Math. Nachr. **171** (1995) 227–258

Complex Interpolation of Hardy-Type Subspaces

By N. J. KALTON[1] of Columbia

(Received August 24, 1993)
(Revised Version May 3, 1994)

Abstract. We consider the problem of complex interpolation of certain Hardy-type subspaces of Köthe function spaces. For example, suppose that X_0 and X_1 are Köthe function spaces on the unit circle T, and let H_{X_0} and H_{X_1} be the corresponding Hardy spaces. Under mild conditions on X_0, X_1 we give a necessary and sufficient condition for the complex interpolation space $[H_{X_0}, H_{X_1}]_\theta$ to coincide with H_{X_θ} where $X_\theta = [X_0, X_1]_\theta$. We develop a very general framework for such results and our methods apply to many more general situations including the vector-valued case.

1. Introduction

Let X be a Köthe function space on the circle $\mathbf{T}$ equipped with its usual Haar measure. Consider the Hardy subspace H_X consisting of all $f \in X \cap N^+$ where N^+ is the Smirnov class or Hardy algebra. Provided $X \subset L_{\log}$ (see Section 2 for the definition) this is a closed subspace. Consider the following two problems:

(1) When is H_X complemented in X by the usual Riesz projection?

(2) If X_0, X_1 are two such Köthe function spaces when is it true that the complex interpolation space $X_\theta = [X_0, X_1]_\theta$ satisfies $H_{X_\theta} = [H_{X_0}, H_{X_1}]_\theta$?

In the case of weighted L_p-spaces, a precise answer to (1) was given by MUCKENHAUPT [26] in terms of the so-called A_p-conditions. In the case $p = 2$, the Helson-Szegö theorem [15] gives an alternative precise criterion; in the same direction COTLAR and SADOSKY [8], [9] gave necessary and sufficient conditions for all weighted L_p-spaces (see also [10]). Subsequently, RUBIO DE FRANCIA extended the Cotlar-Sadosky methods to all 2-convex or 2-concave Köthe function spaces. In the case of L_p-spaces (without weights) (2) is answered by a well-known theorem of JONES [16], [17] (cf. recent proofs by XU [34], MÜLLER [27] and PISIER [29]). For weighted L_p-spaces (2) has recently been studied by CWIKEL, MCCARTHY and WOLFF [11], and KISLIAKOV and XU [20], [21] (who also consider vector-valued analogues). See also [33].

[1] Research supported by NSF grant DMS-9201357

In this paper we will develop a very general approach to question (2) by relating it to (1). We will be able to answer (2) completely under some mild restrictions on the spaces. In fact our approach uses very little specific information about Hardy spaces or properties of analytic functions and we give our results in a rather general setting, which includes abstract Hardy spaces generated by weak*-Dirchlet algebras and certain vector-valued cases as studied by KISLIAKOV and XU.

We limit our discussion in the introduction to the case of the circle. Let us say that a Köthe function space X is BMO-regular if and only if there exist constants (C, M) so that given $0 \leq f \in X$ there exists $g \geq f$ with $\|g\|_X \leq M \|f\|_X$ and $\|\log g\|_{BMO} \leq C$. A weighted L_p-space, $L_p(w)$ is BMO-regular if and only if $\log w \in BMO$. The concept of BMO-regularity appears implicity first in the work of COTLAR and SADOSKY [9] and also in RUBIO DE FRANCIA [30] in connection with the boundedness of the Hilbert transfrom (it should be noted that in both cases the boundedness of the Hilbert transform is related to the BMO-regular of a space derived from X, not of X itself). We show that a superreflexive space X is BMO-regular if and only if the Riesz projection is bounded on an interpolation space $L_2^{\theta} X^{1-\theta}$ for some $\theta > 0$ (cf. [18] for other conditions equivalent to this property for X).

If X_0, X_1 are super-reflexive and $X_0, X_1, X_0^*, X_1^* \subset L_{\log}$, then we give a necessary and sufficient condition for $H_{X_\theta} = [H_{X_0}, H_{X_1}]_\theta$ where $0 < \theta < 1$ (in this case we say that the Hardy algebra $H = N^+$ is interpolation stable at θ for (X_0, X_1)). Consider first the case when X_0 is BMO-regular; then it necessary and sufficient that X_1 is BMO-regularity. For the general case the necessary and sufficient condition is obtained by "lifting" the direction $X_0 \to X_1$ to create a parallel direction $L_2 \to Z$; the condition is then that Z is BMO-regular. This is precisely explained in Section 5; let us remark that if $X_1 = wX_0$ is obtained by a change of weight, then $Z = w^{1/2} L_2 = L_2(w^{-1})$ so that the condition is simply that $\log w \in BMO$. Our result includes the results of the previous work of KISLIAKOV and XU [21] as special cases and extends, as we have explained, to a very general setting, thus giving also vector-valued applications.

Let us also comment on the methods used. In Section 3 we discuss a very general formulation of question (2); when does the operation of interpolation commute with taking a particular subspace? Our main result is that if this happens, then under appropriate conditions, one can extrapolate the boundedness of a projection onto the subspace. In Section 4 we consider an arbitrary self-adjoint operator T on L_2. We then discuss for which Köthe spaces X it is true that T is bounded on $L_2^{1-\theta} X^{\theta}$ for some $\theta > 0$. If we assume that T is bounded on some L_p where $p \neq 2$, then this can answered in terms of the weighted L_2-spaces on which T is bounded. These results are of course closely related to the earlier work of COTLAR and SADOSKY [9], and RUBIO DE FRANCIA [30]; unlike [30] we do not assume 2-convexity or 2-concavity but our conclusions are somewhat weaker.

We put these ideas together in Section 5, restricting our attention to "Hardy-type" algebras, which we introduce as an abstraction of the Smirnov class; in this case our operator T becomes the orthogonal projection onto H_2. We are then able to relate the results of Section 4 to the notion of BMO-regularity and prove our main results. We discuss further applications in Section 6. At the end of Section 6, we improve the results of KISLIAKOV and XU ([20], [21]) on interpolation of vector-valued Hardy spaces, by giving necessary and sufficient conditions for such interpolation to be "stable" at least in the super-reflexive case.

Let us mention that we use some ideas from [18]; however we have tried to avoid using differential techniques in order to keep our approach as simple as possible. We plan a further paper showing how by using such techniques one can improve and extend these results. We do use freely however, the notion of an indicator function for a Köthe function space as introduced and studied in [18].

We would like to thank MICHAEL CWIKEL, MARIO MILMAN and RICHARD ROCHBERG for discussing this problem with us, and CORA SADOSKY for some very helpful remarks. We also thank QUANHUA XU for several important suggestions which we have incorporated.

2. Köthe function spaces

Let S be a Polish space and let μ be a probability measure on S. Let $L_0(\mu)$ denote, as usual, the space of all equivalence classes of (complex) Borel functions on S with the topology of convergence in measure.

We define a Köthe quasinorm on L_0 to be a lower-semicontinuous functional $f \to \|f\|_X$ defined on L_0 with values in $[0, \infty]$ such that:

(1) $\|f\|_X = 0$ if and only if $f = 0$ a.e.,

(2) $\|f\|_X \leq \|g\|_X$ whenever $|f| \leq |g|$ a.e.,

(3) There exists a constant C so that $\|f + g\|_X \leq C(\|f\|_X + \|g\|_X$ for $f, g \in L_0$,

(4) There exists $u \in L_0$ so that $u > 0$ a.e. and $\|u\|_X < \infty$.

Associated to the Köthe quasi-norm we can associate a maximal quasi-Köthe function space $X = \{f\colon \|f\|_X < \infty\}$. X is then a quasi-Banach space under quasi-norm $f \to \|f\|_X$; furthermore, $B_X = \{f\colon \|f\|_X \leq 1\}$ is closed in L_0 so that X has the Fatou property (cf. [4]). We can also define a minimal quasi-Köthe function space X_0 to be the closure of $L_\infty \cap X$ in X. In this paper, however, we will only deal with maximal spaces (i.e., spaces with the Fatou property). If in (3) $C = 1$, then B_X is convex and X is an Banach space; in this case we say that X is a (maximal) Köthe function space. Henceforward we will adopt the convention that all spaces are maximal.

If X is a Köthe function space and $w \in L_{0,\mathbf{R}}$ with $w > 0$ a.e. we define the weighted space wX by $\|f\|_{wX} = \|fw^{-1}\|_X$. Thus $wL_p(w^{-p})$.

If X is a Köthe function space we will let X^* denote its Köthe dual, i.e., the maximal Köthe function space induced by $\| \ \|_{X^*}$ where $\|f\|_{X^*} = \sup_{g \in B_X} \int |fg| \, d\mu$. It is not difficult to show that X^* is also a Köthe function space. Of course, if X is reflexive as will usually be the case, then X^* is the Banach dual of X.

We recall that a quasi-Köthe function space X is p-convex where $0 < p < \infty$ with constant M if for every, $f_1, \ldots, f_n \in X$ we have that

$$\left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{1/p} \right\|_X \leq M \left(\sum_{k=1}^{n} \|f_k\|_X^p \right)^{1/p},$$

and q-concave $(0 < q < \infty)$ with constant M if for every $f_1, \ldots, f_n \in X$,

$$\left(\sum_{k=1}^{n} \|f_k\|_X^q \right)^{1/q} \leq M \left\| \left(\sum_{k=1}^{n} |f_k|^q \right)^{1/q} \right\|_X.$$

If X is p-convex and q-concave there is an equivalent quasi-norm so that the p-convexity and q-concavity constants are both one. For convenience we will say that X is exactly p-convex or q-concave if the associated constant of convexity or concavity is one. X is a Köthe function space if and only it is 1-convex with constant one. A Köthe function space is super-reflexive if and only if it is p-convex and q-concave for some $1 < p \leq q < \infty$.

For any Köthe function space X we define the quasi-Köthe function space X^α by

$$\|f\|_{X^\alpha} = \| |f|^{1/\alpha}\|_X^\alpha .$$

Then X^α is exactly $1/\alpha$-convex. If X, Y are two Köthe function spaces and $0 < \alpha, \beta < \infty$ we can define a quasi-Köthe function space $Z = X^\alpha Y^\beta$ by setting

$$\|f\|_Z = \inf\{\max(\|g\|_X, \|h\|_Y)^{\alpha+\beta} : |f| = |g|^\alpha |h|^\beta\} .$$

Then Z is exactly $1/(\alpha + \beta)$-convex. It may also be shown easily that, since X, Y are assumed maximal, there is always an optimal factorization $|f| = |g|^\alpha |h|^\beta$.

We now describe a simple method of doing calculations with Köthe function spaces introduced in [18]. We will not need the full force of the results in [18] and thus we will try to keep to description brief. Let us recall [18] that a semi-ideal $\mathscr{I}$ is a subset of $L_{1,+}$ so that if $0 \le f \le g \in \mathscr{I}$ then $f \in \mathscr{I}$; $\mathscr{I}$ is strict if it contains a strictly positive function. For a functional $\Phi : \mathscr{I} \to \mathbf{R}$ we define

$$\Delta_\Phi(f, g) = \Phi(f) + \Phi(g) - \Phi(f + g) .$$

We say that Φ is semilinear if:

(1) If $f \in \mathscr{I}$ and $\alpha > 0$, then $\Phi(\alpha f) = \alpha\Phi(f)$,

(2) There is a constant δ so that for all $f, g \in \mathscr{I}$ we have $\Delta_\Phi(f, g) \le \delta(\|f\|_1 + \|g\|_1)$,

(3) If $f \in \mathscr{I}$ and $0 \le f_n \le f$ with $\|f_n\|_1 \to 0$, then $\lim \Phi(f_n) = 0$.

If X is a Köthe function space we define $\mathscr{I}_X$ to be the set of nonnegative functions f in L_1 so that

$$\sup_{x \in B_X} \int f \log_+ |x| \, d\mu < \infty$$

and there exists $x \in B_X$ so that $f \log |x|$ is integrable. Then $\mathscr{I}_X$ is a strict semi-ideal.

On $\mathscr{I}_X$ we can define the indicator functional

$$\Phi_X = \sup_{x \in B_X} \int f \log |x| \, d\mu .$$

The indicator function Φ_X is semilinear with $\delta \le \log 2$ (see [18], Proposition 4.2). In the special case $X = L_1$ we obtain

$$\mathscr{I}_X = L \log L \quad \text{and} \quad \Phi_{L_1}(f) = \Lambda(f) = \int f \log f \, d\mu .$$

It then may be shown that for general X and $f, g \in \mathscr{I}_X \cap L \log L$, we have

$$0 \le \Delta_{\Phi_X}(f, g) \le \Delta_\Lambda(f, g) .$$

There is a converse to this result ([18]. Theorem 5.2). If Φ is a semilinear map on a strict semi-ideal $\mathscr{I} \subset L \log L$ so that $0 \le \Delta_\Phi(f, g) \le \Delta_\Lambda(f, g)$ for all $f, g \in \mathscr{I}$, then there is a unique Köthe function space X so that $\mathscr{I}_X \supset \mathscr{I}$ and $\Phi(f) = \Phi_X(f)$ for $f \in \mathscr{I}$. Furthermore, X is exactly p-convex and exactly q-concave if and only if

$$\frac{1}{q} \Delta_\Lambda(f, g) \le \Delta_{\Phi_X}(f, g) \le \frac{1}{p} \Delta_\Lambda(f, g) \quad \text{for} \quad f, g \in \mathscr{I}.$$

It is also easy to see that if $Z = X^\alpha Y^\beta$, then

$$\Phi_Z(f) = \alpha\Phi_X(f) + \beta\Phi_Y(f) \quad \text{for} \quad f \in \mathscr{I}_X \cap \mathscr{I}_Y .$$

This enables us to use the indicator functions to calculate spaces. Let us give a simple application which we will later; basically this is a simple generalization of PISIER's extrapolation theorem [28] (cf. [18], Corollary 5.4).

Proposition 2.1. *Suppose that X, Y are Köthe function spaces with Y super-reflexive. Then there exists a super-reflexive Köthe function space Z and $0 < \theta < 1$ so that $Y = X^{1-\theta}Z^{\theta}$ up to equivalence of norm.*

Proof. For convenience, we do all calculations on a strict semi-ideal contained in

$$\mathscr{I}_X \cap \mathscr{I}_Y \cap L \log L .$$

We can assume that Y is exactly p-convex and q-concave where $\frac{1}{p} + \frac{1}{q} = 1$ and $2 < q < \infty$. Let $\alpha = \frac{1}{2q}$. We define $\Phi = \Phi_Y + \alpha(\Phi_Y - \Phi_X)$. Then

$$\Delta_{\Phi} = (1 + \alpha)\, \Delta_{\Phi_Y} - \alpha \Delta_{\Phi_X} \geq \alpha \Delta_{\Lambda} .$$

Similarly,

$$\Delta_{\Phi} \leq (1 + \alpha)\,(1 - 2\alpha)\, \Delta_{\Lambda} \leq (1 - \alpha)\, \Delta_{\Lambda} .$$

Thus we can apply Theorem 5.2 of [18] to find a space Z so that $\Phi_Z = \Phi$ and Z will be $2q$-concave and r-convex where $\frac{1}{r} + \frac{1}{2q} = 1$. ■

Finally let us define $L_{\log}$ to be the Orlicz space of all $f \in L_0$ so that $\int \log_+ |f| \, d\mu < \infty$. Then $L_{\log}$ can be F-normed by

$$f \to \int \log (1 + |f|) \, d\mu .$$

We will especially concerned with the class of Köthe function spaces $\mathscr{X}$ of all X so that $X, X^* \subset L_{\log}$.

