Math. Z. 235, 559–568 (2000)
Digital Object Identifier (DOI) 10.1007/s002090000147

Mathematische Zeitschrift

# A solution to the problem of $L^p-$maximal regularity

**N.J. Kalton**[1], **G. Lancien**[2]

[1] Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA (e-mail: nigel@math.missouri.edu)
[2] Equipe de Mathématiques – UMR 6623, Université de Franche-Comté, F-25030 Besançon cedex, France (e-mail: GLancien@vega.univ-fcomte.fr)

Received June 11, 1999; in final form September 6, 1999 /
Published online September 14, 2000 – 

**Abstract.** We give a negative solution to the problem of the $L^p$-maximal regularity on various classes of Banach spaces including $L^q$-spaces with $1 < q \neq 2 < +\infty$.

*Mathematics Subject Classification (1991):* 47D06, 46B03, 46B52, 46C15

## 1. Introduction

In this paper we consider the following abstract Cauchy problem:

$$\begin{cases} u'(t) + B(u(t)) = f(t) & \text{for } 0 \leq t < T \\ u(0) = 0 \end{cases}$$

where $T \in (0, +\infty)$, $-B$ is the infinitesimal generator of a bounded analytic semigroup on a complex Banach space $X$ and $u$ and $f$ are $X$-valued functions on $[0, T)$. Suppose $1 < p < \infty$. $B$ is said to satisfy $L^p-$*maximal regularity* if whenever $f \in L^p([0, T); X)$ then the solution

$$u(t) = \int_0^t e^{-(t-s)B} f(s)\, ds$$

satisfies $u' \in L^p([0, T); X)$. It is known that $B$ has $L^p$-maximal regularity for some $1 < p < \infty$ if and only if it has $L^p$-maximal regularity for every $1 < p < \infty$ [4], [5], [20]. We thus say simply that $B$ satisfies *maximal regularity (MR)*.

The first author was partially supported by NSF grant DMS-9870027

The question whether $B$ satisfies maximal regularity has been extensively studied. De Simon [4] showed that $B$ always satisfy (MR) if $X$ is a Hilbert space. We also mention the early work of Grisvard [7] using interpolation spaces between $X$ and the domain $D(B)$. More recently, Dore and Venni [6] showed that $B$ satisfies (MR) if $X$ is an UMD Banach space and $B$ admits bounded imaginary powers with an estimate $||B^{is}|| \leq Ke^{\theta|s|}$ for some $0 \leq \theta < \pi/2$. The question we address was first asked by H. Brézis (see [2]) and is the following: under which conditions on the Banach space $X$ does every negative generator of a bounded analytic semigroup on $X$ satisfy (MR). Let us say that $X$ has the *the maximal regularity property (MRP)* if $B$ satisfies (MR) whenever $-B$ is the generator of a bounded analytic semigroup. The result of De Simon cited above implies that Hilbert spaces satisfy (MRP). Note also that by a theorem of Lotz [14] every strongly continuous semigroup on $L^\infty$ is uniformly continuous and this implies $L^\infty$ has (MRP). On the other hand, Coulhon and Lamberton [2] exhibited counterexamples on $X = L^p(\mathbb{R}; E)$ whenever $1 < p < \infty$ and $E$ is not an UMD Banach space. More recently, Le Merdy [9] found counterexamples on other fundamental spaces such as $L^1(\mathbb{T})$, $C(\mathbb{T})$ and $\mathcal{K}(\ell_2)$. For several years it has been an open question whether the spaces $L^p$ have (MRP) or even whether every UMD-space has (MRP).

In this paper, we provide a fairly complete answer to this question. In fact we will show that (MRP), up to isomorphism, characterizes Hilbert spaces among spaces with an unconditional basis or (more generally) separable Banach lattices. We also extend the Coulhon-Lamberton result cited above by showing that $L^p(\mathbb{R}; E)$ for $1 \leq p < \infty$ can never have (MRP) unless $p = 2$ and $E$ is a Hilbert space.

