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ABSTRACT

Recently, Jawerth, Rochberg and Weiss have studied nonlinear maps arising from interpolation theory which satisfy commutator relationships with interpolated linear operators. Here we present a very general result of this type for rearrangement-invariant Banach function spaces.

Key words: Interpolation theory, commutators, twisted sums, Hardy spaces, rearrangement invariant Banach function spaces.

1. Introduction

In two recent papers [7], [21] the authors have explored commutator results obtained by considering the differential of an interpolation family of Banach spaces. In each case the conclusion was that an interpolated linear operator almost commutes with a certain nonlinear functional.

To make these concepts precise let us suppose that E is a Polish space and μ is a nonatomic finite or σ-finite Borel measure on E. Suppose $1 \leq p_0 < p_1 < \infty$ and that T is an operator of strong type $(p_0, p_0), (p_1, p_1)$ i.e. T maps L_{p_j} boundedly into L_{p_j} for $j = 0, 1$. Consider the nonlinear functional

$$\Omega(f) = f \log |f|. \tag{1.1}$$

Then for $p_0 < p < p_1$, Rochberg and Weiss [21] show that

$$\|[T, \Omega]f\|_p \leq C\|f\|_p \tag{1.2}$$

where $[T, \Omega]f = T\Omega(f) - \Omega T(f)$ and C is a constant which is independent of f. This result is obtained from the complex method of interpolation. A simple direct proof for the Hilbert transform or Riesz projection on the circle, without using interpolation theory, is given in [11].

In [7], Jawerth, Rochberg and Weiss use various real methods of interpolation to obtain similar results. For convenience suppose that E is an interval (a, b) (or can be

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identified with an interval); for a measurable function f define

$$r_f(t) = \mu\{s : |f(s)| > |f(t)| \text{ or } |f(s)| = |f(t)|, s \leq t\}$$

and

$$\Omega(f) = f \log r_f \tag{1.3}$$

Under the same hypotheses as above they obtain the commutation relationship (1.2) for this choice of Ω.

The object of this paper is to derive a very general commutator theorem including the above results but for a much wider choice of possible functions Ω, somewhat in the spirit of the Boyd interpolation theorem [1]. The author's interest in such a result was first aroused by connections with the theory of twisted sums of Banach spaces developed in [9],[10],[14],[15] for example. One-dimensional discrete analogues of the Ω-functions (1.1) and (1.3) were used by the author [9] and Ribe [20] to construct examples of non-locally convex quasi-Banach spaces X with a one-dimensional subspace L so that $X/L \cong \ell_1$. Later the author and Peck [14] (see also [8],[10],[17]) studied the Banach space Z_p $(1 < p < \infty)$ of pairs of sequences $(u_n), (v_n)$ such that

$$\|(u_n), (v_n)\| = (\sum |u_n - v_n \log \frac{|v_n|}{\|v\|_p}|^p)^{1/p} + \|v\|_p < \infty$$

where $\|v\|_p = (\sum |v_n|^p)^{1/p}$. The function space analogue of Z_p, (called ZF_p in [11]) is the space of pairs of functions (f, g) in $L_0(E)$ such that

$$\|(f, g)\| = \{\int_E |f - g \log \frac{|g|}{\|g\|_p}|^p d\mu\}^{1/p} + \|g\|_p < \infty$$

where $\|g\|_p = \{\int |g|^p d\mu\}^{1/p}$. Again we require $1 < p < \infty$, and this space is considered by Rochberg and Weiss [21] who show that the commutator relationship (1.1) - (1.2) is equivalent to the requirement that the map $(f,g) \to (Tf,Tg)$ is bounded on ZF_p. The spaces Z_p and ZF_p are examples of twisted sums of, respectively, the spaces ℓ_p and ℓ_p or the spaces L_p and L_p. Since many more examples can be created (cf.[14]) it is natural to ask for a more general result of this nature.

Let us now describe our main result. We suppose X is a separable re-arrangement function space on (E,μ) (e.g. an Orlicz space or Lorentz space). Let $\Omega : X \to L_0(\mu)$ be any map. We shall say that Ω is a *centralizer* if Ω obeys a commutation relationship with every multiplication operator on X. Precisely we require the existence of a function $\delta : \mathbf{R}_+ \to \mathbf{R}_+$ so that if $f \in X$, $u \in L_\infty$ with $\|u\|_\infty \le 1$ then $\Omega(uf) - u\Omega(f) \in X$ and

$$\|\Omega(uf) - u\Omega(f)\|_X \le \delta(\|f\|_X) \tag{1.4}$$

(or $\|[M_u,\Omega]f\|_X \le \delta(\|f\|_X)$ where $M_f = uf$).

If additionally Ω commutes with all re-arrangements then Ω is called a symmetric centralizer. Precisely we require further that

$$\|\Omega(S_\sigma f) - S_\sigma\Omega(f)\|_X \le \delta(\|f\|_X) \tag{1.5}$$

whenever $\sigma : E \to E$ is a measure preserving automorphism and $S_\sigma f = f \circ \sigma$.

Both maps (1.1) and (1.3) obey these conditions with $\delta(\|f\|_X) = C\|f\|_X$ for some constant C, provided the Boyd indices of X are finite (following the notation of Lindenstrauss-Tzafriri [16]). However, many other examples can be created, as described in Section 3, e.g.

$$\Omega(f) = f|\log|f||^a|\log r_f|^b \tag{1.6}$$

where $0 \leq a, b \leq a+b \leq 1$ or

$$\Omega(f) = f((\log|f|)^2+1)^{a/2}((\log r_f)^2+1)^{b/2} \tag{1.7}$$

where $a+b \leq 1$ and a, b can be negative.

Suppose now that the Boyd indices p_X, q_X of X satisfy $p_0 < p_X \leq q_X < p_1$, and X is separable. Our main result (Theorem 6.10) asserts an operator $T : X \to X$ of strong types (p_0, p_0), (p_1, p_1) commutes with any symmetric centralizer $\Omega : X \to L_0$ in the sense that there is a function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\|[T, \Omega]f\|_X \leq \eta(\|f\|_X) \tag{1.8}$$

To be precise (1.8) holds for a dense order-ideal of $f \in X$ in general, but if X is super-reflexive (1.8) holds for all f.

We now briefly describe the layout of the paper. Section 2 is devoted to notation. Centralizers are introduced and examples are studied in Section 3. Section 4 explores the relationship between centralizers and twisted sums; we do not however use much of the general theory of twisted sums and the paper can be read without knowledge of this theory, except for the results of Section 8. We introduce in Section 4 the notion of a 'lattice twisted square' and show that commutator results correspond to results on the boundedness of a certain operator on a lattice twisted square.

Sections 5 and 6 contain our main results. We introduce the symmetrized Hardy class $H_1^{sym}(E)$ in Section 6. This is the space of all complex functions $f \in L_1(E)$ such that

$$\Lambda(f) = \|f\|_1 + \sup_{\phi} \left| \int f\phi(\log r_f)d\mu \right| < \infty$$

where ϕ ranges over all bounded functions $\phi : \mathbf{R} \to \mathbf{R}$ with $\phi(0) = 0$, and $|\phi(s)-\phi(t)| \leq |s-t|$ for all s, t. H_1^{sym} is a quasi-Banach space with quasi-norm Λ. Its relationship to H_1 is contained in a recent result of B. Davis [4] that if $E = \mathbf{T}$, a real function $f \in H_1^{sym}$ if and only if f is the re-arrangement of a function $\tilde{f}$ in $\Re H_1$. We do not use Davis's theorem which is actually a consequence of our results; we also give in Section 7 a reasonably simple direct proof.

In Section 5, we show that if T is of strong types $(p_0, p_0), (p_1, p_1)$ and if $p_0 < p_X \leq q_X < p_1$ then the bilinear form

$$B_T(f, g) = Tf.g - f.T^*g$$

maps $X \times X^*$ boundedly into H_1^{sym}. This critically uses the Boyd interpolation theorem. In Section 6 we show that this in turn implies our main result.

Section 7 contains the "simple" proof of Davis's theorem alluded to above (another non-probabilistic proof due to J.L. Lewis was communicated to the author by A. Baernstein; a vector-valued version is proved in [13]). In fact we prove an earlier characterization of rearrangements of $\Re H_1$-functions due to Ceretelli [2], and show that this is equivalent to Davis's theorem. Our main results for the special cases of the Hilbert transform or Riesz projection can then be obtained directly without use of interpolation theory. In Section 8, we tidy up some loose ends.

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2. Notation

In general we will work over a Polish space E, i.e. a complete separable metric space, and μ will be a non-atomic finite or σ-finite Borel measure on E. For applications, we will take $E = \mathbf{T}$, the unit circle with Haar measure, or E is an open subset of $\mathbf{R}^n$ with Lebesgue measure. For any such measure space (E, μ) there is a measure-preserving Borel isomorphism σ of E onto an interval (a, b) where $-\infty \leq a < b \leq \infty$. Frequently it will be convenient to suppose that E is an interval; however, our main results do not depend on this assumption.

For terminology concerning Kothe function spaces we follow the book of Lindenstrauss-Tzafriri [18]. In general we consider spaces of complex functions. Let $L_0(E, \mu)$ denote the space of Borel functions on E. A Kothe function space X is a Banach space of (equivalence classes of) locally integrable functions $f \in L_0$ such that

(2.1) If B is a Borel set with $\mu(B) < \infty$ then $1_B \in X$

(2.2) If $f \rightarrow X$, $g \rightarrow L_0$ with $|g| \leq |f|$ a.e. then $g \in X$ and $\|g\|_X \leq \|f\|_X$ (where $\|\cdot\|_X$ denotes the norm on X).

X is said to be *minimal* if the closed linear span $[1_B : \mu(B) < \infty]$ is dense in X, and *maximal* if whenever $0 \leq f_n \uparrow f$ a.e. with $f_n \in X$ and $\sup \|f_n\| < \infty$ then $f \in X$ and

$$\|f\|_X = \sup_n \|f_n\|_X$$

(cf. [18] p.118 for the same concepts for r.i. function spaces).

