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Orlicz sequence spaces without local convexity

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1. *Introduction*

In this paper we continue the study of Orlicz sequence spaces initiated by Lindberg (5) and Lindenstrauss and Tzafriri (7), (8) and (9). Our main concern is to investigate features of the theory which occur when the restriction of local convexity is lifted. It is clear that some results will hold with identical proofs, at least when the space is locally bounded. However, we are chiefly interested in the differences which arise. We always assume that the Orlicz function F satisfies the Δ_2-condition.

We essentially consider two topics: (a) the subspaces of l_F and (b) the complemented subspaces of l_F. After some definitions and preliminary results (sections 2–3) we study subspaces in section 4. Here the techniques of Lindenstrauss and Tzafriri work unchanged in locally bounded spaces (as suggested in the introduction of (9)), but the non-locally bounded case seems harder. Thus we are able for locally bounded l_F to classify exactly those Orlicz functions G such that l_G embeds into l_F, and show that l_F always contains a subspace l_p $(0 < p < \infty)$. It is rather more interesting, however, that the result that $l_p(0 < p < \infty)$ embeds into l_F if and only if $\alpha_F \leqslant p \leqslant \beta_F$ is valid even without local boundedness of l_F. In particular l_F contains an infinite-dimensional locally bounded subspace if and only if $\beta_F > 0$.

The structure of complemented subspaces is significantly different from the locally convex case. This difference is caused partly by the failure of averaging projections to be continuous. We study this case in sections 5–8. In section 5 we give an analytic criterion for the inclusion map $l_F \hookrightarrow l_p$ to be strictly singular. If this criterion fails and $l_F \subset l_p$, then l_F contains a complemented subspace isomorphic to l_p. We later use this idea in section 8 to solve problem 1 of Lindenstrauss and Tzafriri (8), by showing the existence of a convex Orlicz function l_F such that l_p is isomorphic to a complemented subspace of l_F but x^p is not equivalent to any function in $E_{F,1}$. After some technical results in section 6, we establish our main results in section 7. We show that if l_G is isomorphic to a complemented subspace of l_F then either G is equivalent to F or G is equivalent to a convex function.

If l_F contains no complemented subspace isomorphic to a locally convex Orlicz sequence space, then l_F has (up to affine equivalence) a unique unconditional basis. In the Banach space case it is well known that precisely three spaces have this property l_1, l_2 and c_0, (see (6) and (10)). Here, however, we can produce many such spaces. If $l_F \subset l_1$ then a necessary and sufficient condition for l_F to have a unique unconditional basis is

$$\lim_{\epsilon\to 0}\ \inf_{0<s\leqslant 1} \frac{1}{\log\left(\dfrac{1}{\epsilon}\right)}\int_\epsilon^1 \frac{F(sx)}{sx^2}\,dx = \infty.$$

In particular, $l_p(0 < p < 1)$ has a unique unconditional basis (solving a problem of Stiles (15)).

In section 8 we give some examples to distinguish between various conditions. In particular we solve a problem of Lindenstrauss and Tzafriri as observed above.

2. *Definitions*

We shall use the term *F-space* to mean a complete metric linear space. A sequence (x_n) in an F-space is called *regular* if there is a neighbourhood V of 0 such that $x_n \notin V$ $(n \in \mathbb{N})$ and (*topologically*) *normalized* if it is regular and bounded.

If (x_n) is a basis of an F-space X, the associated continuous linear functionals will always be denoted by (x_n'); sometimes we refer to the basis as $(x_n; x_n')$. If (u_n) is a basic sequence in X then (u_n) is complemented if its closed linear span is the range of a projection P. In this case there exist linear functionals (u_n') such that $Px = \Sigma\, u_n'(x)\, u_n$ $(x \in X)$. We refer to (u_n, u_n') as a *complemented basic sequence.*

A basic sequence (u_n) is a *block basic sequence* with respect to (x_n) if it takes the form

$$u_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i \quad (n = 1, 2, \ldots),$$

where $p_0 = 0 < p_1 < p_2 \ldots$ (u_n, u_n') is a *complemented block basic sequence* if in addition

$$u_n' = \sum_{i=p_{n-1}+1}^{p_n} b_i x_i'.$$

An Orlicz function F is a non-decreasing function $F: [0, \infty) \to [0, \infty)$ continuous at 0 such that $F(0) = 0$ and $F \not\equiv 0$. The Orlicz sequence space l_F is the vector space of all (real) sequences (x_n) such that for some $\epsilon > 0$ $\Sigma F(|\epsilon x_n|) < \infty$. We define

$$B_F(\epsilon) = \{x : \Sigma F(|x_n|) \leqslant \epsilon\}$$

and then the sets $\{rB_F(\epsilon),\ r > 0,\ \epsilon > 0\}$ form the base for an F-space topology on l_F.

We shall only be interested in those cases when the unit vector basis (e_n) is a basis of l_F. This occurs when (i) $F(x) > 0$ whenever $x > 0$, and (ii) (the Δ_2-condition at 0)

$$\sup_{0 < x \leqslant 1} \frac{F(2x)}{F(x)} < \infty.$$

Then l_F consists of all sequences such that $\Sigma\, F(|x_n|) < \infty$. In fact the behaviour of F outside a neighbourhood of 0 is irrelevant to the definition of l_F, and so we may assume, without loss of generality, that F satisfies the Δ_2-condition on $\mathbb{R}$, i.e.

$$\sup_{0 < x < \infty} \frac{F(2x)}{F(x)} < \infty.$$

Hence forward, by an Orlicz function we shall understand an Orlicz function such that $F(x) > 0$ for $x > 0$ and satisfying the Δ_2-condition on $\mathbb{R}$.

Two Orlicz functions F and G will be called *equivalent* if

$$0 < \inf_{0 < x \leqslant 1} \frac{F(x)}{G(x)} \leqslant \sup_{0 < x \leqslant 1} \frac{F(x)}{G(x)} < \infty.$$

3. *Some basic results*

In this section we list some preliminary results, which will be required at various stages later.

PROPOSITION 3·1. *Let X be an F-space with a basis (x_n, x_n'). Let (y_n) be a regular basic sequence in X such that $\lim_{n\to\infty} x_k'(y_n) = 0$ for each k; then there is a subsequence (y_{n_k}) of (y_n) equivalent to a block basic sequence of (x_n).*

Proof. This has been established by Shapiro (14) when X is locally pseudo-convex. In general it is sufficient to construct (y_{n_k}) and a block basic sequence (z_k) such that $\Sigma||y_{n_k} - z_k|| < \infty$, where $||\cdot||$ is any F-norm defining the topology of X; we omit the details.

If X is an F-space with a separating dual then the Mackey topology on X is the finest locally convex topology on X weaker than the original topology (cf. (13)). The Mackey topology is a metrizable topology. We denote by $\tilde{X}$ the completion of X in the Mackey topology.

PROPOSITION 3·2. *Let (x_n) be a complemented basic sequence in X. Then*

(i) *(x_n) is a complemented basic sequence in $\tilde{X}$;*

(ii) *$a_n x_n \to 0$ if and only if $a_n x_n \to 0$ for the Mackey topology;*

(iii) *(x_n) is topologically normalized if and only if (x_n) is topologically normalized for the Mackey topology.*

Proof. (i) Let $E = \overline{\text{lin}}\,(x_n)$ and $P: X \to E$ be a continuous projection onto E. If $x \in E$, define

$$S_n x = \sum_{i=1}^{n} x_i'(x)\, x_i.$$

We observe that if U is a convex neighbourhood of 0 in X, then $P^{-1}(U)$ is a convex neighbourhood, and hence P is Mackey-continuous. Similarly each $S_n P$ is continuous; furthermore $\bigcap_{n=1}^{\infty} (S_n P)^{-1}(U)$ is a neighbourhood of 0 and hence $(S_n P: n \in \mathbb{N})$ is an equicontinuous collection. Thus the maps $S_n: E \to E$ are equicontinuous for the Mackey topology (of X), and so (x_n) is a basic sequence.

(ii) If $a_n x_n \not\to 0$, then there is a subsequence $a_{n_k} x_{n_k}$ such that $||a_{n_k} x_{n_k}|| \geqslant \epsilon$. As $x_{n_k}'(Px)\, x_{n_k} \to 0$ for $x \in X$, we have $a_{n_k}^{-1} x_{n_k}'(x) \to 0$ for $x \in X$. Thus the linear functionals $a_{n_k}^{-1} x_{n_k}' \circ P$ are equicontinuous on X and hence $x \mapsto \sup_k |a_{n_k}^{-1} x_{n_k}'(Px)|$ is a continuous semi-norm on X and hence also for the Mackey topology. Thus $a_{n_k} x_{n_k} \not\to 0$ in the Mackey topology.

(iii) Follows immediately from (ii).

Let F be an Orlicz function. Then we define $\hat{F}$ to be the largest Orlicz function such that $\hat{F}(x) \leqslant F(x)$ for all x and $\hat{F}$ is convex on $[0, 1]$. It is easy to show that

$$\hat{F}(x) = \inf\left\{\frac{1}{n}\sum_{i=1}^{n} F(x_i) : n \in \mathbb{N},\quad 0 \leqslant x_i \leqslant 1,\quad \frac{1}{n}\Sigma x_i = x\right\}$$

for $0 \leqslant x \leqslant 1$, and
$$\hat{F}(x) = F(x) \quad (x > 1).$$

Clearly $\hat{F}$ is equivalent at 0 to a convex Orlicz function and so $l_{\hat{F}}$ is a Banach space.