Lemma 2.2. *If X is a Köthe function space, then the following conditions are equivalent:*

(1) $X \in \mathscr{X}$.

(2) *If $f \in X$ there exists $g \in X$ with $g \geq |f|$ and* $\log g \in L_1$.

(3) *If $\varepsilon > 0$ and $f \in X$ there exists $g \in X$ with*

$$g \geq |f|, \ \|g\|_X \leq \|f\|_X + \varepsilon \quad \text{and} \quad \log g \in L_1 .$$

(4) $L_\infty \subset \mathscr{I}_X$.

Proof. (1) ⇒ (4). We must show $\chi_S \in \mathscr{I}_X$. If $X \subset L_{\log}$, then it follows from the Closed Graph Theorem that the inclusion is continuous and hence

$$\sup_{x \in B_X} \int f \log_+ |x| \, d\mu < \infty .$$

On the other hand, by a theorem of LOZANOVSKII [25] (cf. [13], [25]) there exist nonnegative $x \in B_X$ and $x^* \in B_{X^*}$ with $xx^* = \chi_S$.

Thus

$$\log|x| = \log_+|x| - \log_+|x^*| \in L_1 .$$

(4) $\Rightarrow$ (3). If $f \in X$, then $\log_+|f| \in L_1$. However there exists $h \in B_X$ with $\log|h| \in L_1$. Now take $g = \max(|f|, \eta|h|)$ for small enough η.

(3) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Clearly the conditions imply $X \subset L_{\log}$. Now suppose $x^* \in X^*$. There exists $x \in X$ with $\log|x| \in L_1$. Now $xx^* \in L_1$ so that $\int \log|xx^*|\,\mathrm{d}\mu < \infty$. Hence $\int \log|x^*|\,\mathrm{d}\mu < \infty$, i.e., $\log_+|x^*| \in L_1$. Thus $X^* \subset L_{\log}$. ■

Remark. In doing calculations with indicator functions we can always restrict to a small enough strict semi-ideal. Later in the paper for economy we will not mention the domains of the indicators in question when doing algebraic manipulations. The reader may wish simply to consider only spaces $X \in \mathscr{X}$ and regard all indicator functions as defined on $L_{\infty,\mathbf{R}}$.

3. Complex interpolation of subspaces

Let us describe a very general setting for complex-type interpolation. We recall that if X is a topological vector space and D is an open subset of the complex plane, then a function $F: D \to X$ is analytic if for each $a \in D$ there exists a neighborhood U of a and a power series $\sum_{n=0}^{\infty} x_n(z-a)^n$ so that $F(z) = \sum x_n(z-a)^n$ for $z \in U$. We will consider a triple $(D, X, \mathscr{F})$ where D is an open subset of the complex plane conformally equivalent to the unit disk Δ, X is a complex topological vector space and $\mathscr{F}$ is subspace of the space $\mathscr{A}(D, X)$ of all X-valued analytic functions on D equipped with a norm $F \to \|F\|_{\mathscr{F}}$ such that:

1. If $F \in \mathscr{A}(D, X)$ and φ is any conformal mapping of D onto Δ, then $\varphi F \in \mathscr{F}$ if and only if $F \in \mathscr{F}$ and $\|\varphi F\|_{\mathscr{F}} = \|F\|_{\mathscr{F}}$.

2. If $z \in D$ and $x \in X$, then $\inf\{\|F\|_{\mathscr{F}} : F(z) = x\} = 0$ if and only if $x = 0$.

Under these assumptions, we define, for $z \in D$, $X_z = \{x : \|x\|_{X_z} < \infty\}$ where

$$\|x\|_{X_z} = \inf\{\|F\|_{\mathscr{F}} : F(z) = x\} .$$

We will call the spaces $\{X_z : z \in D\}$ the interpolation field generated by $\{D, X, \mathscr{F}\}$.

The following elementary lemma will be used repeatedly. If $a, b \in D$ we let $\delta(a, b) = |\varphi(b)|$ where φ is any conformal map of D onto Δ with $\varphi(a) = 0$. Thus if $D = \Delta$ we have

$$\delta(a, b) = \frac{|b-a|}{|1-\bar{a}b|} .$$

If $D = \mathscr{S}$ is the strip $\mathscr{S} = \{z : 0 < \Re z < 1\}$, then for $0 \le s, t \le 1$ we have

$$\delta(s, t) = \frac{\sin\frac{\pi}{2}|s-t|}{\sin\frac{\pi}{2}(s+t)} .$$

Lemma 3.1. *Suppose that $F, G \in \mathscr{F}$ and that $a \in D$. Suppose $F(a) = G(a)$. Then*

$$\|F(z) - G(z)\|_{X_z} \leq \delta(a, z)\, \|F - G\|_{\mathscr{F}}.$$

Proof. Let φ be an conformal map of D onto Δ with $\varphi(a) = 0$. Let $H \in \mathscr{F}$ be defined so that $\varphi H = F - G$. Then $\|H\|_{\mathscr{F}} \leq \|F - G\|_{\mathscr{F}}$ and $\|F(z) - G(z)\|_{X_z} \leq |\varphi(z)|\, \|H\|_{\mathscr{F}}$. The lemma follows. ■

Now suppose that V is a linear subspace of X. Let $\mathscr{F}(V)$ be the space of $F \in \mathscr{F}$ such that $F(z) \in V$ for every $z \in D$. Let (V_z) be the interpolation field generated by $\{D, Y, \mathscr{F}(V)\}$. We will say that V is interpolation stable at $z \in D$ if there is a constant C so that for $v \in V_z$ we have $\|v\|_{V_z} \leq C\|v\|_{X_z}$. The least such constant C we denote by $K(z) = K(z, V)$ where $K(z) = \infty$ if V fails to be interpolation stable.

Theorem 3.2. *Suppose that V is interpolation stable at some $a \in D$; let $K(a) = K$. Then V is interpolation stable at any $z \in D$, with $3K\delta(a, z) < 1$, and $K(z) \leq 4K(1 - 3K\delta)^{-1}$. In particular, the set of $z \in D$ so that V is interpolation stable at z is open.*

Proop. Suppose $\delta = \delta(a, z) < 1/(3K)$. Suppose $v \in V_z$; then for $\varepsilon > 0$ we can pick $F \in \mathscr{F}(V)$ and $G \in \mathscr{F}$ so that $F(z) = G(z) = v$ and $\|F\|_{\mathscr{F}} \leq (1 + \varepsilon)\, \|v\|_{V_z}$ while $\|G\|_{\mathscr{F}} \leq (1 + \varepsilon)\, \|v\|_{X_z}$. Thus by Lemma 3.1,

$$\|F(a) - G(a)\|_{X_a} \leq (1 + \varepsilon)\, \delta(\|v\|_{X_z} + \|v\|_{V_z}).$$

It follows that

$$\|F(a)\|_{X_a} \leq (1 + \varepsilon)\, (1 + \delta)\, \|v\|_{X_z} + \delta\, \|v\|_{V_z}).$$

Now pick $H \in \mathscr{F}(V)$ so that $H(a) = F(a)$ and

$$\|H\|_{\mathscr{F}} \leq (1 + \varepsilon)\, \|F(a)\|_{V_a} \leq (1 + \varepsilon)\, K\, \|F(a)\|_{X_a}.$$

Then

$$\begin{aligned}
\|v\|_{V_z} &\leq \|F(z) - H(z)\|_{V_z} + \|H(z)\|_{V_z} \\
&\leq \delta(\|F\|_{\mathscr{F}} + \|H\|_{F}) + \|H\|_{\mathscr{F}} \\
&\leq \delta\, \|F\|_{\mathscr{F}} + (1 + \delta)\, (1 + \varepsilon)\, K\, \|F(a)\|_{X_a} \\
&\leq \delta(1 + K + K\delta)\, (1 + \varepsilon)\, \|v\|_{V_z} + (1 + \delta)^2\, (1 + \varepsilon)^2\, K\, \|v\|_{X_z}.
\end{aligned}$$

Since $\delta < 1$ and $K \geq 1$, we have that

$$\|v\|_{V_z} \leq 3K\delta\, \|v\|_{V_z} + 4K\, \|v\|_{X_z}$$

whence we conclude that

$$\|v\|_{V_z} \leq \frac{4K}{(1 - 3K\delta)}\, \|v\|_{X_z}$$

and so $K(z) \leq 4K(1 - 3K\delta)^{-1}$. ■

Theorem 3.3. *Suppose that V, W are two subspaces of X which are both interpolation stable at a. Suppose further that $X_a = V_a \oplus W_a$. Then there exists $\eta > 0$ so that if $\delta(a, z) < \eta$ then $X_z = V_z \oplus W_z$.*

Proof. Let $K = \max(K(a, V), K(a, W))$ and let $M = \max(\|P\|, \|Q\|)$ where P, Q are the induced projections from X_a onto V_a and W_a. We let $\eta = 1/(300K^2M)$. Suppose that $z \in D$ satisfies $\delta = \delta(a, z) < \eta$.

First suppose that z is such that $V_z + W_z$ fails to be dense in X_z. Then there exists $x \in X_z$ so that $\|x\|_{X_z} = 1$ and $\|x - (v + w)\|_{X_z} \geq \frac{1}{2}$ whenever $v \in V_z$ and $w \in W_z$ Pick any $F \in \mathscr{F}$ with $F(z) = x$ and $\|F\|_{\mathscr{F}} \leq 2$. Then there exist $G \in \mathscr{F}(V)$ and $H \in \mathscr{F}(W)$ so that $\|G\|_{\mathscr{F}}, \|H\|_{\mathscr{F}} \leq 4KM$ and $G(a) = PF(a)$, $H(a) = QF(a)$. Then

$$\begin{aligned} \tfrac{1}{2} &\leq \|F(z) - G(z) - H(z)\|_{X_z} \\ &\leq \delta(2 + 2(4KM)) \\ &\leq 10KM\eta\,. \end{aligned}$$

This contradiction immediately leads to the conclusion that $V_z + W_z$ is dense in X_z.

To complete the proof, suppose $v \in V_z$, $w \in W_z$ satisfying $\|v + w\|_{X_z} = 1$. Let $\gamma = \max(1, \|v\|_{X_z}, \|w\|_{X_z})$. We will show that $\gamma \leq 8KM$ and this will complete the proof. We choose $F \in \mathscr{F}$ with $\|F\|_{\mathscr{F}} \leq 2$ and $F(z) = v + w$. Notice that $K(z, V), K(z, W) \leq 8K$ by Theorem 3.2. We therefore pick $G \in \mathscr{F}(V)$, $H \in \mathscr{F}(W)$ so that $\|G\|_{\mathscr{F}} \leq 10K\gamma$ and $\|H\|_{\mathscr{F}} \leq 10K\gamma$, and $G(z) = v$, $H(z) = w$.

Now we have, by Lemma 3.1, $\|F(a) - G(a) - H(a)\|_{X_a} \leq 30K\gamma\delta$ (where $\delta = \delta(a, z)$) and hence $\|PF(a) - G(a)\|_{X_a} \leq 30KM\gamma\delta$ and $\|QF(a) - H(a)\|_{X_a} \leq 30KM\gamma\delta$. However $\|PF(a)\|_{X_a} \leq M\,\|F(a)\|_{X_a} \leq 2M$ and so obtain an estimate

$$\|G(a)\|_{X_a} \leq 2M + 30KM\gamma\delta)$$

with a similar estimate for $\|H(a)\|_{X_a}$. Thus

$$\|G(a)\|_{V_a} \leq 2KM + 30K^2M\gamma\delta < 2KM + \frac{\gamma}{10}$$

and we can find $E \in \mathscr{F}(V)$ with $E(a) = G(a)$ and $\|E\|_{\mathscr{F}} \leq 3KM + \frac{1}{10}\gamma$. Now

$$\|v\|_{X_z} \leq \|E(z)\|_{X_z} + \|E(z) - G(z)\|_{X_z}.$$

Thus

$$\begin{aligned} \|v\|_{X_z} &\leq (1 - \delta)\,\|E\|_{\mathscr{F}} + \delta\,\|G\|_{\mathscr{F}} \\ &\leq 3KM(1 + \delta) + \tfrac{1}{5}\gamma + 10K\gamma\delta \\ &\leq 4KM + \tfrac{1}{2}\gamma\,. \end{aligned}$$

With a similar estimate on w we obtain

$$\gamma \leq (4KM) + \tfrac{1}{2}\gamma$$

and so $\gamma \leq 8KM$ as promised. ∎

Let us now give a simple application. Obviously one special case of the above construction is the usual Calderòn method of complex interpolation. To be more precise, note that if (X_0, X_1) is a Banach couple, then if we take $D = \mathscr{S}$, $X = X_0 + X_1$ and $\mathscr{F}$ to be the space of functions $F \in \mathscr{A}(\mathscr{S}, X)$ so that F is bounded on $\mathscr{S}$ and extends continuously to the closure of $\mathscr{S}$ so that F is X_j-continuous on the line $\Re z = j$ for $j = 0, 1$ then the interpolation field generated is given by $X_z = [X_0, X_1]_\theta$ where $\theta = \Re z$.

Now if X is a Banach space and $1 \leq p < \infty$, we consider the space $L_p(X) = L_p(\mathbf{T}, X)$ of all Bochner measurable functions $f: \mathbf{T} \to X$ so that

$$\|f\|_p = \left[\int_0^{2\pi} \|f(e^{it})\|^p \frac{dt}{2\pi}\right]^{1/p} < \infty .$$

The subspace $H_p(X)$ consists of all $f \in L_p(X)$ so that $\int_0^{2\pi} f(e^{it}) \frac{dt}{2\pi} = 0$. It is well-known that $H_p(X)$ is complemented in $L_p(X)$ by the vector-valued Riesz projection if and only if $1 < p < \infty$ and X is UMD-space (see [3], [4]).

Theorem 3.4. *Suppose that (X_0, X_1) is a Banach couple. Suppose that for some $0 < \theta < 1$, we have that $[X_0, X_1]_\theta$ is a UMD-space and $[H_2(X_0), H_2(X_1)]_\theta = H_2[X_0, X_1]_\theta$. Then there exists $\eta > 0$ so that if $|\phi - \theta| < \eta$, then $[X_0, X_1]_\phi$ is also UMD.*

Remark. Blasco and Xu [2] show if X_0 and X_1 are UMD-spaces then

$$[H_2(X_0), H_2(X_1)]_\theta = H_2(X_\theta).$$

This results is therefore a converse to their result. They also present an example of Pisier to show that $(H_2(X_0), H_2(X_1)]_\theta$ need not coincide with $H_2(X_\theta)$. We remark that in [18] we construct an example where $[X_0, X_1]_{1/2} = L_2$ but X_θ is not UMD for any $\theta \neq \frac{1}{2}$, thus giving another counterexample.