We invariably also suppose that X is *separable* and thus automatically minimal. Note we exclude $L_\infty(\mu)$. If X is separable then X is also maximal if and only if X contains no isomorphic copy of c_0. If X is separable and $f_n, f \to X$ with $|f_n| \leq |f|$ a.e. and $f_n \to g$ a.e. then $\|f_n - g\|_X \to 0$. This fact, which follows from the order-continuity of X, will be used frequently.

When X is separable, the dual space X^* of X can be identified with a Kothe function space which is maximal. We denote by X_0^* the closed linear span $[1_B : \mu(B) < \infty]$ in X^*. The dual of X_0^* is the *maximal hull* X_{max} of X.

If $f_1 \in L_0(E_1, \mu)$ and $f_2 \in L_0(E_2, \mu_2)$ we write $f_1 \sim f_2$ if for every Borel subset B of $\mathbf{C} \setminus \{0\}$, we have

$$\mu_1(f_1^{-1}(B)) = \mu_2(f_2^{-1}(B)).$$

We also write f^* for the decreasing re-arrangement of $|f|$, where $f \in L_0(E, \mu)$ i.e. $f^* : (0, \infty) \to \mathbf{R}$ is defined by

$$f^*(t) = \inf_{\mu B = t} \sup_{s \in E \setminus B} |f(s)|$$

A Kothe function space X is said to be a *re-arrangement invariant function space* (r.i. function space) if X is either maximal or minimal and if whenever $f \in X, g \in L_0$ with $g^* \leq f^*$ then $g \in X$ and

$$\|f\|_X \leq \|f\|_X$$

(see [18] p.114 onwards). This is equivalent to requiring that for every measure-preserving (Borel) automorphism $\sigma : E \to E$, S_σ is an isometry where

$$S_\sigma f = f \circ \sigma,$$

or again to the assertion that $f \sim g$ implies $\|f\|_X = \|g\|_X$.

If X is an r.i. function space it will be convenient to define $X(a,b)$ for any interval (a,b) with $b-a\le \mu E$ by requiring that $f\in X(a,b)$ if and only if there exists $g\in X$ with $f\sim g$ and then setting $\|f\|_X=\|g\|_X$. In particular, if $L=\mu E$ then X is isomorphic to $X(0,L)$. It will be convenient to use X to denote any such equivalent space.

On $X(0,L)$ we can define dilation operators $D_s : X(0,L)\to X(0,L)$ by

$$\begin{aligned} D_s f(t) &= f(t/s) \qquad t\le L\min(1,s)\\ &= 0 \qquad\qquad Ls<t\le L. \end{aligned}$$

The *Boyd indices* p_X and q_X of X are defined by

$$p_X=\lim_{s\to\infty}\frac{\log s}{\log\|D_s\|}$$

$$q_X=\lim_{s\to 0}\frac{\log s}{\log\|D_s\|}$$

and then $1\le p_X\le q_X\le\infty$, and

$$\frac{1}{p_X}+\frac{1}{q_{X^*}}=1,$$

$$\frac{1}{q_X}+\frac{1}{p_{X^*}}=1$$

(see [18] p.131).

Before leaving the topic of r.i. function spaces let us mention one easy lemma which is surely well known; we include a proof for completeness.

LEMMA 2.1. *Suppose $f,g,h\in L_0(E,\mu)$ and $f^*g^*h^*\in L_1[0,\infty)$. Then $fgh\in L_1$ and*

$$|\int_E fgh\,d\mu|\le\int_0^\infty f^*g^*h^*dt.$$

For two functions this is to be found in Hardy, Littlewood and Polya [5], Theorem 368. For three functions it suffices to consider the following question. Suppose $a_1 > a_2 \ldots > a_n > 0$, $b_1 > \ldots > b_n > 0$ and $c_1 > c_2 > \ldots > c_n > 0$; we claim the maximum of

$$\sum_{k=1}^{n} a_{\sigma(k)} b_{\tau(k)} c_k$$

over all permutations σ, τ of $\{1, 2, \ldots, n\}$ is attained at $\sigma = \tau =$ identity. For the optimal arrangement, if $k < \ell$, we have $c_k > c_\ell$ and hence (Theorem 368 of [5]), $a_{\sigma(k)} b_{\tau(k)} \geq a_{\sigma(\ell)} b_{\tau(\ell)}$. Thus either $a_{\sigma(k)} > a_{\sigma(\ell)}$ or $b_{\tau(k)} > b_{\tau(\ell)}$ or both. Assume for simplicity the former inequality. Then $a_{\sigma(k)} c_k > a_{\sigma(\ell)} c_\ell$ and hence again by [5], $b_{\tau(k)} \geq b_{\tau(\ell)}$. It follows that σ and τ are both monotone increasing and hence the identity.

We recall that a Banach space X has type p if for some constant C we have for every $x_1, \ldots, x_n \in X$

$$\mathcal{E}(\|\sum_{i=1}^{n} \epsilon_i x_i\|) \leq C(\sum_{i=1}^{n} \|x_i\|^p)^{1/p} \tag{2.3}$$

where the ϵ_i are independent random variables on some probability space satisfying $P(\epsilon_i = +1) = P(\epsilon_i = -1) = 1/2$. We say X has non-trivial type if it is of type p for some $p > 1$. This is equivalent to the statement that X is B-convex or that ℓ_1^n is not finitely representable in X.

Similarly X has cotype p if for some c we have

$$\mathcal{E}(\|\sum_{i=1}^{n} \epsilon_i x_i\|) \geq c(\sum_{i=1}^{n} \|x_i\|^p)^{1/p} \tag{2.4}$$

X has finite cotype if ℓ_∞^n is not finitely representable in X.

A separable Kothe function space X has finite cotype if and only if it satisfies a q-concavity condition for some $q < \infty$ i.e.

$$\|(\sum_{i=1}^{n}|f_i|^q)^{1/q}\|_X \geq c(\sum_{i=1}^{n}\|f_i\|_X^q)^{1/q} \tag{2.5}$$

for $f_1, ..., f_n \in X$.

A Kothe function space has non-trivial type if and only if it is super-reflexive, if and only if it satisfies both a q-concavity condition (2.5) and a p-convexity condition where $p > 1$

$$\|(\sum_{i=1}^{n}|f_i|^p)^{1/p}\|_X \leq C(\sum_{i=1}^{n}\|f_i\|_X^p)^{1/p} \tag{2.6}$$

See [18] for details.

We shall make very little direct use of the theory of twisted sums of Banach and quasi-Banach spaces developed in [9],[14] and [15], although as we indicate in Section 4 below our results are closely related to this theory. In Section 8, we do utilize some ideas which we now explain.

Let X_0 be a normed space and let Y be any Banach space. A map $\Phi : X_0 \to Y$ is called *quasi-linear* if there is a constant $\beta < \infty$ such that

$$\Phi(\alpha x) = \alpha\Phi(x) \qquad \alpha \in \mathbf{C}, x \in X \tag{2.7}$$

$$\|\Phi(x_1 + x_2) - \Phi(x_1) - \Phi(x_2)\| \leq \beta(\|x_1\| + \|x_2\|) \qquad x_1, x_2 \in X \tag{2.8}$$

A Banach space X is called a $\mathcal{K}$-space [15] if there is a constant $C = C(X)$ so that whenever $X_0 \subset X$ is a dense subspace and $\Phi : X_0 \to \mathbf{C}$ is a quasi-linear map satisfying (2.7), (2.8) then there is a linear, but not necessarily continuous, map $\phi : X_0 \to \mathbf{C}$ with

$$|\Phi(x) - \phi(x)| \leq C\beta\|x\| \qquad x \in X_0.$$

It turns out that every Banach space with non-trivial type is a $\mathcal{K}$-space [9], while also every quotient of an $\mathcal{L}_\infty$-space is a $\mathcal{K}$-space [16]. $\mathcal{K}$-spaces have the important property that if Y is any quasi-Banach space and N is a closed subspace of Y so that N is locally convex and Y/N is locally convex and a $\mathcal{K}$-space then Y is also locally convex (i.e. a Banach space).

Convention. Throughout the paper we use C for a constant independent of f, g, Ω, T etc., but depending on X, p, q,etc., which may vary from line to line.

3. Centralizers and symmetric centralizers

We shall suppose E is a separable complete metric space and that μ is a finite or σ-finite measure on E. Let X be a separable Kothe space on E. A map $\Omega : X \to L_0(E)$ will be called a *centralizer* (on X) if there is a function $\delta : \mathbf{R}_+ \to \mathbf{R}_+$ so that if $f \in X$, $u \in L_\infty$ with $\|u\|_\infty \leq 1$ then

$$\|\Omega(uf) - u\Omega(f)\|_X \leq \delta(\|f\|_X). \tag{3.1}$$

It is implicit in equation (3.1) and similar equations that $\|\phi\|_X < \infty$ implies $\phi \in X$.

It will be useful to introduce two stronger notions. We say Ω is a *strong centralizer* if there is a least constant $\Delta = \Delta(\Omega)$ so that

$$\|\Omega(uf) - u\Omega(f)\|_X \leq \Delta\|f\|_X \tag{3.2}$$

for $f \in X$, $\|u\|_\infty \leq 1$.

Ω is a *homogeneous centralizer* if

$$\Omega(\alpha f) = \alpha\Omega(f) \qquad \alpha \in \mathbf{C}, f \in X \tag{3.3}$$

Homogeneous centralizers are automatically strong centralizers.

If further X is rearrangement invariant then a centralizer Ω is symmetric if there is a function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ so that whenever $\sigma : E \to E$ is a measure-preserving Borel automorphism then

$$\|\Omega(S_\sigma f) - S_\sigma(\Omega(f))\|_X \leq \eta(\|f\|_X) \tag{3.4}$$

If Ω is symmetric and homogeneous we denote by $\Delta^*(\Omega)$ the least constant so that $\Delta^*(\Omega) \geq \Delta(\Omega)$ (defined prior to (3.2)) and, for every σ,

$$\|\Omega(S_\sigma f) - S_\sigma(\Omega(f))\|_X \leq \Delta^*(\Omega)\|f\|_X \tag{3.5}$$

We shall say that a map $\Omega : X \to L_0$ satisfying

$$\sup_{\|f\| \leq r} \|\Omega(f)\|_X < \infty \tag{3.6}$$

for every $r > 0$, is a *null-centralizer*. In general if Ω_1 and Ω_2 are any two maps satisfying $\Omega_1 - \Omega_2$ is a null-centralizer then Ω_1 and Ω_2 are *equivalent*. Clearly if the properties of being a centralizer or a symmetric centralizer are preserved under equivalence.

