REMARKS ON SUBSPACES OF H_p WHEN $0<p<1$

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1. Introduction. Let $\mathbf{T}$ be the unit circle in the complex plane and let Δ be the open unit disc. As usual H_p, $0<p<1$ denotes the quasi-Banach space of all functions $f:\Delta\to C$ analytic in Δ such that

$$\|f\|_p^p=\sup_{0<r\leqslant 1}\int_{\mathbf{T}}|f(rw)|^p\,dm(w)<\infty$$

where m is normalized Lebesgue measure on the circle. By considering boundary values H_p can be identified with a closed subspace of $L_p(\mathbf{T})$.

In this paper we give a number of results on the closed subspaces of H_p. Our first result is to show that H_p can have no complemented locally convex subspaces; this answers a question of Shapiro (see [7]). Indeed, we show that H_p cannot have any locally convex subspaces with the Hahn-Banach Extension Property (HBEP). A closed subspace M of a quasi-Banach space X has HBEP if every continuous linear functional on M can be extended to a continuous linear functional on X.

Next we consider special subspaces of the type $H_p(M)$ where M is a set of non-negative integers. Then $H_p(M)$ is the closed linear span of $\{z^m:m\in M\}$. We show that $H_p(M)$ can only have HBEP if it is thick in the sense that if

$$M=\{m_n:n=1,2\ldots\}\quad\text{where}\quad m_1<m_2<m_3\ldots$$

then $m_n\leqslant cn$ for some constant c. This again answers a question raised by Shapiro; Duren, Romberg and Shields [3] observed that $H_p(M)$ fails to have HBEP when M is a Hadamard gap sequence.

We also show that $H_p(M)$ is the range of a translation-invariant projection if and only if M is a finite union of arithmetic progressions modulo a finite set.

In the last section we discuss the nature of Banach subspaces of H_p. We conjecture that every Banach subspace of H_p has the Radon-Nikodym Property and show this is true for translation-invariant subspaces.

2. Preliminaries. We recall that a complex quasi-normed linear space X is called a quasi-Banach space and that if for some p, $0<p\leqslant 1$, the quasi-norm obeys the law

$$\|x_1+x_2\|^p\leqslant\|x_1\|^p+\|x_2\|^p\quad x_1,x_2\in X$$

then X is called a p-Banach space. The dual space of X will be denoted by X^*. If X^* separates the points of X, then the Mackey topology on X is the finest locally convex topology on X with the same dual space. This topology is a norm topology generated by $\text{co}(U)$ where $U=\{x:\|x\|\leqslant 1\}$ is the unit ball of X. Let $|||\cdot|||$ be the associated

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norm, i.e. $|||x||| = \inf(\lambda : \lambda^{-1}x \in \operatorname{co} U)$. Then the completion of X with respect to $|||\cdot|||$ is denoted by $\hat{X}$ and is called the containing Banach space of X.

One result we shall need later is the following lemma due to N. T. Peck [12].

LEMMA 2.1. *Suppose X is a real n-dimensional p-Banach space; then* $\|x\| \leqslant n^{1/p-1}|||x|||$ *for* $x \in X$.

The containing Banach space of H_p was determined by Duren, Romberg and Shields [3]. Let λ be normalized planar measure on the open unit disc Δ in the complex plane and for $0<p<q\leqslant 1$ define $B_{p,q}$ to be the space of analytic functions $f: \Delta \to C$ such that $\|f\|_{p,q}^q = \int_\Delta |f(z)|^q (1-|z|)^{q/p-2}\, d\lambda(z) < \infty$. Then $(B_{p,q}\|\cdot\|_{p,q})$ is a q-Banach space.

The following inclusion results are due to Hardy and Littlewood [5] and Shapiro [14]. Theorem 2.3 is due to Duren, Romberg and Shields [3].

THEOREM 2.2. *If $0<p<q<r\leqslant 1$ then $H_p \subset B_{p,q} \subset B_{p,r}$ and the inclusion maps are continuous.*

THEOREM 2.3. *$B_{p,1}$ is the containing Banach space of H_p.*

Here the identification is not an isometry (i.e. the norm of $B_{p,1}$ is not the containing Banach space norm for H_p).

