(A)$ is equivalent to the boundedness of the Fourier multiplier $f(i\xi)$. It then follows from the Hörmander-Mikhlin conditions that if $1 < p < \infty$ the operator $A$ has an $H^\infty(\Sigma_\phi)$−calculus where $\phi > \frac{\pi}{2}$. Thus $\omega_H(A) = \frac{\pi}{2}$.

*Example 2.* Now consider the vector-valued analogue $Af = f'$ on $L_p(\mathbf{R}; X)$ where $X$ is a Banach space. Then the same arguments show that $A$ is sectorial with $\omega(A) = \frac{\pi}{2}$, but for $1 < p < \infty$, one only obtains an $H^\infty$-calculus in those spaces where the vector-valued analogues of the Hörmander-Mikhlin conditions give boundedness of Fourier multipliers. A Banach space $X$ with this property is called a *(UMD)-space* for *unconditional martingale differences.* This class of spaces was introduced by Burkholder [6], and the Fourier multiplier result we need was proved by McConnell [26]. For our purposes it is simplest to use a characterization of Bourgain [4] to define (UMD)-spaces: a Banach space $X$ has (UMD) if and only for some (and hence for every) $1 < p < \infty$ the vector-valued Hilbert transform is bounded on $L_p(\mathbf{R};X)$. Readers who are not so familiar with concept may like to note the important classical spaces $L_p$ for $1<p<\infty$ are (UMD) spaces.

It is worth reproducing a simple argument to show that if $A$ has an $H^\infty(\Sigma_\phi)$-calculus for some $\phi<\pi$ (or even (BIP)) then indeed $X$ is (UMD). In fact it is enough to consider imaginary powers. In fact the boundedness of $A^{2it}$ implies that the multiplier $m_1(\xi)=\exp(-t\pi\text{sgn }\xi)|\xi|^{2it}$ is bounded on $L_p(X)$. However if $A$ has $H^\infty(\Sigma_\phi)$-calculus then so has $-A$ and the same reasoning leads to the boundedness of the multiplier $m_2(\xi)=\exp(t\pi\text{sgn }\xi)|\xi|^{2it}$. Taking $t=1$ we can deduce the boundedness of the multiplier $m_3(\xi)=|\xi|^{2i}\text{sgn }\xi$ while from $t=-1$ we obtain the boundedness of the multiplier $m_4(\xi)=|\xi|^{-2i}$. Combining gives us the boundedness of the multiplier sgn $\xi$ i.e. the Hilbert transform.

*Example 3.* Now let us consider an example closer in spirit to Banach space theory. Suppose $X$ has a Schauder basis $(e_n)$ and let $(a_n)_{n=1}^\infty$ be an increasing real sequence with $a_1>0$. Let us define

$$A(\sum_{n=1}^\infty c_ne_n)=\sum_{n=1}^\infty a_nc_ne_n.$$

Here the domain of $A$ is the set of $x=\sum_{n=1}^\infty c_ne_n$ so the series $\sum_{n=1}^\infty a_nc_ne_n$ converges. Such examples were first considered by Baillon and Clément [2]. It is easy to show that $A$ is sectorial and $\omega(A)=0$.

If the basis $(e_n)$ is unconditional then $A$ has an $H^\infty$-calculus and $\omega_H(A)=0$. If we take $a_n$ be an interpolating sequence for $H^\infty(\Sigma_\phi)$ where $\phi>0$ then the converse is true; an example is $a_n=2^n$.

It is clear these ideas can be extended to Schauder decompositions. If $(E_n)$ is a Schauder decomposition of $X$ we can define, in an exactly similar way,

$$A(\sum_{n=1}^\infty x_n)=\sum_{n=1}^\infty a_nx_n$$

where $x_n\in E_n$ and $\mathcal{D}(A)$ is again the set where the right-hand series converges,

## 3. The maximal regularity problem

Let us now suppose that $A$ is sectorial with $\omega(A)<\frac{\pi}{2}$. For $0<T<\infty$ consider the Cauchy problem:

$$\frac{dx}{dt}+Ax=h(t)\quad 0\le t\le T\quad x(0)=0. \tag{3.1}$$

This can be formally solved by

$$x(t)=\int_0^t e^{-A(t-s)}h(s)ds. \tag{3.2}$$

Now suppose $h\in L_p([0,T];X)$ where $1<p<\infty$. Then one may easily check that $x\in L_p([0,T];X)$. We say that $A$ has $L_p$*–maximal regularity* if it also follows (for any $T$) that $dx/dt\in L_p(X)$; this clearly equivalent to the $L_p$-boundedness of the operator

$$h\to\int_0^t Ae^{-A(t-s)}h(s)ds.$$

Although this definition apparently depends on $p$ in fact $L_p$-maximal regularity for some $1 < p < \infty$ implies $L_p$-maximal regularity for every $p$ when $1 < p < \infty$; see [13]. It is also not difficult to see that this definition is independent of $T$. We can therefore refer just to the problem of maximal regularity.