Proof. We consider the Banach couple $(L_2(X_0), L_2(X_1))$. Let V be the subspace of $L_2(X_0) + L_2(X_1) \subset L_2(X_0 + X_1)$ of all f so that

$$\int_0^{2\pi} e^{int} f(e^{it})\, dt = 0$$

for $n > 0$. Let W be the space of all f so that

$$\int_0^{2\pi} e^{int} f(e^{it})\, dt = 0$$

for $n \leq 0$. Our assumptions guarantee that V, W are interpolation stable at θ and that

$$[L_2(X_0), L_2(X_1)]_\theta = L_2([X_0, X_1]_\theta = V_\theta \oplus W_\theta .$$

By Theorem 3.3. we obtain a similar decomposition for $|\phi - \theta| < \eta$ which implies the result. ■

Let us now discuss the case of Köthe function spaces. Suppose that S is a Polish space and that μ is a probability measure on S. As in [18] we consider the class $\mathcal{N}^+$ of all functions $F: \Delta \to L_0$ of the form $F(z)(s) = F_s(z)$ where F_s is in the Smirnov class N^+ for almost every $s \in S$. Then $\mathcal{N}^+(\mathcal{S})$ is the class of maps $F: \mathcal{S} \to L_0$ where $F \circ \varphi \in \mathcal{N}^+$ with $\varphi: D \to \mathcal{S}$ any conformal mapping. If $F \in \mathcal{N}^+(\mathcal{S})$ we can extend F to the lines $z = j + it$ $(j = 0, 1)$ so that $F(j + it) = \lim_{s \to j} F(s + it)$ in L_0, for a.e. t.

Suppose that X_0, X_1 are Köthe function spaces (assumed maximal so that $f \to \|f\|_{X_j}$ is lower-semi-continuous on L_0). Consider the space $\mathscr{F} = \mathscr{F}(X_0, X_1)$ of all $F \in \mathscr{N}^+(\mathscr{S})$ so that

$$\|F\|_{\mathscr{F}} = \max_{j=0,1} \{\operatorname{ess\,sup} \|F(j + it)\|_{X_j}\} < \infty .$$

Then $\mathscr{F}$ generates an interpolation field X_z for $z \in \mathscr{S}$ so that $X_z = X_0^{1-\theta} X^\theta$ where $\theta = \Re z$. Now suppose that Z is a separable Köthe function space which contains both X_0 and X_1. It is essentially shown in [18] that if $F \in \mathscr{F}(X_0, X_1)$, then $F: \mathscr{S} \to Z$ is analytic and $\lim_{s \to j} F(s + it) = F(j + it)$ in the space Z (so that we can work in Z in place of L_0); see Lemma 2.2. of [18].

Now suppose that V is a linear subspace of L_0 so that for some separable Köthe function space $Z \supset X_0, X_1$ the space $V \cap Z$ is closed in Z. Then $V_j = V \cap X_j$ is closed in X_j for $j = 0, 1$. Furthermore, if $F \in \mathscr{F}(X_0, X_1; V) = \mathscr{F}(X_0, X_1)(V)$, then F has boundary values in V_j along the line $z = j + it$, $-\infty < t < \infty$. The method of interpolation generated this way is not precisely the complex method introduced by Calderón, but we now make some remarks which establish that under reasonable hypotheses we obtain the same result.

The usual interpolation spaces $[V_0, V_1]_\theta$ are induced by considering the subspace $\mathscr{F}_c(X_0, X_1; V)$ of all F so that

(a) F is analytic into $V_0 + V_1$,
(b) $\lim_{s \to j} F(s + it)$ exist a.e. in $V_0 + V_1$

and

(c) $t \to F(j + it)$ is Bochner measurable into V_j for $j = 0, 1$.

In fact, only condition (c) is required; this is a consequence of the following lemma.

Lemma 3.5. *If $G \in \mathscr{F}(X_0, X_1; V)$, then $G \in \mathscr{F}_c(X_0, X_1; V)$ if and only if for each j the map $t \to G(j + it)$ has essentially separable range in X_j.*

Proof. This is essentially proved in Lemma 2.2. of [18], although our assumptions are a little less strict; we sketch the argument. Observe first that $t \to G(j + it)$ is Bochner measurable into X_j for each j. Let $\varphi: \Delta \to \mathscr{S}$ be a conformal mapping. Suppose $G \in \mathscr{F}(X_0, X_1; L_0)$. Then $F = G \circ \varphi \in \mathscr{N}^+$ and has L_0-boundary values $F(e^{it})$ for a.e. $0 \le t < 2\pi$. Then $t \to F(e^{it})$ is $V_0 + V_1$ and $X_0 + X_1$-measurable. It is thus Bochner integrable in both $X = X_0 + X_1$, and $W = V_0 + V_1$.

Now suppose that $w \in X^*$ is strictly positive. It follows that

$$\int_0^{2\pi} \int_S |F_s(e^{it})| \, w(s) \, d\mu(s) \frac{dt}{2\pi} < \infty .$$

Then if $F_s(z) = \sum_{n \ge 0} a_n(s) z^n$ for $|z| < 1$ we observe that $t \to F_s(e^{it}) \in N^+ \cap L_1 = H_1$ for a.e. s. Hence of $n \ge 0$, we have, μ-a.e.,

$$\int_0^{2\pi} F_s(e^{it}) \, e^{-int} \frac{dt}{2\pi} = a_n(s) .$$

Similarly if $n < 0$ we have

$$\int_0^{2\pi} F_s(e^{it})\, e^{-int} \frac{dt}{2\pi} = 0\,.$$

Hence we can also evaluate the Bochner integrals in $L_1(w)$

$$\int_0^{2\pi} F_s(e^{it})\, e^{-int} \frac{dt}{2\pi} = a_n$$

when $n \geq 0$ and

$$\int_0^{2\pi} F_s(e^{it})\, e^{-int} \frac{dt}{2\pi} = 0$$

when $n < 0$. These integrals have the same values in W and X and it thus follows easily that $F : \Delta \to W$ is analytic and has a.e. boundary values $F(e^{it})$. This implies that $G \in \mathscr{F}_c(X_0, X_1; V)$. ■

Remark: It follows that if X_0, X_1 are separable Köthe function spaces (which are as usual assumed to have the Fatou property), then $[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta$ (cf. [5]).

Proposition 3.6. *Suppose that X_0, X_1 are Köthe function spaces and that V is a linear subspace of L_0 so that $V \cap Z$ is closed in Z for some separable Köthe function space Z containing X_0, X_1. Suppose* $0 < \theta < 1$. Suppose that either

(a) X_0 *and* X_1 *are both separable or*
(b) $X_\theta = X_0^{1-\theta} X_1^\theta$ *is reflexive.*

Then V is interpolation stable at θ for the interpolation method generated by $\mathscr{F}(X_0, X_1)$ if an only if $[V_0, V_1]_\theta = V \cap X_\theta$ up to equivalence of norm.

Proof. It is immediately clear that $[V_0, V_1]_\theta = V \cap X_\theta$ implies interpolation stability of V. In the other direction consider first (a). In this case Lemma 3.5 implies that $\mathscr{F}_c(X_0, X_1; V) = \mathscr{F}(X_0, X_1; V)$ and the conclusion is immediate.

Now consider (b); let V_θ be the space induced by the method $\mathscr{F}(X_0, X_1; V)$ and let $W = [V_0, V_1]_\theta$. Let B be the closed unit ball of V_θ and let B' be the closed unit ball of W. Then $B' \subset B$; it follows from the Open Mapping Theorem since both spaces are complete that if we can show that B' is dense in B then W and V_θ coincide. Suppose that B'' is the V_θ-closure of B'. Since V_θ is a closed subspace of X_θ, B'' is weakly compact. If $B'' \neq B$ there exist $v \in V_\theta$ with $\|v\|_{V_\theta} \leq 1$ and $v \notin B''$. Since B''. Since B'' is weakly compact in Z there exist $\phi \in Z^*$ so that $|\phi(v)| > 1$ but $\sup_{w \in B'} |\phi(w)| \leq 1$. Let $|\phi(v)| + \varrho^2$ where $\varrho > 1$. Pick any $F \in \mathscr{F}(X_0, X_1; V)$ with $\|F\|_{\mathscr{F}} \leq \varrho$ and $F(\theta) = v$.

Now for $\tau > 0$ let

$$G_\tau(z) = \frac{1}{\tau} \int_0^\tau F(z + it)\,\mathrm{d}t\,,$$

for $z \in \mathscr{S}$, where the integrals are computed in Z. It is clear that the boundary values of G_τ are given by the same formula; hence $t \to G_\tau(j + it)$ is continuous in X_j for $j = 0, 1$. Since clearly $G_\tau \in \mathscr{F}(X_0, X_1; V)$, Lemma 3.5 can be applied to give that $G_\tau \in \mathscr{F}_c(X_0, X_1; V)$. Hence $\varrho^{-1}G_\tau(\theta) \in B'$. We conclude that $|\phi(G_\tau(\theta))| \le \varrho$ and letting $\tau \to 0$ gives $|\phi(v)| \le \varrho$, a contradiction. ■

In the situations when we will apply this result we will consider a closed subspace V of $L_{\log}$ and Köthe function spaces X_0, $X_1 \in \mathscr{X}$. The following lemma then shows that Propositoin 3.6 can be used.

Lemma 3.7. *If $X \in \mathscr{X}$, then there is a separable Köthe function $Z \supset X$ with $Z \in \mathscr{X}$.*

Proof. Simply pick $0 \le w \in X^*$ with $\log w \in L_1$ and let $Z = L_1(w)$. ■

4. Operators on Köthe function spaces

Now suppose that S is a Polish space and that μ is a probabilty measure on S. We suppose that $T: L_2(\mu) \to L_2(\mu)$ is a bounded self-adjoint operator with $\|T\| \le 1$. Now suppose that X is a Köthe function space on (S, μ). We define

$$\|T\|_X = \sup\{\|Tf\|_X : f \in L_2 \cap X,\ \|f\|_X \le 1\}\,.$$

If X is a separable Köthe function space (with the Fatou property), then $L_2 \cap X$ is dense in X and so T extends to a bounded operator $T: X \to X$ if and only if $\|T\|_X < \infty$.

The following remarks are elementary.

Lemma 4.1. (1) *For any separable Köthe function space X, we have $\|T\|_X = \|T\|_{X^*}$.*
(2) *If X, Y are separable Köthe function spaces and $0 < \theta < 1$, then* $\|T\|_{X^\theta Y^{1-\theta}} \le \|T\|_X^\theta \|T\|_Y^{1-\theta}$.

Proof. (1) is a trivial deduction from duality and the self-adjointness of T (the Banach space adjoint of T is the complex conjugate of the Hilbert space adjoint of T).
(2) is immediate from complex interpolation. ■

Let us now say that X is a T-direction (space) if there exists $0 < \theta < 1$ so that $\|T\|_{X^\theta L_2^{1-\theta}} < \infty$. Note that if $0 < \theta < 1$, then the space $X^\theta L_2^{1-\theta}$ is p-convex and q-concave where $\dfrac{1}{p} = 1 - \dfrac{1}{q} = \dfrac{1-\theta}{2} + \theta$. It is thus super-reflexive and hence separable. Clearly, by duality, X is a T-direction if and only if X^* is a T-direction.

If $w \in L_{0,\mathbb{R}}(\mu)$ we will that w is a T-weight direction if there exists $\alpha > 0$ so that T is bounded on $L_2(\mathrm{e}^{\alpha w})$. Thus w is T-weight direction if and only if $L_2(\mathrm{e}^w)$ is a T-direction. The space of all T-weight directions will be denoted by $\mathscr{D} = \mathscr{D}(T)$. We define

$$\|w\|_{\mathscr{D}} = \inf\{t > 0 : \|T\|_{L_2(\mathrm{e}^{w/t})} \le \mathrm{e}\}\,.$$

By complex interpolation it is clear that $\|w\|_{\mathscr{D}} < \infty$ if $w \in \mathscr{D}$.

Lemma 4.2. **(1)** *The set* $\{w : \|w\|_{\mathscr{D}} \leq 1\}$ *is a closed absolutely convex subset of* $L_{0,\mathbf{R}}$.
(2) *For any* $f \in L_2$ *and* $w \in \mathscr{D}$ *such that* $f, wf \in L_2$ *we have*

$$\|T(wf) - wTf\|_2 \leq \mathrm{e}\, \|w\|_{\mathscr{D}}\, \|f\|_2 .$$

In particular, $\|w\|_{\mathscr{D}} = 0$ *if and only if* $T(wf) = wT(f)$ *whenever* $f, wf \in L_2$.

Proof. Note that $\|w\|_{\mathscr{D}} \leq 1$ if and only if

$$\int |Tf|^2\, \mathrm{e}^{w/t}\, \mathrm{d}\mu \leq \mathrm{e}^2 \int |f|^2\, \mathrm{e}^{w/t}\, \mathrm{d}\mu$$

whenever $f \in L_2(1 + \mathrm{e}^w)$ and $t > 1$. It then follows from the Dominated Convergence Theorem that $\|w\|_{\mathscr{D}} \leq 1$ if and only if

$$\int |Tf|^2\, \mathrm{e}^{w}\, \mathrm{d}\mu \leq \mathrm{e}^2 \int |f|^2\, \mathrm{e}^{w}\, \mathrm{d}\mu$$

for $f \in L_2(1 + \mathrm{e}^w)$.

Now suppose that w_n is a sequence with $w_n \to w$ a.e. and $\|w_n\|_{\mathscr{D}} \leq 1$. Let $u = 1 + \sup \mathrm{e}^{w_n}$. Then if $f \in L_2(u)$ we clearly have $\|Tf\|_{L_2(\mathrm{e}^w)} \leq \mathrm{e}\, \|f\|_{L_2(\mathrm{e}^w)}$. By a density argument this estimate extends to $L_2(1 + \mathrm{e}^w)$. Hence $\|w\|_{\mathscr{D}} \leq 1$.

Convexity of the set $\{w : \|w\|_{\mathscr{D}} \leq 1\}$ follows from the fact that $L_2(u)^\theta L_2(v)^{1-\theta} = L_2(u^\theta v^{1-\theta})$. Symmetry follows from the fact that $L_2(u)^* = L_2(u^{-1})$.

(2) Finally suppose $\|w\|_{\mathscr{D}} \leq 1$. Then for any real $-1 \leq t \leq 1$ we have $\|T\|_{L_2(\mathrm{e}^{tw})} \leq \mathrm{e}$. Suppose $f, g \in \bigcap_{n \in \mathbf{Z}} L_2(\mathrm{e}^{nw})$. Then the maps $z \to \mathrm{e}^{zw} f$ and $z \to \mathrm{e}^{zw} g$ are entire L_2-valued functions. It follows that the map $\varphi(z) = \int T(\mathrm{e}^{zw} f)\, \mathrm{e}^{-zw} g\, \mathrm{d}\mu$ is an entire function. However if $z = x + iy$ with $-1 \leq x \leq 1$,

$$\begin{aligned} |\varphi(z)| &\leq \left(\int |T(\mathrm{e}^{zw} f)|^2\, \mathrm{e}^{2xw}\, \mathrm{d}\mu\right)^{1/2} \left(\int |\mathrm{e}^{-zw}|^2\, \mathrm{e}^{2xw}\, |g|^2\, \mathrm{d}\mu\right)^{1/2} \\ &\leq \mathrm{e}\, \|f\|_2\, \|g\|_2 \end{aligned}$$

and so by Cauchy's theorem, $|\varphi'(0)| \leq \mathrm{e}\, \|f\|_2\, \|g\|_2$. This implies that

$$\left|\int (T(wf) = wTf)\, g\, \mathrm{d}\mu\right| \leq \mathrm{e}\, \|f\|_2\, \|g\|_2 .$$

By varying g we see that $T(wf) - wTf \in L_2$ and $\|T(wf) - wTf\|_2 \leq \mathrm{e}\, \|f\|_2$ whenever $f \in \bigcap_{n \in \mathbf{Z}} L_2(\mathrm{e}^{nw})$. A simple approximation argument completes the proof that this holds under the weaker hypothesis that $f, wf \in L_2$.