THEOREM 3·3. *Let F be an Orlicz function (satisfying the Δ_2-condition). Then the Mackey topology of l_F is that induced by $\hat{F}$.*

Proof. Since $l_F \subset l_{\hat{F}}$, it is easy to see that the topology induced by $\hat{F}$ is weaker than the Mackey topology. Conversely let $||\cdot||$ be any continuous semi-norm on l_F. Then there exists a $0 < a \leqslant F(1)$ such that if

$$\sum_{i=1}^{\infty} F(|x_i|) \leqslant a$$

then $||x|| \leqslant 1$. Also the set $\{e_n : n \in \mathbb{N}\}$ is bounded and hence there exists $M < \infty$ such that

$$||e_n|| \leqslant M \quad (n \in \mathbb{N}).$$

Now for any $x \in l_F$ with $\sup_n |x_n| \leqslant 1$, let $A \subset \mathbb{N}$ be the set of $i \in \mathbb{N}$ such that $a < F(|x_i|) \leqslant 1$. Then

$$||\sum_{i \in A} x_i e_i|| \leqslant M|A|$$

and

$$a|A| \leqslant \sum_{i \in A} F(|x_i|) \leqslant \sum_{i=1}^{\infty} F(|x_i|).$$

Hence

$$||\sum_{i \in A} x_i e_i|| \leqslant \frac{M}{a} \sum_{i=1}^{\infty} F(|x_i|).$$

The set $\mathbb{N} - A$ may be decomposed into a finite number of subsets $\sigma_1 \ldots \sigma_m$ such that

$$\tfrac{1}{2}a < \sum_{i \in \sigma_j} F(|x_i|) \leqslant a$$

and one remainder set σ_{m+1} with

$$0 \leqslant \sum_{i \in \sigma_{m+1}} F(|x_i|) \leqslant \tfrac{1}{2}a.$$

Clearly

$$\tfrac{1}{2}ma \leqslant \sum_{i=1}^{\infty} F(|x_i|)$$

and

$$||\sum_{i \in \sigma_j} x_i e_i|| \leqslant 1 \quad (j = 1, 2, \ldots, m+1)$$

so that

$$||\sum_{i \in \mathbb{N}-A} x_i e_i|| \leqslant m+1 \leqslant \frac{2}{a} \sum_{i=1}^{\infty} F(|x_i|) + 1.$$

Hence

$$||x|| \leqslant \frac{M+2}{a} \sum_{i=1}^{\infty} F(|x_i|) + 1.$$

Now suppose $x \in l_F$ has finite support and that

$$\sum_{i=1}^{\infty} \hat{F}(|x_i|) \leqslant a.$$

Then for some large enough choice of N there exist sequences $(y_i^{(k)})$ $k = 1, 2, \ldots, N$ with $0 \leqslant y_i^{(k)} \leqslant 1$, such that

$$\frac{1}{N}(y_i^{(1)} + \ldots + y_i^{(N)}) = |x_i|,$$

and

$$\frac{1}{N} \sum_{k=1}^{N} \sum_{i=1}^{\infty} F(y_i^{(k)}) \leqslant \sum_{i=1}^{\infty} \hat{F}(|x_i|) + a \leqslant 2a.$$

Define $$z_i^{(k)} = (\operatorname{sgn} x_i)\, y_i^{(k)}.$$

Then $$x = \frac{1}{N}(z^{(1)} + \ldots + z^{(N)})$$

and $$\|z^{(k)}\| \leqslant \frac{M+2}{a} \sum_{i=1}^{\infty} F(y_i^{(k)}) + 1.$$

Hence $$\|x\| \leqslant \frac{M+2}{a}(2a) + 1 = 2M+5.$$

By density $\|x\| \leqslant 2M+5$ for any $x \in l_F$ with

$$\sum_{i=1}^{\infty} \hat{F}(|x_i|) \leqslant a.$$

Hence $\|\cdot\|$ is continuous for the $\hat{F}$-topology on l_F.

Remark. The Mackey topology of $L_F(0, 1)$ is the topology of the convex minorant of F over the whole real line. This follows from results in (4).

COROLLARY. *Every complemented basic sequence in an Orlicz sequence space l_F is normal.*

Proof. By Proposition 3·2 and Theorem 3·3 (since $l_{\hat{F}}$ is a Banach space).

4. *Subspaces of Orlicz spaces*

We shall denote by I the unit interval $[0, 1]$ and by I_0 the half-open interval $(0, 1]$. $C(I)$ and $C(I_0)$ will denote the spaces of continuous real valued functions on I and I_0 in each case with compact convergence.

If F is an Orlicz function we define $T_t F = F_t \in C(I_0)$ for $0 < t \leqslant 1$ by

$$F_t(x) = \frac{F(tx)}{F(t)} \quad (0 < x \leqslant 1).$$

Note. Throughout sections 4–7 we shall assume that every Orlicz function F has the property that $xF(x)$ is convex. Every Orlicz function F is equivalent to an Orlicz function G satisfying this condition, e.g. let $G(0) = 0$ and

$$G(x) = \frac{1}{x}\int_0^x F(t)\,dt \quad (x > 0).$$

(Note we assume the Δ_2-condition.) This assumption is technically convenient (see Lemma 4·1) and is necessary in Theorems 4·5, 4·6 and 5·1. However, all the other main theorems are preserved under equivalence of Orlicz functions, and hence the assumption is redundant.

We shall also use the notation

$$|x|_F = \sum_{i=1}^{\infty} F(|x_i|) \quad (x \in l_F).$$

Under the Δ_2-condition, if $u_n \in l_F$ then $u_n \to 0$ if and only if $|u_n|_F \to 0$.

LEMMA 4·1. *$\{F_t: t \in I_0\}$ is relatively compact in $C(I_0)$.*

Proof. If we set $G(x) = xF(x)$, then G is convex and the set $\{G_t: t \in I_0\}$ is equicontinuous at each $x \in I$ (see (7), p. 382). However, $F_t(x) = x^{-1}G_t(x)$ and hence $\{F_t: t \in I_0\}$ is equicontinuous at each $x \in I_0$. As $F_t(x) \leqslant 1$ for all t, x the set $\{F_t: t \in I_0\}$ is relatively compact in $C(I_0)$.

We may therefore extend the map $t \to F_t(I_0 \to C(I_0))$ to a continuous map $\tau \to F_\tau$ $(\beta I_0 \to C(I_0))$. Each F_τ is increasing in x and so we may define

$$F_\tau(0) = \lim_{x \to 0} F_\tau(x).$$

The Δ_2-condition implies that $F_\tau(x) > 0$ whenever $x \in I_0$. F_τ is (the restriction of) an Orlicz function if and only if $F_\tau(0) = 0$.

We also define (cf. (9))

$$\alpha_F = \sup\left\{p: \sup_{0<x,t\leqslant 1} \frac{F(tx)}{F(t)\,x^p} < \infty\right\}$$

$$\beta_F = \inf\left\{p: \inf_{0<x,t\leqslant 1} \frac{F(tx)}{F(t)\,x^p} > 0\right\}.$$

Clearly $0 \leqslant \alpha_F \leqslant \beta_F < \infty$. Note that α_F and β_F are preserved under equivalence of Orlicz functions.

PROPOSITION 4·2. *The following conditions on F are equivalent:*

(i) *l_F is locally bounded;*

(ii) *$\alpha_F > 0$;*

(iii) *the map $(\tau, x) \mapsto F_\tau(x)$ $(\beta I_0 \times I \to I)$ is jointly continuous;*

(iv) *the functions $(F_t: t \in I_0)$ are equicontinuous at 0;*

(v) *there exist a, u with $0 < a, u < 1$ such that $F(ux) \leqslant aF(x)$, $0 \leqslant x \leqslant 1$.*

Most of these equivalences are essentially known (cf. (11), (16) and (12), ch. III). The remainder are not difficult to verify.

Now let $Z_F = \{\tau \in \beta I_0: F_\tau(0) = 0\}$, and $E_{F,1} = \{F_\tau: \tau \in Z_F\}$. Observe that Z_F is a Borel subset of βI_0. If μ is a probability measure on βI_0 with $\mu(\beta I_0 - Z_F) = 0$, let

$$F_\mu(x) = \int_{\beta I_0} F_\tau(x)\, d\mu(\tau),$$

and let $C_{F,1}$ be the set of such F_μ. Also let C_F be the set of F_μ where μ is a probability measure on βI_0 such that $\mu(I_0) = \mu(\beta I_0 - Z_F) = 0$.

l_F is locally bounded if and only if $Z_F = \beta I_0$; if so $E_{F,1}$ is compact in $C(I)$ and $C_{F,1} = \overline{\mathrm{co}}\, E_{F,1}$ (see (5), (7) and (8)). The results in the locally bounded case are essentially trivial generalizations of results of Lindenstrauss and Tzafriri. In the more general non-locally bounded case, our results are incomplete.

PROPOSITION 4·3. *Let (u_n) be a topologically normalized block basic sequence with respect to (e_n) in l_F. Then there is a subsequence of (u_n) equivalent to the unit vector basis of some l_G, where $G \in C_{F,1}$.*

Proof. Since (u_n) is bounded.

$$\lim_{x\to 0} |xu_n|_F = 0 \quad \text{uniformly in } n.$$

Hence we may find $\gamma > 0$ such that $|\gamma u_n|_F \leqslant 1$ for all n. Then let $\psi_n(x) = |x\gamma u_n|_F$, i.e. if $v_n = \gamma u_n$

$$\psi_n(x) = \sum_{v_n(k)>0} F(|v_n(k)|)\, F_{|v_n(k)|}(x),$$

where

$$\sum_{v_n(k)>0} F(|v_n(k)|) = \psi_n(1) \leqslant 1.$$

Hence $(\psi_n : n \in \mathbb{N})$ is equicontinuous on I_0 by Lemma 4·1, and thus also on I (since $\psi_n(x) \to 0$ uniformly in n). Hence there is a subsequence $\psi_{n_k} \to \psi$ uniformly on I. Clearly $\psi(1) = \lim \psi_{n_k}(1) > 0$ by regularity of (u_n). By passing to a further subsequence we may suppose

$$\sum_k ||\psi - \psi_{n_k}||_\infty < \infty,$$

where $||\cdot||_\infty$ is the norm in $C(I)$. Thus $\Sigma t_k u_{n_k}$ converges if and only if $\Sigma\psi(|t_n|) < \infty$. But

$$\psi_{n_k}(x) = \int_{\beta I_0} F_\tau(x)\, d\mu_k(\tau) \quad (0 < x \leqslant 1),$$

where μ_k is a positive Borel measure satisfying $\mu_k(\beta I_0) = \psi_{n_k}(1)$.

If ν is any weak*-cluster point of μ_k,

$$\psi(x) = \int_{\beta I_0} F_\tau(x)\, d\nu(\tau) \quad (0 < x \leqslant 1)$$

and $\nu(\beta I_0) = \psi(1)$. As $\lim_{x\to 0} \psi(x) = 0$, it follows by the Monotone Convergence Theorem that $\nu(\beta I_0 \backslash Z_F) = 0$. Writing $G = \psi(1)^{-1}\psi$ we obtain the result.

COROLLARY. *Let (u_n) be a topologically normalized symmetric basic sequence in l_F. Then (u_n) is equivalent to the unit vector basis of some l_G, $G \in C_{F,1}$.*

Proof. Use the same argument as (5), corollary 3·9 (using Proposition 3·1 of this paper).

We now consider the converse problem: if $G \in C_{F,1}$, does l_G embed into l_F? We require first a simple lemma.