The objective of this first section is to give examples of centralizers and symmetric centalizers.

Let $\phi : \mathbf{R} \to \mathbf{R}$ be any Lipschitz map. We denote by $L(\phi)$ the Lipschitz constant of ϕ i.e.

$$L(\phi) = \sup \frac{|\phi(x_1) - \phi(x_2)|}{|x_1 - x_2|}.$$

We shall also denote by $\mathcal{L}_1(\mathcal{L}_1^b)$ the collections of all (bounded) Lipschitz maps $\phi : \mathbf{R} \to \mathbf{R}$ with $L(\phi) \leq 1$ and $\phi(0) = 0$.

Let us first note that for any Lipschitz map ϕ and any separable Kothe function space the map

$$\Omega(f) = f\phi(\log|f|)$$

is a strong centralizer. In fact

$$\begin{aligned}\|\Omega(uf) - u\Omega(f)\|_X &= \|uf(\phi(\log|f|) - \phi(log|f| + log|u|))\|_X \\ &\leq L(\phi)\|u\log|u|f\|_X \\ &\leq \frac{1}{e}L(\phi)\|f\|_X\end{aligned}$$

so that $\Delta(\Omega) \leq \frac{1}{e}L(\phi)$. Furthermore if X is re-arrangement invariant Ω is symmetric and $\Delta^*(\Omega) \leq e^{-1}L(\phi)$. The special case

$$\Omega(f) = f\log|f|$$

is studied in [21].

We now proceed to a much more general type of result. Now we will assume that E is either an interval (a,b) or $E = \mathbf{R}$ (with μ Lebesgue measure). For any $f \in L_0$ we define the *rank function* r_f by

$$r_f(t) = \mu\{s : |f(s)| > |f(t)| \text{ or } s \leq t \text{ and } |f(s)| = |f(t)|\}$$

Note that $0 \leq r_f \leq \infty$ and $r_f = 0$ only on a set of measure zero; also if $f \in X$ where X is a r.i. function space with finite Boyd indices, it is easy to see that $r_f = \infty$ only when $f = 0$.

THEOREM 3.1. *Let X be a separable r.i. function space on E whose Boyd indices are finite. Then there is a constant $C = C(X)$ with the following property. Let $\psi : \mathbf{R}^2 \to \mathbf{R}$ be a Lipschitz function so that*

$$|\psi(x_1,y_1) - \psi(x_2,y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)$$

Define for $f \in X$

$$\Omega_\psi(f) = f\psi(\log|f|, \log|r_f|)$$

[Here $\Omega_\psi(f)(t) = 0$ when $f(t) = 0$]. Then Ω_ψ is a symmetric strong centralizer on X and

$$\Delta(\Omega_\psi) \leq CL \tag{3.7}$$

$$\|\Omega_\psi(S_\sigma f) - S_\sigma \Omega_\psi(f)\|_X \leq CL\|f\|_X \tag{3.8}$$

for every measure preserving $\sigma : E \to E$.

PROOF: Since the Boyd indices of X are finite we have $\|D_s\| \leq Cs^\alpha$ for $0 < s < 1$ and some $\alpha > 0$.

Let $f \in X$ and set $g = uf \in X$ where $|u| \leq 1$ a.e. Let $\xi_1 = \log|f|$, $\eta_1 = \log|r_f|$, $\xi_2 = \log|g|$, $\eta_2 = \log|r_f|$. Then

$$u|\Omega_\psi(f) - \Omega_\psi(uf) = uf(\psi(\xi_1, \eta_1) - \psi(\xi_2, \eta_2))$$

so that

$$\|u\Omega_\psi(f) - \Omega_\psi(uf)\|_X \leq \||uf|(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|)\|_X.$$

Now

$$\begin{aligned} |uf||\xi_1 - \xi_2| &= |uf| \log \frac{1}{|u|} \\ &\leq e^{-1}|f| \end{aligned}$$

so that

$$\||uf||\xi_1 - \xi_2|\|_X \leq e^{-1}\|f\|_X.$$

We now estimate $(uf)(\eta_1 - \eta_2)$. Since $|uf| = 0$ if either η_1 or $\eta_2 = \infty$ and $\eta_1 = -\infty$ or $\eta_2 = -\infty$ occur only on sets of measure zero we may estimate

$$\|g|\eta_1 - \eta_2|\|_X \leq \sum_{m \in \mathbf{Z}} \|g|\eta_1 - \eta_2|1_{A_m}\|_X$$

where

$$A_m = \{t : \eta_1 + m < \eta_2 \leq \eta_1 + m + 1\}.$$

If $m \geq 0$ and $t \in A_m$

$$\begin{aligned} |g(t)| &= g^*(r_g(t)) \\ &\leq g^*(e^m r_f(t)) \\ &\leq f^*(e^m r_f(t)) \end{aligned}$$

so that

$$\begin{aligned} \|g\ 1_{A_m}\|_X &\leq \|D_{e^{-m}} f\|_X \\ &\leq Ce^{-m\alpha}\|f\|_X \end{aligned}$$

and

$$\|g\ 1_{A_m}|\eta_1 - \eta_2|\|_X \leq C(m+1)e^{-m\alpha}\|f\|_X.$$

If $m < 0$, and $t \in A_m$

$$\begin{aligned} |g(t)| &\leq |f(t)| \\ &= f^*(r_f(t)) \\ &\leq f^*(e^{-m-1} r_g(t)) \end{aligned}$$

so that

$$\begin{aligned} \|g\ 1_{A_m}\| &\leq \|D_{e^{m+1}} f\|_X \\ &\leq Ce^{\alpha(m+1)}\|f\|_X \end{aligned}$$

and

$$\|g\ 1_{A_m}|\eta_1 - \eta_2|\|_X \leq (-m)e^{\alpha(m+1)}\|f\|_X.$$

Combining we obtain

$$\begin{aligned}\|g|\eta_1 - \eta_2|\|_X &\le Ce^{\alpha} \sum_{m\in\mathbf{Z}} (|m|+1)e^{-\alpha|m|}\|f\|_X \\ &\le C\|f\|_X\end{aligned}$$

(where $C = C(X)$). This leads to

$$\|u\Omega_\psi(f) - \Omega_\psi(uf)\|_X \le CL\|f\|_X.$$

Now suppose $\sigma : E \to E$ is a measure-preserving Borel automorphism. Define

$$\Omega'_\psi(f) = S_\sigma^{-1}\Omega_\psi(S_\sigma f).$$

Then Ω'_ψ is also a strong centralizer, and if f takes no value on a set of positive measure we have

$$\Omega'_\psi(f) = \Omega_\psi(f).$$

For general f pick g with $|g| \ge |f|$ so that $\|g\|_X \le 2\|f\|_X$ and g assumes no value on a set of positive measure. Then

$$\|\Omega'_\psi(f) - fg^{-1}\Omega'_\psi(g)\|_X \le 2CL\|f\|_X$$

$$\|\Omega_\psi(f) - fg^{-1}\Omega_\psi(g)\|_X \le 2CL\|f\|_X$$

and so

$$\|\Omega'_\psi(f) - \Omega_\psi(f)\|_X \le 4CL\|f\|_X$$

or

$$\|S_\sigma\Omega_\psi(f) - \Omega_\psi(S_\sigma f)\|_X \le 4CL\|f\|_X$$

A special class is of great importance to us. If $\phi : \mathbf{R} \to \mathbf{R}$ is a Lipschitz map we shall denote by Γ_ϕ the homogeneous centralizer

$$\Gamma_\phi(f) = f\phi(\log r_f)$$

COROLLARY 3.2. *If X is a separable r.i. function space over E with finite Boyd indices then there is a constant $C = C(X)$ so that if $\phi : \mathbf{R} \to \mathbf{R}$ is Lipschitz then Γ_ϕ is a symmetric homogeneous centralizer on X with*

$$\Delta^*(\Gamma_\phi) \leq CL(\phi).$$

In order to recognize more general centralizers we must identify null centralizers.

THEOREM 3.3. *Let X be a separable r.i. function space over E with finite Boyd indices $1 \leq p_X \leq q_X < \infty$. Suppose $0 < \alpha < p_X \leq q_X < \beta < \infty$ and let $K(\alpha,\beta) = \{(x,y) : x > 0, \alpha x + y < 0 < \beta x + y\}$.*

Suppose $\psi : \mathbf{R}^2 \to \mathbf{R}$ is continuous and satisfies

(3.9) For some $C, k > 0$,

$$\psi(x,y) \leq C(1 + |x| + |y|)^k$$

(3.10) ψ is bounded on $\pm K(\alpha,\beta)$

Then $\Omega_\psi(f) = f\psi(\log|f|, \log r_f)$ is a null centralizer on X. If E has finite measure (3.10) may be replaced by

(3.11) ψ is bounded on $K(\alpha,\beta)$.

PROOF: We shall prove only the first case. Fix p_0, q_0 so that $\alpha < p_0 < p_X$, $q_X < q_0 < \beta$.

Now for any $f \in X$ we have

$$f^*(t) \leq Ct^{-1/p_0}\|f\|_X \qquad t \leq 1 \tag{3.12}$$

$$\leq Ct^{-1/q_0}\|f\|_X \qquad t \geq 1$$

Conversely if

$$g(t) = \begin{cases} t^{-1/q_0}, & \text{for } t \leq 1 \\ t^{-1/p_0}, & \text{for } t > 1 \end{cases} \tag{3.13}$$

then $g \in X$. We omit the elementary proofs of these statements.

Now suppose $M \geq 1$ and $f \in X$ with $\|f\|_X \leq M$. Let $\xi = \log|f|$, $\eta = \log r_f$. Then by (3.12)

$$p_0\xi + \eta \leq p_o \log CM \qquad \eta < 0 \tag{3.14}$$

$$q_0\xi + \eta \leq q_0 \log CM \qquad \eta \geq 0 \tag{3.15}$$

We split E into three sets. Let $A_1 = \{t : \xi > 0,\ \beta\xi + \eta > 0\}$, $A_2 = \{\xi \leq 0,\ \alpha\xi + \eta > 0$ and $A_3 = E \setminus (A_1 \cup A_2)$.