THEOREM 2.4. *$B_{p,1}$ is isomorphic to l_1.*

This result is due to Lindenstrauss and Pelczynski [10]. Following this Kwapien and Pelczynski [9] note a result of Shapiro that any complemented subspace of H_p which is locally convex must be isomorphic to l_1, and conjecture that there is no such complemented subspace. This will be a deduction from our first results in the next section.

3. Subspaces of H_p with HBEP. In order to prove our main result it will be necessary to show that $B_{p,q}$ is isomorphic to a subspace of l_q for $p<q<1$. We first give a simple proof of this fact and then show how recent deeper results of Coifmann and Rochberg [1] show that $B_{p,q} \cong l_q$. The proof of this proposition is similar to Theorem 6.2 of [10].

PROPOSITION 3.1. *Suppose (Ω, Σ, μ) is a probability measure space and $0<p<1$. Let X be a closed subspace of $L_p(\Omega, \Sigma, \mu)$ with the property that given $\epsilon>0$ there exists $B \in \Omega$ with $\mu(\Omega \backslash B) < \epsilon$ and such that the set of functions $\{f \cdot 1_B;\ \|f\|_p \leqslant 1, f \in X\}$ is a relatively compact subset of $L_1(\Omega, \Sigma, \mu)$. Then X is isomorphic to a subspace of l_p.*

REMARK. Here 1_B is the indicator function of $B \in \Sigma$.

Proof. Partition Ω into countably many disjoint sets Ω_n such that if $K_n = \{f \cdot 1_{\Omega_n};\ f \in X,\ \|f\|_p \leqslant 1\}$ then K_n is relatively compact. Fix ϵ, $0<\epsilon<1$ and choose $\epsilon_n > 0$ so that $\sum_{n=1}^\infty \epsilon_n^p < (\epsilon/4)^p$. For each n choose $g_{1,n}, \ldots, g_{m(n),n} \in X$ with $\|g_{i,n}\| \leqslant 1$ and such that if $f \in X$ with $\|f\|_p \leqslant 1$ then for some i, $1 \leqslant i \leqslant m(n)$, $\int_{\Omega_n} |g_{i,n} - f|\, d\mu \leqslant \epsilon_n$. Then choose simple functions $h_{i,n}$ supported on Ω_n so that

$\int_{\Omega_n} |g_{i,n} - h_{i,n}|\, d\mu \leq \epsilon_n$. There is a sub-$\sigma$-algebra Σ_0 of Σ generated by countably many atoms such that each $(h_{i,n} : 1 \leq i \leq m(n), 1 \leq n < \infty)$ is Σ_0-measurable. Let E be the natural projection of $L_1(\Omega, \Sigma, \mu)$ onto $L_1(\Omega, \Sigma_0, \mu)$ i.e.

$$Ef = \sum_{n=1}^{\infty} \frac{1}{\mu(A_n)} \left(\int_{A_n} f\, d\mu \right) 1_{A_n}$$

where $(A_n)_{n=1}^{\infty}$ are the atoms of Σ_0. Then for $f \in X$ with $\|f\| \leq 1$, and $n \in \mathbf{N}$, choose $h_{i,n}$ with $\int_{\Omega_n} |f - h_{i,n}|\, d\mu \leq 2\epsilon_n$ so $\int_{\Omega_n} |Ef - h_{i,n}|\, d\mu \leq 2\epsilon_n$ i.e. $\int_{\Omega_n} |f - Ef|\, d\mu \leq 4\epsilon_n$. Hence $\int_{\Omega_n} |f - Ef|^p\, d\mu \leq (4\epsilon_n)^p$ and so if we define $T : X \to L_p(\Omega, \Sigma_0, \mu)$ by $Tf = \sum_{n=1}^{\infty} E(f \cdot 1_{\Omega_n})$, then $\|Tf - f\|^p \leq \sum_{n=1}^{\infty} (4\epsilon_n)^p < \epsilon^p$ and T is an isomorphic embedding. As $L_p(\Omega, \Sigma_0, \mu) \cong l_p$, the result is proved. □

COROLLARY 3.2. *For* $p < q < 1$, $B_{p,q}$ *is isomorphic to a subspace of* l_q.