LEMMA 4·4. *Let $\{f_\alpha\}$ be a net in $C_{F,1}$ such that $f_\alpha \to f$ pointwise on I_0, where $f \in C_{F,1}$. Then $||f_\alpha - f||_\infty \to 0$.*

Proof. Since $C_{F,1}$ is compact as a subset of $C(I_0)$, $f_\alpha \to f$ uniformly on compact subsets of I_0. However, for given $\epsilon > 0$, choose δ such that $f(\delta) < \frac{1}{3}\epsilon$ (f is continuous at 0 since $f \in C_{F,1}$) and then α_0 such that for $\alpha \leqslant \alpha_0$, $|f(\delta) - f_\alpha(\delta)| \leqslant \frac{1}{3}\epsilon$. Then for $\alpha \geqslant \alpha_0$ and $0 \leqslant x \leqslant \delta$, $|f(x) - f_\alpha(x)| \leqslant f(x) + f_\alpha(x) \leqslant f(\delta) + f_\alpha(\delta) \leqslant \epsilon$. As $f_\alpha \to f$ uniformly on $[\delta, 1]$, there exists α_1 such that $||f_\alpha - f|| \leqslant \epsilon$ for $\alpha \geqslant \alpha_1$.

For $0 < A < \infty$, we shall say that $G \in C_{F,1}$ is A-accessible if there is a sequence (u_n) of elements of l_F of finite support such that

$$A^{-1}G(t) - h_n(t) \leqslant |tu_n|_F \leqslant AG(t) + k_n(t) \quad (0 \leqslant t \leqslant 1),$$

where $h_n, k_n \in C(I)$ and $\Sigma||h_n||_\infty + \Sigma||k_n||_\infty < \infty$. If G is A-accessible for some A, $0 < A < \infty$ then l_G is isomorphic to a subspace of l_F (a space spanned by such a sequence (u_n) with disjoint support). Obviously it is sufficient to assume that $||h_n||_\infty$ and $||k_n||_\infty \to 0$, in the definition of A-accessibility. If Γ_A is the set of A-accessible $G \in C_{F,1}$, then Γ_A is clearly closed in $C_{F,1}$ under uniform convergence on I, and hence by Lemma 4·4 under pointwise convergence on I_0.

Let μ be a Borel measure on βI_0 whose support is contained in $[a, b] \subset I_0$, and suppose θ, $0 < \theta < 1$, fixed. Define

$$\lambda_k = \int_{\theta^{k+1}b+}^{\theta^k b} \frac{1}{F(t)} d\mu(t) \quad (k = 0, 1, 2, \ldots),$$

and let u_μ be an element of l_F taking the value $\theta^{k+1}b$ precisely $[\lambda_k]$ times. Then

$$\begin{aligned} |xu_\mu|_F &= \sum_{k=0}^{\infty} [\lambda_k] F(\theta^{k+1}bx) \\ &\leqslant \sum_{k=0}^{\infty} \lambda_k F(\theta^{k+1}bx) \\ &\leqslant \int_a^b \frac{F(tx)}{F(t)} d\mu(t) = F_\mu(x). \end{aligned} \tag{1}$$

Conversely

$$\begin{aligned} |xu_\mu|_F &\geqslant \tfrac{1}{2} \sum_{\lambda_k \geqslant 1} \lambda_k F(\theta^{k+1}bx) \\ &= \tfrac{1}{2} \sum_{k=0}^{\infty} \lambda_k F(\theta^{k+1}bx) - \tfrac{1}{2} \sum_{\lambda_k<1} \lambda_k F(\theta^{k+1}bx) \\ &= \tfrac{1}{2} \sum_{k=0}^{\infty} \int_{\theta^{k+1}b+}^{\theta} \frac{F(\theta^{k+1}bx)}{F(t)} d\mu(t) - \tfrac{1}{2} \sum_{\lambda_k<1} \lambda_k F(\theta^{k+1}bx) \\ &\geqslant \frac{1}{2K} \int_a^b \frac{F(tx)}{F(t)} d\mu(t) - \tfrac{1}{2} \sum_{\lambda_k<1} \lambda_k F(\theta^{k+1}bx) \\ &= \frac{1}{2K} F_\mu(x) - \tfrac{1}{2} \sum_{\lambda_k<1} \lambda_k F(\theta^{k+1}bx) \end{aligned} \tag{2}$$

(where $K = K(\theta)$ is a constant such that $F(x) \leqslant KF(\theta x)$, $0 \leqslant x \leqslant 1$). Thus

$$|xu_\mu|_F \geqslant \frac{1}{2K} F_\mu(x) - \frac{1}{2} \sum_{k=0}^{\infty} (\theta^{k+1}bx). \tag{3}$$

THEOREM 4·5. *Let F be an Orlicz function satisfying $\sum_{n=0}^{\infty} F(2^{-n}) < \infty$. Then if $G \in C_F$, l_G is isomorphic to a subspace of l_F.*

Proof. Let

$$G(x) = \int_{\beta I_0} F_\tau(x)\, d\mu(\tau),$$

where $\mu(I_0) = 0$.

Then there is a net (ν_α) of probability measures with supports contained in $(0, b_\alpha] \subset I_0$ such that $b_\alpha \to 0$ and $\nu_\alpha \to \mu$ in the weak*-topology of $C(\beta I_0)^*$. Hence $F_{\nu_\alpha}(x) \to G(x)$ uniformly on I by Lemma 4·4, and so there is a sequence ν_n with support, contained in $(0, b_n]$ such that $F_{\nu_n}(x) \to G(x)$ uniformly and $b_n \to 0$. Fix $\theta = \frac{1}{2}$, and let $u_n = u_{\nu_n}$; then for $0 \leqslant x \leqslant 1$, by equations (1) and (3),

$$|xu_n| \leqslant G(x) + (F_{\nu_n}(x) - G(x))$$

$$|xu_n| \geqslant \frac{1}{2K} F_{\nu_n}(x) - \frac{1}{2} \sum_{k=0}^{\infty} F((\tfrac{1}{2})^{k+1} b_n x)$$

$$\geqslant \frac{1}{2K} G(x) + \frac{1}{2K}(G(x) - F_{\nu_n}(x)) - \frac{1}{2} \sum_{k=0}^{\infty} F((\tfrac{1}{2})^{k+1} b_n).$$

Now $G(x) - F_{\nu_n}(x) \to 0$ uniformly on I and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} F((\tfrac{1}{2})^{k+1} b_n) = 0.$$

Hence G is $2K$-accessible.

THEOREM 4·6. *Suppose l_F is locally bounded. Then*
(i) *l_G is isomorphic to a subspace of l_F if and only if G is equivalent to a function in $C_{F,1}$;*
(ii) *l_F contains a subspace isomorphic to l_p for some p, $0 < p < \infty$.*

Proof. (i) This is a straightforward generalization of results of Lindenstrauss and Tzafriri.

By 4·5 it is only necessary to consider $G \in C_{F,1} \backslash C_F$, i.e.

$$G(x) = \int_{\beta I_0} F_\tau(x)\, d\mu(\tau)$$

where $\mu(I_0) > 0$. Hence

$$G(x) > \alpha F(x) \quad \text{for some } \alpha \quad (0 < \alpha < 1).$$

Since l_F is locally bounded there exist $0 < \theta < 1$, $0 < a < \frac{1}{2}$ such that $F(\theta x) \leqslant aF(x)$ $(0 \leqslant x \leqslant 1)$. Choose measures ν_n supported in I_0 such that $F_{\nu_n} \to G$ uniformly on I. Then for θ chosen above, and $b = 1$, let $u_n = u_{\nu_n}$. Then

$$|xu_n|_F \leqslant G(x) + (F_{\nu_n}(x) - G(x))$$

and by (3),

$$|xu_n|_F \geqslant \frac{1}{2K} F_{\nu_n}(x) - \frac{1}{2}\frac{a}{1-a} F(x) \geqslant \frac{1}{2K} F_{\nu_n}(x) - \tfrac{1}{2} F(x).$$

Put $v_n = u_n + e_{m(n)}$ where $u_{n,m(n)} = 0$. Then

$$|xv_n|_F \leqslant G(x) + F(x) + (F_{\nu_n}(x) - G(x))$$

$$\leqslant (1 + \alpha^{-1})\, G(x) + (F_{\nu_n}(x) - G(x))$$

while

$$|xv_n|_F \geqslant \frac{1}{2K}[G(x) + (F_{\nu_n}(x) - G(x))]$$

and G is $\max(2K, 1 + \alpha^{-1})$-accessible.

(ii) Let $S: C(I) \to C(I)$ be defined by $Sf(x) = xf(x)$. Then $SC_{F,1} = C_{SF,1}$ and SF is convex and satisfies the Δ_2-condition. Hence by results of (7) $SC_{F,1}$ contains x^p for some $p \geqslant 1$. Hence $x^{p-1} \in C_{F,1}$, and since $C_{F,1}$ is equicontinuous at 0, $p > 1$.

THEOREM 4·7. *$l_p (p > 0)$ is isomorphic to a subspace of l_F if and only if $\alpha_F \leqslant p \leqslant \beta_F$.*

Remark. Note that this theorem holds without the assumption that l_F is locally bounded.

Proof. The necessity of $\alpha_F \leqslant p \leqslant \beta_F$ follows as in (9), Theorem 1. If $\alpha_F = \beta_F$, then l_F is locally bounded and the result follows from Theorem 4·6. More generally if $\alpha_F < \beta_F$ (including the case $\alpha_F = 0$), then, proceeding as in (9), if $\alpha_F < p < \beta_F$ and $f(x) = F(x)/x^p$, there exist $0 < u_n < v_n < w_n < 1$ such that $w_n \to 0$ and $nf(u_n) < f(v_n)$ and $nf(w_n) < f(v_n)$. Letting $a_n = u_n/w_n$ put

$$G_n(x) = C_n^{-1} \int_{a_n}^1 F(tw_n x)\, t^{-p-1}\, dt,$$

where
$$C_n = \int_{a_n}^1 F(tw_n)\, t^{-p-1}\, dt.$$

Then $G_n \in C_{F,1}$ and $G_n(x) \to x^p$ pointwise (and hence uniformly).