On A_1, (ξ, η) belongs to the union of $K(\alpha, \beta)$ and a fixed bounded region in $\mathbf{R}^2$ where $\alpha\xi + \eta \geq 0$ but (3.14) and (3.15) hold. Thus there is a constant B_1 independent of f so that

$$|\psi(\xi, \eta)| \leq B_1 \qquad t \in A_1.$$

Similarly on A_2, (ξ, η) belongs to the union of $-K(\alpha, \beta)$ and a fixed bounded region, so that

$$|\psi(\xi, \eta)| \leq B_2 \qquad t \in A_2.$$

Pick $\delta > 0$ small enough so that

$$(1-\delta)\alpha^{-1} - \delta > p_0^{-1}$$

$$(1+\delta)\beta^{-1} + \delta < q_0^{-1}$$

$$\delta < q_0^{-1}$$

Then there is a constant C_0 so that

$$\log|\psi| \leq C_0 + \delta(|\xi| + |\eta|)$$

(by (3.9)).

For $t \in A_3$, if $\eta \geq 0$ we must have $\xi \leq 0$ and $\alpha\xi + \eta \leq 0$. Thus

$$\begin{aligned}\log|\Omega_\psi(f)| &\leq \xi + \log|\psi| \\ &\leq C_0 + \xi + \delta(-\xi + \eta) \\ &\leq C_0 - ((1-\delta)\alpha^{-1} - \delta)\eta \\ &\leq C_0 - p_0^{-1}\eta\end{aligned}$$

If $\eta < 0$ and $\xi \leq 0$ then

$$\begin{aligned}\log|\Omega_\psi(f)| &\leq C_0 + (1-\delta)\xi - \delta\eta \\ &\leq C_0 - \delta\eta \\ &\leq C_0 - q_0^{-1}\eta\end{aligned}$$

If $\eta < 0$ and $\xi > 0$ then $\beta\xi + \eta \leq 0$

$$\begin{aligned}\log|\Omega_\psi(f)| &\leq C_0 + (1+\delta)\xi - \delta\eta \\ &\leq C_0 - (\frac{1}{\beta}(1+\delta) + \delta)\eta \\ &\leq C_0 - q_0^{-1}\eta\end{aligned}$$

Thus

$$|\Omega_\psi(f)\cdot 1_{A_3}| \le e^{C_0} g(r_f(t))$$

where g is given by (3.13). Hence

$$\|\Omega_\psi(f)\cdot 1_{A_3}\| \le e^{C_0}\|g\|_X \le B_3.$$

Combining

$$\|\Omega_\psi(f)\|_X \le MB_1 + MB_2 + B_3$$

and Ω_ψ is the null centralizer.

Theorem 3.3 now allows to perturb Theorem 3.1.

THEOREM 3.4. *Under the same notation as Theorem 3.3 suppose $\psi : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function satisfying (3.9) and For some $M > 0$, $L > 0$ we have*

$$|\psi(x_1,y_1) - \psi(x_2,y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|) \tag{3.14}$$

for (x_1,y_1), $(x_2,y_2) \in \pm K(\alpha,\beta)$ and $|x_1| + |y_1|$, $|x_2| + |y_2| \le M$.

Then Ω_ψ is a symmetric centralizer on X.

If E has finite measure then (3.14) may be replaced by For some $M > 0$, $L > 0$ we have

$$|\psi(x_1,y_1) - \psi(x_2,y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|) \tag{3.15}$$

for (x_1,y_1), $(x_2,y_2) \in K(\alpha,\beta)$, $|x_1| + |y_1|$, $|x_2| + |y_2| \ge M$.

PROOF: Simply find a uniformly Lipschitz function ψ_1 agreeing with ψ on the set $\pm K(\alpha,\beta) \cap \{(x,y) : |x| + |y| \ge M\}$. Then $\Omega_\psi - \Omega_{\psi_1}$ is a null centralizer by Theorem 3.3.

EXAMPLES: (1) Suppose $a \geq 0,\ b \geq 0,\ a + b \leq 1$. Set $\psi(x,y) = |x|^a\ |y|^b$. Then

$$\Omega_\psi(f) = f|\log|f||^a\ |\log r_f|^b$$

is a centralizer on any r.i. function space with finite Boyd indices.

Here

$$|\frac{\partial\psi}{\partial x}| = a|y|^b\ |x|^{a-1} = a(\frac{|y|}{|x|}^b\ |x|^{a+b-1}$$
$$|\frac{\partial\psi}{\partial y}| = b|y|^{b-1}\ |x|^a = b(\frac{|x|}{|y|})^a\ |y|^{a+b-1}$$

on both bounded on any set $\pm K(\alpha,\beta) \cap \{(x,y) : |x| + |y| \geq M\}$.

(2) More generally for possibly negative values of a, b with $a + b \leq 1$ we can take

$$\psi(x,y) = (x^2+1)^{a/2}(y^2+1)^{b/2}$$

$$\Omega_\psi(f) = f((\log|f|)^2+1)^{a/2}(|\log r_f|^2+1)^{b/2}$$

(3) For a more complicated example take

$$\psi(x,y) = x\log\frac{y^2+1}{x^2+1}$$

(4) We conclude with the remark that there are centralizers on L_p, $1 \leq p < \infty$ which are not even equivalent to the type of centralizer given by Theorem 3.4. As the proof is somewhat involved we shall not pursue this remark here.

4. Lattice twisted squares and commutators

We first establish that the equivalence class of every centralizer contains a homogeneous centralizer.

PROPOSITION 4.1. *Let $\Omega : X \to L_0(E)$ be a centralizer. Then there is a homogeneous centralizer $\Omega' : X \to L_0$ which is equivalent to Ω and satisfies*

$$\Omega'(uf) = u\Omega(f)$$

if $|u| = 1$ a.e., $f \in X$.

PROOF: We define

$$\Omega'(f) = \|f\|_X \operatorname{sgn} f \Omega(|f| \cdot \|f\|_X^{-1})$$

for $f \neq 0$ and $\Omega'(0) = 0$. Here

$$\operatorname{sgn}\ f(t) = \begin{cases} f(t)|f(t)|^{-1}, & \text{for } f(t) \neq 0 \\ 0, & \text{for } f(t) = 0 \end{cases}$$

Suppose Ω satisfies (3.1). Then if $\|f\|_X \leq 1$

$$\|\Omega'(f) - \Omega(f)\|_X \leq \delta(1).$$

If $\|f\|_X \geq 1$ then

$$\|\Omega(|f|\|f\|_X^{-1}) - \|f\|_X^{-1}\overline{\operatorname{sgn}\ f}\Omega(f)\|_X \leq \delta(\|f\|_X)$$

and so

$$\|\Omega'(f) - 1_{\operatorname{supp}\ f}\Omega(f)\|_X \leq \|f\|_X \delta(\|f\|_X).$$

However

$$\|\Omega(f) - 1_{\mathrm{supp}\, f}\Omega(f)\|_X \leq \delta(\|f\|_X)$$

so that

$$\|\Omega'(f) - \Omega(f)\|_X \leq (\|f\|_X + 1)\delta(\|f\|_X).$$

This implies that Ω' is a centralizer equivalent to Ω, and Ω' is clearly homogeneous.

LEMMA 4.2. *Suppose $\Omega : X \to L_0$ is a strong centralizer satisfying (3.2). Then for $f_1, f_2 \in X$*

$$\|\Omega(f_1 + f_2) - \Omega(f_1) - \Omega(f_2)\|_X \leq 3\Delta(\|f_1\|_X + \|f_2\|_X) \tag{4.1}$$

where $\Delta = \Delta(\Omega)$.

PROOF: Let $g = |f_1| + |f_2|$ and write $f_i = u_i g$ where $|u_i| \leq 1$ a.e.$(i = 1, 2)$ and $|u_1 + u_2| \leq 1$ a.e. Then by (3.2)

$$\begin{aligned}\|\Omega(f_1 + f_2) - \Omega(f_1) - \Omega(f_2)\|_X &\leq 3\Delta\|g\|_X \\ &\leq 3\Delta(\|f_1\|_X + \|f_2\|_X).\end{aligned}$$

Now suppose Ω_0 is any centralizer on X. We may find a centralizer Ω equivalent to Ω_0 which is homogeneous and hence satisfies (4.1). Let $X \oplus_{\Omega_0} X$ be the space of all $(f, g) \in L_0 \oplus L_0$ so that $g \in X$ and $f - \Omega_0(g) \in X$. Then $X \oplus_{\Omega_0} X = X \oplus_\Omega X$ can be quasinormed by

$$\|(f, g)\|_{X,\Omega} = \|f - \Omega(g)\|_X + \|g\|_X \tag{4.2}$$

The fact that (4.2) defines a quasi-norm follows easily from (4.1). Now $X \oplus_\Omega X$ is a twisted sum of X with itself (cf.[14]). Precisely if we define $j : X \to X \oplus_\Omega X$ by

$$jf = (f, 0)$$

and $q : X \oplus_\Omega X \to X$ by

$$q(f, g) = g$$

then j is an isomorphic embedding and q is an open mapping. Thus the sequence

$$0 \longrightarrow X \xrightarrow{j} X \oplus_\Omega X \xrightarrow{q} X \longrightarrow 0$$

is exact. This also proves that $X \oplus_\Omega X$ is complete, i.e. a quasi-Banach space. If X is super-reflexive ($=$ B-convex for lattices) then $X \oplus_\Omega X$ is isomorphic to a Banach space by the results of [14], but in general $X \oplus_\Omega X$ is non-locally convex.