Note that we have restricted our problem to a finite interval. It is possible also to consider the case when $T = \infty$ which leads to a slightly stronger form of maximal regularity; let us refer to this as *strong maximal regularity.*

It is an old result of de Simon [11] that if $X$ is a Hilbert space then *every* sectorial operator $A$ with $\omega(A) < \frac{\pi}{2}$ has maximal regularity. Let us say that a Banach space has the *maximal regularity property* or (MRP) if it satisfies this condition that every sectorial operator with $\omega(A) < \frac{\pi}{2}$ has maximal regularity. It is not too difficult to find counter-examples in certain Banach spaces (cf.[21]), but it was conjectured that *at least* the classical Banach spaces $X = L_p$ where $1 < p < \infty$ have (MRP). This conjecture is usually attributed to Brézis (around 1980) although it may have been around before that time. It is based on the fact that for all concrete examples arising from partial differential equations this seems to be the case. Subsequently this conjecture crystallized into the form that any space with (UMD) has (MRP); we will discuss this further below. It should perhaps be remarked that the space $L_\infty$ has (MRP) for somewhat trivial reasons: any sectorial operator which generates a bounded semigroup is already a bounded operator by a theorem of Lotz [25].

Let us now fix $T = 2\pi$ and suppose that $A^{-1}$ is bounded; this second assumption is not necessary but allows us to make a convenient alternative formulation; in fact we can always reduce to this case by replacing $A$ by $I + A$. It is shown in [17] that, under the this assumption, we can transfer the maximal regularity problem to the circle. Effectively we can replace the boundary condition $x(0) = 0$ by the boundary condition:

$$\int_0^{2\pi} x(s)ds = 0.$$

In this case we can expand the solution in a Fourier series:

$$x(t) \sim -\sum_{n \neq 0} R(-in, A)\hat{h}(n)e^{int}.$$

Then maximal regularity becomes equivalent to the boundedness of the operator

$$h \to \sum_{n \neq 0} AR(-in, A)\hat{h}(n)e^{int} \tag{3.3}$$

on the space $L_p(\mathbf{T}; X)$ where $\mathbf{T}$ denotes the unit circle with normalized Haar measure $dt/2\pi$ and

$$\hat{h}(n) = \frac{1}{2\pi}\int_0^{2\pi} h(t)e^{-int}dt.$$

Taking, as we may, $p = 2$, we summarize this in the statement that $A$ has maximal regularity if and only if there is a constant $C$ so that for all finitely non-zero sequences $(x_n)_{n\in\mathbf{Z}}$ we have:

$$\left(\int_0^{2\pi} \|\sum_{n\in\mathbf{Z}} AR(-in, A)x_n e^{int}\|^2 \frac{dt}{2\pi}\right)^{\frac{1}{2}} \le C\left(\int_0^{2\pi} \|\sum_{n\in\mathbf{Z}} x_n e^{int}\|^2 \frac{dt}{2\pi}\right)^{\frac{1}{2}}. \tag{3.4}$$

At this point one can see why de Simon's theorem must hold. We can take $p = 2$. By using Parseval's identity when $X$ is a Hilbert space we have:

$$\|\sum_{n\neq 0} AR(in, A)x_n e^{int}\| = (\sum_{n\neq 0} \|AR(in, A)x_n\|^2)^{\frac{1}{2}}.$$

Thus all we need is uniform boundedness of the operators $\{AR(in, A) : n \neq 0\}$ and this is almost exactly the assumption that $\omega(A) < \frac{\pi}{2}$. This also suggests that such a theorem is really rather unlikely to hold in a general Banach space, because such properties are very special.

Let us now explain the role of the (UMD) assumption. (UMD)-spaces became important in Banach space theory during the 1980's after their introduction by Burkholder. Two results from the 1980's strongly suggested this was the right assumption for maximal regularity.

First suppose $L_p(\mathbf{T}; X)$ has (MRP) where $1 < p < \infty$. Then consider the subspace $L_p^0(\mathbf{T}; X)$ of all functions $h$ with mean zero, i.e. $\hat{h}(0) = 0$. Then we have an example of a sectorial operator $A$ by taking

$$Ah \sim \sum_{n\neq 0} |n|\hat{h}(n)e^{int}$$

on a its natural domain. (We need to restrict to functions of mean zero to prevent $A$ having non-trivial kernel). In fact $\omega(A) = 0$. Now by maximal regularity the operator on $L_p(\mathbf{T}^2; X)$ given by

$$\sum_{m,n} \hat{h}(m,n)e^{ims+int} \to \sum_{m,n} \frac{im}{im + |n|}\hat{h}(m,n)e^{ims+int}$$

is bounded. But if we consider the subspace of functions of the form $h(s - t)$ we quickly see that the Hilbert transform is bounded on $L_p(\mathbf{T}; X)$ i.e. $X$ has (UMD). This example is a version of a result of Coulhon-Lamberton [8].