Note that
$$G_n(x) = A_n^{-1} \int_{u_n}^{w_n} F(tx)\, t^{-p-1}\, dt,$$

where
$$A_n = \int_{u_n}^{w_n} F(t)\, t^{-p-1}\, dt.$$

Thus $G_n = F_\mu$, where $\int f\, d\mu = A_n^{-1} \int_{u_n}^{w_n} f(t)\, F(t)\, t^{-p-1}\, dt.$

Put $\theta = \frac{1}{2}$ and $b = w_n$ and proceed as in the discussion preceding Theorem 4·5. Fix N so that $2^{-N} \geqslant a_n > 2^{-(N+1)}$; then

$$\lambda_k = A_n^{-1} \int_{2^{-(k+1)}w_n}^{2^{-k}w_n} t^{-p-1}\, dt = \frac{1}{pA_n w_n^p} (2^{(k+1)p} - 2^{kp}) \quad (k = 0, 1, 2, \ldots, N-1)$$

and
$$\lambda_k = 0 \quad (k = N+1, N+2, \ldots).$$

Let M be the largest integer such that

$$\frac{1}{pA_n w_n^p}(2^p - 1)\, 2^{Mp} < 1.$$

Then
$$\begin{aligned}\sum_{\lambda_k<1} \lambda_k F(2^{-(k+1)} w_n x) &\leqslant \sum_{k=0}^{M} 2^{kp} \left(\frac{2^p - 1}{pA_n w_n^p}\right) F(2^{-(k+1)} w_n x) + F(2^{-(N+1)} w_n x) \\ &\leqslant \left(1 + \sum_{k=0}^{M} 2^{(k-M)p}\right) F(w_n) \quad (0 \leqslant x \leqslant 1) \\ &\leqslant \frac{2^{p+1} - 1}{2^p - 1} F(w_n).\end{aligned}$$

We now appeal to equation (2); there exists an element u_n of finite support such that

$$|xu_n|_F \geqslant \frac{1}{2K} G_n(x) - \frac{2^{p+1} - 1}{2^{p+1} - 2} F(w_n)$$

and by equation (1)
$$|xu_n|_F \leqslant G_n(x).$$

As $w_n \to 0$ and $G_n(x) \to x^p$ uniformly, x^p is $2K$-accessible for $0 < p < \beta_F$. Then x^{β_F} is also $2K$-accessible and the result is proved.

COROLLARY. *l_F contains an infinite-dimensional locally bounded (resp. Banach) subspace if and only if $\beta_F > 0$ ($\beta_F \geqslant 1$)* (cf. (16), p. 34, corollary 2 and p. 98).

Proof. If $X \subset l_F$ is locally bounded (resp. Banach) and infinite-dimensional, then by results of (3) and Proposition 3·1, X contains a basic sequence equivalent to a topologically normalized block basic sequence. Hence X contains a subspace Y isomorphic to a locally bounded (Banach) Orlicz sequence space and hence a subspace l_p with $p > 0$ ($p \geqslant 1$). Thus $\beta_F > 0$ ($\beta_F \geqslant 1$). The converse is the preceding theorem.

5. *Strict singularity of the inclusion map*

Recall that an operator $T: X \to Y$ between two F-spaces is strictly singular if it fails to be an isomorphism on any infinite-dimensional subspace. Suppose F and G are Orlicz functions such that $l_F \subset l_G$, i.e. $G(x)/F(x)$ is bounded on I_0. Let

$$w(t) = G(t)/F(t) \quad (t \in I_0)$$

and w also denote its unique extension to βI_0.

THEOREM 5·1. *Suppose $l_F \subset l_G$ and l_G is locally bounded. Then the inclusion map is an isomorphism on some infinite-dimensional subspace of l_F if and only if there exists $C < \infty$ and a probability measure μ on βI_0 such that*

$$\int F_\tau(x)\, d\mu(\tau) \leqslant C \int w(\tau)\, G_\tau(x)\, d\mu(\tau) \quad (0 \leqslant x \leqslant 1).$$

Proof. (*a*) *Necessity.* By a standard gliding hump argument the inclusion map is an isomorphism on the closed linear span of some topologically normalized block basic sequence (u_n). As in the proof of Proposition 4·3 we may assume that $|u_n|_F \leqslant 1$ and $|u_n|_G \leqslant 1$, and that $\lim_{n\to\infty} |xu_n|_F = H_1(x)$ and $\lim_{n\to\infty} |xu_n|_G = H_2(x)$ exist uniformly. By passing to a further subsequence we may assume that

$$\sum_{n=1}^{\infty} \sup_{0 \leqslant x \leqslant 1} \left| |xu_n|_F - H_1(x) \right| < \infty$$

and

$$\sum_{n=1}^{\infty} \sup_{0 \leqslant x \leqslant 1} \left| |xu_n|_G - H_2(x) \right| < \infty.$$

As (u_n) is regular in l_F and l_G, $H_1(x) > 0$ and $H_2(x) > 0$ for $x > 0$. Thus $\Sigma \alpha_n u_n$ converges in l_F if and only if $\Sigma H_1(|\alpha_n|) < \infty$ and in l_G if and only if $\Sigma H_2(|\alpha_n|) < \infty$. Hence H_1 and H_2 are equivalent Orlicz functions; as they clearly satisfy the Δ_2-condition we have $H_1(x) \leqslant CH_2(x)$ ($0 \leqslant x \leqslant 1$) for some $C < \infty$.

Now,

$$|xu_n|_F = \int_{\beta I_0} F_\tau(x)\, d\mu_n(\tau),$$

where μ_n is a positive measure with finite support contained in I_0 and $\|\mu_n\| \leqslant 1$. Thus

$$H_1(x) = \int_{\beta I_0} F_\tau(x)\, d\mu(\tau) \quad (0 < x \leqslant 1),$$

where $\mu \neq 0$ is any weak* cluster point of (μ_n). As

$$|xu_n|_G = \int_{\beta I_0} G_\tau(x)\, w(\tau)\, d\mu_n(\tau)$$

we have

$$H_2(x) = \int_{\beta I_0} G_\tau(x)\, w(\tau)\, d\mu(\tau) \quad (0 < x \leqslant 1).$$

Replacing μ by $||\mu||^{-1}\mu$ we obtain the result.

(*b*) *Sufficiency*. Let (a_n) be any sequence in I_0 such that $a_n \to 0$ if $\mu(I_0) = 0$ and $a_n \equiv 1$ if $\mu(I_0) > 0$. Then there exist probability measures μ_n with support in $(0, a_n]$ such that

$$\int F_\tau(x)\, d\mu_n(\tau) \to \int F_\tau(x)\, d\mu(\tau),$$

$$\int G_\tau(x)\, w(\tau)\, d\mu_n(\tau) \to \int G_\tau(x)\, w(\tau)\, d\mu(\tau),$$

for all rational $x \in I_0$. Equicontinuity of the functions F_τ and $w(\tau)\, G_\tau$ on I_0 implies the convergence is pointwise on I_0. However, since l_G is locally bounded $G_\tau(0) = 0$ for all τ and hence $\int F_\tau(0)\, d\mu(\tau) = 0$. Thus $\mu(\beta I_0 - Z_F) = 0$ and we also have convergence at 0 and so convergence is uniform on I (see Lemma 4·4).

Since l_G is locally bounded, there exists $\theta < 1$, $a < \frac{1}{2}$ such that $G(\theta x) < aG(x)$, $0 \leqslant x \leqslant 1$. Let $\nu_n = w\mu_n$; then as in the discussion preceding Theorem 4·5, let (u_n) be an element of l_F taking the value $\theta^{k+1} a_n$, $[\lambda_k]$ times where

$$\lambda_k = \int_{\theta^{k+1} a_n +}^{\theta^k a_n} \frac{1}{G(t)}\, d\nu_n(t).$$

Then as in the proofs of Theorems 4·5 and 4·6, by picking a subsequence we may suppose that either (u_n) or (v_n) where $v_n = u_n + e_{m(n)}$, (where $u_{n,m(j)} = 0$ for all j) is equivalent in l_G to the unit vector basis of l_H where

$$H(x) = \lim_{n\to\infty} \int_{\beta I_0} G_\tau(x)\, d\nu_n(\tau) = \int_{\beta I_0} G_\tau(x)\, w(\tau)\, d\mu(\tau).$$

$\left(\text{Note here that } \int_{I_0} w(t)\, d\mu(t) = 0 \text{ if and only if } \mu(I_0) = 0.\right)$

Now

$$\begin{aligned} |xu_n|_F &= \sum_{k=0}^{\infty} [\lambda_k]\, F(\theta^{k+1} a_n x) \\ &\leqslant \sum_{k=0}^{\infty} \left\{ \int_{\theta^{k+1} a_n +}^{\theta^k a_n} \frac{1}{G(t)}\, d\nu_n(t) \right\} F(\theta^{k+1} a_n x) \\ &\leqslant \sum_{k=0}^{\infty} \left\{ \int_{\theta^{k+1} a_n +}^{\theta^k a_n} \frac{1}{F(t)}\, d\mu_n(t) \right\} F(\theta^{k+1} a_n x) \\ &\leqslant \int_0^{a_n} F_t(x)\, d\mu_n(t) \to \int_{\beta I_0} F_\tau(x)\, d\mu(\tau). \end{aligned}$$

Thus by picking a further subsequence we may suppose that $\Sigma H(|\alpha_n|) < \infty$ implies $\Sigma \alpha_n u_n$ converges in l_F. This establishes the isomorphism in the case when $\mu(I_0) = 0$ and (u_n) is equivalent to the unit vector basis of l_H. In the other case $\mu(I_0) > 0$ and

$H(x) \geqslant \alpha F(x)$ where $\alpha > 0$. Thus $\Sigma H(|\alpha_n|) < \infty$ implies $\Sigma \alpha_n v_n$ converges again, establishing the required isomorphism.