We would like to mention that our constructions have been initially inspired by a very useful transference principle for maximal regularity proved by C. Le Merdy [9], although we only use a very simple version of it (see our Proposition 2.1).

This work was done during a visit of the first author to the Department of Mathematics at the Université de Franche-Comté in spring 1999; he would like to thank the Department for its warm hospitality.

## 2. Notation and basic facts

We will mainly adopt the notation introduced in [9].

Let $X$ be a complex Banach space, $1 < p < \infty$ and $0 < T < \infty$. We denote by $\mathcal{A}_T$ the differentiation $d/dt$ on $L^p([0,T); X)$ with domain $D(\mathcal{A}_T) = W_0^{1,p}([0,T); X) = \{u \in W^{1,p}([0,T); X) : \ u(0) = 0\}$. Let now $-B$ be the generator of a bounded analytic semigroup on $X$. The operator $I_{L^p} \otimes B$ defined on $L^p([0,T)) \otimes D(B)$ is closable and we denote by $\mathcal{B}$ its closure. $\mathcal{B}$ can also be described by $D(\mathcal{B}) = L^p([0,T);D(B))$ and $(\mathcal{B}u)(t) = B(u(t))$ for $u$ in $D(\mathcal{B})$. We say that $B$ satisfies $L^p$-maximal regularity (on $[0,T)$) if the operator $\mathcal{A}_T+\mathcal{B}$ with domain $D(\mathcal{A}_T)\cap D(\mathcal{B})$ has a bounded inverse; this is equivalent to the formulation in the introduction, and obviously independent of $T$. By a result in [3], $L^p-$maximal regularity is also equivalent to the statement that $\mathcal{A}_T + \mathcal{B}$ is closed, and also to an inequality of the form

$$\|\mathcal{A}_T u\| \leq C\|\mathcal{A}_T u + \mathcal{B}u\|$$

for $u \in D(\mathcal{A}_T) \cap D(\mathcal{B})$. As remarked in the introduction, it is known (see [4], [5], [20]) that this property does not depend on $1 < p < \infty$. Thus we will simply say that $B$ satisfies (MR) and work only on $L^2([0,T);X)$. Then we say that a Banach space $X$ has the *maximal regularity property (MRP)*, if every $B$ such that $-B$ generates a bounded analytic semigroup on $X$ satisfies (MR).

We will also use the following terminology. A closed densely defined operator $B$ on a Banach space $X$ is said to be sectorial of type $\omega$, where $0 < \omega < \pi$, if the spectrum $\sigma(B)$ of $B$ is included in $\overline{\Sigma_\omega}$, where $\Sigma_\omega = \{z \in \mathbb{C} : \ |\mathrm{Arg}(z)| < \omega\}$ and for every $\omega < \theta < \pi$ there exists $C_\theta > 0$ so that for any $\lambda \notin \overline{\Sigma_\theta}$ we have $\|(\lambda - B)^{-1}\| \leq C_\theta|\lambda|^{-1}$. Notice that $-B$ generates a bounded analytic semigroup on $X$ if and only if $B$ is sectorial of type $\omega$, for some $\omega < \pi/2$ (see [21] or [10] for details).

Our next result can be regarded as a form of transference to the circle. For $f \in L^1([0,2\pi);X)$ we define the Fourier coefficients $\hat{f}(n)$ in the usual way

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$$

for $n \in \mathbb{Z}$.

**Proposition 2.1.** *Let $X$ be a Banach space and let $-B$ be an invertible generator of a bounded analytic semigroup on $X$. Assume that $B$ satisfies (MR). Then there is a constant $C$ so that for any $X$-valued trigonometric polynomial $f(t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}$ we have*

$$\left(\int_0^{2\pi} \|\sum_{n\in\mathbb{Z}} in(in+B)^{-1}\hat{f}(n)e^{int}\|^2 \frac{dt}{2\pi}\right)^{1/2} \leq C\left(\int_0^{2\pi} \|f(t)\|^2 \frac{dt}{2\pi}\right)^{1/2}.$$