The second result was the spectacular and important result of Dore-Venni (1987)[14]:

**Theorem 3.1** *Suppose $X$ is a (UMD)-space and $A$ is a sectorial operator whose imaginary powers $A^{it}$ are bounded and satisfy an estimate*

$$\|A^{it}\| \leq Ke^{\theta|t|} \tag{3.5}$$

*where $\theta < \frac{\pi}{2}$. Then $X$ has maximal regularity.*

This is deduced from a more general result which we will discuss later. Let us note that (3.5) is stronger than $\omega(A) < \frac{\pi}{2}$. Thus the maximal regularity conjecture essentially reduces to whether the assumption (3.5) is really necessary.

## 4. Solution of the maximal regularity problem

We will now describe the approach taken by the author and Gilles Lancien in [17] to resolve the maximal regularity problem. The most important observation is that we should look at examples in the spirit of Example 3 of §2, rather than examples arising from partial differential equations, which have a habit of behaving well. In this manner we can essentially transport basis theory in Banach spaces, much of which was developed in the period 1960-1975, and make it immediately applicable in this new setting.

Let us first look more carefully at (3.4). It is immediately tempting to *restrict* the range of summation to some sequence such as $\{2^n\}$ which forms a Sidon set, because then a result of Pisier [29] implies that have an estimate

$$\left(\int_0^{2\pi} \|\sum_{n=1}^{\infty} x_n e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{\frac{1}{2}} \approx (\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n x_n\|^2)^{\frac{1}{2}}$$

where $(\epsilon_n)$ is an independent sequence of Rademacher-type random variables.

We thus see that if $A$ has maximal regularity we have an inequality of the type;

$$(\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n R(i2^n, A) x_n\|^2)^{\frac{1}{2}} \leq C(\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n x_n\|^2)^{\frac{1}{2}}. \tag{4.1}$$

This inequality turns out to have far reaching ramifications, which we shall return to later. In fact for the results of this section we do not really need to use (4.1), but clearly it motivated our work.

Now let us suppose $X$ is a Banach space with (MRP) and suppose $(E_n)$ is a Schauder decomposition of $X$. We can use the idea of Example 3. Let us take $A$ corresponding to the sequence

$$(a_n)_{n=1}^{\infty} = \{1, 2, 2, 4, 4, \cdots\}.$$

Suppose $x_n \in E_n$ is finitely non-zero. Then we have

$$\left(\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n (\frac{2^{n-1}}{2^{n-1} + i2^n} x_{2n-1} + \frac{2^n}{2^n + i2^n} x_{2n})\|^2\right)^{\frac{1}{2}} \leq C(\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n (x_{2n-1} + x_{2n})\|^2)^{\frac{1}{2}}.$$

This inequality clearly implies an estimate

$$(\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n x_{2n}\|^2)^{\frac{1}{2}} \leq C'(\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n (x_{2n-1} + x_{2n})\|^2)^{\frac{1}{2}}. \tag{4.2}$$

This leads to the following conclusion:

**Lemma 4.1** *Let $X$ be a Banach space with (MRP). If $X$ has a Schauder decomposition $(E_n)$ so that the blocked decomposition $(E_{2n_1} + E_{2n})$ is unconditional then the subspace $\sum_{n=1}^{\infty} E_{2n}$ is complemented (and hence $(E_n)$ is unconditional).*

This Lemma follows from (4.2) because unconditionality allows us to estimate

$$\|\sum x_n\| \approx (\mathbf{E}\|\sum_{n=1}^{\infty} \epsilon_n (x_{2n-1} + x_{2n})\|^2)^{\frac{1}{2}}.$$

Once we have Lemma 4.1 the problem reduces to some classical results in Banach space theory. Let us first suppose $X$ has an unconditional basis $(e_n)$ and suppose $(y_n)$ is a normalized block basic sequence i.e.

$$y_n = \sum_{k=r_{n-1}+1}^{r_n} c_k e_k$$

where $0 = r_0 < r_1 < r_2 < \cdots$. Then a Lemma of Zippin [32] allows us to make a new basis $(e'_n)$ so that $[e'_k : r_{n-1} < k \le r_n] = [e_k : r_{n-1} < k \le r_n]$ and $e_{r_n} = y_n$. Then let $E_{2n-1} = [e'_k : r_{n-1} < k < r_n]$ and $E_{2n} = [y_n]$. It follows from the Lemma that $[y_n]$ is a *complemented* subspace of $X$. Thus *every block basic sequence spans a complemented subspace.*