THEOREM 5·2. *Suppose $l_F \subset l_G$ and l_G is locally bounded. Then the following conditions are equivalent*:

(i) *The inclusion map is strictly singular.*

(ii) *For any $C > 0$ there exist distinct points $x_1 \ldots x_n \in I_0$ and $a_1 \ldots a_n > 0$ such that*

$$\sum_{i=1}^{n} a_i F(tx_i) \geqslant C \sum_{i=1}^{n} a_i G(tx_i) \quad (0 \leqslant t \leqslant 1).$$

(iii) *For any $C > 0$, there exists $a > 0$ and a positive Borel measure $\mu \neq 0$ with support contained in $[a, 1]$ such that*

$$\int F(tx)\, d\mu(x) \geqslant C \int G(tx)\, d\mu(x) \quad (0 \leqslant t \leqslant 1).$$

Proof. (i) $\Rightarrow$ (ii). For any $C > 0$, and any probability measure μ on βI_0, by Theorem 5·1.

$$\int (F_\tau(x) - Cw(\tau)\, G_\tau(x))\, d\mu(\tau) > 0 \quad \text{for some} \quad x \in I_0.$$

Let $S = \overline{\text{co}}\,\{F_\tau - w(\tau)\, G_\tau : \tau \in \beta I_0\}$ in $C(I_0)$. Then S is compact (apply Lemma 4·1) and hence also compact for the topology ρ of pointwise convergence. If

$$T = \{f \in C(I_0) : f(x) \leqslant 0 \quad x \in I_0\}$$

then T is a closed convex set in $(C(I_0), \rho)$ and $S \cap T = \varnothing$. Hence there is a ρ-continuous linear functional L on $C(I_0)$ such that

$$\sup_{f \in T} L(f) < \inf_{f \in S} L(f).$$

Then
$$L(f) = \sum_{i=1}^{n} a_i f(x_i),$$

where $x_1 \ldots x_n \in I_0$ are distinct and $a_1 \ldots a_n \neq 0$. Clearly $a_1 \ldots a_n \geqslant 0$ and $\sup_{f \in T} L(f) = 0$. Thus

$$\sum_{i=1}^{n} a_i F_\tau(x_i) \geqslant C \sum_{i=1}^{n} a_i w(\tau)\, G_\tau(x_i) \quad (\tau \in \beta I_0).$$

Restricting to I_0 we have

$$\sum_{i=1}^{n} a_i \frac{F(tx_i)}{F(t)} \geqslant C \sum_{i=1}^{n} a_i \frac{G(t)}{F(t)} \frac{G(tx_i)}{G(t)} \quad (0 < t \leqslant 1).$$

Thus
$$\sum_{i=1}^{n} a_i F(tx_i) \geqslant C \sum_{i=1}^{n} a_i G(tx_i) \quad (0 \leqslant t \leqslant 1).$$

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Suppose ν is a probability measure on βI_0 such that

$$\int_{\beta I_0} F_\tau(x)\, d\nu(\tau) \leqslant C \int w(\tau)\, G_\tau(x)\, d\nu(\tau) \quad (0 \leqslant x \leqslant 1),$$

where $C > 0$. Pick μ a positive non-zero measure on $[a, 1]$, where $a > 0$ such that

$$\int F(tx)\,d\mu(x) \geqslant 2C\int G(tx)\,d\mu(x) \quad (0 \leqslant t \leqslant 1).$$

Then

$$\int F_t(x)\,d\mu(x) \geqslant 2C\int w(t)\,G_t(x)\,d\mu(x) \quad (0 < t \leqslant 1)$$

and hence by continuity

$$\int F_\tau(x)\,d\mu(x) \geqslant 2C\int w(\tau)\,G_\tau(x)\,d\mu(x) \quad (\tau \in \beta I_0).$$

Thus

$$\int_{\beta I_0}\int_a^1 F_\tau(x)\,d\mu(x)\,d\nu(\tau) \geqslant 2C\int_{\beta I_0}\int_a^1 w(\tau)\,G_\tau(x)\,d\mu(x)\,d\nu(\tau)$$

$$\geqslant 2\int_{\beta I_0}\int_a^1 F_\tau(x)\,d\mu(x)\,d\nu(\tau).$$

Hence $F_\tau(x) = 0$ for $x \in \operatorname{supp}\mu$, $\tau \in \operatorname{supp}\nu$. This is impossible, since, as observed at the beginning of section 4, $F_\tau(x) > 0$ for $\tau \in \beta I_0$, $x \in I_0$.

THEOREM 5·3. *Suppose $l_F \subset l_p$ $(0 < p < \infty)$. Then the inclusion map is strictly singular if and only if*

$$\lim_{\epsilon\to 0}\ \inf_{0<s\leqslant 1} \frac{1}{\log\left(\frac{1}{\epsilon}\right)}\int_\epsilon^1 \frac{F(su)}{s^p u^{p+1}}\,du = \infty. \tag{$*$}$$

Proof. If $(*)$ is satisfied, then given C, there exists $\epsilon > 0$ such that

$$\int_\epsilon^1 \frac{F(su)}{s^p u^{p+1}}\,du \geqslant C\log\left(\frac{1}{\epsilon}\right) = C\int_\epsilon^1 \frac{du}{u}.$$

Hence

$$\int_\epsilon^1 \frac{F(su)}{u^{p+1}}\,du \geqslant C\int_\epsilon^1 \frac{(su)^p}{u^{p+1}}\,du$$

establishing (iii) of Theorem 5·2.

Conversely, if the map is strictly singular, then for any $C > 0$, there exist $0 < x_1 < \ldots < x_n < 1$ and $a_1 \ldots a_n > 0$ such that

$$\sum_{i=1}^n a_i\,F(stx_i) \geqslant 2C\sum_{i=1}^n a_i\,s^p t^p x_i^p \quad (0 \leqslant s, t \leqslant 1).$$

For $0 < \epsilon < x_1^2$

$$\int_{\epsilon/x_1}^1 \sum_{i=1}^n a_i\,F(stx_i)\frac{dt}{t^{p+1}} \geqslant 2C\sum_{i=1}^n a_i\,s^p x_i^p\int_{\epsilon/x_1}^1 \frac{dt}{t}$$

$$= 2Cs^p\left(\sum_{i=1}^n a_i\,x_i^p\right)\log\left(\frac{x_1}{\epsilon}\right).$$

However,

$$\int_{\epsilon/x_1}^1 \sum_{i=1}^n a_i\,F(stx_i)\frac{dt}{t^{p+1}} = \sum_{i=1}^n a_i\int_{\epsilon/x_1}^1 \frac{F(stx_i)}{t^{p+1}}\,dt$$

$$= \sum_{i=1}^n a_i\,x_i^p\int_{\epsilon x_i/x_1}^{x_i} \frac{F(su)}{u^{p+1}}\,du$$

$$\leqslant \left(\sum_{i=1}^n a_i\,x_i^p\right)\int_\epsilon^1 \frac{F(su)}{u^{p+1}}\,du.$$

Hence $$\frac{1}{\log(1/\epsilon)}\int_\epsilon^1 \frac{F(su)}{s^p u^{p+1}}\,du \geqslant 2C\,\frac{\log(1/\epsilon)-\log(1/x_1)}{\log(1/\epsilon)} \geqslant C.$$

Remark. If we put $\phi_p(x) = e^{px}F(e^{-x})$ $(0 \leqslant x < \infty)$, then $(*)$ is equivalent to

$$\lim_{l\to\infty}\ \inf_{u\geqslant 0} \frac{1}{l}\int_u^{u+l} \phi_p(x)\,dx = \infty,$$

i.e. ϕ_p is 'almost convergent' to $+\infty$.

THEOREM 5·4. *Suppose $l_F \subset l_p$ where $p \geqslant 1$ and l_F has no complemented subspace isomorphic to l_p. Then $(*)$ is satisfied.*

Proof. If $(*)$ is not satisfied the inclusion map $J: l_F \to l_p$ is an isomorphism on some infinite-dimensional closed subspace X. Then $J(X)$ contains a subspace $Y \simeq l_p$ which is complemented in l_p, with projection $P: l_p \to Y$. This means that $J^{-1}PJ$ is a projection of l_F onto $J^{-1}(Y) \simeq l_p$, which is a contradiction.

In section 8 we shall use this result to resolve a problem of Lindenstrauss and Tzafriri ((9), problem 1) by showing that l_p can be complemented in l_F without x^p being equivalent to a function in $E_{F,1}$.

6. *Main technical results*

In this section we suppose that (u_n, u_n') is a complemented unconditional basic sequence in l_F. By Theorem 3·3, Corollary, we may assume that (u_n) is topologically normalized. We shall call (u_n) *essential* if

$$\inf_n \sup_k |u_n'(e_k)\, e_k'(u_n)| = 0$$

and otherwise *inessential.*

Our first result is that if (u_n) is inessential it is equivalent to (e_n). The proof of this could be considerably simplified if we make the assumption that F is convex. However, our main results are of interest only if F is not convex. We start with an inequality of Paley–Zygmund type (cf. (2), p. 24). For this purpose we denote by $\{r_n: n = 1, 2, \ldots\}$ the sequence of Rademacher functions on $[0, 1]$ (or, equivalently, any sequence of independent random variables taking the values ± 1 with probabilities $\frac{1}{2}$).

LEMMA 6·1. *Let m denote Lebesgue measure on I, and $m\times m$ the product measure on $I\times I$. If (a_{ij}) is an $n\times n$ matrix then*

$$(m\times m)\left\{(s,t): \left|\sum_{i=1}^n \sum_{j=1}^n a_{ij}\, r_i(s)\, r_j(t)\right| \geqslant \frac{1}{10}\sqrt{\left(\sum_{i=1}^n\sum_{j=1}^n |a_{ij}|^2\right)}\right\} \geqslant \frac{1}{10}.$$

Proof. We are grateful to the referee for suggesting the following argument. Our original proof gave the same result with $\frac{1}{10}$ replaced by $\frac{1}{16}$. The argument is similar to that of (2), theorem 3, p. 24.

Let $$\phi_i(t) = \sum_{j=1}^n a_{ij}\, r_j(t) \quad (0 \leqslant t \leqslant 1),$$

and $$X(s,t) = \left(\sum_{i=1}^n r_i(s)\,\phi_i(t)\right)^2 \quad (0 \leqslant s, t \leqslant 1).$$

Considered as a random variable, the expectation $E(X)$ of X is given by

$$E(X) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_0^1 r_i(s)\,r_j(s)\,ds\right)\left(\int_0^1 \phi_i(t)\,\phi_j(t)\,dt\right)$$
$$= \sum_{i=1}^{n}\int_0^1 (\phi_i(t))^2\,dt = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2.$$

We also have

$$E(X^2) = \sum_{i,j,k,l}\left(\int_0^1 r_i(s)\,r_j(s)\,r_k(s)\,r_l(s)\,ds\right)\left(\int_0^1 \phi_i(t)\,\phi_j(t)\,\phi_k(t)\,\phi_l(t)\,dt\right)$$

(where the suffices i, j, k, l take the values $1, 2, \ldots, n$).