Clearly now if $\|w\|_{\mathscr{D}} = 0$ we obtain the conclusion that $T(wf) = wTf$ under the same hypotheses. Conversely if $T(wf) = wTf$ for all f such that $f, wf \in L_2$ it is easy to reverse the argument to show that $\|T\|_{L_2(\mathrm{e}^{tw})} \leq 1$ for all real t. ∎

Now if X is a Köthe function space, we will say that X satifies the T-weight condition if there exist constants (C, M) so that if $0 \leq f \in B_X$, then there exists $g \geq f$ with $\|g\|_X \leq M$ and $\|\log g\|_{\mathscr{D}} \leq C$. We then say that X satisfies the T-weight condition with constants (C, M).

Theorem 4.3. **(1)** *Suppose that* X_j *for* $j = 0, 1$ *are Köthe function space with the T-weight condition with constants* (C_j, M_j). *Then if* $0 < \theta < 1$, $X_0^{1-\theta} X_1^\theta$ *has the T-weight condition with constants* $((1 - \theta)\, C_0 + \theta C_1, M_0^{1-\theta} M_1^\theta)$.

(2) *Suppose* $0 < \theta < 1$. *Then for any Köthe function space* X, X *has the T-weight condition with constants* (C, M) *if and only* $X^\theta (= L_\infty^{1-\theta} X^\theta)$ *has the T-weight condition with constants* $(\theta C, M^\theta)$.

Proof. (1) Suppose that X_j satisfies the T-weight condition with constants (C_j, M_j). Suppose

$$0 \le f \in X_\theta = X_1^{1-\theta} X_1^\theta \quad \text{with} \quad \|f\|_{X_\theta} \le 1 .$$

We may factor $f = f_0^{1-\theta} f_1^\theta$ where $0 \le f_j \in B_{X_j}$ for $j = 1, 2$. Then pick $g_j \in X_j$ with $0 \le f_j \le g_j$ and $\|g_j\|_{X_j} \le M_j$ so that $\|\log g_j\|_{\mathscr{D}} \le C_j$. Then $f \le g = g_0^{1-\theta} g_1^\theta$, and clearly $\|g\|_{X_\theta} \le M_0^{1-\theta} M_1^\theta$ and $\|\log g\|_{\mathscr{D}} \le (1-\theta) C_0 + \theta C_1$.

Before proceeding we will need a technical lemma.

Lemma 4.4. *Suppose that* X *is a Köthe function space with the property that there exist constants* $0 < c < 1$, C, M *so that if* $0 \le f \in X$ *there exists a Borel set* $A \subset S$, *and* $g \ge f_{\chi_A}$ *such that:*

(1) $\|f - f_{\chi_A}\|_X \le c \|f\|_X$,
(2) $\|g\|_X \le M \|f\|_X$,
(3) $\|\log g\|_{\mathscr{D}} \le C$.

Then X *satisfies the T-weight condition with constants* (C', M') *where* $C' = \max(1, C)$ *and for suitable* M'.

Proof. Suppose $f = f_0 \in B_X$. We inductively define Borel sets $(A_n)_{n=1}^\infty$, and sequences $(f_n)_{n\ge 1}, (g_n)_{n\ge 1}$ in X_+ so that for $n \ge 1$,

$$\|f_{n-1} - f_{n-1}\chi_{A_n}\| \le c \|f_{n-1}\|_X ,$$
$$g_n \ge f_{n-1}\chi_{A_n}$$
$$\|g_n\|_X \le M \|f_{n-1}\|_X$$
$$\|\log g_n\|_{\mathscr{D}} \le C$$
$$f_n = f_{n-1} - f_{n-1}\chi_{A_n} .$$

Then, by construction, $\|f_n\|_X \le c^n$ and $f - f_n = f\chi_{B_n}$ where $B_n = \bigcup_{k \le n} A_k$. It follows that $f = \max_{n \ge 1} f_{n-1}\chi_{A_n} \le \max_{n\ge 1} g_n$.

Now for $0 < p \le 1$ we have

$$\left\| \left(\sum g_n^p\right)^{1/p} \right\|_X \le \left(\sum \|g_n\|_X^p\right)^{1/p} \le M \left(\sum_{n=0}^\infty c^{np}\right)^{1/p}$$

by p-convexity of X. Thus

$$\left\| \left(\sum g_n^p\right)^{1/p} \right\|_X \le M(1 - c^p)^{-1/p} .$$

Choose $p = \min(1, 1/C)$, and let $g = \left(\sum g_n^p\right)^{1/p}$. Then $\|\log g_n^p\|_{\mathscr{D}} < 1$ and so that if $h \in L_2(1 + g)$ then

$$\int |Th|^2 g_n^p \, d\mu \le e^2 \int |h|^2 g_n^p \, d\mu .$$

On adding we see that

$$\int |Th|^2\, g^p\, d\mu \le e^2 \int |h|^2\, g^p\, d\mu\,.$$

Thus $\|\log g\|_{\mathscr{D}} \le \dfrac{1}{p}$, and the result follows. ■

Theorem 4.5. *Suppose that X_0 is a Köthe function space satisfying the T-weight condition and that X_1 is an arbitrary Köthe function space. Suppose that for some $0 < \phi \le 1$ the space $X_0^{1-\phi}X_1^{\phi}$ satisfies the T-weight condition. Then, for any $0 < \theta < 1$, the space $X_0^{1-\theta}X_1^{\theta}$ satisfies the T-weight condition. If X_1 is super-reflexive, then we also have that X_1 satisfies the T-weight condition.*

Proof. If $0 < \theta \le \phi$ this follows immediately from Theorem 4.3 We therefore suppose $0 < \phi < \theta < 1$. We will write $X_\tau = X_0^{1-\tau}X_1^{\tau}$. Suppose that X_0 satisfies the T-weight condition with costants (C_0, M_0) and that X_ϕ satisfies the T-weight condition with constants (C_ϕ, M_ϕ). We will verify the condition of Lemma 4.4. for the space X_θ. Fix a contant L so that $L^{\phi/\theta-\phi} = 2M_0M_\phi$.

Suppose $f = f_0 \ge 0$ and $\|f_\theta\|_{X_\theta} \le 1$. Then we can write $f_\theta = f_0^{1-\theta}f_1^{\theta}$ where $f_j \ge 0$, $\|f_j\|_{X_j} \le 1$ for $j = 0, 1$. Let $f_\phi = f_0^{1-\phi}f_1^{\phi}$ so that $\|f_\phi\|_{X_\phi} \le 1$. Then there exists $g_\phi \ge f_\phi$ with $\|g_\phi\|_{X_\phi} \le M_\phi$ and $\|\log g_\phi\|_{\mathscr{D}} \le C_\phi$.

We thus write $g_\phi = g_0^{1-\phi}g_1^{\phi}$ where $g_j \ge 0$ and $\|g_j\|_{X_j} \le M_\phi$. Then there exists $h_0 \ge g_0$ with $\|h_0\| \le M_0M_\phi$ and $\|\log h_0\|_{\mathscr{D}} \le C_0$. Next we define

$$h_\theta = h_0(g_\phi h_0^{-1})^{\alpha} \quad \text{where} \quad \alpha = \theta/\phi > 1 \quad \text{and} \quad (0/0) = 0\,.$$

Then

$$h_\theta = g_\phi(g_\phi h_0^{-1})^{\alpha-1} \le g_0^{1-\theta}g_1^{\theta}\,.$$

Thus $\|h_\theta\|_{X_\theta} \le M_\phi$. Let $h' = Lh_\theta$. We note that $\|h'\|_{X_\theta} \le LM_\phi$ and

$$\begin{aligned}\|\log h'\|_{\mathscr{D}} &= \|\log h_\theta\|_{\mathscr{D}}\\ &= \|(1-\alpha)\log h_0 + \alpha \log g_\phi\|_{\mathscr{D}}\\ &\le (\alpha - 1)\,C_0 + \alpha C_\phi\,.\end{aligned}$$

Let $A = \{s : f(s) \le Lh_\theta(s)\}$ and let $B = S \setminus A$. Then

$$f\chi_A \le h' \quad \text{and} \quad \|f - \chi_A\|_{X_\theta} \le \|f_0\chi_B\|_{X_0}^{1-\theta}\,.$$

Now if $s \in B$ we have

$$\begin{aligned}\frac{f(s)}{f_\phi(s)} &\ge L\,\frac{h_\theta(s)}{g_\phi(s)}\\ &= L\left(\frac{g_\phi(s)}{h_0(s)}\right)^{\alpha-1}.\end{aligned}$$

Hence

$$
\begin{aligned}
f_0(s) &= f_\phi(s)\left(\frac{f_\phi(s)}{f(s)}\right)^{-\phi/\theta-\phi} \\
&\le L^{-(\phi/\theta-\phi)} g_\phi(s)\left(\frac{g_\phi(s)}{h_0(s)}\right)^{-1} \\
&\le \frac{1}{2} M_0^{-1} M_\phi^{-1} h_0(s).
\end{aligned}
$$

We conclude that $\|f_0\chi_B\|_{X_0} \le \frac{1}{2}$ and hence that $\|f - f\chi_A\|_{X_\theta} \le (\frac{1}{2})^{1-\theta} = c < 1$ say. Thus the hypotheses of Lemma 4.4. are verified and X_θ has the T-weight condition.

For the last assertion, if X_1 is super-reflexive, we may suppose that there is a Köthe function space Y so that $X_1 = X_0^{1-\tau} Y^\tau$ for $0 < \tau < 1$. The above argument then gives the conclusion. ■

Let us draw a simple conclusion from Theorem 4.5.

Theorem 4.6. *Suppose that X is q-concave for some $q < \infty$ and that $\Phi_X = \sum_{j=1}^{n} a_j \Phi_{X_j}$ where $\alpha_j \in \mathbf{R}$ and each X_j satisfies the T-weight condition. Then X satisfies the T-weight condition.*

Proof. Since we may replace X by X^α where $0 < \alpha < 1$ we consider only the case when X is super-reflexive. It clearly also suffices to establish this theorem when $n = 2$. It follows directly from Theorem 4.3 when $\alpha_1, a_2 \ge 0$. If $\alpha_1, \alpha_2 \le 0$, then $-\Phi_X$ is convex so that Φ_X is linear when $X = wL_\infty$ for some weight w which contradicts super-reflexivity. We may thus suppose that α_1 and α_2 have opposite signs and by Theorem 4.3 we need only consider the case $\alpha_1 = 1$ and $\alpha_2 < 0$. Define $Y_1 = X^{1/2}$ and then let Y_0 be defined by $Y_0 = X_1^{1/2} X_2^{1/4}$. Then Y_0 is an interpolation space between X and $X_1^{1/2} X_2^{1/2}$ and so is super-reflexive. By Theorem 4.3, Y_0 satisfies the T-weight condition; but for an appropriate $\phi > 0$ we have $Y_0^{1-\phi} Y_1^\phi = X_1^{1/2}$ which also satisfies the T-weight condition. Now by Theorem 4.5 we conclude that $X^{1/2}$ satisfies the T-weight condition and thus Theorem 4.3 completes the proof. ■

Lemma 4.7. *Let X be an exactly 2-convex Köthe function space and let $Y = (X^2)^*$. Then:*
(1) *If Y satisfies the T-weight condition with constants $(1, M)$ then $\|T\|_X < \infty$.*
(2) *If $\|T\|_X < \infty$, then Y satisfies the T-weight condition.*

Proof. (1) Suppose $f \in L_2 \cap B_X$. Suppose $0 \le u \in Y$ with $\|u\|_Y \le 1$. Then there exists $v \ge u$ so that $\|v\|_Y \le M$ and $\|\log v\| \le 1$. Thus (cf. [30], Theorem A'),

$$\int |Tf|^2 u \, d\mu \le e^2 \int |f|^2 v \, d\mu \le M e^2 \, \| |f|^2 \|_{X^2}$$

and henced $\|Tf\|_X \le M^{1/2} e \, \|f\|_X$.

(2) This follows from a result of Rubio de Francia ([30], Theorem A'); in the case when X is a weighted L_p-space for $p > 2$ it was shown by Cotlar and Sadosky [9]. In fact by Therorem A' of [30] there is a constant M so that if $0 \le u \in B_Y$, there exists $v \ge u$ with $\|T\|_{L_2(v)} \le M$ and $\|v\|_Y \le 2 \|u\|_y$. Now by interpolation $\|\log v\|_{\mathscr{D}} \le \log M$ and we are done. ■

Lemma 4.8. *Let X be an exactly 2-convex Köthe function space and let $Y = (X^2)^*$. Then Y satisfies the T-weight condition if and only if X is a T-direction space.*

Proof. Suppose that Y satisfies the T-weight condition with constants (C, M). Choose $\theta > 0$ so that $\theta C < 1$. Consider the space $Z = L_2^{1-\theta} X^\theta$. Then

$$\Phi_Z = \frac{(1-\theta)}{2} \Lambda + \theta \Phi_X$$

and so $2\Phi_Z + \theta\Phi_Y = \Lambda$ so that $Y^\theta = (Z^2)^*$. By applying Lemma 4.7, is bounded on Z.

Conversely, if X is a T-direction space there exists $\theta > 0$ so that $\|T\|_Z < \infty$ where $Z = L_2^{1-\theta} X^\theta$ and so by Lemma 4.7, Y^θ satisfies the T-weight condition. Theorem 4.3 completes the proof. ■

We are now finally able to state our main result of this section.

Theorem 4.9. *Suppose that $T: L_2 \to L_2$ is a self-adjoint operator with $\|T\| \le 1$. Suppose that L_∞ is a T-direction space (i.e., there exists $p > 2$ so that $\|T\|_{L_p} < \infty$). Then*

(1) *If X satisfies the T-weight condition, then X is a T-direction space.*

(2) *If X is q-concave for some $q < \infty$, then X is a T-direction space if and only if X satisfies the T-weight condition.*

Remark. Note that L_∞ is always a T-weight space. In general our assumption that T is bounded at some L_p where $p > 2$ is equivalent to the requirement that L_2 satisfies the T-weigth condition by Lemma 4.8. This shown that the assumption is necessary for the theorem to hold.