*Proof.* Denote by $(e^{-tB})_{t>0}$ the semigroup generated by $-B$. For any trigonometric polynomial $f$ define $g \in L^2([-2\pi,2\pi);X)$ by:

$$\begin{cases} g(s) = (1-e^{-2\pi B})^{-1}f(s+2\pi) & \text{for } -2\pi \leq s \leq 0 \\ g(s) = f(s) & \text{for } 0 < s \leq 2\pi \end{cases}$$

Since $B$ has a negative exponential type, the definition of $g$ makes sense. Moreover there is a constant $C > 0$ independent of $f$ so that

$$||g||_{L^2([-2\pi,2\pi);X)} \leq C||f||_{L^2([0,2\pi);X)}.$$

Now we solve

$$\begin{cases} u'(t) + B(u(t)) = g(t) \quad \text{for } -2\pi \leq t < 2\pi \\ u(-2\pi) = 0 \end{cases}$$

Elementary calculations show that $u' = \sum_{n\in\mathbb{Z}} in(in + B)^{-1}\hat{f}(n)e^{int}$ on $(0, 2\pi)$. Now, the fact that $B$ has (MR) yields the result. □

Next, is an elementary lemma about (MRP) that we will use extensively:

**Lemma 2.2.** *Let $X$ be a Banach space and $Y$ be a complemented subspace of $X$. Assume that $X$ has the (MRP) then $Y$ has the (MRP).*

*Proof.* Assume $X = Y \oplus Z$ and that $B$ is a sectorial operator of type $< \pi/2$ on $Y$ which fails (MR). Then the operator $B'$, defined on $X$ by $D(B') = D(B)\oplus Z$ and $B'x = By$ when $x = y+z$ with $(y, z) \in D(B)\times Z$ provides a counterexample to (MR) on $X$. □

The operators that we will use will be multipliers associated with various Schauder decompositions. Let us introduce some notation for that purpose. If $F \subset X$, we denote by $[F]$ the closed linear span of $F$. Let $(E_n)_{n\geq1}$ be a sequence of closed subspaces of $X$. Assume that $(E_n)_{n\geq1}$ is a Schauder decomposition of $X$ and let $(P_n)_{n\geq1}$ be the associated sequence of projections from $X$ onto $E_n$. For convenience we will also denote this Schauder decomposition by $(E_n, P_n)_{n\geq1}$. Notice that the spaces $Z_n = P_n^*(X^*)$ form a Schauder decomposition of the subspace $Z = [\cup_{n=1}^{\infty} Z_n]$ of $X^*$.

Let now $(b_n)_{n\geq1}$ be a sequence of complex numbers. We define the (possibly unbounded) operator $M(b_n; E_n)$ with domain $D(M(b_n; E_n)) = \{x \in X$ such that $\sum b_nP_nx$ converges in $X\}$ by $M(b_n; E_n)x = \sum b_nP_nx$.

The following lemma is elementary (see [1] or [22] for a proof in the case of a Schauder basis).

**Lemma 2.3.** *(i) $M(b_n; E_n)$ is a closed densely defined operator.*
*(ii) If $b_1 > 0$ and $(b_n)_{n\geq1}$ is an increasing sequence of reals, then $M(b_n; E_n)$ is invertible and sectorial of type $\omega$ for any $\omega \in (0, \pi)$.*

## 3. The results

We first establish a necessary condition for spaces with a Schauder decomposition to have the maximal regularity property.