If $u \in L_\infty$, let M_u be the multiplication operator on X given by $M_u f = uf$. Notice that for each $u \in L_\infty$ the operator

$$\tilde{M}_u(f, g) = (M_u f, M_u g)$$

is bounded on $X \oplus_\Omega X$. In fact, if $|u| \leq 1$ a.e.

$$\begin{aligned}
\|(uf, ug)\|_{X,\Omega} &= \|uf - \Omega(ug)\|_X + \|ug\|_X \\
&= \|u(f - \Omega(g)\|_X + \|u\Omega(g) - \Omega(ug)\|_X + \|ug\|_X \\
&\leq \|(f, g)\|_{X,\Omega} + \Delta(\Omega)\|g\|_X \\
&\leq (1 + \Delta(\Omega))\|(f, g)\|_{X,\Omega}.
\end{aligned}$$

The map $u \to \tilde{M}_u$ is then bounded algebra homomorphism of $L_\infty(E)$ into $\mathcal{L}(X \oplus_\Omega X)$ so that the diagram below commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & X \oplus_\Omega X & \longrightarrow & X & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle M_u} & & \big\downarrow{\scriptstyle \tilde{M}_u} & & \big\downarrow{\scriptstyle M_u} & & \\
0 & \longrightarrow & X & \longrightarrow & X \oplus_\Omega X & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Thus we can interpret $X \oplus_\Omega X$ as a module over L_∞ in such a way that

$$0 \to X \to X \oplus_\Omega X \to X \to 0$$

is a short exact sequence of L_∞-modules. Thus $X \oplus_\Omega X$ is a twisted sum of X with itself X as an L_∞-module.

We shall demonstrate that every such L_∞-module twisted sum is of the form $X \oplus_\Omega X$. In order to do this we require some preparatory results. The first of these will be used frequently; it implies that strong centralizers have certain continuity properties.

LEMMA 4.3. *Let X be a separable Kothe function space and let I be a dense order-ideal in X. Let $\Omega : I \to L_0$ be a map satisfying (3.2). Suppose $h \in I$, $h \geq 0$ and $\Omega(h) \in X$. Suppose $f_n \in I$ with $|f_n| \leq h$ and $f_n \to g$ in measure. Then*

(a) If $g = 0$ $\quad \lim_{n\to\infty} \|\Omega(f_n)\|_X = 0$

(b) If $g \neq 0$ $\quad \limsup_{n\to\infty} \|\Omega(g) - \Omega(f_n)\|_X \leq 6\Delta\|g\|_X$.

PROOF: (a) Let $g_m = \sup_{k\geq m} |f_k|$. For $n \geq m$

$$\|\Omega(f_n) - f_n g_m^{-1}\Omega(g_m)\|_X \leq \Delta\|g_m\|_X$$

Now $\Omega(g_m) - g_m h^{-1}\Omega(h) \in X$ and so $\Omega(g_m) \in X$. As $f_n g_m^{-1} \to 0$ in measure and X is separable

$$\lim_{n\to\infty} \|f_n g_m^{-1}\Omega(g_m)\|_X = 0$$

and so

$$\limsup_{n\to\infty} \|\Omega(f_n)\|_X \leq \Delta\|g_m\|_X$$

Now $\|g_m\|_X \to 0$ and the conclusion follows.

(b) Now $|f_n - g| \leq 2|h|$ and so

$$\lim_{n\to\infty} \|\Omega(f_n - g)\|_X = 0$$

Now

$$\|\Omega(f_n) - \Omega(g)\|_X \leq \|\Omega(f_n - g)\|_X + 3\Delta(\|f_n\|_X + \|g\|_X)$$

by Lemma 4.2. Thus as $\|f_n\|_X \to \|g\|_X$ we obtain the conclusion.

PROPOSITION 4.4. *Let X be a separable maximal Kothe function space on E and let I be a dense order-ideal in X. Let $\Omega_0 : I \to L_0$ be a map satisfying equation (3.2). Then there is a strong centralizer $\Omega : X \to L_0$ so that $\Omega|I = \Omega_0$.*

If Ω_0 is homogeneous, Ω can be chosen homogeneous.

PROOF: For each $f \in X \setminus I$ we choose a partition A_n of E into Borel sets so that $f1_{A_n} \in I$ and

$$\|f1_{A_n}\|_X \leq 2^{-n}\|f\|_X.$$

We then set

$$\Omega(f) = \sum_{n=1}^{\infty} 1_{A_n}\Omega_0(f1_{A_n}) \qquad \text{(summation in } L_0)$$

and define $\Omega(f) = \Omega_0(f)$ if $f \in I$.

Note in passing that if Ω_0 is homogeneous then the selection of (A_n) can be made so that Ω is homogeneous. It must now be verified that Ω is a strong centralizer. Let us suppose $f \in X \setminus I$ and $g = uf$ where $|u| \leq 1$ a.e. First consider the case $g \in I$.

By simple induction we can prove from Lemma 4.2 which applies to Ω_0 that

$$\begin{aligned}\|\Omega_0(g1_{A_1\cup\ldots\cup A_n}) - \sum_{k=1}^{n}\Omega_0(g1_{A_k})\|_X \\ &\leq 3\Delta\sum_{k=1}^{n} k\|g1_{A_k}\|_X \\ &\leq 3\Delta\sum_{k=1}^{n} k2^{-k}\|f\|_X\end{aligned}$$

Further

$$\begin{aligned}\|\Omega_0(g1_{A_k}) - u1_{A_k}\Omega_0(f1_{A_k})\|_X &\le \Delta\|f1_{A_k}\| \\ &\le 2^{-k}\Delta\|f\|_X.\end{aligned}$$

Hence

$$\begin{aligned}\|\Omega_0(g1_{A_1\cup\ldots\cup A_n}) &- u1_{A_1\cup\ldots\cup A_n}\Omega(f)\|_X \\ &\le \Delta(\sum_{k=1}^{n}(3k+1)2^{-k})\|f\|_X \\ &\le 7\Delta\|f\|_X.\end{aligned}$$

Now

$$\begin{aligned}\limsup_{n\to\infty}\|\Omega_0(g1_{A_1\cup\ldots\cup A_n}) - \Omega_0(g)\|_X &\le 6\Delta\|g\|_X \\ &\le 6\Delta\|f\|_X\end{aligned}$$

and so

$$\|\Omega_0(g) - u\Omega(f)\|_X \le 13\Delta\|f\|_X$$

or

$$\|\Omega(uf) - u\Omega(f)\|_X \le 13\Delta\|f\|_X.$$

Next suppose $g \in X \setminus I$. We may select v_n with $0 \le v_n \uparrow 1$ a.e. so that $v_n g \in I$. By the above argument,

$$\|\Omega_0(v_n g) - v_n\Omega(g)\|_X \le 13\Delta\|g\|_X$$

and

$$\|\Omega_0(v_n g) - v_n u\Omega(f)\|_X \le 13\Delta\|f\|_X.$$

Hence

$$\|v_n(\Omega(g) - u\Omega(f))\|_X \leq 26\Delta\|f\|_X$$

and letting $n \to \infty$ we obtain, since X is maximal,

$$\|\Omega(uf) - u\Omega(f)\|_X \leq 26\Delta\|f\|_X.$$

Now we return to the consideration of a general L_∞-module twisted sum. We suppose that Y is a quasi-Banach space so that we have a short exact sequence

$$0 \longrightarrow X \xrightarrow{j} Y \xrightarrow{q} X \longrightarrow 0.$$

We further suppose the existence of a bounded algebra homomorphism $u \to \tilde{M}_u$ $(L_\infty(E) \to \mathcal{L}(Y))$ so that the diagram below commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{q} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M_u} & & \downarrow{\scriptstyle \tilde{M}_u} & & \downarrow{\scriptstyle M_u} & & \\ 0 & \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{q} & X & \longrightarrow & 0 \end{array}$$

Under these circumstances we say that Y is a *lattice twisted square* of X. For convenience denote the L_∞-action on Y by $\tilde{M}_u y = uy$.

THEOREM 4.5. *Let X be a separable maximal Kothe function space on E and let Y be a lattice twisted square of X. Then there is a homogeneous centralizer $\Omega : X \to L_0$ and an isomorphism $J : Y \to X \oplus_\Omega X$ so that if $u \in L_\infty$, $y \in Y$*

$$J(uy) = uJ(y)$$

(i.e. J is an L_∞-module isomorphism).

PROOF: Let $w \in X$ be such that $w > 0$ a.e. and let I be the principal order-ideal generated by w. Let $\rho : X \to Y$ be a map such that $q\rho(f) = f$ for $f \in X$ and

$$\rho(\alpha f) = \alpha\rho(f) \qquad \alpha \in \mathbf{C}, f \in X$$

$$\|\rho(f)\| \leq C\|f\| \qquad f \in X$$

where C is some fixed constant (this is possible since q is open). For $f \in I$, $fw^{-1} \in L_\infty$ and we define $\Omega_0 : I \to X$ by

$$\Omega_0(f) = j^{-1}(\rho(f) - fw^{-1}\rho(w))$$

To see this definition makes sense, note that $q\rho(f) - q(fw^{-1}\rho(w)) = f - fw^{-1}q\rho(w) = 0$, since q commutes with the L_∞-actions on Y and X. Ω_0 is homogeneous. If $|u| \leq 1$ a.e. then

$$\Omega_0(uf) = j^{-1}(\rho(uf) - ufw^{-1}\rho(w))$$

$$u\Omega_0(f) = j^{-1}(u\rho(f) - ufw^{-1}\rho(w))$$

so that

$$\Omega_0(uf) - u\Omega_0(f) = j^{-1}(\rho(uf) - u\rho(f))$$

Now

$$\|\rho(uf) - u\rho(f)\|_Y \leq C'\|f\|_X$$

where C' is independent of f. Thus Ω_0 verifies (3.2) and by Proposition 4.4 extends to a homogeneous centralizer $\Omega : X \to L_0$.

To define J, suppose $y \in Y$ satisfies $q(y) \in I$. This is a dense subspace of Y. For such y let $Jy = (f, g)$ where

$$f = j^{-1}(y - (q(y)w^{-1})\rho(w))$$

$$g = q(y)$$

Then

$$\begin{aligned}\|Jy\|_{X,\Omega} &= \|j^{-1}(y-\rho q(y))\|_X + \|q(y)\|_X \\ &\le C\|y\|\end{aligned}$$

for some C is independent of Y.