Proof. We assume that $p > 2$ and $p' < 2$ are conjugate indices so that $\|T\|_{L_p} = \|T\|_{L_{p'}} < \infty$. We first notice that L_2 much satisfy the T-weight condition. Indeed, by Lemma 4.7, L_r satisfies the T-weight conditions when $\dfrac{1}{r} + \dfrac{2}{p} = 1$ and hence by Theorem 4.3 L_2 satisfies the T-weight condition.

We will now prove (2) under the stronger hypothesis that X is super-reflexive.

We next show that, in general, if X is super-reflexive and satisfies the T-weight condition, then X^* also satisfies the T-weight condition. In fact, $L_2 = X^{1/2}(X^*)^{1/2}$ and so it follows from Theorem 4.5 that X^* has the T-weight condition.

We now proceed to the proof of the theorem. Assume first that X is super-reflexive and satisfies the T-weight condition. We now may select $0 < \alpha < 1$ small enough so that $(X^*)^\alpha$ has the T-weight condition with constants $(1, M)$ for suitable M. Now by Lemma 4.7 T is bounded on the space Z where

$$2\Phi_Z = \Lambda - \alpha\Phi_{X^*}, \quad \text{or} \quad \Phi_Z = \tfrac{1}{2}(1-\alpha)\Lambda + \tfrac{1}{2}\alpha\Phi_X .$$

Now $\Delta_{\Phi_Z} \ge \frac{1}{2}(1-\alpha)\Delta_\Lambda$ so that Z has nontrivial concavity and is thus super-reflexive. It follows that T is also bounded on any space $Y = L_{p'}^{1-\beta} Z^\beta$ where $0 < \beta < 1$. We select β so that Y is an interpolation space between L_2 and X. In fact

$$\Phi_Y = \left((1-\beta)\left(1 - \frac{1}{p}\right) + \frac{\beta}{2}(1-\alpha)\right)\Lambda + \frac{1}{2}\alpha\beta\Phi_X$$

and the conclusion is obtained by choosing β so that

$$2(1-\beta)\left(1-\frac{1}{p}\right)+\beta(1-\alpha)+\frac{1}{2}\alpha\beta=1$$

or

$$\beta=\frac{1-\dfrac{2}{p}}{\dfrac{\alpha}{2}+1-\dfrac{2}{p}}.$$

Now T is bounded at Y and so X is a T-direction space.

Now suppose, conversely, that X is a T-direction space. Then for suitable $\theta>0$, T is bounded at $L_2^{1-\theta}X^\theta$. Interpolating with L_p we see that T is also bounded at any space Z where

$$\Phi_Z=\left(\frac{1-\alpha}{p}+\frac{\alpha}{2}(1-\theta)\right)\Lambda+\theta\alpha\Phi_X$$

with $0<\alpha<1$. Notice that

$$\Delta_{\Phi_Z}\le\left(\frac{1-\alpha}{p}+\frac{\alpha(1+\theta)}{2}\right)\Delta_\Lambda,$$

and so by choosing α small enough we can suppose that Z is 2-convex. Let us put

$$\Phi_Z=\beta\Lambda+\gamma\Phi_X\quad\text{where}\quad 0<\beta,\gamma\quad\text{and}\quad\beta+\gamma<\tfrac{1}{2}.$$

Thus Y satisfies the T-weight condition where $\Phi_Y=\Lambda-2\Phi_Z$. We can now solve for Φ_X in form

$$\Phi_X=\frac{1}{2\gamma}\left((1-2\beta)\Lambda-\Phi_Z\right).$$

An applications of Theorem 4.6 now completes the proof for the case when X is super-reflexive.

Now consider (1). If X satisfies the T-weight condition, then so does $L_2^{1/2}X^{1/2}$ by Theorem 4.6 (or 4.5). This space is super-reflexive and so it is also a T-direction space; hence X is T-direction space.

Finally we complete the proof of (2) when X is q-concave for some finite q and is a T-direction space. Then $Y=L_2^{1/2}X^{1/2}$ is a T-direction space and is super-reflexive; hence it satisfies the T-weight condition. Since $\Phi_X=2\Phi_Y-\frac{1}{2}\Lambda$, we complete the proof by Theorem 4.6. ■

5. Interpolation of Hardy spaces

We again consider a probability measure μ on a Polish space S. Consider the Orlicz algebra $L_{\log}$, and let $\mathscr{X}$ be the collection of all Köthe function spaces X so that $X, X^*\subset L_{\log}$. Consider a closed subalgebra H of $L_{\log}$ (which is always assumed to contain the constants). We define for every $X\in\mathscr{X}$ the Hardy space $H_X=H\cap X$ so that H_X is a closed subspace of X. In particular, we define $H_p=L_p\cap X$ when $1\le p\le\infty$.

We will say that H is of Dirichlet type if for every invertible $f \in L_{\log}$ there exists $g \in H$ which is invertible in H so that $|g| = |f|$ a.e.; equivalently, H is a Dirichlet-type algebra if for every real $u \in L_1$ there exists an invertible $g \in H$ with $|g| = e^u$ a.e..

The simplest example of such a Dirichlet-type algebra is the Smirnov class N^+ (or Hardy algebra) considered as a subalgebra of $L_{\log}(\mathbf{T})$. In this way one generates the standard Hardy spaces. More generally suppose that A is a subalgebra of $L_\infty(S, \mu)$ so that $f \to \int f \, d\mu$ is a multiplicative linear functional and $\Re A$ is weak*-dense in $L_{\infty,\mathbf{R}}$. Thus A is a weak*-Dirichlet algebra (cf. [1], [12], [14]). Let H be the closure of A in $L_{\log}$; then H has the Dirichlet property and the standard abstract Hardy spaces are obtained. The reader may consult Gamelin [12] for details when A is generated by a Dirichlet algebra: see also Barbey-König [1].

Another example is obtained when one considers $(\mathbf{T} \times S, \lambda \times \mu)$ and defines H to be the space of all functions $f(t, s)$ so that $f \in L_{\log}$ and for a.e. $s \in S$ the function $f_s \in N^+$ where $f_s(t) = f(t, s)$. In this way we can treat vector-valued problems.

Notice that, in each case, one can always replace the measure μ by a measure $w \, d\mu$ as long as $w, \log w \in L_1$. This will not change $L_{\log}$ or H but will alter the space H_2. This change of density allows one to study skew projections.

Lemma 5.1. *Let H be any closed subalgebra of $L_{\log}$. If $f \in H_1$ the $e^f \in H$.*

Proof. The series $\sum_{n \geq 0} \frac{f^n}{n!}$ converges in $L_{\log}$ since it converges a.e. and

$$\sum_{n \geq 0} \frac{|f|^n}{n!} = e^{|f|} \in L_{\log}. \quad \blacksquare$$

Lemma 5.2. *Suppose that H is a Dirichlet-type algebra. Then if $f \in H$ and $v \in L_1$ there is a sequence $g_n \in H$ so that $|g_n| \leq \min(n\, e^v, |f|)$ and $g_n \to f$ in measure (and hence in $L_{\log}$).*

Proof. First consider the subspace G of $L_{1,\mathbf{R}} \times L_{0,\mathbf{R}}$ of all (u, v) so that $e^{\pm(u+iv)} \in H$. It is easy to check that G is closed. Hence by application of the Open Mapping Theorem if $\|u_n\|_1 \to 0$ there exist $v_n \to 0$ in L_0 so that $e^{u_n + iv_n} \in H$.

Now pick any $h \in L_{1,\mathbf{R}}$ with $h \geq |f|$. There exists an invertible $\psi \in H$ with $|\psi| = e^h$. Now let

$$u_n = h - \min(h, v + \log n).$$

Then $\|u_n\|_1 \to 0$ and so there exist $v_n \to 0$ in measure so that $e^{\pm(u_n + iv_n)} \in H$. Let $g_n = \psi\, e^{-(u_n + iv_n)} f$ and the result follows easily. $\blacksquare$

Suppose that H is a closed subalgebra of $L_{\log}$. We define V to the subspace of $L_{\log}$ of all f so that $\int \bar{f} g \, d\mu = 0$ whenever $g \in H$ and $fg \in L_1$. For $X \in \mathscr{X}$ we set $V_X = V \cap X$. It is trivial to see that if $f \in V$ and $g \in H$, then $f\bar{g} \in V$. We will V_p for V_{L_p} when $1 \leq p \leq \infty$.

Lemma 5.3. *Assume that H is a Dirichlet-type algebra.*

(1) *$f \in V$ if and only if there exists an invertible $g \in H$ so that $fg \in L_1$ and $\int \bar{f} gh \, d\mu = 0$ for evey $h \in H_\infty$.*

(2) *V is a closed subspace of $L_{\log}$.*

(3) *If $X \in \mathscr{X}$, then $X \cap H_\infty$ is dense in H_X and $X \cap V_\infty$ is dense in V_X.*

Proof. (1) Suppose $\psi \in H$ and $f\psi \in L_1$. Then by Lemma 5.2 there exists $\psi_n \in H$ so that

$$|\psi_n| \leq \min(n|g|, |\psi|) \quad \text{and} \quad \psi_n \to \psi \quad \text{in measure.}$$

Then $\int \bar{f}\psi_n \, d\mu = 0$ and the conclusion follows from Dominated Convergence.

(2) Suppose $f_n \to f$ in $L_{\log}$ where $f_n \in V$. By passing to a subsequence we can suppose that $F = \sup_n |f_n| \in L_{\log}$. Choose any invertible $g \in H$ so that $|g| \geq F$. Then

$$\int \bar{f} g^{-1} h \, d\mu = 0 \quad \text{for every} \quad h \in H_\infty .$$

Hence by (1), $f \in V$.

(3) Suppose $f \in X$; then (Lemma 2.2) there exists $w \geq |f|$ with $\log w \in L_1$. The there exists an invertible $g \in H$ with $|g| = w$ a.e. and by Lemma 5.2 a sequence $g_n \in H_\infty$ with $|g_n| \leq |g|$ so that $g_n \to g$ in measure. If $f \in H$, then the sequence $(fg^{-1}g_n)$ is in $H_\infty \cap X$, converges in measure to f and is lattice bounded by $|f|$. Hence it convergence also in X. If $f \in V$ we use a similar argument on $f\bar{g}^{-1}\bar{g}_n$. ■

From now on, we suppose that H is a Dirichlet-type algebra. We define $\mathscr{R}$ to be the orthogonal projetion of L_2 onto H_2; it follows from the preceding lemma that the kernel of $\mathscr{R}$ is V_2. Further, if $X \in \mathscr{X}$, then $\mathscr{R}$ is bounded at X if and only if $X = H_X \oplus V_X$. We will say that H is a Hardy-type algebra if $L_p = H_p \oplus V_p$ for all $1 < p < \infty$. Note that all the examples quoted are of Hardy type.

If $w \in L_{1,\mathbb{R}}$ we will say that $w \in BMO$ if $w \in H_1 + L_\infty$ and we define the BMO-norm by

$$\|w\|_{BMO} = \inf\{\|w - h\|_\infty : h \in H_1\} .$$

Let us note in passing that the infimum is attained. Indeed, if $h_n \in H_1$ is such that $\|w - h_n\|_\infty \to \|w\|_{BMO}$, then by KOMLOS's theorem [22], since (h_n) is L_1-bounded, we can pass to a sequence of convex combinations (g_n) of (h_n) which converge a.e. to some g. However it is easily seen that $\|g_n - g\|_p \to 0$ when $p < 1$ and so $g \in H$ since H is closed in $L_{\log}$.

Proposition 5.4. *If $w \in L_1$, then $w \in BMO$ if and only if w is an $\mathscr{R}$-weight direction. Further, there is a constant C so that if $w \in BMO$ then*

$$C^{-1} \|w\|_{BMO} \leq \|w\|_{\mathscr{D}(\mathscr{R})} \leq C \|w\|_{BMO} ,$$

Proof. First suppose $w \in \mathscr{D} \cap L_1$. By Lemma 4.2, if $f \in V_2$ then

$$\|\mathscr{R}(wf)\|_2 \leq e \|w\|_{\mathscr{D}} \|f\|_2 .$$

Now suppose $f \in V_\infty$ with $wf \in L_2$. Then for $\varepsilon > 0$ there exists an invertible $g \in H$ so that $|g| = |f|^{1/2} + \varepsilon$ a.e. Then $f\bar{g}^{-1} \in V_2$ and $g \in H_2$, and so

$$\int fw \, d\mu = \int \bar{g}(wf\bar{g}^{-1}) \, d\mu = \int \bar{g}\mathscr{R}(wf\bar{g}^{-1}) \, d\mu .$$

Hence we have

$$\left|\int fw \, d\mu\right| \leq e \|w\|_{\mathscr{D}} \|fg^{-1}\|_2 \|g\|_2 .$$

Letting $\varepsilon \to 0$ we obtain

$$\left|\int fw \, d\mu\right| \leq e \|w\|_{\mathscr{D}} \|f\|_1 .$$

Now by the Hahn-Banach theorem there exists $\psi \in L_\infty$ so that

$$\|\psi\|_\infty \leq \mathrm{e}\,\|w\|_{\mathscr{D}}$$

and

$$\int f(w - \overline{\psi})\,\mathrm{d}\mu = 0 \quad \text{for} \quad f \in V_\infty \cap L_2(|w|^2).$$

Now for any $f \in V_\infty$ we can find, utilizing Lemma 5.2, with $v = -(\log_+ |f| + \log_+ |w|)$, an invertible $g_n \in H$ so that

$$|g_n| \leq \min(1, n\,|f|^{-1}\,w^{-1}) \quad \text{and} \quad g_n \to 1 \quad \text{a.e.}$$

Then $\overline{g_n f} \in V_\infty \cap L_2(w^2)$ and by the Dominated Convergence Theorem we have $\int f(w - \overline{\psi})\,\mathrm{d}\mu = 0$.

It now follows, again from the Hahn-Banach theorem, that $w - \psi \in H_1$ and hence that $\|w\|_{BMO} \leq \mathrm{e}\,\|w\|_{\mathscr{D}}$.

Now conversely suppose $w \in BMO$, with $\|w\|_{BMO} \leq 1$. Let

$$X_0 = L_2(\mathrm{e}^{2w}) \quad \text{and} \quad X_1 = L_2(\mathrm{e}^{-2w}).$$

Then if $X_\theta = [X_0, X_1]_\theta$ we have $X_{1/2} = L_2$. We claim that H is interpolation stable at 1/2 and further, there is a universal constant C so that $K(\frac{1}{2}, H) \leq C$. In fact, there exists $h \in H_\infty$ so that $\|w - h\|_\infty \leq 1$. Suppose $f \in H_2$. Then we define a map $F : \mathscr{S} \to H$ by

$$F(z) = \mathrm{e}^{-1 + 4z^2}\,\mathrm{e}^{(1-2z)h} f.$$

It is clear that F is analytic into H and

$$\begin{aligned}\int |F(it)|^2\,\mathrm{e}^{-2w}\,\mathrm{d}\mu &= \mathrm{e}^{-2-8t^2} \int |\mathrm{e}^{2(1-2it)(h-w)}|\,|f|^2\,\mathrm{d}\mu \\ &= \mathrm{e}^{4|t|-8t^2}\,\|f\|_2^2 \leq \mathrm{e}^{1/2}\,\|f\|_2^2,\end{aligned}$$

while

$$\begin{aligned}\int |F(1+it)|^2\,\mathrm{e}^{-2w}\,\mathrm{d}\mu &\leq \mathrm{e}^{6-8t^2} \int |\mathrm{e}^{2(-1-2it)(h-w)}|\,|f|^2\,\mathrm{d}\mu \\ &\leq \mathrm{e}^{4+4|t|-8t^2}\,\|f\|_2^2 \leq \mathrm{e}^{9/2}\,\|f\|_2^2.\end{aligned}$$

It follows that H is interpolation stable at 1/2 with $K(1/2, H) \leq \mathrm{e}^{5/4}$. Now it follows from Theorem 3.3. and its proof that $L_2(\mathrm{e}^{w/t}) = H_2(\mathrm{e}^{w/t}) \oplus V_2(\mathrm{e}^{w/t})$ if $|t| \leq C$ for some absolute consant C. Thus $w \in \mathscr{D}(\mathscr{R})$ and $\|w\|_{\mathscr{D}} \leq C$. ■

We will now say that a Köthe function space $X \in \mathscr{X}$ is *BMO*-regular (for H) if there are constants (C, M) so that if $0 \leq f \in X$ with $\|f\|_X \leq 1$, then there exists $g \in X$ with $g \geq f$, $\|g\|_X \leq M$ and $\|\log g\|_{BMO} \leq C$.