**Theorem 3.1.** *Let $(E_n, P_n)_{n\geq 1}$ be a Schauder decomposition of the Banach space $X$. Let $Z_n = P_n^* X^*$ and $Z = [\cup_{n=1}^{\infty} Z_n]$. Assume $X$ has (MRP). Then there is a constant $C > 0$ so that whenever $(u_n)_{n=1}^N$ are such that $u_n \in [E_{2n-1}, E_{2n}]$ and $(u_n^*)_{n=1}^N$ are such that $u_n^* \in [Z_{2n-1}, Z_{2n}]$ then*

$$\left(\int_0^{2\pi} \|\sum_{n=1}^N P_{2n} u_n e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2} \leq C \left(\int_0^{2\pi} \|\sum_{n=1}^N u_n e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2}$$

*and*

$$\left(\int_0^{2\pi} \|\sum_{n=1}^N P_{2n}^* u_n^* e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2} \leq C \left(\int_0^{2\pi} \|\sum_{n=1}^N u_n^* e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2}.$$

*Proof.* Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences defined by

$$a_{2n-1} = b_{2n-1} = b_{2n} = 2^{n-1} \text{ and } a_{2n} = 2^n.$$

We let $A = M(a_n; E_n)$ and $B = M(b_n; E_n)$. It is easy to see that

$$(i2^n + A)^{-1} u_n = (i2^n + 2^{n-1})^{-1} P_{2n-1} u_n + (i2^n + 2^n)^{-1} P_{2n} u_n$$

while

$$(i2^n + B)^{-1} u_n = (i2^n + 2^{n-1})^{-1} u_n.$$

Hence

$$i2^n (i2^n + B)^{-1} u_n - i2^n (i2^n + A)^{-1} u_n = \frac{i}{(i+1)(2i+1)} P_{2n} u_n.$$

If we assume that $X$ has (MRP) then both $A$ and $B$ satisfy (MR) and so we can apply Proposition 2.1 to each in turn for the polynomial $f(t) = \sum_{n=1}^N u_n e^{i2^n t}$. Subtracting gives us the first estimate.

The second estimate follows by duality. More precisely the operators $f \mapsto \sum_{n\in\mathbb{Z}} in(in + A)^{-1} \hat{f}(n) e^{int}$ and $f \mapsto \sum_{n\in\mathbb{Z}} in(in + B)^{-1} \hat{f}(n) e^{int}$ can be extended to bounded linear operators on $L^2([0, 2\pi); X)$. Taking adjoints and restricting to the subspace $L^2([0, 2\pi); Z)$ one easily obtains similar estimates in the dual. □

We first examine two important examples.

**Corollary 3.2.** *$c_0$ and $\ell_1$ fail the (MRP).*

*Proof.* Denote by $(x_n)_{n\geq 1}$ the canonical basis of $c_0$ and let $s_n = x_1+..+x_n$. $(s_n)_{n\geq 1}$ is a Schauder basis of $c_0$ which is usually called the summing basis of $c_0$. We now apply Theorem 3.1 with the sequence of projections $(P_n)$ associated with the Schauder basis $(s_n)$ and $u_n = s_{2n} - s_{2n-1}$. Then we obtain that there is $C > 0$ so that for every $N \geq 1$,:

$$\left(\int_0^{2\pi} \|\sum_{n=1}^{N} s_{2n} e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2}$$
$$\leq C \left(\int_0^{2\pi} \|\sum_{n=1}^{N} (s_{2n} - s_{2n-1}) e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2}.$$

The right-hand side is equal to $C$ but, considering only the first co-ordinate of the left-hand side with respect to the canonical basis, we have

$$\left(\int_0^{2\pi} \|\sum_{n=1}^{N} s_{2n} e^{i2^n t}\|^2 \frac{dt}{2\pi}\right)^{1/2} \geq N^{1/2}.$$

This is a contradiction.

Assume now that $\ell_1$ has the (MRP). Let $(v_n)_{n\geq 1}$ be the coordinate functionals associated with the summing basis $(s_n)$ of $c_0$. The closed linear space $Y$ spanned in $\ell_1$ by the sequence $(v_n)$ is of codimension 1 in $\ell_1$ and hence is isomorphic to $\ell_1$. The bi-orthogonal functionals $(v_n^*)$ in $Y^*$ are equivalent to the summing basis of $c_0$. Hence using the same calculation as above and the second inequality of Theorem 3.1 we again get a contradiction. □

We now explain the consequences of Theorem 3.1 when $X$ admits an unconditional basis.