However, more is clearly true: we can apply this argument to any permutation of the original basis, so any block basic sequence of any permutation spans a complemented subspace of $x$. This is exactly the hypothesis of a theorem of Lindenstrauss and Tzafriri [23],[24] Theorem 2.a.10:

**Theorem 4.2** *If $X$ has a normalized unconditional basis $(e_n)$ such that every block basic sequence of every permutation spans a complemented subspace, then $(e_n)$ is equivalent to the canonical basis of one the spaces $\ell_p$ where $1 \le p < \infty$ or $c_0$.*

Thus any Banach space $X$ with (MRP) and an unconditional basis is one of these spaces. But it is clear that we can eliminate the spaces $\ell_p$ when $p \neq 1, 2$. For in these spaces Pełczyński showed that it is possible to find an unconditional basis which is not equivalent to the standard one [28], and working with that basis leads to a contradiction. We are now left with $c_0, \ell_1$ and $\ell_2$ ; these are the precise three spaces with a unique unconditional basis by beautiful results of Lindenstrauss-Pełczyński and Lindenstrauss-Zippin, [24] Theorem 2.b.10. It would be nice to conclude by the argument by saying these three spaces have (MRP); but unfortunately that is false. The cases of $c_0$ and $\ell_1$ can be eliminated by working with the summing basis of $c_0$ (see [17] for details).

Thus we have proved:

**Theorem 4.3** *[17] Let $X$ be a Banach space with an unconditional basis and the maximal regularity property. Then $X$ is isomorphic to a Hilbert space.*

Since the spaces $L_p$ when $1 < p < \infty$ have unconditional bases this implies the original conjecture is false. It is possible to refine this argument to give the following results:

**Theorem 4.4** *[17] Let $X$ be a Banach space with (MRP). If either (a) $X$ is an order-continuous Banach lattice or (b) $X = L_p(Y)$ for some $1 \le p < \infty$ then $X$ is a Hilbert space.*

Note here that (b) completes the result of Coulhon-Lamberton cited in §3. It is possible to ask whether every separable Banach space with (MRP) is a Hilbert space or at least every separable Banach space with a basis. This seems harder, but some partial results are obtained in [18]. One final remark is in order: one may wonder why these techniques fail in $L_\infty$; the answer is that $L_\infty \sim \ell_\infty$ has no Schauder decompositions [12].

## 5. Rademacher boundedness and maximal regularity

Since Theorems 4.3 and 4.4 effectively end the discussion of the maximal regularity property for Banach spaces, one is naturally led to seek conditions on a sectorial operator so that it has maximal regularity. Let us look again at (4.1) which is clearly (at least if

$A^{-1}$ is bounded) a necessary condition. It turns out (although we did not immediately know this) that this condition embodies a concept which has made sporadic appearances in the literature over the last 20 years. Let us say that a family of operators $\mathcal{F} \subset \mathcal{L}(X)$ is *Rademacher-bounded* or *R-bounded* if there is a constant $C$ so that if $T_1, \cdots, T_n \in \mathcal{F}$ then

$$(\mathbf{E}\|\sum_{k=1}^{n} \epsilon_k T_k x_k\|^2)^{\frac{1}{2}} \leq C(\mathbf{E}\|\sum_{k=1}^{n} \epsilon_k x_k\|^2)^{\frac{1}{2}}. \tag{5.1}$$

This concept seems to date implicitly to a paper of Bourgain [4]. It was subsequently used by Berkson and Gillespie [3] and a comprehensive study was undertaken by Clément, de Pagter, Sukochev and Witvliet [7]. Let us observe that in the spaces $L_p$ when $1 \leq p < \infty$ (or indeed in any Banach lattice with nontrivial cotype) R-boundedness is equivalent to a square-function estimate:

$$\|(\sum_{k=1}^{n} |T_k x_k|^2)^{\frac{1}{2}}\| \leq C\|(\sum_{k=1}^{n} |x_k|^2)^{\frac{1}{2}}\|.$$

From (4.1) we obtain the fact that if $A$ has maximal regularity then the sequence $\{AR(\pm i2^n, A)\}_{n=1}^{\infty}$ is R-bounded; a slight variation gives us that for $\frac{1}{2} \leq s \leq 1$ the sequences $\{AR(\pm is2^n, A)\}_{n=1}^{\infty}$ are uniformly Rademacher-bounded (i.e. with the same constant $C$.). From this it is possible to see that the sets $\{AR(it, A) : |t| \geq 1\}$ are R-bounded, based on simple properties of the resolvent. If $A^{-1}$ is bounded then using the analyticity of the resolvent one can actually show that the set $\{AR(\lambda, A) : \Re\lambda \leq 0\}$ is R-bounded.