Lemma 5.5. *Suppose $X \in \mathscr{X}$. Then X satisfies the $\mathscr{R}$-weight condition if and only if X is BMO-regular.*

Proof. One direction is obvious. For the other, note that if X is the $\mathscr{R}$-weight direction then given $f \in X_+$ with $\|f\|_X = 1$ there exists $f' \geq f$ with $\|f'\|_X f_X \leq 2$ and $\log f' \in L_1$ by Lemma 2.2. Thus if X satisfies the $\mathscr{R}$-weight condition with constants (C, M) there exists $g \geq f'$ with $\|g\|_X \leq 2M$ and $\|\log g\|_{\mathscr{D}} \leq C$. But then also $\log g \in L_1$ so that $\|\log g\|_{BMO} \leq C'$ for a suitable constant C'.

Proposition 5.6. *Let H be a Hardy-type algebra. If $X \in \mathcal{X}$ is super-reflexive, then X is BMO-regular if and only if X is an $\mathcal{R}$-direction. In particular, each L_p is BMO-regular.*

Proof. This is simply Theorem 4.9. ∎

Theorem 5.7. *Suppose that H is a Dirichlet-type algebra. Suppose that $X_0, X_1 \in \mathcal{X}$ are both BMO-regular and that $0 < \theta < 1$. Let $X_\theta = X_0^{1-\theta} X_1^\theta$. Suppose either that* (a) *both X_0, X_1 are separable or* (b) *X^θ is reflexive. Then H is interpolation stable at θ for (X_0, X_1), i.e., $[H_{X_0}, H_{X_1}]_\theta = H_{X_\theta}$.*

Proof. We suppose that, for $j = 0, 1, X_j$ are BMO-regular with constants (C_j, M_j). Suppose $f \in H_{X_\theta}$ with $\|f\|_{X_\theta} = 1$; then we can factor $|f| = f_0^{1-\theta} f_1^\theta$ where $0 \leq f_0, f_1$ and $\|f_j\|_{X_j} = 1$ for $j = 0, 1$. Pick $f'_j \geq f_j$ so that $\|f'_j\|_{X_j} \leq M_j$ and $\|\log f'_j\|_{BMO} \leq C_j$. Then pick $h_j \in H_1$ so that $\|\log f'_j - h_j\|_\infty \leq C_j$. We consider the following function for $z \in \mathcal{S}$,

$$F(z) = e^{z^2 - \theta^2} e^{(z-\theta)(h_1 - h_0)} f .$$

F is continuous into H and $F(\theta) = f$. Further if $z = j + it$ where $j = 0, 1$

$$|F(j + it)| \leq e^{j^2 - \theta^2 - t^2} e^{(|j-\theta| + |t|)(C_0 + C_1)} |f'_j| .$$

Hence $F \in \mathcal{F}(X_0, X_1; H)$ (see Section 3). Thus we get an estimate

$$\|F(j + it)\|_{X_j} \leq C' ,$$

where $C' = C'(C_0, C_1, M_0, M_1, \theta)$. We can now appeal to Proposition 3.6 to deduce that $\|f\|_{[H_{X_0}, H_{X_1}]_\theta} \leq C'$ and this proves the theorem.

Remark. In the case when $S = \mathbf{T}$ and $H = N^+$ then the spaces L_p satisfy the BMO-condition. This is immediate from Proposition 5.6 but there is an amusing alternative argument. It suffices to consider the case $p = 2$. The Hardy-Littlewood maximal function $\mathcal{M}$ is bounded on L_2 (cf. [31]) and for any $f \in L_2, \log \mathcal{M} f \in BMO$ by a result of Coifman-Rochberg [6] with an appropriate bound. Combining these facts shows that L_2 and every L_p satisfies the BMO-condition. Notice that this then implies an immediate proof of a well-known theorem of P. Jones [16], [17] that $[H_\infty, H_1]_\theta = H_p$ where $p = \dfrac{1}{\theta}$. Interpolation with H_p when $p < 1$ can be handled in the same way.

To understand the picture for interpolation in general, we need two further lemmas.

Lemma 5.8. *Suppose that $X_0, X_1 \in \mathcal{X}$ are separable Köthe function spaces and that $0 < \theta < 1$ is such that H is interpolation stable at θ for (X_0, X_1). Suppose that $Y_0, Y_1 \in \mathcal{X}$ are also separable Köthe function spaces so that for Köthe function space W we have $Y_j \approx X_j W$ for $j = 0, 1$. Then H is interpolation stable for (Y_0, Y_1).*

Proof. As usual let $X_\theta = X_0^{1-\theta} X_1^\theta$ and $Y_\theta = Y_0^{1-\theta} Y_1^\theta$. Suppose that K is the constant of interpolation stability at θ for (X_0, X_1). Suppose $f \in H_{Y_\theta}$, and $\|f\|_{Y_\theta} = 1$. Then we can factorize

$$f = bw \quad \text{where} \quad \|b\|_{X_\theta} = \|w\|_W = 1 .$$

Now pick

$$b' \geq |b| \quad \text{so that} \quad \log b' \in L_1 \quad \text{and} \quad \|b'\|_{X_\theta} \leq 2 .$$

The there exists an invertible $g \in H$ with $|g| = |b'|$. Hence there exists

$$F \in \mathcal{N}^+(\mathcal{S}) \quad \text{with} \quad F : \mathcal{S} \to H$$

so that

$$F(\theta) = g \quad \text{and (a.e.)} \quad \|F(j + it)\|_{X_j} \le 2K .$$

Define

$$G : \mathcal{S} \to H \quad \text{by} \quad G(z) = F(z)\, fg^{-1} .$$

It is easy to see that $\|G(j + it\|_{Y_j} \le 2K$ (a.e.) and $G(\theta) = f$. H is stable at θ for (Y_0, Y_1). ■

Lemma 5.9. *Suppose that $X_0, X_1 \in \mathcal{X}$ are separable Köthe function spaces such that H is interpolation stable at θ for (X_0, X_1). Then V is also interpolation stable at θ.*

Proof. Suppose $f \in X_\theta = X_0^{1-\theta} X_1^\theta$ and $f \in V$ with $\|f\|_{X_\theta} = 1$. Pick any

$$f' \in X_\theta \quad \text{so that} \quad f' \ge |f|,\ \|f'\|_{X_\theta} \le 2 \quad \text{and} \quad \log f' \in L_1 .$$

Then pick $g \in H$ so that $|g| = f'$. There exists an $F \in \mathcal{N}^+(\mathcal{S})$ with $F : \mathcal{S} \to H$ so that

$$F(\theta) = g \quad \text{and} \quad \|F(j + it)\|_{X_j} \le 2K$$

almost everywhere. Define

$$\overline{F}(z) = \overline{F(\overline{z})} \quad \text{and consider} \quad G(z) = f\overline{g}^{-1}\,\overline{F}(z) .$$

Then G is also admissible but has range in V, $G(\theta) = f$ and $\|G(j + it)\|_{X_j} \le 2K$ a.e. so V is also interpolation stable at θ. ■

Lemma 5.10. *Suppose that $X_0, X_1 \in \mathcal{X}$ are separable Köthe function spaces so that H is interpolation stable at some $0 < \theta < 1$ for (X_0, X_1). If $\mathcal{R}$ is bounded at $X_\theta (= X_0^{1-\theta} X_1^\theta)$, then there exists $\eta > 0$ so that $\mathcal{R}$ is also bounded on X_ϕ if $|\phi - \theta| \le \eta$.*

Proof. This follows directly from Theorem 3.3. and Lemma 5.9. ■

Remark. Let us note that this implies that if L_2 is BMO-regular then since H must be interpolation-stable at $\theta = \frac{1}{2}$ for $(L_{3/2}, L_3)$, then $\mathcal{R}$ is bounded on L_p for some $p > 2$. This provides a weak converse to Proposition 5.6.

For the remainder of this section we require that H is of Hardy type, i.e., the Riesz projection is bounded on L_p for $1 < p < \infty$.

Proposition 5.11. *Suppose that H is of Hardy type and $X \in \mathcal{X}$ is q-concave for some $q < \infty$. Then X is a $\mathcal{R}$-direction space if and only if X is BMO-regular.*

Proof. By Theorem 4.9 and Lemma 5.5. we obtain the result for super-reflexive X. In the general case if X is BMO-regular, then so is $L_2^{1/2} X^{1/2}$ and this must therefore be a $\mathcal{R}$-direction space, which implies that X is an $\mathcal{R}$-direction space. Conversely, if X is an $\mathcal{R}$-direction space, then $L_2^{1/2} X^{1/2}$ is BMO-regular. But then Theorem 4.6 implies that $X^{1/2}$ is BMO-regular since it is super-reflexive. This in turn implies that X is BMO-regular. ■

To state our main theorem we introduce the idea of a BMO-direction. If $X_0, X_1 \in \mathscr{X}$ we define a Köthe function space Z by $\Phi_Z = \frac{1}{2}(\Lambda + \Phi_{X_1} - \Phi_{X_0})$. We say that $X_0 \to X_1$ is a BMO-direction if Z is an $\mathscr{R}$-direction space. If either X_0 is p-convex where $p > 1$ or X_1 is q-concave where $q < \infty$, then Z has nontrivial concavity and so this is the same as requiring that Z is BMO-regular. If, for example, both spaces are super-reflexive, and X_0 is already BMO-regular, then $X_0 \to X_1$ is a BMO-direction if and only if X_1 is BMO-regular; this follows immediately from Theorems 4.6 and 4.9. On an intuitive level, $X_0 \to X_1$ is a BMO-direction if and only if the parallel complex interpolation scale through L_2 only passes through BMO-regular spaces.

Theorem 5.12. *Suppose that H is a Hardy-type algebra and that $X_0, X_1 \in \mathscr{X}$ are super-reflexive Köthe function spaces. Then, for any $0 < \theta < 1$, H is interpolation stable at θ for (X_0, X_1) if and only if $X_0 \to X_1$ is a BMO-direction.*

In particular, if X_0 is BMO-regular then H is interpolation stable at θ for (X_0, X_1) if and only if X_1 is BMO-regular.

Proof. We may suppose that both X_0, X_1 are p-convex and q-concave (with constant one) where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and $1 < p \le q < \infty$. Let $\varepsilon = \dfrac{1}{2q}$. Let $X_\tau = X_0^{1-\tau} X_1^\tau$ for $0 < \tau < 1$.

We start with some remarks on the implications of H being of Hardy type. In this situation we can apply Proposition 5.11: a super-reflexive $X \in \mathscr{X}$ is BMO-regular if and only if X is a $\mathscr{R}$-direction space. Note that L_2 is BMO-regular and further that X is BMO-regular if and only if X^* is BMO-regular.

Let us first suppose that H is interpolation stable at θ. Now X_θ is p-convex; furthermore, there is a Köthe function space W defined by

$$\Phi_W = \frac{1}{p} \Lambda - \Phi_{X_\theta}.$$

Now consider the quasi-Köthe spaces Y_ϕ defined by

$$\Phi_{Y_\phi} = \frac{1}{p} + \Phi_{X_\phi} - \Phi_{X_\theta}.$$

Clearly,

$$\left| \Delta_{\Phi_{Y_\phi}} - \frac{1}{p} \Delta_\Lambda \right| \le |\phi - \theta| \, \Delta_\Lambda .$$

Hence if $|\phi - \theta| \le \varepsilon = \dfrac{1}{2q}$, then Y_ϕ is a super-reflexive Köthe function space. We set

$$\phi_0 = \theta - \varepsilon \quad \text{and} \quad \phi_1 = \theta + \varepsilon .$$

Then $L_p = Y_\theta = Y_{\phi_0}^{1/2} Y_{\phi_1}^{1/2}$.

Now H is interpolation stable at $\frac{1}{2}$ for (X_{ϕ_0}, X_{ϕ_1}) since it is also interpolation stable at θ for (X_0, X_1). By Lemma 5.8 H is interpolation stable at $\frac{1}{2}$ also for (Y_{ϕ_0}, Y_{ϕ_1}). However $Y_{\phi_0}^{1/2}, Y_{\phi_1}^{1/2} = L_p$ and $\mathscr{R}$ is by assumption bounded at L_p.

By Lemma 5.10, we conclude that there exists $\eta > 0$ so that $Y_\phi = H_{Y_\phi} \oplus V_{Y_\phi}$ for $|\phi - \theta| \le 2\eta$. In particular $\mathcal{R}$ is bounded on $B = Y_{\theta+2\eta}$; hence B is BMO-regular. Now

$$\Phi_B = \frac{1}{p}\Lambda + \eta(\Phi_{X_1} - \Phi_{X_0}) = \left(\frac{1}{p} - \eta\right)\Lambda + 2\eta\Phi_Z .$$

Applying Theorem 4.6 gives that Z is BMO-regular.

Now we consider the converse; assume that Z is BMO-regular. We note that if $-\varepsilon < \tau < 1 + \varepsilon$, then there is a Köthe function space X_τ defined by

$$\Phi_{X_\tau} = \Phi_{X_0} + \tau(\Phi_{X_1} - \Phi_{X_0}),$$

and further, each such space is p'-convex and $2q$-concave where $\dfrac{1}{p'} + \dfrac{1}{2q} = 1$.

We show first that if $0 \le \tau_0 \le 1$, then H is interpolation stable at all $0 < \sigma < 1$ for $\left[X_{\tau_0 - \frac{1}{2}\varepsilon}, X_{\tau_0 + \frac{1}{2}\varepsilon}\right]$. To this end note that X_{τ_0} is q-concave and so there is a Köthe function space W defined by

$$\Phi_W = \Phi_{X_{\tau_0}} - \tfrac{1}{2}\varepsilon\Lambda .$$

Now we also have that Z^* is BMO-regular. Hence both $Y_0 = Z^\varepsilon$ and $Y_1 = (Z^*)^\varepsilon$ are BMO-regular. Now H is interpolation stable at all $0 < \sigma < 1$ for (Y_0, Y_1) by Theorem 5.5. But

$$\Phi_{Y_j} = \tfrac{1}{2}\varepsilon(\Lambda + (2j-1)(\Phi_{X_1} - \Phi_{X_0})).$$

Hence

$$\Phi_{Y_j} + \Phi_W = \Phi_{X_{\tau_0}} + (j - \tfrac{1}{2})\varepsilon(\Phi_{X_1} - \Phi_{X_0}) = \Phi_{X_{\tau_0 + \varepsilon\left(j - \frac{1}{2}\right)}}.$$

Thus by Lemma 5.8, H is interpolation stable at all $0 < \sigma < 1$ for $\left(X_{\tau_0 - \frac{1}{2}\varepsilon}, X_{\tau_0 + \frac{1}{2}\varepsilon}\right)$.