We also note that $\|Jy\| \ge \|q(y)\|$ and

$$\begin{aligned}\|Jy\| &\ge \|j^{-1}\|^{-1}\|y-\rho q(y)\| \\ &\ge \|j^{-1}\|^{-1}(\|y\| - C\|q(y)\|)\end{aligned}$$

so that J is an isomorphic embedding. J has dense range in $X \oplus_\Omega X$ since it includes all $(f,0)$ for $f \in X$ and qJ has dense range in X. Thus J extends to an isomorphism between Y and $X \oplus_\Omega X$.

If $y \in Y$ and $q(y) \in I$, then for $u \in L_\infty$

$$J(uy) = (f,g)$$

where

$$\begin{aligned}f &= j^{-1}(uy - (q(uy)w^{-1})\rho(w)) \\ g &= q(uy)\end{aligned}$$

Thus $J(uy) = uJ(y)$ and this extends to all $y \in Y$.

Now let X be any separable Kothe function space and suppose $\Omega : X \to L_0$ is a centralizer. We define $D(\Omega) = \{f : \Omega(f) \in X\}$. $D(\Omega)$ is an order-ideal in X. It is also dense since if $f \in X$ we can find v with $0 < v \le 1$ a.e. so that $v\Omega(f) \in X$. Then $vf \in D(\Omega)$.

Suppose $T : X \to X$ is a bounded linear operator so that T maps $D(\Omega)$ into $D(\Omega)$. Then for $f \in D(\Omega)$ we define the commutator $[T, \Omega]$ by

$$[T,\Omega](f) = T\Omega(f) - \Omega T(f). \tag{4.3}$$

We shall say that T and Ω *commute* if $T(D(\Omega)) \subset D(\Omega)$ and for some function $\delta : \mathbf{R}_+ \to \mathbf{R}_+$ we have

$$\|[T,\Omega](f)\|_X \leq \delta(\|f\|_X).$$

If Ω is homogeneous this becomes

$$\|[T,\Omega](f)\|_X \leq k\|f\|_X \tag{4.5}$$

for some constant k. Notice that if T and Ω commute and Ω' is equivalent to Ω then T and Ω' commute. This via Proposition 4.1 reduces commutation problems to the homogeneous case.

THEOREM 4.6. *Let X be a separable Kothe function space and let $t : X \to X$ be a bounded linear operator. Let $\Omega : X \to L_0$ be a homogeneous centralizer. Then the following are equivalent:*

(1) T and Ω commute

(2) There is a dense order-ideal I so that $T(I) \subset D(\Omega)$ and

$$\sup_{f\in I}\sup_{\|f\|\leq 1} \|T\Omega(f) - \Omega T(f)\|_X = M < \infty$$

(3) There is a bounded operator $\tilde{T} : X \oplus_\Omega X \to X \oplus_\Omega X$ so that if $f \in D(\Omega), g \in X$

$$\tilde{T}(f,g) = (Tf, Tg).$$

PROOF: (1)$\Rightarrow$(2) is obvious. For (2)$\Rightarrow$(3) define for $f \in I, g \in X$

$$T_0(f,g) = (Tf, Tg).$$

We show T_0 extends to a bounded operator on $X \oplus_\Omega X$. In fact

$$\begin{aligned}\|T_0(f,g)\|_{X,\Omega} &= \|Tf - \Omega(Tg)\|_X + \|Tg\|_X \\ &= \|T(f-\Omega(g))\|_X + \|[T,\Omega]g\|_X + \|Tg\|_X \\ &\leq (\|T\| + M)\|(f,g)\|_{X,\Omega}.\end{aligned}$$

Denote by $\tilde{T}$ the extension of T_0 to $X \oplus_\Omega X$. If $f \in D(\Omega)$ and $g \in X$ we choose $f_n \in I$ with $f_n \to f$ a.e. and $|f_n| \leq |f|$. Then

$$\|(f - f_n, 0)\|_{X,\Omega} \to 0$$

and so

$$\tilde{T}(f,g) = \lim_{n\to\infty} T_0(f_n, g) = \lim_{n\to\infty}(Tf_n, Tg)$$

Now

$$\|T(f-f_n), 0\|_{X,\Omega} \to 0$$

so that $\tilde{T}(f,g) = (Tf, Tg)$.

(3)$\Rightarrow$(1) is proved by a calculation very similar to the one given above.

We close with some remarks. First note that every centralizer Ω commutes with every multiplication operator M_u. Second, notice that Ω is symmetric if and only if it commutes (uniformly) with every rearrangement operator S_σ. Equation (3.4) reads exactly

$$\|[S_\sigma, \Omega](f)\|_X \leq \eta(\|f\|_X)$$

while (3.1) is

$$\|[M_u,\Omega](f)\|_X \leq \delta(\|f\|_X)$$

if $\|u\|_\infty \leq 1$.

5. A preliminary commutator theorem

Our next result is an important key to the methods of the paper. We first recall a theorem of Lozanovskii [19] that if X is a Kothe function space on E and $h \in L_1$ then for every $\epsilon > 0$ we can find factorization $h = fg$ with $f \in X, g \in X^*$ and

$$\|f\|_X \ \|g\|_{X^*} \leq (1+\epsilon)\|h\|_1 \tag{5.1}$$

(An alternative treatment of the discrete case is in [6]).

THEOREM 5.1. *Let X be a separable Kothe function space and let Ω be a homogeneous centralizer on X. Then there is a homogeneous centralizer $\Omega^{[1]}$ on L_1 so that*

$$\Delta(\Omega^{[1]}) \leq 36\Delta(\Omega) \tag{5.2}$$

$$\|\Omega^{[1]}(fg) - \Omega(f)g\|_1 \leq 18\Delta(\Omega)\|f\|_X\|g\|_{X^*} \tag{5.3}$$

for $f \in X, g \in X^*$.

Furthermore if Ω' is any homogeneous centralizer on L_1 so that for some $C < \infty$

$$\|\Omega'(fg) - \Omega(f)g\|_1 \leq C\|f\|_X\|g\|_{X^*} \tag{5.4}$$

then Ω' and $\Omega^{[1]}$ are equivalent.

PROOF: Suppose $f_1, f_2 \in X$ and $g_1, g_2 \in X^*$ with $f_1g_1 = f_2g_2 = h \in L_1$. Let $F = |f_1| + |f_2| \in X$. Then

$$\|\Omega(f_1) - f_1F^{-1}\Omega(F)\|_X \leq \Delta\|F\|_X$$

(where $\Delta = \Delta(\Omega)$). Hence

$$\|\Omega(f_1)g_1 - hF^{-1}\Omega(F)\|_1 \leq \Delta\|F\|_X\|g_1\|_{X^*}$$

and therefore

$$\|\Omega(f_1)g_1 - \Omega(f_2)g_2\|_1 \leq 2\Delta(\|f_1\|_X + \|f_2\|_X)(\|g_1\|_{X^*} + \|g_2\|_{X^*})$$

If we replace by f_2, g_2 by αf_2 and $\alpha^{-1}g_2$ where $\alpha > 0$ and minimize the right-hand side over α we obtain

$$(5.5) \qquad \|\Omega(f_1)g_1 - \Omega(f_2)g_2\|_1 \leq 2\Delta(\|f_1\|_X^{1/2}\|g_1\|_{X^*}^{1/2} + \|f_2\|_X^{1/2}\|g_2\|_{X^*}^{1/2})^2$$

Now by Lozanovskii's result [19], we select for each $h \in L_1$ a factorization $h = f_0 g_0$ where

$$(5.6) \qquad \|f_0\|_X\|g_0\|_{X^*} \leq 2\|h\|_1$$

in such a way that if $\alpha \in \mathbf{C}$, with $\alpha \neq 0$, αh is factored as $(\alpha f_0).g_0$. Then define

$$(5.7) \qquad \Omega^{[1]}(h) = \Omega(f_0)g_0$$

Now for any $f \in X, g \in X^*$, if $h = fg$

$$\|\Omega^{[1]}(fg) - \Omega(f)g\|_X \leq 2\Delta(\|f\|_X^{1/2}\|g\|_{X*}^{1/2} + \sqrt{2}\|h\|_1^{1/2})^2$$

by (5.5) so that we obtain (5.3).

If $|u| \leq 1$ a.e. and $h \in L_1$ then if $h = f_0 g_0$ in (5.6) and (5.7) we obtain

$$\begin{aligned}\|\Omega(f_0)ug_0 - \Omega^{[1]}(uh)\|_1 &\leq 18\Delta\|f_0\|_X\|ug_0\|_{X^*} \\ &\leq 36\Delta\|h\|_1\end{aligned}$$

so that

$$\|u\Omega^{[1]}(h) - \Omega^{[1]}(uh)\|_1 \leq 36\Delta\|h\|_1$$

and this gives (5.2).

Uniqueness is also clear. If $h = f_0 g_0$ is the factorization used in (5.6) and (5.7) then if Ω' satisfies (5.4)

$$\begin{aligned}\|\Omega'(h) - \Omega^{[1]}(h)\|_1 &\leq C\|f_0\|_X\|g_0\|_{X^*}\\ &\leq 2C\|h\|_1.\end{aligned}$$

Thus $\Omega^{[1]}$ and Ω' are equivalent.

COROLLARY 5.2. *If Ω is symmetric then $\Omega^{[1]}$ is symmetric and $\Delta^*(\Omega^{[1]}) \leq 38\Delta^*(\Omega)$.*

PROOF: Let $\sigma : E \to E$ be a measure preserving automorphism. If $h \in L_1$ is factored according to (5.6) and (5.7) then

$$S_\sigma h = (S_\sigma f_0)(S_\sigma g_0)$$

and so

$$\begin{aligned}\|\Omega^{[1]}(S_\sigma h) - \Omega(S_\sigma f_0)S_\sigma g_0\|_1 &\leq 18\Delta\|f_0\|_X\|g_0\|_{X^*}\\ &\leq 36\Delta\|h\|_1\end{aligned}$$

However

$$\|\Omega(S_\sigma f_0) - S_\sigma\Omega(f_0)\|_X \leq \Delta^*\|f_0\|_X$$

so that

$$\|\Omega^{[1]}(S_\sigma h) - S_\sigma(\Omega(f_0).g)\|_X \leq 36\Delta\|h\|_1 + \Delta^*\|f_0\|_X\|g_0\|_{X^*}$$

or

$$\|\Omega^{[1]}(S_\sigma h) - S_\sigma \Omega^{[1]}(h)\|_X \leq 38\Delta^*\|h\|_1$$

There is a partial converse to Theorem 5.1 which we will discuss later in Section 8. We shall need to identify $\Gamma_\phi^{[1]}$ where Γ_ϕ is the special homogeneous centralizer introduced in section 3, i.e.

$$\Gamma_\phi(f) = f\phi(\log r_f)$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is Lipschitz (see Corollary 3.2). To this end we prove first a lemma on the form of the factorization given by the Lozanovskii theorem for a symmetric norm.