The remarkable fact is that this condition is actually sufficient for maximal regularity in a (UMD)-space. This was discovered by Lutz Weis [31] and myself (unpublished) independently in the early summer of 1999. My own argument was a sledgehammer technique using the unconditionality of the Haar series expansion of any $h \in L_p([0, T]; X)$. Weis instead proved an elegant and more general Fourier multiplier result which implied the same conclusion. Let us state Weis's theorem:

**Theorem 5.1** *Let $X$ be a Banach space with (UMD) and suppose $A$ is a sectorial operator with $\omega(A) < \frac{\pi}{2}$. Then $A$ has strong maximal regularity if and only if $\{AR(\lambda, A) : \Re\lambda \leq 0\}$ is Rademacher-bounded.*

*Remark.* For maximal regularity one only needs that $\{AR(\lambda, A) : \Re\lambda \leq -\delta\}$ is Rademacher-bounded for some $\delta > 0$.

This suggest a new concept of *Rademacher-sectoriality.* We say that a sectorial operator is Rademacher-sectorial or R-sectorial for some angle $\phi$ if the set $\{R(\lambda, A) : |\arg\lambda| \geq \phi\}$ is R-bounded. We can denote by $\omega_R(A)$ the infimum of all such angles $\phi$. Clearly $\omega_R(A) \geq \omega(A)$. The content of the results of §4 is that a Banach space with unconditional basis always admits a sectorial operator which is not Rademacher-sectorial for any angle.

At this point Weis and I got in contact and decided to pool our resources and work on these ideas together, starting in the fall of 1999 and continuing in the spring and summer of 2000.

## 6. The joint functional calculus

For specialists in the area, the maximal regularity problem is but one aspect of a more general problem concerning the sum of two closed commuting operators. Let us suppose $A, B$ are two sectorial operators on a Banach space $X$ which commute in the sense that their resolvents commute. We can consider the sum $A+B$ on $\mathcal{D}(A) \cap \mathcal{D}(B)$; however it is not immediately clear that with this domain $A+B$ is closed. One sufficient condition is that $\|Ax\| \leq C\|Ax + Bx\|$ for $x \in X$. This can be viewed as the problem of whether $A(A+B)^{-1}$ can be defined as a bounded operator. The problem of maximal regularity is exactly of this type (cf.[10]). We consider $D$ on $L_p([0,T];X)$ to be the operator $Df = f'$ on the set of all $f \in L_p$ of the form $f(t) = \int_0^t g(s)ds$ where $g \in L_p$. We then consider the equation

$$(D + \tilde{A})f = h$$

where $\tilde{A}f(t) = A(f(t))$ and $\tilde{A}$ has domain of all $f$ such that $f(t) \in \mathcal{D}(A)$ almost everywhere. Maximal regularity is exactly the requirement that $A(D+A)^{-1}$ is bounded or that $D+A$ is closed [10].

From this more general viewpoint the Dore-Venni Theorem (Theorem 3.1) reads [14]:

**Theorem 6.1** *Suppose $X$ is a Banach space with (UMD) and that $A, B$ are commuting sectorial operators with bounded imaginary powers satisfying*

$$\|A^{it}\| \leq Ke^{\theta_A|t|} \qquad \|B^{it}\| \leq Ke^{\theta_B|t|}.$$

*Then if $\theta_A + \theta_B < \pi$ then $A+B$ is closed on $\mathcal{D}(A) \cap \mathcal{D}(B)$ and $A(A+B)^{-1}$ extends to a bounded operator on $X$.*

To derive Theorem 3.1 one need only observe that when $X$ has (UMD) and $1 < p < \infty$ $D$ has (BIP) with any $\theta_D > \frac{\pi}{2}$. In fact, we saw in Example 2 of §2 that $X$ has (UMD) if and only if $D$ has an $H^\infty$-calculus (or even (BIP)), and that in this case it always follows that $\omega_H(D) = \frac{\pi}{2}$.

Thus it is natural for us to consider the general problem of developing a joint functional calculus for two commuting sectorial operators. Suppose $\phi_A > \omega(A)$ and $\phi_B > \omega(B)$. Then if $f \in H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})$ we make a formal definition of $f(A,B)$ by modifying (2.1):

$$f(A,B)x = \frac{-1}{4\pi^2} \int_{\Sigma_{\nu_A}} \int_{\Sigma_{\nu_B}} f(\zeta_1, \zeta_2) R(\zeta_1, A) R(\zeta_2, A) x \, d\zeta_2 d\zeta_1. \tag{6.1}$$

The general question is then to give conditions so that $f(A,B)$ is a bounded operator. In the particular case when $f(z_1, z_2) = z_1(z_1 + z_2)^{-1}$ then we will require of course that $\omega(A) + \omega(B) < \pi$ to avoid the singularities of $f$. It is natural to assume that one of the operators, say $A$, has an $H^\infty$-calculus.