Proof. If $\Delta_r = \{z : |z| \leq r\}$, then the set $(f \cdot 1_{\Delta_r};\ \|f\|_{p,q} \leq 1)$ is compact in $C(\Delta_r)$ (it is a normal family) and thus also in $L_1(\Delta_r, \lambda)$. □

Now we sketch the deeper result obtainable from the work of Coifman and Rochberg [1].

THEOREM 3.3. $B_{p,q} \cong l_q$.

Proof. Coifman and Rochberg show the existence of bounded linear operators $T : l_q \to B_{p,q}$; $V : B_{p,q} \to l_q$ so that $\|TVf - f\| < \epsilon \|f\|$ where $\epsilon < 1$. Thus TV is an automorphism of $B_{p,q}$ and if $T_1 = (TV)^{-1} T$ then $T_1 V = I$ on $B_{p,q}$. Thus $B_{p,q}$ is isomorphic to a complemented subspace of l_q, and a theorem of Stiles [15] gives the result. □

THEOREM 3.4. *Let* X *be a closed infinite dimensional subspace of* H_p *with* HBEP. *Then* X *cannot be* q*-convex for any* $q > p$.

REMARK. By definition X is q-convex if it can be equivalently quasi-normed to be a q-Banach space.

Proof. Suppose X is q-convex where $q > p$ and choose r so that $p < r < q$. We consider the inclusion map $J : X \to B_{p,r}$. By the preceding results $B_{p,r}$ is isomorphic at least to a subspace of l_r, but X is q-convex where $q > r$. Thus J is compact (see [15] and [16]). Hence the inclusion map $J : X \to B_{p,1}$ is compact (use Theorem 2.2). Clearly this means the induced map $J : \hat{X} \to B_{p,1}$ is compact and so the adjoint $J^* : B_{p,1}^* \to X^*$ is compact. However J^* is a surjection since X has HBEP in H_p, and $B_{p,1}$ is the containing Banach space of H_p. Thus $\dim X^* < \infty$ and we have a contradiction. □

COROLLARY 3.5. *If* $0 < p < 1$, H_p *has no complemented locally convex subspace* (*or even a locally convex subspace with* HBEP).

The proof of our next corollary would take us too far afield. We merely note that it is possible to prove that a closed subspace of L_p which is not q-convex for any $q > p$ contains a copy of l_p. (A proof can be obtained from [8] and certain ultra-product arguments.)

COROLLARY 3.6. *If X is a complemented subspace of H_p, $(0<p<1)$ then X contains a copy of l_p.*

PROBLEM. Does every complemented subspace of H_p contain a complemented copy of l_p for $0<p<1$?

4. HBEP and complementation for translation-invariant subspaces. Let

$$\mathbf{Z}_+ = \{n : n \geqslant 0\} \subset \mathbf{Z}.$$

If $M \subset \mathbf{Z}_+$ we denote by $H_p(M)$ the closed linear span of $(e_m : m \in M)$ where $e_m(z) = z^m$. Note that $H_p(M) = \{f \in H_p : \hat{f}(n) = 0 \text{ for } n \notin M\}$ where $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ is the Taylor series expansion of f.

We shall require first a lemma which has some independent interest.