LEMMA 5.3. *Let $\|\cdot\|$ be a symmetric norm on $\mathbf{R}^n$ and suppose $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ with $x_1 \geq \ldots \geq x_n \geq 0$. Then there exist $u = (u_1, \ldots, u_n) \in \mathbf{R}^n, v = (v_1, \ldots, v_n) \in \mathbf{R}^n$ with*

$$u_i v_i = x_i \qquad 1 \leq i \leq n \tag{5.8}$$

$$\|u\|\|v\|^* = \sum_{i=1}^n x_i = \|x\|_1 \tag{5.9}$$

and $u_1 \geq u_2 \geq \ldots \geq u_n \geq 0$ with $u_i = u_{i+1}$ whenever $x_i = x_{i+1}$, and $u_i = 0$ if and only if $x_i = 0$.

Here $\|\cdot\|^*$ denotes the dual norm on $\mathbf{R}^n$.

PROOF: By the Lozanovskii theorem and a compactness argument we can find a factorization $x = uv$ satisfying (5.8) and (5.9), and with $u \geq 0$, $v \geq 0$. Amongst all such factorizations let us suppose we choose the factorization $x = uv$ where $\|u\|_1 = 1$ and $\|u\|_2 = \sqrt{\sum |u_i|^2}$ is minimized. We shall show that for this factorization we have $u_1 \geq \ldots \geq u_n \geq 0$. Indeed, suppose not; then there exists k so that $u_k < u_{k+1}$.

We now define

$$\bar{u} = 1/2(u + u') \tag{5.10}$$

where $u' = (u_1, \ldots, u_{k-1}, u_{k+1}, u_k, \ldots u_n)$. Clearly $\|\bar{u}\| \leq \|u\|$, $\|\bar{u}\|_1 = 1$ and $\|\bar{u}\|_2 < \|u\|_2$.

Now define $\bar{v}$ by $\bar{v}_i = v_i$ for $i \notin (k, k+1)$ and

$$\bar{v}_k = x_k \bar{u}_k^{-1} \tag{5.11}$$

$$\bar{v}_{k+1} = x_{k+1} \bar{u}_{k+1}^{-1} \tag{5.12}$$

Consider the vectors $(\bar{v}_k, \bar{v}_{k+1})$ and (v_k, v_{k+1}) in $\mathbf{R}^2$. We claim there exist λ_1, λ_2 with $0 \leq \lambda_1, \lambda_2 \leq 1,\ \lambda_1 + \lambda_2 \leq 1$ and

$$(\bar{v}_k, \bar{v}_{k+1}) = \lambda_1 (v_k, v_{k+1}) + \lambda_2 (v_{k+1}, v_k). \tag{5.13}$$

This can be checked using Proposition 2.a.5 of [18] p.124 (due to Hardy, Littlewood and Polya). We check that $\bar{v}_k \geq \bar{v}_{k+1}$ and $v_k \geq v_{k+1}$ (since $u_k < u_{k+1}$). Then we require

$$\bar{v}_k \leq v_k$$

$$\bar{v}_k + \bar{v}_{k+1} \leq v_k + v_{k+1}.$$

The first inequality is clear since $\bar{u}_k \geq u_k$. For the second inequality we note

$$(u_{k+1} - u_k)(v_k - v_{k+1}) \geq 0 \tag{5.14}$$

and so

$$\begin{aligned}(u_{k+1} + u_k)(v_k + v_{k+1}) &\geq 2(u_k v_k + u_{k+1} v_{k+1}) \\ &= 2(x_k + x_{k+1}).\end{aligned}$$

Thus

$$\begin{aligned}(v_k + v_{k+1}) &\geq 2(\frac{x_k + x_{k+1}}{u_k + u_{k+1}}) \\ &= \bar{v}_k + \bar{v}_{k+1}.\end{aligned}$$

Hence $\|\bar{v}\|^* \leq \|v\|^*$. This establishes a contradiction since $\|\bar{u}\| \, \|\bar{v}\|^* \leq \|u\| \, \|v\|^* = \|x\|_1$. We conclude that $u_1 \geq u_2 \geq \ldots \geq u_n \geq 0$.

It is clear further that $u_k = 0$ whenever $x_k = 0$. Also if $x_k = x_{k+1}$ then we can similarly define $\bar{u}, \bar{v}$, by (5.10)-(5.12) and demonstrate that (5.13) holds since clearly (5.14) holds.

The deduction in this case is that $\|\bar{u}\|_2 = \|u\|_2$ and so $\bar{u} = u$ i.e. $u_k = u_{k+1}$.

THEOREM 5.4. *Let X be a separable r.i. function space with finite Boyd indices defined on E where $E = \mathbf{R}$ or $E \subset \mathbf{R}$ is an interval. Then there is a constant $C = C(X)$ so that $\phi : \mathbf{R} \to \mathbf{R}$ is a Lipschitz map and $f \in X,\ g \in X^*$*

$$\|\Gamma_\phi(fg) - \Gamma_\phi(f)g\|_1 \leq CL(\phi)\|f\|_X \|g\|_{X^*}.$$

PROOF: We compute $\Gamma^{[1]}$ via Theorem 5.1. Suppose $h \in L_1$ is a simple function of the form

$$h = \sum_{j=1}^{n} a_j \ 1_{B_j}$$

where $|a_1| \geq |a_2| \geq \ldots \geq |a_n| > 0$ and $\mu(B_1) = \mu(B_2) = \ldots = \mu(B_n) < \infty$. It now follows from Lemma 5.3 that we may find $u_1 \geq \ldots \geq u_n \geq 0$ and $v_l, \ldots, v_n \geq 0$ so that

$$\|\sum_{j=1}^{n} u_j \, 1_{B_j}\|_X \|\sum_{j=1}^{n} v_j \, 1_{B_j}\|_{X^*} = \|h\|_1$$

$u_j v_j = |a_j|$ for $1 \leq j \leq n$, $u_j = u_{j+1}$ whenever $|a_j| = |a_{j+1}|$ and $u_j = 0$ whenever $a_j = 0$.

By making a small perturbation we can then factorize $h = fg$ so that

$$\|f\|_X \, \|g\|_{X^*} \leq 2\|h\|_1$$

and

$$f = \sum_{j=1}^{n} u_j' \, 1_{B_j}$$

where $u_1' \geq u_2' \geq \ldots \geq u_n' \geq 0$, $u_j' = u_{j+1}'$ if and only if $a_j = a_{j+1}$, and $u_j' = 0$ whenever $a_j' = 0$.

Now

$$\Gamma_\phi(f)g = \Gamma_\phi(h)$$

and so

$$\begin{aligned}\|\Gamma_\phi(h) - \Gamma_\phi^{[1]}(h)\| &\leq CL(\phi)\|f\|_X\|g\|_{X*} \\ &\leq CL(\phi)\|h\|_1\end{aligned}$$

by Theorem 5.1.

For general $f \in L_1$ we may find a sequence h_n with $0 \leq |h_n| \leq |f|$, $h_n \to f$ a.e., with each h_n of the form described above.

Suppose $h_n = u_n f$; then

$$\|\Gamma_\phi(u_n f) - \Gamma_\phi^{[1]}(u_n f)\|_1 \leq CL(\phi)\|f\|_1$$

and so

$$\|(\Gamma_\phi(f) - \Gamma_\phi^{[1]}(f))u_n\|_1 \leq CL(\phi)\|f\|_1$$

By a limiting argument,

$$\|\Gamma_\phi(f) - \Gamma_\phi^{[1]}(f)\|_1 \leq CL(\phi)\|f\|_1$$

and now Theorem 5.1 implies the result.

Now let us relate our result to commutators. If $T : X \to X$ is a bounded linear operator we induce a bilinear form $B_T : X \times X^* \to L_1$ by

$$B_T(f,g) = Tf.g - f.T^*g$$

and $\|B_T\| \leq 2\|T\|$.

THEOREM 5.5. *Let X be a separable Kothe function space and suppose Ω is a homogeneous centralizer on X. Let $T : X \to X$ be a bounded linear operator. Then T and Ω commute if and only if there is a dense order-ideal $I \subset X$, and a dense order-ideal $J \subset X_0^*$ so that*

$$T(I) \subset D(\Omega), \qquad I \subset D(\Omega) \tag{5.15}$$

and for some constant K, and for all $f \in I$, $g \in J$,

$$\left|\int_E \Omega^{[1]}(B_T(f,g))d\mu\right| \leq K\|f\|_X \|g\|_{X^*} \tag{5.16}$$

PROOF: In fact, if we assume (5.15) then B_T maps $I \times X^*$ into $D(\Omega^{[1]})$ by an easy application of Theorem 5.1, so that the integral is defined.

Suppose (5.15),(5.16) hold. Then

$$\|\Omega^{[1]}(B_T(f,g)) - \Omega^{[1]}(Tf.g) - \Omega^{[1]}(f.T^*g)\|_1$$
$$\leq 3\Delta(\Omega^{[1]})\|T\| \; \|f\|_X \; \|g\|_{X^*}$$
$$\leq 108\Delta\|T\| \; \|f\|_X \; \|g\|_{X^*}$$

where $\Delta = \Delta(\Omega)$.