Thus
$$E(X^2) = \sum_{i=1}^{n}\int_0^1 |\phi_i(t)|^4\,dt + 6\sum_{1\leqslant i<j\leqslant n}\int_0^1 |\phi_i(t)\,\phi_j(t)|^2\,dt$$
$$\leqslant 3\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1 |\phi_i(t)\,\phi_j(t)|^2\,dt.$$

However,

$$\int_0^1 |\phi_i(t)\,\phi_j(t)|^2\,dt = \sum_k a_{ik}^2 a_{jk}^2 + 2\sum_{k\neq l} a_{ik}\,a_{il}\,a_{jk}\,a_{jl} + \sum_{k\neq l} a_{ik}^2\,a_{jl}^2$$
$$\leqslant 2\sum_{k,l} a_{ik}\,a_{il}\,a_{jk}\,a_{jl} + \sum_{k,l} a_{ik}^2\,a_{jl}^2$$
$$= 2\big(\sum_k a_{ik}\,a_{jk}\big)^2 + \big(\sum_k a_{ik}^2\big)\big(\sum_l a_{jl}^2\big)$$
$$\leqslant 3\big(\sum_k a_{ik}^2\big)\big(\sum a_{jl}^2\big).$$

Hence
$$E(X^2) \leqslant 9\big(\sum_{i,k} a_{ik}^2\big)^2$$

Now by (2), p. 6, inequality II,

$$(m\times m)\{(s,t): |X(s,t)| \geqslant \tfrac{1}{100}\Sigma\Sigma|a_{ij}|^2\} \geqslant \tfrac{1}{10}$$

and the lemma follows.

LEMMA 6·2. *Let F be an Orlicz function satisfying the Δ_2-condition with constant K. Then there are constants C_1, C_2 and $C_3 > 0$ depending only on K such that*

$$C_1 F\left(\sqrt{\left(\sum_{i=1}^{n}|a_i|^2\right)}\right) \leqslant \int_0^1 F\left(\left|\sum_{i=1}^{n} a_i\,r_i(t)\right|\right)dt \leqslant C_2 F\left(\sqrt{\left(\sum_{i=1}^{n}|a_i|^2\right)}\right)$$

whenever $a_1 \ldots a_n \in \mathbb{R}$ and $n \in \mathbb{N}$, and

$$C_3 F\left(\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2}\right) \leqslant \int_0^1\int_0^1 F\left(\left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,r_i(s)\,r_j(t)\right|\right)ds\,dt$$

whenever (a_{ij}) is a real $n\times n$ matrix.

Proof. By the Paley–Zygmund inequality

$$\int_0^1 F\left(\left|\sum_{i=1}^{n} a_i\,r_i(t)\right|\right)dt \geqslant \tfrac{1}{4}F\big(\tfrac{1}{4}\sqrt{(\Sigma|a_i|^2)}\big)$$
$$\geqslant \frac{1}{4K^2}F\big(\sqrt{\Sigma|a_i|^2}\big).$$

Conversely choose p so that $2^p = K$ and let $\alpha = \sqrt{(\Sigma a_i^2)}$. Then for $x > \alpha$

$$F(x) \leqslant K^2 F(x_0),$$

where $\frac{1}{2}\alpha < x_0 \leqslant \alpha$, and l is an integer such that $2^l x_0 = x$. Hence

$$\frac{F(x)}{x^p} \leqslant \frac{F(x_0)}{x_0^p} \leqslant 2^p \frac{F(\alpha)}{\alpha^p}.$$

Hence

$$\begin{aligned}\int_0^1 F\left(\left|\sum_{i=1}^n a_i r_i(t)\right|\right) dt &= \int_{\Sigma a_i r_i(t) > \alpha} F\left(\left|\sum_{i=1}^n a_i r_i(t)\right|\right) dt + \int_{\Sigma a_i r_i(t) \leqslant \alpha} F\left(\left|\sum_{i=1}^n a_i r_i(t)\right|\right) dt \\ &\leqslant \frac{2^p F(\alpha)}{\alpha^p} \int_0^1 \left|\sum_{i=1}^n a_i r_i(t)\right|^p dt + F(\alpha) \\ &\leqslant \left(2^p \left(\frac{p}{2}+1\right)^{\frac{1}{2}p} + 1\right) F(\alpha)\end{aligned}$$

by Khintchin's inequality ((1), p. 131).

The second inequality follows similarly from Lemma 6·1.

Lemma 6·3. *Let (u_n, u_n') be a complemented topologically normalized unconditional basic sequence in l_F. Suppose $(A_n : n \in \mathbb{N})$ is a sequence of mutually disjoint subsets of $\mathbb{N}$ such that*

$$\sum_{k \in A_n} u_n'(e_k)\, e_k'(u_n) = \theta_n$$

for $n \in \mathbb{N}$. Suppose $|\theta_n| \geqslant \theta > 0$ $(n \in \mathbb{N})$. Define

$$v_n = \sum_{k \in A_n} e_k'(u_n)\, e_k,$$

$$v_n' = \theta_n^{-1} \sum_{k \in A_n} u_n'(e_k)\, e_k'.$$

Then (v_n, v_n') is a complemented block basic sequence equivalent to (u_n).

Proof. Clearly $v_i'(v_j) = \delta_{ij}$ so that (v_n, v_n') is biorthogonal. Since (u_n, u_n') is complemented and unconditional, it follows that if

$$\eta(\delta) = \sup_{n \in \mathbb{N}} \sup_{\epsilon_1 \ldots \epsilon_n = \pm 1} \sup_{|x|_F \leqslant \delta} \left|\sum_{i=1}^n \epsilon_i u_i'(x)\, u_i\right|_F$$

then $\lim_{\delta \to 0} \eta(\delta) = 0$. (Of course $\eta(\delta)$ can be $+\infty$ for some $\delta > 0$.) The proof involves demonstrating the continuity of two operators:

(1) $S: \operatorname{lin}(u_n) \to \operatorname{lin}(v_n)$, $\quad Sx = \sum_{n=1}^{\infty} u_n'(x) v_n$.

(2) $T: \operatorname{lin}(e_n) \to \operatorname{lin}(u_n)$, $\quad Tx = \Sigma v_n'(x)\, u_n$.

Once we have shown these operators continuous, they may be extended to maps $S: \overline{\operatorname{lin}}\,(u_n) \to \overline{\operatorname{lin}}\,(v_n)$ and $T: l_F \to \overline{\operatorname{lin}}\,(u_n)$. Then $Su_n = v_n$ and $Tv_n = u_n$ and so (u_n) and (v_n) are equivalent. Further ST is the projection $x \mapsto \Sigma v_n'(x)\, v_n$ of l_F into $\overline{\operatorname{lin}}\,(v_n)$.

For (1), let $x = \sum_{i=1}^{n} \xi_i u_i \in \operatorname{lin}(u_i)$. Then if $|x|_F \leqslant \delta$,

$$\left| \sum_{i=1}^{n} \xi_i r_i(t) u_i \right|_F \leqslant \eta(\delta)$$

and so
$$\int_0^1 \sum_{j=1}^{\infty} F\left(\left| \sum_{i=1}^{n} r_i(t)\, \xi_i\, e_j'(u_i) \right| \right) dt \leqslant \eta(\delta).$$

By Lemma 6·2
$$C_1 \sum_{j=1}^{\infty} F\left(\sqrt{\sum_{i=1}^{n} |\xi_i|^2 |e_j'(u_i)|^2} \right) \leqslant \eta(\delta).$$

Hence
$$\sum_{j=1}^{n} \sum_{k \in A_j} F\left(\sqrt{\sum_{i=1}^{n} |\xi_i|^2 |e_k'(u_i)|^2} \right) \leqslant C_1^{-1} \eta(\delta),$$

$$\sum_{j=1}^{n} \sum_{k \in A_j} F(|\xi_j e_k'(u_j)|) \leqslant C_1^{-1} \eta(\delta),$$

i.e.
$$\left| \sum_{j=1}^{n} \xi_j v_j \right|_F \leqslant C_1^{-1} \eta(\delta),$$

$$|Sx|_F \leqslant C_1^{-1} \eta(\delta).$$

As $\lim_{\delta \to 0} \eta(\delta) = 0$, S is continuous.

For (2) let $x = \sum_{i=1}^{\infty} \xi_i e_i \in \operatorname{lin}(e_i)$, where (ξ_n) is finitely non-zero and $|x|_F = \delta$. Then for $0 \leqslant t \leqslant 1$, $n \in \mathbb{N}$,

$$\left| \sum_{k=1}^{n} r_k(t) \sum_{i \in A_k} \xi_i e_i \right|_F \leqslant \delta$$

and hence for $0 \leqslant s \leqslant 1$

$$\left| \sum_{j=1}^{n} r_j(s)\, u_j' \left(\sum_{k=1}^{n} r_k(t) \sum_{i \in A_k} \xi_i e_i \right) u_j \right|_F \leqslant \eta(\delta).$$

Thus
$$\int_0^1 \int_0^1 \sum_{l=1}^{\infty} F\left(\left| \sum_{j=1}^{n} \sum_{k=1}^{n} r_j(s)\, r_k(t) \sum_{i \in A_k} \xi_i\, u_j'(e_i)\, e_l'(u_j) \right| \right) ds\, dt \leqslant \eta(\delta)$$

and by Lemma 6·2

$$\sum_{l=1}^{\infty} F\left(\sqrt{\left(\sum_{j=1}^{n} \sum_{k=1}^{n} \left(\sum_{i \in A_k} \xi_i\, u_j'(e_i)\, e_l'(u_j) \right)^2 \right)} \right) \leqslant C_3^{-1} \eta(\delta).$$

Hence
$$\sum_{l=1}^{\infty} F\left(\sqrt{\left(\sum_{k=1}^{n} \left(\sum_{i \in A_k} \xi_i\, u_k'(e_i)\, e_l'(u_k) \right)^2 \right)} \right) \leqslant C_3^{-1} \eta(\delta).$$

Thus
$$\sum_{l=1}^{\infty} F\left(\sqrt{\sum_{k=1}^{n} |\theta_k\, v_k'(x)\, e_l'(u_k)|^2} \right) \leqslant C_3^{-1} \eta(\delta).$$

However, $|\theta_k| \geqslant \theta$, for all k and so there is a constant M such that

$$\sum_{l=1}^{\infty} F\left(\sqrt{\left(\sum_{k=1}^{n} |v_k'(x)\, e_l'(u_k)|^2 \right)} \right) \leqslant M \eta(\delta).$$

Consider
$$\int_0^1 \left| \sum_{k=1}^n r_k(t)\, v_k'(x)\, u_k \right|_F dt = \sum_{l=1}^{\infty} \int_0^1 F\left(\left| \sum_{k=1}^n r_k(t)\, v_k'(x)\, e_l'(u_k) \right| \right) dt$$
$$\leqslant C_2 \sum_{l=1}^{\infty} F\left(\sqrt{\left(\sum_{k=1}^n |v_k'(x)\, e_l'(u_k)|^2 \right)} \right)$$
$$\leqslant MC_2\, \eta(\delta).$$

Hence for some t, $0 \leqslant t \leqslant 1$ with $r_k(t) = \pm 1$ for $k = 1, 2, \ldots, n$,
$$\left| \sum_{k=1}^n r_k(t)\, v_k'(x)\, u_k \right|_F \leqslant MC_2\, \eta(\delta)$$
and so
$$\left| \sum_{k=1}^n v_k'(x)\, u_k \right|_F \leqslant \eta(MC_2\, \eta(\delta))$$
so that T is also continuous and the proof is complete.