Now if $I = [\alpha, \beta]$ is a closed sub-interval of $(-\varepsilon, 1 + \varepsilon)$ we will say that I is acceptable if H is interpolation stable at all $0 < \sigma < 1$ for (x_α, X_β). Suppose that I, J are two acceptable intervals which intersect in a non-trivial interval; then we claim that $I \cup J$ is acceptable. In fact, excluding the trivial cases when $I \subset J$ or $J \subset I$ we can suppose that $I = [\alpha_1, \beta_1]$ and $J = [\alpha_2, \beta_2]$ where $\alpha_1 < \alpha_2 < \beta_1 < \beta_2$. Then we have

$$H_{X_{\beta_1}} = [H_{X_{\alpha_2}}, H_{X_{\beta_2}}]_\sigma \quad \text{where} \quad \beta_1 = (1-\sigma)\alpha_2 + \sigma\beta_2$$

and similarly

$$H_{X_{\alpha_2}} = [H_{X_{\alpha_1}}, H_{X_{\beta_1}}]_{\sigma'} \quad \text{where} \quad \alpha_2 = (1-\sigma')\alpha_1 + \sigma'\beta 1 .$$

By applying Wolff's theorem [32] we obtain

$$H_{X_{\alpha_2}} = [H_{X_{\alpha_1}}, H_{X_{\beta_2}}]_\varrho \quad \text{where} \quad \alpha_2 = (1-\varrho)\alpha_1 + \varrho\beta_2 .$$

It then follows from the re-iteration theorem that we actually have that H is interpolation stable at any $0 < \sigma < 1$ for $(X_{\alpha_1}, X_{\beta_2})$.

Now by simple induction we can obtain that $[0, 1]$ is acceptable and this implies the result. ■

Remarks. QUANHUA XU has pointed out that it follows from Theorem 5.12 that if X_0 is p-convex for some $p > 1$ and if H is interpolation stable at some $0 < \theta < 1$, then $X_0 \to X_1$ is a BMO-direction. In fact, the proof of Theorem 5.12 essentially yields this fact since that direction of the argument only uses that X_θ is r-convex, for some $r > 1$.

Theorem 5.13. *Suppose that H is a Hardy-type algebra and that $X_0, X_1 \in \mathscr{X}$. Suppose that X_0 is p-convex for $p > 1$ and is BMO-regular. Suppose that X_1 is q-concave for some $q < \infty$. Then, for any $0 < \theta < 1$, H is interpolation stable at θ (i.e., $[H_{X_0}, H_{X_1}]_\theta = H_{X_\theta}$) where $X_\theta = X_0^{1-\theta} X_1^\theta$, if and only if X_1 is BMO-regular.*

Proof. First note that every X_θ is super-reflexive and that Proposition 3.6 can be invoked to show the equivalence of the parenthetical statement with interpolation stability. One direction of the proof is simply Theorem 5.7. Conversely, if H is interpolation stable at some $0 < \theta < 1$, then we may pick $0 < \tau < \theta$ and H is interpolation stable at $1/2$ for $(X_{\theta-\tau}, X_{\theta+\tau})$. Hence $(X_{\theta-\tau} \to X_{\theta+\tau})$ is a BMO-direction. Thus if $\Phi_Z = \frac{1}{2}\Lambda + \tau(\Phi_{X_1} - \Phi_{X_0})$ then Z is BMO-regular. Theorems 4.6 and 4.9 allow us to conclude that X_1 is BMO-regular. ■

Let us mention at this stage that, in the case of the standard Hardy spaces on **T**, pairs X_0, X_1 for which $X_0 \to X_1$ is a BMO-direction, can be characterized neatly by using extended indicators. As in [18] it is possible to extend the indicator Φ_X to any complex $f \in L_1$ with $|f| \in \mathscr{I}_X \cap L \log L$ by setting $\Phi_X(f) = \int_{\mathbf{T}} f \log x \, d\lambda$ where $|f| = xx^*$ is the Lozanovskii factorization of $|f|$, i.e., the unique pair $x, x^* \geq 0$, so that supp x, x^* = supp f and $\|x\|_X = 1$, $\|x^*\|_{X^*} = \|f\|_1$. The extended Φ_X is a quasilinear map with constant 4/e (see Lemma 5.6 of [18]). The following theorem follows almost directly from Theorem 9.8 of [18]. We will not give a formal proof here, as we plan a more detailed investigation in a subsequent paper.

Theorem 5.14. *Suppose that $S = \mathrm{T}$ and $H = N^+$ is the Smirnov class. If $X_0, X_1 \in \mathscr{X}$, then $X_0 \to X_1$ is a BMO-direction if and only if there is a constant C so that for any $f \in H_\infty$.*

$$|\Phi_{X_1}(f) - \Phi_{X_0}(f)| \leq C \|f\|_1 .$$

6. Skew projections

We now establish some results on "skew" projections. We suppose that H is a closed subalgebra of $L_{\log}$ of Hardy type (of course our principal example of interest is the Smirnov class). If $w > 0$ a.e. and $\log w \in L_1$, then we define $\mathscr{R}_w$ to be the orthogonal projecion of the weighted Hilbert space $L_2(w)$ onto its subspace $H \cap L_2(w) = H_2(w)$.

Theorem 6.1. *Suppose that H is of Hardy-type. Suppose that $X \in \mathscr{X}$ is super-reflexive and that $0 \leq v, w \in L_1$ satisfy $\log v, \log w \in L_1$. Then if $\mathscr{R}_v, \mathscr{R}_w$ are both bounded at X, then $\log v - \log w \in BMO$.*

Proof. Clearly by duality, $\mathscr{R}_v$ is also bounded at $v^{-1}X^*$ and at $L_2(v)$. It then follows easily that H is interpolation stable at any $0 < \theta < 1$ for $(L_2(v), v^{-1}X^*)$. By Theorem 5.9,

Z_1 is BMO-regular where for $0 \le f \in L_\infty$, we have

$$\begin{aligned}\Phi_{Z_1}(f) &= \tfrac{1}{2}\left(\Lambda(f) - \Phi_{L_2(v)}(f) + \Phi_{v^{-1}X^*}\right)\\ &= \tfrac{1}{4}\left(\Lambda(f) + 2\Phi_{X^*}(f) - \textstyle\int f \log v \,\mathrm{d}\lambda\right).\end{aligned}$$

By similar reasoning, H is interpolation stable at any $0 < \theta < 1$ for $(L_2(w), X)$ and hence Z_2 is BMO-regular where

$$\Phi_{Z_2}(f) = \tfrac{1}{4}\left(\Lambda(f) + 2\Phi_X(f) + \textstyle\int f \log w \,\mathrm{d}\lambda\right).$$

Thus $Y = Z_1^{1/2} Z_2^{1/2}$ is BMO-regular. But

$$\Phi_Y(f) = \tfrac{1}{2}\Lambda(f) + \tfrac{1}{8}\textstyle\int f(\log w - \log v)\,\mathrm{d}\mu .$$

Hence $L_2((vw^{-1})^{1/4})$ is BMO-regular so that $\log v - \log w \in BMO$. ■

The following theorem is suggested by a result of COIFMAN-ROCHBERG [7] on boundedness of skew projections on weighted L_2-spaces. We observe that although we consider more general Köthe spaces, our result is here restricted to projections on Hardy subspaces; however, we plan to investigate more general results of this type in a forthcoming paper.

Theorem 6.2. *Suppose that H is of Hardy type. Suppose that $X_0, X_1 \in \mathscr{X}$ are super-reflexive and that $0 \le v, w \in L_1$ with $\log w \in L_1$. Suppose that $\mathscr{R}_v, \mathscr{R}_w$ are both bounded on X_0. If $\mathscr{R}_v$ is also bounded on X_1, then there exists $\eta > 0$ so that $\mathscr{R}_w$ is bounded on $X_0^{1-\theta} X_1^\theta$ for $0 < \theta \le \eta$.*

Proof. Since X_0, X_1 are super-reflexive, we may suppose that both are p-convex and q-concave where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and $1 < p \le q < \infty$. As in Theorem 5.12 if $\varepsilon = \dfrac{1}{2q}$ we can define super-reflexive spaces X_τ for $-\varepsilon \le \tau \le 1 + \varepsilon$.

Since $\mathscr{R}_v$ is a bounded on X_0 and X_1, it is easy to see that H is interpolation stable for any $0 < \theta < 1$ and (X_0, X_1). Thus $X_0 \to X_1$ is a BMO-direction by Theorem 5.12 and so also is $X_{-\varepsilon} \to X_1$. Hence H is interpolation stable at $\theta = \dfrac{\varepsilon}{1+\varepsilon}$ for $(X_{-\varepsilon}, X_1)$; the corresponding interpolation space is X_0.

Without loss of generality we can suppose that $\mathrm{d}v = w\,\mathrm{d}\lambda$ is a probability measure. Then $L_{\log}(v) = L_{\log}(\lambda)$ and so we can consider H as a Dirichlet-type algebra on (S, v). It follows from Lemma 5.10, since $\mathscr{R}_w$ is bounded on X_0, that there exists $\eta > 0$ so that $\mathscr{R}_w$ is also bounded on X_θ for all $|\theta| \le \eta$. ■

7. The vector-valued case

Finally let us point out an application to the vector-valued case. Let S be a Polish space and μ be a probability measure on S. Let X be a Köthe function space on S and let Y be a Köthe function space on $\mathbf{T}$. We denote by $Y(X)$ the Köthe function space on $\mathbf{T} \times S$ with measure $\lambda \times \mu$ given by $\|f\|_{Y(X)} = \|F\|_Y$ where $F(t) = \|f(t,\cdot)\|_X$.

Lemma 7.1. *Suppose $X \in \mathscr{X}(S)$ and $Y \in \mathscr{X}(\mathbf{T})$. Then for $0 \le f \in L_\infty(\mathbf{T} \times S)$ we have*

$$\Phi_{Y(X)}(f) = \Phi_Y(F) + \int_{\mathbf{T}} \Phi_X(f_t)\,\mathrm{d}\lambda(t),$$

where $F(t) = \int_S f(t,s)\,\mathrm{d}\mu(s)$ and $f_t(s) = f(t,s)$.

Proof. Let us suppose first that $\|f\|_1 = 1$ and that f is a simple function of the form

$$f = \sum_{j=1}^{n} c_j \chi_{A_j \times B_j}.$$

Suppose that the Lozanovskii factorization of F for (Y, Y^*) is given by $F = GH$. Then for each t suppose that

$$f(t,s)\,F(t)^{-1} = u(t,s)\,v(t,s)$$

is the Lozanovskii factorization for (X, X^*). Then the Lozanovskii factorization for $(Y(X), Y(X)^*)$ is given by $f = gh$ where $g(t,s) = G(t)\,u(t,s)$ and $h(t,s) = H(t)\,v(t,s)$. Thus

$$\begin{aligned}\Phi_{Y(X)}(f) &= \int_{\mathbf{T}} \int_S f(t,s)\,(\log G(t) + \log u(t,s))\,\mathrm{d}\mu(s)\,\mathrm{d}\lambda(t)\\ &= \Phi_Y(F) + \int_{\mathbf{T}} \Phi_X(f_t)\,\mathrm{d}\lambda(t).\end{aligned}$$

For general f the measurability of the integrand and the same formula follows by a simple continuity argument (cf. [18], Lemma 4.3). ∎

If X is a super-reflexive Köthe funtion space in $\mathscr{X}(S)$ and Y is a super-reflexive Köthe function space on $\mathbf{T}$ with $Y \in \mathscr{X}(\mathbf{T})$, then we set $H_Y(X)$ to be the closed subspace of $Y(X)$ of all functions $f(\cdot, s) \in N^+$ for μ-a.e. $s \in S$.

We will denote the Riesz projection on $L_2(\mathbf{T})$ by $\mathscr{R}$ and the vector-valued Riesz projection on $L_2(\mathbf{T} \times S)$ by $\widetilde{\mathscr{R}}$.

We are in effect studying the Hardy-type algebra $\mathscr{H}$ consisting of all $f \in L_{\log}(\mathbf{T} \times S)$ with $f^s = f(\cdot, s) \in N^+$ for a.e. $s \in S$. For this algebra $\mathscr{H}_1$ consists of all $f \in L_1(\mathbf{T} \times S)$ so that $f^s \in H_1(\mathbf{T})$ for a.e. $s \in S$. The corresponding BMO-space we denote $\mathscr{BMO}$.

In the vector-valued case we must consider the notion of UMD-spaces as introduced and studied initially by BURKHOLDER [4]. In fact a result of BOURGAIN [3] implies that if $X \in \mathscr{X}(S)$ then X is a UMD-space if and only if the Riesz projection $\widetilde{\mathscr{R}}$ is bounded on $L_2(X)$. This characterization will be all that require.

Now let us say that a Köthe function space $X \in \mathscr{X}(S)$ is UMD-regular if for some $0 < \theta < 1$ the space $L_2^\theta X^{1-\theta}$ is a UMD-space. If X_0, X_1 are two Köthe function spaces on S we say that $X_0 \to X_1$ is a UMD-direction if the space Z is UMD-regular where

$$\Phi_Z = \tfrac{1}{2}\Lambda + \tfrac{1}{2}(\Phi_{X_1} - \Phi_{X_0}).$$

Proposition 7.2. *If $f \in L_1(\mathbf{T} \times S)$, then $f \in \mathscr{BMO}$ if and only if $f^s \in BMO$ for a.e. $s \in S$ with $\|f\|_{\mathscr{BMO}} = \|\,\|f^s\|_{\mathscr{BMO}}\|_\infty < \infty$.*

Proof. If $f \in L_1$ the map $s \to \|f^s\|_{BMO}$ is easily seen to be measurable, and it is trivial to check that $\|f\|_{\mathscr{BMO}} \ge \|\,\|f^s\|_{BMO}\|_\infty$. For the converse it is enough to note that the set K

of (ϕ, ψ) in $L_1(\mathbf{T}) \times L_1(\mathbf{T})$ such that $\phi \in H_1$ and $\|\phi - \psi\|_\infty \le 1$ is a Borel set. It follows by standard selection theorems that there is a universally measurable map $\psi \to \tilde{\psi}$ from

$$\{\psi \in L_1; \|\psi\|_{BMO} \le 1\} \quad \text{to} \quad H_1 \quad \text{so that} \quad \|\psi - \tilde{\psi}\|_\infty \le 1 .$$

It follows easily that if $f \in L_1(\mathbf{T} \times S)$ with $\|f^s\|_{BMO} \le 1$ for a.e. s, then there exists $g \in \mathscr{H}_1$ with $\|f - g\|_\infty \le 1$.