In [19] we discovered a very general such theorem:

**Theorem 6.2** *Suppose $X$ is any Banach space and that $A, B$ are commuting sectorial operators on $X$. Suppose that $\sigma > \omega_H(A)$ and $\sigma' > \omega(B)$ and that $f \in H^\infty(\Sigma_\sigma \times \Sigma_{\sigma'})$. Suppose further that the set $\{f(w, B) : w \in \Sigma_\sigma\}$ consists of bounded operators and is $R$-bounded. Then $f(A,B)$ is bounded.*

We should emphasize that this theorem is really quite easy to prove; this is significant because it can be used to replace quite delicate and involved arguments in some existing theorems in the literature. A second point is that the R-boundedness assumption is too strong; one can replace it by $U$-boundedness, where we say that $\mathcal{F}$ is U-bounded if for some $C$ and all $T_1, \cdots, T_n \in \mathcal{F}$, $x_1, \cdots, x_n \in X$ and $x_1^*, \cdots, x_n^* \in X^*$ we have

$$\sum_{k=1}^{n} |\langle T_k x_k, x_k^* \rangle| \leq C \max_{\epsilon_k = \pm 1} \| \sum_{k=1}^{n} \epsilon_k x_k \| \max_{\epsilon_k = \pm 1} \| \sum_{k=1}^{n} \epsilon_k x_k^* \|. \tag{6.2}$$

U-boundedness is significantly weaker than R-boundedness, but, unfortunately for many practical examples it yields nothing new.

Let us put Theorem 6.2 to use by considering two problems. First let us look at the results of Lancien, Lancien and Le Merdy [20] on the existence of a full $H^\infty$–joint calculus (these results extended earlier work by Albrecht, Franks and McIntosh [1], [15] who considered only $L_p$-spaces when $1 < p < \infty$). Suppose $B$ also has an $H^\infty$–calculus and $\phi_A > \omega_H(A)$, $\phi_B > \omega_H(B)$. We say that $(A, B)$ has a joint $H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})$-functional calculus if for every $f \in H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})$ the operator $f(A, B)$ is bounded and then one necessarily has an estimate $\|f(A, B)\| \leq C\|f\|_{H^\infty}$. To apply Theorem 6.2 to a particular $f$ one needs that the family $\{f(w, B) : \ w \in \Sigma_{\phi_A}\}$ is R-bounded; to obtain this for every $f$ we really need that the family

$$\{f(B) : \ f \in H^\infty(\Sigma_{\phi_B}), \ \|f\|_{H^\infty} \leq 1\}$$

is R-bounded. This means that $B$ must have a Rademacher-bounded $H^\infty$–calculus. Thus we are led to the question of classifying Banach spaces where an $H^\infty$–calculus already implies a Rademacher-bounded $H^\infty$-calculus. There is a nice example due to Lancien, Lancien and Le Merdy to show that this is not always the case even in UMD-spaces.

*Example 4.* Consider the Schatten ideal $\mathcal{C}_p$ where $1 \leq p < \infty$. We can consider this as a space of infinite matrices $\mathbf{a} = (a_{jk})_{j,k}$. Now define $A(\mathbf{a}) = (2^j a_{jk})_{j,k}$ and $B(\mathbf{a}) = (2^k a_{jk})_{j,k}$ on their appropriate domains. Both $A$ and $B$ are $H^\infty$–sectorial with $\omega_H(A) = \omega_H(B) = 0$. Now for any suitable bounded analytic $f$ defined on some $\Sigma_{\phi_A} \times \Sigma_{\phi_B}$ it is clear that $f(A, B)$ is simply a Schur multiplier i.e. $f(A, B)\mathbf{a} = (f(2^j, 2^k)a_{jk})_{j,k}$. Now it is pointed out in [20] that $(2^j, 2^k)$ is interpolating for $H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})$ so that if $(A, B)$ have any joint $H^\infty$–calculus then every Schur multiplier $(a_{jk})_{j,k} \to (m_{jk}a_{jk})_{j,k}$ would have to be bounded. That only happens when $p = 2$. However if $1 < p < \infty$ these spaces even have (UMD) [5].

However, under appropriate conditions, one can get the desired conclusion. We say following Pisier [30] that $X$ has property $(\alpha)$ if for some constant $C$ if $(\epsilon_j)$ and $(\epsilon_k')$ are two mutually independent sequences of Rademachers then for any $(x_{jk})_{j,k \leq n}$ in $X$ and scalar $(a_{jk})_{j,k \leq n}$ we have

$$(\mathbf{E}\| \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \epsilon_j \epsilon_k' x_{jk} \|^2)^{\frac{1}{2}} \leq C \max_{j,k} |a_{jk}| (\mathbf{E}\| \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \epsilon_k' x_{jk} \|^2)^{\frac{1}{2}}. \tag{6.3}$$

Then any subspace of a Banach lattice with non-trivial cotype has $(\alpha)$. Then we have [19]:

**Theorem 6.3** *Suppose $X$ has property $(\alpha)$ and that $A$ is an $H^\infty-$sectorial operator on $X$. Then for any $\phi > \omega_H(B)$ the set $\{f(B):\ f \in H^\infty(\Sigma_\phi),\ \|f\|_{H^\infty} \leq 1\}$ is R-bounded.*

From this we deduce immediately by Theorem 6.2, as explained above:

**Theorem 6.4** *[20] Suppose $X$ has property $(\alpha)$ and that $A, B$ are $H^\infty-$sectorial operators on $X$. If $\phi_A > \omega_H(A)$ and $\phi_B > \omega_H(B)$ then $(A, B)$ has an $H^\infty(\Sigma_{\phi_A} \times \Sigma_{\phi_B})-$functional calculus.*

In fact in [20] some weaker conditions on $X$ are given which imply the theorem (e.g. if $X$ is any Banach lattice); these results can be obtained from the more delicate version of Theorem 6.2 using U-boundedness (6.2).

Next we turn to the case when $\omega_H(A) + \omega(B) < \pi$ and consider the function $f(z_1, z_2) = z_1(z_1 + z_2)^{-1}$.

If we check Theorem 6.2 we see that we need that $\{w(w + B)^{-1}\ :\ w \in \Sigma_\sigma\}$ is Rademacher-bounded for some $\sigma > \omega_H(A)$. This exactly means that $\omega_R(B) + \omega_H(A) < \pi$.

We can then ask, as in the previous example if it is sufficient that $\omega_H(A) + \omega_H(B) < \pi$. Example 4 above provides again some guidance for if we take $A, B$ as before, we require the boundedness of the Schur multiplier $(2^j(2^j + 2^k)^{-1})_{j,k}$ on $\mathcal{C}_p$. If one spaces out the rows and columns one quickly deduces that if this multiplier is bounded then the lower triangular projection, corresponding to the Schur multiplier which is one below the diagonal and zero elsewhere is also bounded. In the case when $p = 1$ (the trace-class) this is false. Thus there are examples when $A, B$ are both $H^\infty-$sectorial and with $\omega_H(A) = \omega_H(B) = 0$ and yet $A(A + B)^{-1}$ is unbounded.

It turns out that we can prove a result somewhat analogous to Theorem 6.3. We first say that $X$ has property $(\Delta)$ if it obeys an inequality somewhat weaker than $(\alpha)$:

$$(\mathbf{E}\|\sum_{j=1}^n \sum_{k=j}^n \epsilon_j \epsilon'_k x_{jk}\|^2)^{\frac{1}{2}} \leq C(\mathbf{E}\|\sum_{j=1}^n \sum_{k=1}^n \epsilon_j \epsilon'_k x_{jk}\|^2)^{\frac{1}{2}}. \tag{6.4}$$

Note that this is somewhat like a lower triangular projection as in the preceding discussion. Property $(\Delta)$ is not nearly as restrictive as $(\alpha)$. It is enjoyed by any space with $(\alpha)$, and also by (UMD)-spaces and even spaces with analytic (UMD). It fails in $\mathcal{C}_1$ ([16]).

Now in place of Theorem 6.3 we have:

**Theorem 6.5** *Let $X$ be a Banach space with property $(\Delta)$. Then if $A$ is $H^\infty-$sectorial, then $A$ is also R-sectorial and $\omega_R(A) = \omega_H(A)$.*

As before, we deduce:

**Theorem 6.6** *[19] Let $X$ be a Banach space with property $(\Delta)$ and suppose $A, B$ are $H^\infty-$sectorial operators on $X$ with $\omega_H(A) + \omega_H(B) < \pi$. Then $A(A + B)^{-1}$ is bounded (and $A + B$ is a closed operator on $\mathcal{D}(A) \cap \mathcal{D}(B)$).*

More recently, Le Merdy has improved Theorem 6.6:

**Theorem 6.7** *[22] Under the hypotheses of Theorem 6.6, $A + B$ is $H^\infty-$sectorial and $\omega_H(A + B) \leq \max(\omega_H(A), \omega_H(B))$.*

## 7. Concluding Remarks

We have attempted to give the flavor of recent work in this area, particularly in [17] and [19]. There are many problems left to resolve. Let us mention just two intriguing questions.

We know examples of sectorial which are not R-sectorial. However we do not know any example of an R-sectorial operator for which $\omega_R(A) > \omega(A)$.

The second question is more vague. The maximal regularity conjecture was made because all natural examples on say the spaces $L_p$ have maximal regularity. The problem is to explain why this phenomenon occurs. This would require isolating the properties of a sectorial operator induced by some differential operator which force it to be R-sectorial.