LEMMA 4.1. *Suppose X is a p-Banach space and Y is a closed subspace of codimension n. Suppose ϕ is a continuous linear functional on Y. Then*
(i) *If X is real, ϕ has a linear extension ψ with $\|\psi\| \leqslant (n+1)^{1/p-1}\|\phi\|$.*
(ii) *If X is complex, ϕ has a linear extension ψ with $\|\psi\| \leqslant (2n+1)^{1/p-1}\|\phi\|$.*

Proof. (i) Let N be the kernel of ϕ. Thus $\dim X/N = n+1$. ϕ then factors to a linear functional ϕ_1 on Y/N with $\|\phi_1\| = \|\phi\|$. Since $\dim Y/N = 1$ we can choose $\xi \in Y/N$ with $\|\xi\| = 1$ and $|\phi_1(\xi)| = \|\phi\|$. There is by the Hahn–Banach theorem an extension ψ_1 of ϕ_1 to X/N with $\|\psi\| = \|\phi\| / |||\xi|||$ where $|||\cdot|||$ is the containing Banach space norm in X/N. Thus by Peck's lemma 2.1, $\|\psi_1\| \leqslant (n+1)^{1/p-1}\|\phi\|$ and the result follows by inducing ψ on X.

(ii) Let $X_{\mathbf{R}}$ be the associated real space of X. Applying (i) to the linear functional $\operatorname{Re}\phi$ we can produce a real-linear functional θ on $X_{\mathbf{R}}$ with

$$\theta(y) = \operatorname{Re}\phi(y) \quad y \in Y$$

$$\|\theta\| \leqslant (2n+1)^{1/p-1}\|\phi\|.$$

Define $\psi(\chi) = \theta(\chi) - i\theta(i\chi)$, and proceed as in the complex Hahn–Banach theorem. □

The next theorem answers a question of J. H. Shapiro. It shows for example that $H_p(M)$ fails HBEP if $M = \{m^2 : m \in \mathbf{N}\}$.

THEOREM 4.2. *Suppose $M = \{m_n : n = 1, 2 \ldots\}$ whose $m_1 < m_2 < m_3 \ldots.$ Then if $H_p(M)$ has the Hahn-Banach Extension Property there exists $c < \infty$ such that $m_n \leqslant cn$.*

Proof. We first observe that if $\phi_n : H_p \to \mathbf{C}$ is given by $\phi_n(f) = \hat{f}(n)$, then $\|\phi_n\| \geqslant \alpha n^{1/p-1}$ for some $\alpha > 0$. This follows from the Corollary to Theorem 6.5 of Duren [4] p. 100.

Now fix n and consider the linear functional $\psi_n : H_p\{m_n, m_{n+1}, \ldots\} \to \mathbf{C}$ given by $\psi_n(f) = \hat{f}(m_n)$. Then $\|\psi_n\| = 1$. By the preceding result, ψ_n has an extension ψ_n' to $H^p(M)$ with $\|\psi_n'\| \leqslant (2n-1)^{1/p-1}$.

Since $H_p(M)$ has HBEP there is a constant k independent of n such that ψ_n' has an extension ψ_n'' to H_p with $\|\psi_n''\| \leqslant k\|\psi_n'\| \leqslant k(2n-1)^{1/p-1}$. Now define

$$\theta_n(f) = \int_{\mathbf{T}} w^{-m_n}\psi_n''(f_w)\, dm(w)$$

where $f_w(z) = f(wz)$. Then $\theta_n = \phi_{m_n}$ and $\|\phi_{m_n}\| = \|\theta_n\| \leqslant k(2n-1)^{1/p-1}$. Thus $\alpha m_n^{1/p-1} \leqslant k(2n-1)^{1/p-1}$ and the theorem follows. □

Let us say that a sequence $(a_n : n=0,1,2\ldots)$ or a double sequence $(a_n : n \in \mathbf{Z})$ is *periodic* with period q if $a_{n+q} = a_n$ for all n. We shall say that (a_n) is nearly periodic if it differs from a periodic sequence only in a finite set of indices. For a sequence (a_n) this is equivalent to the existence of N, q such that $a_{n+q} = a_n$ for all $n \geqslant N$. We shall say that a subset M of $\mathbf{Z}_+$ or $\mathbf{Z}$ is a periodic or nearly periodic subset of $\mathbf{Z}_+$ or $\mathbf{Z}$ according as its indicator function

$$\begin{aligned} 1_M(n) &= 1 \quad n \in M \\ &= 0 \quad n \notin M \end{aligned}$$

is periodic or nearly periodic as a sequence.