**Theorem 3.3.** *A Banach space with an unconditional basis has the (MRP) if and only if it is isomorphic to $\ell_2$.*

*Proof.* The idea is to show that if $X$ has the (MRP) and an unconditional basis $(x_n)_{n\geq 1}$, then for every permutation $\pi$ of the integers and for every block basis $(u_j)_{j\geq 1}$ of $(x_{\pi(n)})_{n\geq 1}$ the closed subspace of $X$ spanned by the $u_j$'s is complemented in $X$. Once we have shown this the proof is completed by using a theorem of Lindenstrauss and Tzafriri ([11], see also [12] Theorem 2.a.10) which asserts that $(x_n)_{n\geq 1}$ must be equivalent to the canonical basis of $c_0$ or $\ell_p$ for some $p$ in $[1,\infty)$. Then, by Corollary 3.2, $(e_n)$ is equivalent to the canonical basis of $\ell_p$ for some $p$ in $(1,\infty)$. Now, if $1 < p \neq 2 < \infty$, $\ell_p$ admits an unconditional basis which is not equivalent to any of the canonical bases of $c_0$ or $\ell_q$ where $1 \leq q < \infty$. Indeed Pelczyński [17] showed that, for $1 < p < \infty$, $\ell_p$ is isomorphic to $(\sum_{n\geq 1} \oplus \ell_2^n)_p$.

So assume, as we may, that $(x_n)_{n\geq 1}$ is a normalized 1-unconditional basis of $X$ and that $(u_n)_{n\geq 1}$ is a normalized block basis of $(x_n)_{n\geq 1}$, with

$$\forall n \geq 1,\ u_n = \sum_{r_n+1}^{r_{n+1}} a_j e_j,$$

where $0 = r_1 < r_2 < .. < r_n < r_{n+1} < ..$ and $(a_j)_{j\geq 1} \subset \mathbb{C}$. For $n \geq 1$, let $X_n = [x_{r_n+1}, .., x_{r_{n+1}}]$ and $E_{2n} = [u_n]$. Then $(X_n)$ is an unconditional Schauder decomposition of $X$ with associated projections $P_n$, say. Now, by the Hahn-Banach theorem there is a norm-one projection $R_n : X_n \to E_{2n}$. Let $E_{2n-1} = R_n^{-1}(0)$. Then $(E_n)$ is a Schauder decomposition of $X$ with associated projections $Q_{2n-1} = (I - R_n)P_n$ and $Q_{2n} = R_nP_n$. We now apply Theorem 3.1 and exploit the unconditionality of the Schauder decomposition $(X_n)$. There is a constant $C$ so that if $y$ is in the linear span of the $(x_n)_{n\geq 1}$ then

$$\|\sum_{n=1}^{\infty} Q_{2n}y\| \leq C\|y\|.$$

This implies that $[u_n]_{n\geq 1}$ is complemented in $X$.

Clearly the same reasoning can be applied to any permutation of the basis $(x_n)$ so that the proof is complete. □

Although this will be included in further and more general statements let us point out that this already solves our problem for the spaces $L^p(0,1)$:

**Corollary 3.4.** *Let $1 \leq p \leq \infty$. Then $L^p(0,1)$ has the (MRP) if and only if $p = 2$ or $p = \infty$.*

*Proof.* For $1 < p < \infty$, the Haar system is known to be an unconditional basis of $L^p(0,1)$ ([16], see also [13]). So the result follows from the preceding Theorem. The fact that $L^1$ fails (MRP) was proved by C. Le Merdy in [9]. Notice that $L^1$ contains a complemented copy of $\ell_1$, so this result can be derived from Lemma 2.2 and Corollary 3.2. □

We now extend Theorem 3.3 to the case of a space with an unconditional Schauder decomposition.