THEOREM 6·4. *If (u_n, u_n') is a topologically normalized inessential complemented unconditional basic sequence in l_F, then (u_n) is equivalent to (e_n).*

Proof. Let $\beta = \inf_n \max_k |u_n'(e_k)\, e_k'(u_n)| > 0$. There exists a map $\pi: \mathbb{N} \to \mathbb{N}$ such that
$$|u_n'(e_{\pi(n)})\, e_{\pi(n)}'(u_n)| \geqslant \beta.$$
It follows from the fact that $(e_n: n \in \mathbb{N})$ is bounded that
$$\sup_k \sum_{n=1}^{\infty} |u_n'(e_k)\, e_k'(u_n)| < \infty$$
(using also, of course, that (u_n) is complemented and unconditional). Hence there is a constant l such that $\operatorname{card} \pi^{-1}\{k\} \leqslant l$ for $k \in \mathbb{N}$. Thus it is possible to decompose $\mathbb{N}$ into a finite disjoint union $B_1 \cup \ldots \cup B_l$ of subsequences such that $\pi|B_i$ is injective $(1 \leqslant i \leqslant l)$. It is enough to show that each $(u_n: n \in B_i)$ is equivalent to (e_n). Hence we may suppose that π is injective on $\mathbb{N}$. Now let $A_n = \{\pi(n)\}$ and apply Theorem 6·3. (u_n) is equivalent to $(e_{\pi(n)}'(u_n)\, e_{\pi(n)})$, and since both (u_n) and (e_n) are normalized and unconditional, (u_n) and (e_n) are equivalent.

THEOREM 6·5. *Suppose (u_n) is an essential topologically normalized complemented unconditional basic sequence l_F. Then (u_n) has a subsequence equivalence to the unit vector basis of a locally convex Orlicz sequence space.*

Proof. It is easy to see that we may find increasing sequences $\{p_n: n \geqslant 0\}$ with $p_0 = 0$ and $\{m_n: n \geqslant 1\}$ with $m_1 = 1$ such that
$$\sum_{i=p_{n-1}+1}^{p_n} u_{m_n}'(e_i)\, e_i'(u_{m_n}) = \theta_n \geqslant \tfrac{1}{2}$$
and
$$\lim_{n \to \infty} \max_i |u_{m_n}'(e_i)\, e_i'(u_{m_n})| = 0.$$

Now by Lemma 6·3 (u_{m_n}) is equivalent to the complemented block basic sequence (v_n, v_n') where

$$v_n = \sum_{i=p_{n-1}+1}^{p_n} e_i'(u_{m_n})\, e_i = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i,$$

$$v_n' = \theta_n^{-1} \sum_{i=p_{n-1}+1}^{p_n} u_{m_n}'(e_i)\, e_i' = \sum_{i=p_{n-1}+1}^{p_n} b_i e_i'.$$

Note that $\sum_{p_{n-1}+1}^{p_n} a_i b_i = 1$ and that

$$\beta_n = \max_{p_{n-1}<i\leqslant p_n} |a_i b_i| \to 0.$$

We may further assume (multiplying (v_n) if necessary by a bounded sequence of scalars) that $0 < \inf_n |v_n|_F \leqslant \sup_n |v_n|_F \leqslant 1$, and by extraction of a further subsequence that if $\psi_n(t) = |tv_n|_F$, then there is an Orlicz function H such that if

$$\epsilon_n = \sup_{0\leqslant t\leqslant 1} |H(t) - \psi_n(t)|$$

then $\Sigma \epsilon_n < \infty$ (see Proposition 4·3). Thus (v_n) is equivalent to the unit vector basis of l_H. We shall sho·/ that H is equivalent to a convex Orlicz function. Let h be its convex minorant ov.·r $[0, 1]$ i.e.

$$h(x) = \inf\left\{\frac{1}{n}\sum_{i=1}^{n} h(x_i) : n \in \mathbb{N}, x_1 + \ldots + x_n = nx : 0 \leqslant x_i \leqslant 1\right\} \quad \text{for} \quad 0 \leqslant x \leqslant 1$$

and $h(x) = H(x)$ for $x > 1$. We show H and h are equivalent.

Suppose $0 \leqslant \alpha_n \leqslant 1$ for all n and $\Sigma h(\alpha_n) < \infty$. Then there is an increasing sequence $m(n)$ such that

$$m(n)\,\alpha_n = t_1^{(n)} + \ldots + t_{m(n)}^{(n)} \quad (0 \leqslant t_i^{(n)} \leqslant 1),$$

where

$$\frac{1}{m(n)} \sum_{i=1}^{m(n)} H(t_i^{(n)}) \leqslant h(\alpha_n) + 2^{-n}.$$

Next pick an increasing sequence $l(n)$ such that

$$\beta_{l(n)} \leqslant \frac{1}{2m(n)} \quad (n = 1, 2, \ldots).$$

Then the set $\{p_{l(n)-1}+1, \ldots p_{l(n)}\}$ may be decomposed into $m(n)$ subsets $A_1^n \ldots A_{m(n)}^n$ such that

$$\left|\sum_{i\in A_k^n} a_i b_i - \frac{1}{m(n)}\right| \leqslant \frac{1}{2m(n)} \quad (k = 1, \ldots, m(n)).$$

Let

$$w_k^{(n)} = \sum_{i\in A_k^n} a_i e_i \quad (k = 1, \ldots, m(n))$$

and for each permutation σ of $\{1, \ldots, m(n)\}$ let

$$x_\sigma^{(n)} = \sum_{k=1}^{m(n)} t_{\sigma(k)}^{(n)} w_k^{(n)}$$

$$|x_\sigma^{(n)}|_F = \sum_{k=1}^{m(n)} \sum_{i\in A_k^n} F(t_{\sigma(k)}^{(n)} |a_i|)$$

and summing over all permutations

$$\sum_{\sigma} |x_{\sigma}^{(n)}|_F = (m(n)-1)\,! \sum_{i=p_{l(n)-1}+1}^{p_{l(n)}} \sum_{k=1}^{m(n)} F(t_k^{(n)}|a_i|)$$

and so there exists τ such that

$$\begin{aligned} |x_{\tau}^{(n)}|_F &\leqslant \frac{1}{m(n)} \sum_{k=1}^{m(n)} |t_k^{(n)} v_{l(n)}|_F \\ &\leqslant \frac{1}{m(n)} \sum_{k=1}^{m(n)} H(t_k^{(n)}) + \epsilon_{l(n)} \\ &\leqslant h(\alpha_n) + 2^{-n} + \epsilon_{l(n)}. \end{aligned}$$

Put $y_n = x_{\tau}^{(n)}$. Then Σy_n converges in l_F. Now

$$\begin{aligned} v_i'(y_n) &= 0 \quad \text{if} \quad i \neq l(n), \\ v_{l(n)}'(y_n) &= \sum_{k=1}^{m(n)} \sum_{i \in A_k^n} t_{\tau(k)}^{(n)} a_i b_i \\ &= \sum_{k=1}^{m(n)} t_{\tau(k)}^{(n)} \Big(\sum_{i \in A_k^n} a_i b_i \Big) \\ &\geqslant \frac{1}{2} \sum_{k=1}^{m(n)} t_{\tau(k)}^{(n)} \frac{1}{m(n)} \\ &= \tfrac{1}{2}\alpha_n. \end{aligned}$$

Hence $\Sigma \alpha_n v_{l(n)}$ converges, i.e. $\Sigma H(\alpha_n) < \infty$, and the proof is concluded.

7. *Main results*

THEOREM 7·1. *Suppose l_G is isomorphic to a complemented subspace of l_F. Then either G is equivalent to F, or G is equivalent to a convex function.*

Proof. The usual basis of l_G is equivalent to a symmetric complemented unconditional basic sequence in l_F. Either Theorem 6·2 or Theorem 6·3 applies.

THEOREM 7·2. *Suppose l_F is a non-locally convex Orlicz sequence space. Then the following conditions are equivalent:*

(i) *Any two unconditional bases of l_F are affinely equivalent.*

(ii) *Any two topologically normalized unconditional bases of l_F are equivalent.*

(iii) *l_F contains no complemented subspace isomorphic to some l_G where G is a convex Orlicz function.*

(iv) *Any complemented subspace with an unconditional basis is isomorphic to l_F.*

Proof. (i) $\Rightarrow$ (ii) is trivial; (ii) $\Rightarrow$ (i) follows from Theorem 3·3, Corollary.

(ii) $\Rightarrow$ (iii). If $l_F \cong l_G \oplus X$, then $l_F \simeq l_G \oplus l_G \oplus X \cong l_G \oplus l_F$ and hence has a topologically normalized unconditional basis non-equivalent to the standard basis.

(iii) $\Rightarrow$ (iv). By Theorems 6·2 and 6·3.

(iv) $\Rightarrow$ (ii). By Theorems 6·2 and 6·3.

LEMMA 7·3. *If X is a complemented locally convex subspace of l_F, then the topology of X induced by $|\ |_F$ is equivalent to that induced by $|\ |_{\hat{F}}$.*

Proof. Clearly the identity map $i:(X, |\ |_F) \to (X, |\ |_{\hat{F}})$ is continuous. We show i^{-1} is weakly continuous, which implies i^{-1} is continuous since both topologies are locally convex. If $\psi \in (X, |\ |_F)'$ then $\psi \circ P \in l'_F$ where P is a projection of l_F onto X. Hence by Theorem 3·3, $\psi \circ P$ is $|\ |_{\hat{F}}$ continuous on l_F and hence ψ is $|\ |_{\hat{F}}$-continuous on X. This proves the lemma.

Remarks. In other words X has the Hahn–Banach Extension Property.

THEOREM 7·4. *Suppose for any $C > 0$ there exist $x_1 \dots x_n \in I_0$ and $a_1 \dots a_n > 0$ such that*

$$\sum_{i=1}^{n} a_i F(tx_i) \geqslant C \sum_{i=1}^{n} a_i \hat{F}(tx_i) \quad (0 \leqslant t \leqslant 1).$$

Then l_F has a unique unconditional basis (up to affine equivalence).

Proof. By Lemma 7·3 and Theorem 5·2.

In the case when $\limsup_{x\to 0} \dfrac{x}{F(x)} < \infty$ we can give a necessary and sufficient condition for the uniqueness of the unconditional basis.