If μ is a regular Borel measure on $\mathbf{T}$ then its Fourier transform $\hat{\mu} : \mathbf{Z} \to \mathbf{C}$ is given by

$$\hat{\mu}(n) = \int_{\mathbf{T}} w^n\, d\mu(w^{-1}) \quad n \in \mathbf{Z}.$$

μ is idempotent with respect to the convolution algebra $M(\mathbf{T})$ if and only if $\hat{\mu} = 1_M$ for some subset M of $\mathbf{Z}$. In [6] Helson showed that 1_M is the Fourier transform of some measure μ if and only if M is nearly periodic; 1_M is the transform of a measure μ of the form $\mu = \sum_{j=1}^{\infty} c_j\delta(w_j)$ (where $\delta(w_j)$ is the point mass at $w_j \in \mathbf{T}$ and $\sum|c_j| < \infty$) if and only if M is periodic (i.e. a finite union of arithmetic progressions).

For $0 < p < \infty$, denote by $L_p(M)$ the closed linear span of $\{e_n : n \in M\}$ in $L_p = L_p(\mathbf{T}, m)$; if $M \subset \mathbf{Z}_+$ we use the alternative notation $H_p(M)$. If $0 < p < 1$ we shall say $L_p(M)$ is *full* if $z^n \in L_p(M)$ implies $n \in M$; for $1 \leqslant p < \infty$ every $L_p(M)$ is full.

In [13] Rudin showed that Helson's results imply that $L_1(M)$ is complemented in L_1 if and only if M is a nearly periodic subset of $\mathbf{Z}$. Unfortunately Rudin's argument depends on an averaging technique which fails for $p<1$. However there is a substitute for complementation by a translation-invariant projection P. $P : L_p \to L_p$ or $P : H_p \to H_p$ is said to be *translation-invariant* if $(Pf)_w = Pf_w$, $w \in \mathbf{T}$.

THEOREM 4.3. *Suppose $0<p<1$ and that M is an infinite subset of $\mathbf{Z}$. Then $L_p(M)$ is full and complemented in L_p by a translation-invariant projection if and only if M is a periodic subset of $\mathbf{Z}$.*

Proof. A theorem of Oberlin [11] asserts that any translation-invariant operator $P : L_p \to L_p$ takes the form $Pf = \mu * f$, $f \in L_p$ where μ is a measure of finite p-variation, i.e. $\mu = \sum_{j=1}^{\infty} c_j\delta(w_j)$ where $\sum|c_j|^p < \infty$.

If p is a projection, μ is an idempotent and hence by Helson's results $\hat{\mu} = 1_M$ is periodic, i.e. M is periodic.

For the converse note that M is an arithmetic progression i.e. $M=(an+b:n\in\mathbf{N})$; then a projection P_M onto $L_p(M)$ is given by $P_Mf=a^{-1}\sum_{j=1}^{a}\omega^{-jb}f(\omega^jz)$, where ω is a primitive ath root of unity. It is then easily seen that if M is a periodic set it is a finite disjoint union of arithmetic progressions with the same common difference; a projection can then be built up in the obvious way. □

We now turn to the problem of the existence of translation-invariant projections on subspace of H_p of the form $H_p(M)$. We shall need the following preliminary lemma.

LEMMA 4.4. *Let Γ be the Cantor set $\{0,1\}^{\mathbf{Z}_+}$. Suppose $a\in\Gamma$ and let C be the closure of $(a^{(n)}:n\in\mathbf{Z}_+)\subset\Gamma$ where $a_k^{(n)}=a_{n+k}$ for $k\in\mathbf{Z}_+$. Suppose every accumulation point of C is periodic. Then a is nearly periodic.*

Proof. Let C' be the derived set of C (i.e. the set of accumulation points). Then C' is closed in Γ and so is each of the sets $C_q'=\{b\in C';\ b$ is periodic with period $q\}$. By the Baire Category Theorem there exists $q, b\in C'$ and $m\in\mathbf{N}$ such that if $b'\in C'$ and $b_i'=b_i$, $0\leqslant i\leqslant m-1$, then b' has period q. We may clearly suppose m is a multiple of q and then that $m=q$.