Proposition 7.3. *Suppose that $Y \in \mathscr{X}(\mathbf{T})$ and $X \in \mathscr{X}(S)$.*
(1) *If $\tilde{\mathscr{R}}$ is bounded on $Y(X)$, then $\mathscr{R}$ is bounded on Y.*
(2) *If $\mathscr{R}$ is bounded on Y, then $\tilde{\mathscr{R}}$ is bounded on $Y(L_2)$.*

Proof. (1) Pick any fixed $0 \neq x \in X$ and restrict $\tilde{\mathscr{R}}$ to the space $Y([x])$ where $[x]$ is the one-dimensional space $\mathbf{C}x$.

(2) It follows directly from Krivine's theorem ([23], [24]) that the operator $(x_n) \to (\mathscr{R}x_n)$ is bounded on $Y(\ell_2)$ which implis the result. ■

Proposition 7.4. *Suppose that $Y \in \mathscr{X}(\mathbf{T})$ is super-reflexive and that X is a super-reflexive Köthe function space on S with $X \in \mathscr{X}(S)$. Then the following conditions are equivalent:*
(1) *$Y(X)$ is a $\tilde{\mathscr{R}}$-direction space.*
(2) *Y is $BMO(\mathbf{T})$-regular and X is UMD-regular.*
(3) *There exist constants (C, M) so that if $0 \le f \in Y(X)$ there exists $g \ge f$ with*

$$\|g\|_{Y(X)} \le M \, \|f\|_{Y(X)} \text{ and } \operatorname{ess\,sup} \|\log g^s\|_{BMO} \le C$$

where $g^s(t) = g(t, s)$ for $s \in S$.
(4) *$Y(X)$ is $\mathscr{BMO}$-regular.*

Proof. Of course (3) just restates (4) and so the equivalence of (1), (3) and (4) is just Proposition 5.11. Let us prove that (1) $\Rightarrow$ (2). Since $Y(X)$ is a $\tilde{\mathscr{R}}$-direction space it follows that there exists $\theta > 0$ so that $\tilde{\mathscr{R}}$ is bounded on $Y_\theta(X_\theta)$ where $Y_\theta = L_2^{1-\theta} Y^\theta$ and $X_\theta = L_2^{1-\theta} X^\theta$. Thus by Proposition 7.3, $\mathscr{R}$ is bounded on Y_θ which implies that Y is $BMO(\mathbf{T})$-regular. Further $\tilde{\mathscr{R}}$ is bounded on $Y_\theta(L_2)$ so that this is a $\mathscr{BMO}$-regular space. Hence $Y(L_2)$ is a $\mathscr{BMO}$-regular space. We show that $L_2(X)$ is a $\mathscr{BMO}$-regular space. In fact, if $0 \le f \in L_\infty$

$$\Phi_{L_2(X)}(f) = \tfrac{1}{2} \Lambda(F) + \int_{\mathbf{T}} \Phi_X(f_t) \, d\lambda$$

where F, f_t are as in Lemma 7.1. Thus

$$\Phi_{L_2(X)}(f) = \Phi_{Y(X)}(f) - \Phi_{Y(L_2)}(f) + \Phi_{L_2(L_2)}(f)$$

whence $L_2(X)$ is $\mathscr{BMO}$-regular by Theorem 4.6. This implies that $\mathscr{R}$ is bounded at $L_2(X_\phi)$ for some $\phi > 0$, X_ϕ is UMD and so X is UMD-regular.

In the converse direction we show that (2) implies that $Y(X)$ is $\mathscr{BMO}$-regular. Indeed, if Y is a BMO-regular space, then Proposition 7.3 implies that $Y(L_2)$ is $\mathscr{BMO}$-regular. If X is UMD-regular, then $L_2(X)$ is a $\mathscr{BMO}$-regular space. As in the preceding argument we can then use Theorem 4.6 to get that $Y(X)$ is $\mathscr{BMO}$-regular. ■

Theorem 7.5. *Suppose that (X_0, X_1) are super-reflexive Köthe function spaces in $\mathscr{X}(S)$ and that (Y_0, Y_1) are super-reflexive Köthe function spaces on $\mathbf{T}$ in $\mathscr{X}(\mathbf{T})$. Suppose that $0 < \theta < 1$*

and that $Y_\theta = [Y_0, Y_1]_\theta$ *and* $X_\theta = [X_0, X_1]_\theta$. *Then* $[H_{Y_0}(X_0), H_{Y_1}(X_1)]_\theta = H_{Y_\theta}(X_\theta)$ *if and only if* $Y_0 \to Y_1$ *is a BMO-direction and* $X_0 \to X_1$ *is a UMD-direction.*

Proof. The necessary and sufficient condition of Theorem 5.12 is that $Y_0(X_0) \to Y_1(X_1)$ is a $\mathscr{BMO}$-direction. This means by Lemma 7.1 that $W(Z)$ is a $\mathscr{BMO}$-regular space where

$$\Phi_W = \tfrac{1}{2}(\Lambda + \Phi_{Y_1} - \Phi_{Y_0}) \quad \text{and} \quad \Phi_Z = \tfrac{1}{2}(\Lambda + \Phi_{X_1} - \Phi_{X_0}).$$

The equivalence of this with the fact that W is BMO-regular and Z is UMD-regular is proved in Proposition 7.4. Thus the theorem is immediate. ■

Remark. The restriction that $X_0, X_1 \in \mathscr{X}(S)$ can easily be removed. It is well-known that for general Köthe function spaces there exists weight functions w_j, $j = 0, 1$, so that $L_\infty \subset w_j X_j \subset L_1$. Then if $\tilde{w}_j(s, t) = w_j(s)$,

$$[H_{Y_0}(X_0), H_{Y_1}(X_1)]_\theta = \tilde{w}_0^{1-\theta}\, \tilde{w}_1^\theta [H_{Y_0}(w_0X_0), H_{Y_1}(w_1X_1)]_\theta$$

and this coincides with $\tilde{w}_0^{1-\theta}\tilde{w}_1^\theta H_{Y_0}(w_1^{1-\theta}\, \tilde{w}_1^\theta X_\theta)$ and so on.

We may also given a non-super-reflexive version:

Theorem 7.6. *Suppose that* X *is a Köhte function space in* $\mathscr{X}(S)$ *which is* q*-concave for some* $q < \infty$. *Suppose that* (Y_0, Y_1) *are BMO-regular Köthe function spaces on* $\mathbf{T}$ *in* $\mathscr{X}(\mathbf{T})$. *Suppose that* Y_0 *is* p*-convex where* $p > 1$ *and that* Y_1 *is* q*-concave. Suppose that* $0 < \theta < 1$ *and that* $Y_\theta = [Y_0, Y_1]_\theta$ *and* $X_\theta = X^\theta$. *Then* $[H_{Y_0}(L_\infty), H_{Y_1}(X)]_\theta = H_{Y_\theta}(X_\theta)$ *if and only if* X *is UMD-regular.*

Proof. In fact, the special properties of L_∞ imply that $Y_0(L_\infty)$ is $\mathscr{BMO}$-regular. Thus from Theorem 5.12 we see that the conclusion holds if and only if $Y_1(X)$ is $\mathscr{BMO}$-regular. This occurs if and only if the super-reflexive space $Y_1^{1/2}(X^{1/2})$ is $\mathscr{BMO}$-regular or, by Proposition 7.4, if and only if X is UMD-regular. ■

Let us finally relate our work to that of Kisliakov and Xu ([20], [21]). They introduce a technical condition on a space $L_p(X, w) = w^{-1/p}L_p(X)$ where $w > 0$ is a weight function on $\mathbf{T}$ and consider when such spaces "admit sufficiently many analytic partitions of the unity." Let us say, without defining this concept precisely, that $L_p(X, w)$ has the KX-property. They show that if X^α is UMD for some $\alpha > 0$ and $\log w \in BMO$, then $L_p(X, w)$ has the KX-property. They also show that if X_0, X_1 are both reflexive and $L_{p_0}(X_0, w_0)$ and $L_{p_1}(X_1, w_1)$ have the KX-property, then indeed $\mathscr{H}$ is interpolation stable for evey $0 < \theta < 1$ for $(L_{p_0}(X_0, w_0), L_{p_1}(X_1, w_1))$.

Proposition 7.7. *If* $X \in \mathscr{X}(S)$ *is super-reflexive and* $1 < p < \infty$ *is such that* $L_p(X, w)$ *has the KX-property,* X *is UMD-regular and* $\log w \in BMO$. *In particular, if* X^α *is UMD for some* $\alpha > 0$, *then* X *is UMD-regular.*

Proof. As noted above, if $L_p(X, w)$ has the KX-property then $\mathscr{H}$ is stable at any $0 < \theta < 1$ for $(L_2(L_2), L_p(X, w))$. Thus, by Theorem 7.5, X is UMD-regular and $L_p(w)$ is BMO-regular which implies that $\log w \in BMO$. ■

Note that the assumption $X \in \mathscr{X}(S)$ can easily be removed by a change of weight. Thus our results, at least for super-reflexive spaces, include those of KISLIAKOV and XU; in fact, the conclusion also holds for spaces X with nontrivial concavity by a minor modification.

We also note that UMD-regularity of a super-reflexive Köthe function space is actually an isomorphic invariant; thus if X and Y are two such function spaces which are isomorphic (but not necessarily as lattices), then it may be shown that X is UMD-regular if and only if Y is UMD-regular. This can be done by methods of [19]. Let us conclude by remarking that in [18] we construct a super-reflexive Köthe function space which is not UMD-regular. However we do not know any example of a super-reflexive UMD-regular space which is not already a UMD-space (although L_1 is UMD-regular and not UMD).

References

[1] K. BARBEY and H. KÖNIG, Abstract Analytic Function Theory and Hardy Algebras, Springer Lecture Notes Vol. 593, Berlin, 1977

[2] O. BLASCO and Q. XU, Interpolation between Vector-valued Hardy Spaces. J. Funct. Anal. **102** (1991), 331–359

[3] J. BOURGAIN, Some Remarks on Banach Spaces in which Martingale Differences are Unconditional, Ark. Mat. **21** (1983), 163–168

[4] D. L. BURKHOLDER, A Geometrical Characterization of Banach Spaces in which Martingale Difference Sequences are Unconditional, Ann. Prob. **9** (1991), 997–1011

[5] A. P. CALDERÓN, Intermediate Spaces and Interpolation, the Complex Method, Studia Math. **24** (1964), 113–190

[6] R. R. COIFMAN and R. ROCHBERG, Another Characterisation of BMO, Proc. Amer. Math. Soc. **79** (1980), 249–254

[7] R. R. COIFMAN and R. ROCHBERG, Projections in Weighted Spaces, Skew Projections and Inversion of Toeplitz Operators, Int. Equns and Operator Theory **5** (1982), 145–159

[8] M. COTLAR and C. SADOSKY, On the Helson-Szegö Theorem and Related Class of Toeplitz Kernels, Proc. Sympos. Pure Math. **35** (1979), 383–407

[9] M. COTLAR and C. SADOSKY, On Some L^p Versions of the Helson-Szegö Theorem, Conference on harmonic analysis in honor of A. ZYGMUND, 305–317, edited by W. BECKNER, A. P. CALDERÓN, R. FEFFERMAN and P. W. JONES, Wadsworth 1982

[10] M. COTLAR and C. SADOSKY, The Helson-Szegö Theorem in L^p of the Bidimensional Torus, Cont. Math. **107** (1990), 19–37

[11] M. CWIKEL, J. E. MCCARTHY and T. H. WOLFF, Interpolation between Weighted Hardy Spaces, Proc. Amer. Math. Soc. **116** (1992), 381–388

[12] T. W. GAMELIN, Uniform Algebras, Chelsea Publishing Co., New York, 1984

[13] T. A. GILLESPIE, Factorization in Banach Function Spaces, Indag. Math. **43** (1981), 287–300

[14] I. I. HIRSCHMAN and R. Rochberg, Conjugate Function Theory in Weak*-Dirichlet Algebras, J. Funct. Anal. **16** (1974), 359–371

[15] H. HELSON and G. SZEGÖ A Problem in Prediction Theory, Ann. Mat. Pura Appl. **51** (1960), 107–138

[16] P. W. JONES, L^∞-estimates for the $\bar\partial$ Problem in the Half Plane, Acta Math. **150** (1983), 137–152

[17] P. W. JONES, On Interpolation between H_1 and H_∞, Springer Lecture Notes Vol. 1070, 143–151, 1984

[18] N. J. KALTON, Differentials of Complex Interpolation Processes for Köthe Function Spaces, Trans. Amer. Math. Soc. **333** (1992), 479–529

[19] N. J. KALTON, Remarks on Lattice Structure in ℓ_p and L_p when $0 < p < 1$, Israel Math. Conf. Series **5** (1992), 121–130

[20] S. V. KISLIAKOV and Q. XU, Sur l'Interpolation des Espaces H^p, C.R. Acad. Sci. (Paris) **313** (1991), 249–254

[21] S. V. KISLIAKOV and Q. XU Interpolation of Weighted and Vector-valued Hardy Spaces, Trans. Amer. Math. Soc. to appear

[22] J. Komlos, A Generalization of a Problem of Steinhaus, Acta Math. Acad. Sci. Hungar. **18** (1967), 217 – 229
[23] J. L. Krivine, Théorèmes de Factorisation dans les Espaces Réticulés, Exposés 22 – 23, Séminaire Maurey-Schwartz, Ecole Polytechnique, Paris 1973 – 4
[24] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces, II, Function Spaces, Ergeb. Math. Grengzeb., Vol. 97, Springer-Verlag, Berlin, Heidelberg and New York, 1979
[25] G. Y. Lozanovskii, On some Banach Lattices, Siberian Math. J. **10** (1969), 419 – 430
[26] B. Muckenhaupt, Weighted Norm Inequalities for the Hardy Maximal Function, Trans. Amer. Math. Soc. **165**, (1972), 207 – 226
[27] P. F. X. Müller, Holomorphic Martingales and Interpolation between Hardy Spaces, J. d'Analyse Math. to appear
[28] G. Pisier, Some Applications of the Complex Interpolation Method to Banach Lattices, J. d'Analyse Math. **35** (1979), 264 – 281
[29] G. Pisier, Interpolation between Hardy Spaces and Noncommutative Generalizations, Pacific J. Math. **55** (1992), 341 – 368.
[30] J. L. Rubio de Francia, Operators in Banach Lattices and L^2-inequalities, Math. Nachr. **133** (1987), 197 – 209.
[31] A. Torchinsky, Real variable methods in harmonic analysis. Academic Press,d 1986
[32] T. H. Wolff, A Note on Interpolation Spaces, Springer Lecture Notes Vol. 908, 199 – 204, 1982
[33] Q. Xu, Notes on Interpolation of Hardy Spaces, Ann. Inst. Fourier (Grenoble) **42** (1992), 875 – 889
[34] Q. Xu, Elementary Proofs of two Theorems of P. W. Jones on Interpolation between Hardy Spaces, to appear.

Department of Mathematics
University of Missouri-Columbia
Columbia, Mo. 65211
U.S.A.