**Theorem 3.5.** *Let $X$ be a Banach space with an unconditional decomposition $(F_n, P_n)_{n\geq 1}$. Assume that $X$ has the (MRP).*

*Then $X$ is isomorphic to $(\sum_{n=1}^{\infty} \oplus F_n)_{\ell_2}$.*

*Proof.* It suffices to show that if $u_n \in F_n$ with $\|u_n\| = 1$ then $\sum a_n u_n$ converges if and only if $\sum |a_n|^2 < \infty$. As above in the proof of Theorem

3.5, let $R_n$ be a norm-one projection of $F_n$ onto $[u_n]$. Then let $E_{2n} = [u_n]$ and $E_{2n-1} = R_n^{-1}(0)$. Reasoning exactly as in Theorem 3.5 gives that $[u_n]_{n\geq 1}$ is complemented in $X$. But this subspace has an unconditional basis and so Theorem 3.5 yields that $(u_n)$ is equivalent to the canonical basis of $\ell_2$. $\square$

Our next theorem completes the important counterexamples obtained by Coulhon and Lamberton [2].

**Theorem 3.6.** *Suppose $X$ is a Banach space and $1 \leq p < \infty$. Then the Banach space $L^p((0,1);X)$ has the (MRP) if and only if $p = 2$ and $X$ is isomorphic to a Hilbert space.*

*Proof.* We first note that $L^p[0,1]$ is complemented in $L^p((0,1);X)$ so that if the latter has (MRP) then $p = 2$ by Corollary 3.4. Assume that $L^2((0,1);X)$ has the (MRP); we will show that $X$ is isomorphic to a Hilbert space (the opposite implication is due to de Simon [4]). By [2], $X$ must have the UMD property. In particular, $X$ does not contain the $\ell_1^n$'s uniformly. Hence by Pisier's theorem [18], the space $\mathrm{Rad}(X) = \overline{[\varepsilon_n]_{n\geq 1} \otimes X}$ is complemented in $L^2((0,1);X)$ (here $\varepsilon_n$ is a standard Rademacher function). Therefore, by Lemma 2.2, $\mathrm{Rad}(X)$ has the (MRP). Now, $(E_n)_{n\geq 1} = (\varepsilon_n \otimes X)_{n\geq 1}$ is an unconditional Schauder decomposition of $\mathrm{Rad}(X)$. So, by Theorem 3.5, $\mathrm{Rad}(X)$ must be isomorphic to $(\sum \oplus (\varepsilon_n \otimes X))_{\ell_2}$. Finally, it follows from Kwapien's theorem [8] that $X$ is isomorphic to a Hilbert space. $\square$

We now extend Theorem 3.3 and Corollary 3.4 to the setting of Banach lattices. All the notions on Banach lattices that we will use can be found in [13] Chapters 1.a and 1.b.

**Theorem 3.7.** *An order continuous Banach lattice has the (MRP) if and only if it is isomorphic to a Hilbert space.*

*Proof.* Let $X$ be an order continuous Banach lattice. By a result of L. Tzafriri (see for instance [13], Lemma 1.b.13), it is enough to show that every normalized sequence of disjoint elements of $X$ is equivalent to the canonical basis of $\ell_2$. So let $(f_n)_{n\geq 1}$ be such a sequence in $X$. Then $X$ admits an unconditional Schauder decomposition $(E_n)_{n\geq 1}$ such that the $E_n$'s are ideals of $X$ and for all $n \geq 1$, $f_n \in E_n$. Now, by Theorem 3.5, $X$ is isomorphic to $(\sum \oplus E_n)_{\ell_2}$ and $(f_n)$ is equivalent to the canonical basis of $\ell_2$. $\square$

**Corollary 3.8.** *A separable Banach lattice has the (MRP) if and only if it is isomorphic to a Hilbert space.*

*Proof.* Let $X$ be a separable Banach lattice which is not order continuous. Then $X$ is not $\sigma$-complete (see [13] Proposition 1.a.7) and by a result of P.