Choose $u(n)\to\infty$ so that $a_{u(n)+i}=b_i$, $0\leqslant i\leqslant q-1$ (possible since $b\in C'$).

If a is not nearly periodic there is for each n a largest $r(n)$ so that $a_{u(n)+i}=b_i$, $0\leqslant i\leqslant qr(n)-1$. Clearly $r(n)\to\infty$. By passing to a subsequence we may suppose $r(n)\geqslant 1$ for all n and $\lim_{n\to\infty}a_{u(n)+qr(n)-q+i}=d_i$ exists for $i\in{}'''\,{}_{+}$. Now $d_i=b_i$ for $0\leqslant i\leqslant q-1$ and so $d\in C_q'$. Hence for large enough n, $a_{u(n)+qr(n)-q+i}=d_i$, $0\leqslant i\leqslant 2q-1$ and so $a_{u(n)+i}=b_i$, $0\leqslant i\leqslant qr(n)+q-1$, contradicting the choice of $r(n)$. □

THEOREM 4.5. *Suppose $0<p<1$ and that M is an infinite subset of $\mathbf{Z}_+$. Then there is a translation-invariant projection of H_p onto $H_p(M)$ if and only if M is a nearly periodic subset of $\mathbf{Z}_+$.*

Proof. As in the proof of Theorem 4.3 we see that if $M\subset\mathbf{Z}_+$ is periodic then there is a translation-invariant projection onto $H_p(M)$; the same is clearly true if M is finite. As any two translation-invariant projections commute it quickly follows that $H_p(M)$ is complemented in H_p by a translation-invariant projection if M is nearly periodic.

Conversely suppose $H_p(M)$ is complemented by the projection $P_M:H_p\to H_p(M)$ given by $P_M(e_n)=a_ne_n$, $n\in\mathbf{Z}_+$ where $a_n=1_M(n)$. It is clear that this is the form of a translation-invariant projection.

We apply Lemma 4.4 to the point $a=(a_n)\in\Gamma$. Let b be an accumulation point of the set C so that for some $m(n)\to\infty$, $\lim_{n\to\infty}a_{m(n)+i}=b_i$, $i\in\mathbf{Z}_+$. By passing to a subsequence we may suppose that these limits exist for $i\in\mathbf{Z}$, when of course the sequences are defined only eventually. For $\gamma_j\in\mathbf{C}$, $-N\leqslant j\leqslant N$,

$$\begin{aligned}\int_{\mathbf{T}}\left|\sum_{j=-N}^{N}\gamma_jb_jz^j\right|^p dm(z) &= \lim_{n\to\infty}\int_{\mathbf{T}}\left|\sum_{j=-N}^{N}\gamma_ja_{m(n)+j}z^j\right|^p dm(z)\\ &= \lim_{n\to\infty}\int_{\mathbf{T}}\left|\sum_{j=m(n)-N}^{m(n)+N}\gamma_{j-m(n)}a_jz^j\right|^p dm(z)\\ &\leqslant \|P_M\|^p\int_{\mathbf{T}}\left|\sum_{j=-N}^{N}\gamma_jz^j\right|^p dm(z).\end{aligned}$$

Hence there is a bounded linear operator $Q: L_p \to L_p$ so that $Q(e_n) = b_n e_n$ $(n \in \mathbf{Z})$. Q is a translation-invariant projection and so by Theorem 4.3, b is periodic. Now by Lemma 4.4, a is nearly periodic, and the result is proved. □

5. Locally convex subspaces of H_p. The main theorem of this section (Theorem 5.2) represents an attempt to use the topology of uniform convergence on compact subsets (τ). As noted in [14], any bounded subset B of H_p is relatively τ-compact.