THEOREM 7·5. *Suppose that $\limsup_{x\to 0} \dfrac{x}{F(x)} < \infty$ and $\liminf_{x\to 0} \dfrac{x}{F(x)} = 0$. Then the following conditions on l_F are equivalent:*

(i) *Any two unconditional bases of l_F are affinely equivalent.*
(ii) *l_F contains no complemented subspace isomorphic to l_1.*
(iii) $\displaystyle\lim_{\epsilon\to 0} \inf_{0<s\leqslant 1} \frac{1}{\log(1/\epsilon)} \int_{\epsilon}^{1} \frac{F(su)}{su^2}\, du = \infty.$
(iv) *If $\phi(x) = e^x F(e^{-x})$ for $x \geqslant 0$, then*

$$\lim_{l\to\infty} \inf_{v} \frac{1}{l} \int_{v}^{v+l} \phi(x)\, dx = \infty.$$

Proof. (i) $\Leftrightarrow$ (ii). Theorem 7·2.
(ii) $\Rightarrow$ (iii). Theorem 5·4.
(iii) $\Leftrightarrow$ (iv). Put $u = e^{-x}$ in (iii), (see Remark following Theorem 5·3).
(iii) $\Rightarrow$ (ii). By Theorem 5·3 and Lemma 7·3.

Remark 1. If $\liminf_{x\to 0} \dfrac{x}{F(x)} > 0$ then $F(x)$ is equivalent to x and $l_F = l_1$. l_1 has a unique unconditional basis (modulo affine equivalence) by a theorem of Lindenstrauss and Pełczynski (6).

Remark 2. In section 8 we show that condition (iii) above is distinct from the conditions $\lim_{x\to 0} \dfrac{x}{F(x)} = 0$. Clearly we have the following:

THEOREM 7·6. *If $F(x)/x \to \infty$ as $x \to 0$, l_F has a unique unconditional basis (up to affine equivalence).*

Proof. Let $\frac{F(x)}{x} = \lambda(x)$ and $\bar{\lambda}(x) = \inf_{0<y\leqslant x} \lambda(y)$

$$\begin{aligned}\frac{1}{\log(1/\epsilon)}\int_\epsilon^1 \frac{F(su)}{su^2}du &= \frac{1}{\log(1/\epsilon)}\int_\epsilon^1 \lambda(su)\frac{du}{u}\\ &\leqslant \frac{1}{\log(1/\epsilon)}\int_\epsilon^1 \bar{\lambda}(u)\frac{du}{u}\\ &= \infty \quad \text{since} \quad \bar{\lambda}(u)\to\infty \quad \text{as} \quad u \to 0.\end{aligned}$$

In particular $l_p(0 < p \leqslant 1)$ has a unique unconditional basis solving a problem of Stiles (15).

8. *Examples*

Suppose $\{a_n\}$ is a strictly increasing sequence of positive integers and $\{\theta_n\}$ is any increasing sequence of positive numbers satisfying $0 < \theta_n < 2^{a_n-3}$. Define ψ_n on $[0, \infty)$ to be a function of period 2^{a_n} such that

$$\begin{aligned}\psi_n(x) &= 0 \quad (0 \leqslant x \leqslant 2^{a_n} - 4\theta_n)\\ &= \tfrac{1}{2}(x - 2^{a_n}) + 2\theta_n \quad (2^{a_n} - 4\theta_n \leqslant x \leqslant 2^{a_n} - 2\theta_n)\\ &= \tfrac{1}{2}(2^{a_n} - x) \quad (2^{a_n} - 2\theta_n \leqslant x \leqslant 2^{a_n}).\end{aligned}$$

Let $\psi(x) = \sup_{n\in\mathbb{N}} \psi_n(x)$. We observe that $\psi(0) = 0$ and

$$|\psi(x) - \psi(y)| \leqslant \tfrac{1}{2}|x-y| \quad (x, y \geqslant 0),$$

$$\psi(x) = \max(\psi_1(x), \psi_2(x) \ldots \psi_n(x)) \quad (x \leqslant 2^{a_n}).$$

If $p > \frac{1}{2}$ we define

$$\begin{aligned}&F(x) = x^p \exp(\psi(-\log x)) \quad (0 < x \leqslant 1),\\ &F(0) = 0.\end{aligned}$$

Then F is the principal part of an Orlicz function, satisfying the Δ_2-condition. In fact

$$F(2x) \leqslant 2^{p+\frac{1}{2}}F(x) \quad (0 < x \leqslant \tfrac{1}{2}).$$

If $\phi_p(x) = e^{px}F(e^{-x})$ then of course $\phi_p(x) = e^{\psi(x)}$.

Example 1. *There is an Orlicz function F such that*

$$0 = \liminf_{x\to 0}\frac{x}{F(x)} < \limsup_{x\to 0}\frac{x}{F(x)} < \infty$$

and l_F has (up to affine equivalence) a unique unconditional basis.

Let $a_n = n$ and $\theta_n = \alpha n^2$ where $\alpha > 0$ is chosen so that $\theta_n < 2^{n-3}$ $(n \geqslant 1)$ and $\sum_{n=1}^{\infty} 2^{-n}\theta_n < \frac{1}{8}$. Let $p = 1$.

Then $$m\{x: 0 \leqslant x \leqslant 2^n, \psi(x) \neq 0\} \leqslant \sum_{k=1}^{n} m\{x: 0 \leqslant x \leqslant 2^n, \psi_k(x) \neq 0\}$$
$$= \sum_{k=1}^{n} 2^{n-k}(4\theta_k)$$
$$< 2^{n-1}.$$

Here for each n, there exists x, $2^{n-1} < x < 2^n$ such that $\psi(x) = 0$. Thus $\limsup_{x\to 0} \frac{x}{F(x)} = 1$. Since ψ is unbounded $\liminf_{x\to 0} \frac{x}{F(x)} = 0$.

If $2^n \leqslant l < 2^{n+1}$
$$\frac{1}{l}\int_v^{v+l} \phi_1(x)\,dx \geqslant \frac{1}{2^{n+1}}\int_v^{v+2^n} \phi_1(x)\,dx$$
$$\geqslant \frac{1}{2^{n+1}}\int_v^{v+2^n} e^{\psi_n(x)}\,dx$$
$$= \frac{1}{2^{n+1}}\int_0^{2^n} e^{\psi_n(x)}\,dx$$
$$= \frac{1}{2^{n+1}}(2^n + 4(e^{\theta_n} - \theta_n - 1))$$
$$\to \infty \quad \text{as} \quad n \to \infty.$$

By Theorem 7·5, l_F has a unique unconditional basis.

Example 2. *There is a convex Orlicz function G such that for some $p > 1$, l_p is complemented in l_G and x^p is not equivalent to any function in $E_{G,1}$.*

Remark. This solves negatively a problem of Lindenstrauss & Tzafriri (8) (Problem 1).

Choose $a_n = 2^{2n}$ and $\theta_n = 2^n$; fix $p > \frac{3}{2}$. Then $F(x)/x$ is increasing and so F is equivalent to a convex Orlicz function G.
$$\frac{1}{2^{a_n}}\int_0^{2^{a_n}} \phi_p(x) = \frac{1}{2^{a_n}}\int_0^{2^{a_n}} e^{\psi(x)}\,dx$$
$$= 1 + \frac{1}{2^{a_n}}\int_0^{2^{a_n}} (e^{\psi(x)} - 1)\,dx$$
$$\leqslant 1 + \frac{1}{2^{a_n}}\sum_{k=1}^{n}\int_0^{2^{a_n}} (e^{\psi_k(x)} - 1)\,dx$$
$$= 1 + \frac{1}{2^{a_n}}\sum_{k=1}^{n} 2^{a_n - a_k}\int_0^{2^{a_k}} (e^{\psi_k(x)} - 1)\,dx$$
$$= 1 + 4\sum_{k=1}^{n} 2^{-a_k}(e^{\theta_k} - 1 - \theta_k)$$
$$\leqslant 1 + 4\sum_{k=1}^{\infty} 2^{-2^{2k}}(e^{2^k} - 1 - 2^k)$$
$$< \infty$$

and by Theorem 5·3, l_G has a complemented subspace isomorphic to l_p.

Now suppose x^p equivalent to a function in $E_{G,1}$. Since G and F are equivalent, there is a constant $A \geqslant 1$, such that for any $l > 0$ there exists $u = u(l)$ with
$$e^{-A}x^p \leqslant \frac{F(e^{-u}x)}{F(e^{-u})} \leqslant e^A x^p \quad (e^{-l} \leqslant x \leqslant 1)$$

or equivalently $\quad |\psi(u+v)-\psi(u)| \leqslant A \quad (0 \leqslant v \leqslant l).$

Hence $\quad |\psi(x)-\psi(y)| \leqslant 2A \quad (u \leqslant x, y \leqslant u+l).$

Now pick $k \geqslant 64A^2$ and let $l = 3.2^k$, and let $u = u(l)$. For $u+2^k \leqslant x \leqslant u+2.2^k$ we define $N(x)$ to be the least N such that $\psi(x) = \psi_N(x)$.

First suppose $0 < \psi(x) < 2^{N(x)}-2A$. Then there exists $y \in [u, u+l]$ such that $\psi_{N(x)}(y) > \psi(x)+2A$, since $\frac{1}{2}.2^k > 2A$. This is a contradiction and so we conclude that if $\psi(x) > 0$, $\psi(x) > 2^{N(x)}-2A$.

Next suppose that for some $x_0 \in [u+2^k, u+2^{k+1}]$, $2^{N(x_0)} > 8A$. Then

$$2^{N(x_0)-1} < 2^{N(x_0)}-2A < 2^{N(x_0)}+2A < 2^{N(x_0)+1}$$

and it follows from the above remarks that $N(x) = N(x_0)$ for $u+2^k \leqslant x \leqslant u+2^{k+1}$. Then $\psi(x) = \psi_{N(x_0)}(x)$ for $n+2^k \leqslant x \leqslant u+2^{k+1}$ and so there exist x, y in this range so that

$$|\psi(x)-\psi(y)| \geqslant \min\left(\tfrac{1}{4}\,2^{N(x_0)}, \tfrac{1}{4}\,2^k\right) > 2A.$$

Thus we conclude $2^{N(x)} < 8A$ for all x. Let M be the largest integer such that $2^M < 8A$. Then $\psi = \max(\psi_1, \ldots, \psi_M)$ in the range $u+2^k \leqslant x \leqslant u+2.2^k$ and this function has period $2^{2^{2M}} < 2^{64A^2} \leqslant 2^k$. Clearly ψ takes both the values 0 and 2^M in its period. However, $2^M > 4A$, and we have a contradiction.

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