Meyer-Nieberg ([15], see also [13] Theorem 1.a.5) $X$ contains a subspace isomorphic to $c_0$. Since $X$ is separable, it follows from Sobczyk's Theorem [19] that this subspace is complemented in $X$. So, by Lemma 2.2 and Corollary 3.2, $X$ does not have the (MRP). Then, the preceding Theorem concludes our proof. □

## 4. Final remarks

1) One can also consider the problem of the $L^p$-maximal regularity on the half line $[0, +\infty)$, which has in general a different answer (see [9] for an example). But it follows from Theorem 2.4. in [5] that all our results remain valid in this slightly different setting.

2) We do not know if there is a non-Hilbertian subspace of an $L^p$-space $(1 \leq p < \infty)$ with (MRP).

3) We do not know if every space with a basis and (MRP) is isomorphic to a Hilbert space.

## References

1. J.B. Baillon and P. Clément, Examples of unbounded imaginary powers of operators, J. Funct. Anal. **100** (1991), 419–434
2. T. Coulhon and D. Lamberton, Régularité $L^p$ pour les équations d'évolution, Séminaire d'Analyse Fonctionnelle Paris VI-VII (1984–85), 155–165
3. G. Da Prato and P. Grisvard, Sommes d'opérateurs linéaires et équations différentielles opérationnelles, J. Math. Pures Appl. **54** (1975), 305–387
4. L. De Simon, Un' applicazione della theoria degli integrali singolari allo studio delle equazioni differenziali lineare astratte del primo ordine, Rend. Sem. Mat., Univ. Padova (1964), 205–223
5. G. Dore, $L^p$ regularity for abstract differential equations (In "Functional Analysis and related topics", editor: H. Komatsu), Lect. Notes in Math. 1540, Springer Verlag (1993).
6. G. Dore, A. Venni, On the closedness of the sum of two closed operators, Math. Z. **196** (1987), 189–201
7. P. Grisvard, Equations différentielles abstraites, Ann. Sci. Ecole Norm. Sup. (4) **2** (1969), 311–395
8. S. Kwapien, Isomorphic characterizations of inner product spaces by orthogonal series with vector coefficients, Studia Math. **44** (1972), 583–595
9. C. Le Merdy, Counterexamples on $L^p$-maximal regularity, Math. Z. **230** (1999), 47–62
10. C. Le Merdy, $H^\infty$-functional calculus and applications to maximal regularity, Publications Mathématiques de Besançon, Fasicule 16 (1999).
11. J. Lindenstrauss and L. Tzafriri, On the complemented subspaces problem, Israel J. Math. **9** (1971), 263–269
12. J. Lindenstauss and L. Tzafriri, Classical Banach spaces I, Springer, Berlin 1977
13. J. Lindenstauss and L. Tzafriri, Classical Banach spaces II, Springer, Berlin 1979
14. H.P. Lotz, Uniform convergence of operators on $L^\infty$ and similar spaces, Math. Z. **190** (1985), 207–220

15. P. Meyer-Nieberg, Charakterisierung einiger topologischer und ordnungstheoritischer Eigenschaften von Banachverbanden mit Hilfe disjunkte Folgen, Arch. Math. **24** (1973), 640–647
16. R.E. Paley, A remarkable sequence of orthogonal functions, Proc. London Math. Soc. **34** (1932) 241–264
17. A. Pelczyński, Projections in certain Banach spaces, Studia Math. **19** (1960), 209–228
18. G. Pisier, Holomorphic semigroups and the geometry of Banach spaces, Annals of Math. (2) **115** (1982), 375–392
19. A. Sobczyk, Projection of the space $m$ on its subspace $c_0$, Bull. Amer. Math. Soc. **47** (1941), 938–947
20. P.E. Sobolevskii, Coerciveness inequalities for abstract parabolic equations (translations), Soviet Math. Dokl. **5** (1964), 894–897
21. H. Tanabe, Equations of evolution, Pitman, London, San Francisco, Melbourne 1979
22. A. Venni, A counterexample concerning imaginary powers of linear operators, (In "Functional Analysis and related topics", editor: H. Komatsu), Lect. Notes in Math. 1540, Springer, 1993