## REFERENCES


1. D. Albrecht, E. Franks and A. McIntosh, Holomorphic functional calculi and sums of commuting operators. Bull. Austral. Math. Soc. 58 (1998), 291–305.
2. J.-B. Baillon and Ph. Clément, Examples of unbounded imaginary powers of operators. J. Funct. Anal. 100 (1991), 419–434.
3. E. Berkson and T.A. Gillespie, Spectral decompositions and harmonic analysis on UMD spaces, Studia Math. 112 (1994) 13-49.
4. J. Bourgain, Some remarks on Banach spaces in which martingale differences are unconditional, Arkiv Math. 21 (1983) 163-168.
5. J. Bourgain, Vector-valued singular integrals and $H^1-BM)$ duality, 1-19 in *Probability theory and harmonic analysis,* (edited by J.-A. Chao and W. Woyczynski), Marcel Dekker, New York, 1986.
6. D.L. Burkholder, A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. Ann. Probab. 9 (1981), 997–1011.
7. P. Clément, B. de Pagter, F. Sukochev and H. Witvliet, Schauder decomposition and multiplier theorems, Studia Math. 138 (2000), 135–163.
8. T. Coulhon and D. Lamberton, Régularité $L^p$ pour les équations d'évolution, Séminaire d'Analyse Fonctionnelle Paris VI-VII (1984-85),155-165.
9. M. Cowling, I. Doust, A. McIntosh and A. Yagi, Banach space operators with a bounded $H^\infty$-calculus, J. Austral. Math. Soc. 60 (1996) 51-89.
10. G. Da Prato and P. Grisvard, Sommes d'opérateurs linéaires et équations différentielles opérationnelles, J. Math. Pures Appl. 54 (1975), 305-387.
11. L. De Simon, Un' applicazione della theoria degli integrali singolari allo studio delle equazioni differenziali lineare astratte del primo ordine, Rend. Sem. Mat., Univ. Padova (1964), 205-223.
12. D.W. Dean, Schauder decompositions in $(m)$. Proc. Amer. Math. Soc. 18 (1967) 619–623.
13. G. Dore, $L^p$-regularity for abstract differential equations, in *Functional analysis and related topics,* H. Komatsu, editor, Springer Lecture Notes 1540, 1993.
14. G. Dore, A. Venni, On the closedness of the sum of two closed operators, Math. Z. 196 (1987), 189-201.
15. E. Franks and A. McIntosh, Discrete quadratic estimates and holomorphic functional calculi in Banach spaces. Bull. Austral. Math. Soc. 58 (1998), 271–290.

16. U. Haagerup and G. Pisier, Factoriztion of analytic functions with values in non-commutative $L_1$-spaces and applications, Canad. J. Math. 41 (1989) 882-906.
17. N.J. Kalton and G. Lancien, A solution to the problem of $L^p-$maximal regularity, Math. Zeit. 235 (2000) 559-568.
18. N.J. Kalton and G. Lancien, $L^p-$maximal regularity on Banach spaces with a Schauder basis, Arch. Math. to appear.
19. N.J. Kalton and L. Weis, The $H^\infty-$calculus and sums of closed operators, Math. Ann. to appear.
20. F. Lancien, G. Lancien and C. Le Merdy, A joint functional calculus for sectorial operators with commuting resolvent, Proc. London Math. Soc. 77 (1998) 387-414.
21. C. Le Merdy, Counterexamples on $L^p$-maximal regularity, Math. Zeit. 230 (1999), 47-62.
22. C. Le Merdy, Two results about $H^\infty$ functional calculus on analytic UMD Banach spaces, to appear.
23. J. Lindenstrauss and L. Tzafriri, On the complemented subspaces problem, Israel J. Math. 9 (1971), 263-269.
24. J. Lindenstauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Berlin (1977).
25. H.P. Lotz, Uniform convergence of operators on $L^\infty$ and similar spaces, Math. Z. 190 (1985), 207-220.
26. T. McConnell, On Fourier multiplier transformations of Banach-valued functions. Trans. Amer. Math. Soc. 285 (1984), 739–757.
27. A. McIntosh, Operators which have an $H_\infty$ functional calculus. Miniconference on operator theory and partial differential equations (North Ryde, 1986), 210–231, Proc. Centre Math. Anal. Austral. Nat. Univ., 14, Austral. Nat. Univ., Canberra, 1986.
28. A. Pełczyński, Projections in certain Banach spaces, Studia Math. 19 (1960), 209-228.
29. G. Pisier, Les inégalités de Kahane-Khintchin d'après C. Borell, Séminaire sur la géometrie des éspaces de Banach, Ecole Polytechnique, Palaiseau, Exposé VII, 1977-78.
30. G. Pisier, Some results on Banach spaces without local unconditional structure, Comp. Math. 37 (1978) 3-19.
31. L. Weis, Operator-valued Fourier multiplier theorems and maximal regularity, Math. Ann. to appear.
32. M. Zippin, A remark on bases and reflexivity in Banach spaces, Israel J. Math. 6 (1968) 74-79.