Recall that a Banach space X has the Radon-Nikodym Property (RNP) if and only if for each continuous linear operator $T: L_1(0,1) \to X$ there is a $g \in L_\infty((0,1), X)$ so that $T(f) = \int_0^1 f(s)g(s)\, ds$ holds for each $f \in L_1(0,1)$. In Banach space theory, a weaker topology on X in which norm-bounded sets are relatively compact plays a large role in determining that X has RNP (see Chapter III of [2]). Thus we conjecture that each locally convex subspace X of H_p has RNP. Theorem 5.2 represents a partial answer. (In an earlier proof of 5.2 we made more use of the properties of τ. Note that $g_n(s) \to g(s)$ in τ.) We first note

THEOREM 5.1. *L_1 does not embed into H_p when $0 < p < 1$.*

Proof. The argument given in [7] can easily be modified to show that L_1 does not embed into any separable quasi-Banach space admitting a Hausdorff vector topology in which the unit ball is compact (e.g. H_p).

THEOREM 5.2. *Suppose X is a locally convex subspace of H_p which is weakly closed. Then X has the Radon-Nikodym Property.*

Proof. Let $T: L_1(0,1) \to X$ be a bounded linear operator. Note that $H_p \hookrightarrow B_{p,1}$ and $B_{p,1} \cong l_1$ has the Radon-Nikodym Property. Hence T takes the form $Tf = \int_0^1 f(s)g(s)\, ds$ where $g: (0,1) \to B_{p,1}$ is an essentially bounded measurable map.

If for $0 \leqslant k < 2^n$, $\chi_{n,k}$ is the indicator function of the interval $(k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$ then we define

$$g_n(s) = 2^n T\chi_{n,k} \quad k \cdot 2^{-n} \leqslant s < (k+1) \cdot 2^{-n}.$$

Then in $B_{p,1}$, $g_n(s) \to g(s)$ a.e. However, in H_p, $\|g_n(s)\| \leqslant \|T\|$ for all n, s.

Let $A = \{s : g_n(s) \to g(s) \text{ in } B_{p,1}\}$. For $s \in A$, $g_n(s; rw) \to g(s; rw)$, $0 \leqslant r < 1$, $w \in \mathbf{T}$, and so

$$\int_{\mathbf{T}} |g(s; rw)|^p\, dm(w) \leqslant \limsup_{n \to \infty} \int_{\mathbf{T}} |g_n(s; rw)|^p\, dm(w) \leqslant \|T\|^p.$$

Hence for $s \in A$, $g(s) \in H_p$. However $g_n(s) \to g(s)$ weakly in H_p for $s \in A$ and so $g(s) \in X$.

Since $s \mapsto x^*(g(s))$ is measurable for $x^* \in H_p^*$, $s \mapsto x^*(g(s))$ is measurable for all $x^* \in X^*$ (X is separable) and by the Pettis measurability theorem, $g: A \to X$ is measurable. Clearly g can be extended arbitrarily to $(0,1)$ to be essentially bounded in X and then $Tf = \int_0^1 f(s)g(s)\, ds$ in X. Thus X has the Radon-Nikodym Property. □

COROLLARY 5.3. *A locally convex translation-invariant subspace of H_p has the Radon-Nikodym Property.*

Proof. If X is translation-invariant and locally convex, let $M = \{m : e_m \in X\}$. Thus $H_p(M) \subset X$. If $H_p(M) \neq X$ there exists $\phi \in X^*$ with $\phi(e_m) = 0$, $m \in M$ and $\phi(f) \neq 0$ for some $f \in X$. Now for some $m \in \mathbf{Z}$, $\int_{\mathbf{T}} w^{-m}\phi(f_w)\, dm(w) \neq 0$. Since X is locally convex, $\int_{\mathbf{T}} w^{-m} f_w \, dm(w) = \hat{f}(m) e_m$, and so $m \in M$ and $\phi(e_m) \neq 0$.

Thus $H_p(M) = X$ and hence X is weakly closed. □

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