Further

$$\|\Omega^{[1]}(Tf.g) - \Omega(Tf)g\|_1 \leq 18\Delta\|T\| \; \|f\|_X \; \|g\|_{X^*}$$

$$\|\Omega^{[1]}(f.T^*g) - \Omega(f).T^*g\|_1 \leq 18\Delta\|T\| \; \|f\|_X \; \|g\|_{X^*}$$

so that

$$\left|\int_E (\Omega(Tf).g - \Omega(f).T^*g)d\mu\right| \leq K'\|f\|_X \; \|g\|_{X^*}$$

for some K', all $f \in I$ and all $g \in J$. Hence since $\Omega(Tf) - T(\Omega(f)) \in X$ for $f \in I$ we conclude

$$\|\Omega(Tf) - T(\Omega(f))\|_X \leq K'\|f\|_X$$

and Theorem 4.6 yields that T and Ω commute.

For the converse we take $I = D(\Omega), \; J = X_0^*$. Then

$$\|[T,\Omega]f\|_X \leq K_1\|f\|_X \qquad f \in I$$

implies

$$\left|\int_E (\Omega(Tf).g - \Omega(f).T^*g)d\mu\right| \leq K_1\|f\|_X \; \|g\|_{X^*}$$

and the argument given above can be reversed.

We investigate now the special case of $\Omega = \Gamma_\phi$ where ϕ is a Lipschitz function.

THEOREM 5.6. *Suppose E is an interval or $E = \mathbf{R}$. Suppose $1 < p_0 < p_1 < \infty$ and that X is an r.i. function space whose Boyd indices satisfy $p_0 < p_X < q_X < p_1$. Then there is a constant $C = C(X)$, so that whenever $T : X \to X$ is an operator of strong type (p_0, p_0) and strong type (p_1, p_1) and $\phi : \mathbf{R} \to \mathbf{R}$ is a Lipschitz map then for any $f,\ g \in L_\infty$ whose supports have finite measure*

$$|\int \Gamma_\phi(B_T(f,g))d\mu| \leq CL(\phi)\gamma(T)\|f\|_X\ \|g\|_{X^*}$$

where $\gamma(T) = \max(\|T\|_{L_{p_0}}, \|T\|_{L_{p_1}})$.

Remark. Obviously we can relax the assumption on T to weak type by the Marcinkiewicz interpolation theorem.

PROOF: Let us first observe that it suffices to prove the theorem for ϕ increasing since every Lipschitz ϕ can be expressed as the difference of two increasing functions. Let G be the space of all L_∞-functions of support of finite measure. Since $T : L_p \to L_p$ where $p > 1$ it is easy to verify that $B_T(G \times G)$ is contained in $D(\Gamma_\phi)$ for every ϕ. This observation allows us to restrict attention to the case when ϕ is bounded. The general case then follows by approximation. Finally we shall suppose $L(\phi) \leq 1$ and argue by homogeneity.

We introduce some notation. Let $\ell = \mu E$ $(0 < \ell \leq \infty)$. Let $w = \phi(\log t)$. Pick r_0, r_1 so that $p_0 < r_0 < p_X$, $q_X < r_1 < p_1$. Then there is a constant K so that the dilation operators D_s on X and X^* verify

$$\begin{aligned} \|D_s\|_X &\leq K\ s^{1/r_1} && s < 1 \\ &\leq K\ s^{1/r_0} && s \geq 1 \\ \|D_s\|_{X^*} &\leq K\ s^{1-1/r_0} && s < 1 \\ &\leq K\ s^{1-1/r_1} && s \geq 1 \end{aligned}$$

Fix $\delta > 0$ so that $\delta < \min(\frac{1}{p_1}, \frac{1}{r_1} - \frac{1}{p_1}, 1 - \frac{1}{p_0}, \frac{1}{p_0} - \frac{1}{r_0})$.

Now for $0 \leq \theta \leq \delta$ we introduce an equivalent norm on X

$$\|f\|_\theta = \sup_{\|g\|_{X^*} \leq 1} \sup_{h \in W} \left| \int_E fgh d\mu \right|$$

where W is the set of $h \in L_0$ with

$$h^* \leq e^{-\theta w}$$

Let us note immediately that

$$\begin{aligned} \|f\|_\theta &= \sup_{\|g\|_{X^*} \leq 1} \int_0^\ell f^* g^* e^{-\theta w} \, dt \\ &= \|f^* e^{-\theta w}\|_X \end{aligned}$$

(Here we utilize Lemma 2.1 and identify X with $X(0, \ell)$).

We shall need some basic facts concerning $\| \ \|_\theta$. First if $f \in X$ we shall estimate $\|f^* \, e^{\theta w}\|_\theta$. To do this let us define

$$f_1(t) = \alpha \int_{1/2}^\infty f^*(st) s^{1/p_1 - 1} ds \qquad 0 \leq t < \ell$$

where

$$\alpha \int_{1/2}^1 s^{1/p_1 - 1} ds = 1.$$

We then have $0 \leq f^* \leq f_1$, and $f_1 \in X(0, \ell)$ can be written as a Bochner integral

$$f_1 = \alpha \int_{1/2}^\infty (D_{s^{-1}} f^*) s^{1/p_1 - 1} ds$$

giving the estimate

$$\begin{aligned} \|f_1\|_X &\leq \alpha \left(\int_{1/2}^\infty \|D_{s^{-1}}\|_X s^{1/p_1 - 1} ds \right) \|f\|_X \\ &\leq \left(\|D_2\|_X + \alpha \int_1^\infty K s^{1/p_1 - 1/r_1 - 1} ds \right) \|f\|_X \\ &\leq C \|f\|_X \end{aligned}$$

where $C = C(X)$. Now if $\tau > 0$, $e^\tau t < \ell$,

$$\begin{aligned} f_1(e^\tau t) &= \alpha \int_{1/2}^\infty f^*(e^\tau st)s^{1/p_1-1}ds \\ &= \alpha e^{-\tau/p_1} \int_{1/2e^\tau}^\infty f^*(st)s^{1/p_1-1}ds \\ &\le e^{-\tau/p_1} f_1(t). \end{aligned}$$

Thus

$$\begin{aligned} e^{\theta w(e^\tau t)} f_1(e^\tau t) &\le e^{\tau(\theta - 1/p_1)} f_1(t) \\ &\le e^{\tau(\delta - 1/p_1)} f_1(t). \end{aligned}$$

We conclude that $e^{\theta w} f_1$ is decreasing and so

$$\begin{aligned} \|e^{\theta w} f^*\|_\theta &\le \|e^{\theta w} f_1\|_\theta \\ &\le \|f_1\|_X \end{aligned}$$

i.e.

$$\|e^{\theta w} f^*\|_\theta \le C\|f\|_X \tag{5.17}$$

The other calculation we need is of the norms of the dilation operators on $(X, \| \|_\theta)$. Here if $f \in X$, $0 < s < \infty$

$$\begin{aligned} \|D_s f\|_\theta &= \|D_s f^*\|_\theta \\ &= \|e^{-\theta w} D_s f^*\|_X \\ &\le \|D_s\|_X \|e^{-\theta(D_{s^{-1}} w)} f^*\|_X \end{aligned}$$

Now if $s > 1$ we have

$$e^{-\theta(D_{s^{-1}} w)} \le e^{-\theta w}$$

and so $\|D_s\|_\theta \leq \|D_s\|_X$. If $s \leq 1$

$$e^{-\theta(D_{s^{-1}} w)} \leq s^{-\theta} e^{-\theta w}$$

Combining we have

$$\begin{aligned} \|D_s\|_\theta &\leq K\ s^{1/r_0} & s > 1 \\ &\leq K\ s^{1/r_1 - \delta} & s \leq 1 \end{aligned}$$

Since $r_0 > p_0$ and $(\frac{1}{r_1} - \delta)^{-1} < p_1$, the Boyd interpolation theorem can be applied uniformly to the spaces $(X, \|\ \|_\theta)$ (uniformly also over choices of ϕ) (see [1],[18] p.145). We conclude that there is a constant C depending only on X so that for $0 \leq \theta \leq \delta$, $f \in X$

$$\|Tf\|_\theta \leq C\gamma(T)\|f\|_\theta \tag{5.18}$$

Very similar calculations can be performed on X^*. We define for $g \in X^*$

$$\|g\|_{*,\theta} = \sup_{\|f\|_X \leq 1} \sup_{h \in W} \int_E fgh d\mu$$

and as before $\|g\|_{*,\theta} = \|g^* e^{-\theta w}\|_{X^*}$. Analogously we define

$$g_1(t) = \alpha^* \int_{1/2}^\infty g^*(st) s^{-1/p_0} ds \qquad 0 \leq t < \ell$$

where

$$\alpha^* \int_{1/2}^1 s^{-1/p_0} ds = 1.$$

Arguing in the same way, we conclude that $0 \leq g^* \leq g_1$, and $\|g_1\|_{X^*} \leq C\|g\|_{X^*}$, but for $\tau > 0$, $e^\tau t < \ell$,

$$g_1(e^\tau t) \leq e^{(1/p_0 - 1)\tau} g_1(t)$$

This leads to a conclusion dual to (5.17),

$$\|e^{\theta w}g^*\|_{*,\theta} \le C\|g\|_{X^*} \tag{5.19}$$

for $0 \le \theta \le \delta$.

We also estimate the dilation operator norms

$$\begin{aligned} \|D_s\|_{*,\theta} &\le Ks^{1-1/r_1} && s > 1 \\ &\le Ks^{1-1/r_0-\delta} && s \le 1 \end{aligned}$$

Again our original choice of δ allows us to use the Boyd interpolation theorem on T^* which maps L_{q_0} to L_{q_0} and L_{q_1} to L_{q_1} where (q_0, q_1) are the conjugate indices of (p_0, p_1). In the same way we obtain uniform estimates on $\|T^*\|_{*,\theta}$ so that

$$\|T^*g\|_{*,\theta} \le C\gamma(T)\|g\|_{X^*,\theta} \tag{5.20}$$

for $0 \le \theta \le \delta$.

Now we are in a position to complete the proof. Fix $f, g \in G$ and let $u = \phi(\log r_f)$, $v = \phi(\log r_g)$ where $u = 0$ when $r_f = 0$ or ∞ and $v = 0$ similarly if $r_g = 0$ or ∞. For $z \in \mathcal{C}$ we consider the analytic function

$$F(z) = \int_E T(e^{zu}f).e^{-zv}g\,d\mu \tag{5.21}$$

We shall bound F for $|\Re z| < \delta$. Suppose $\Re z = \theta \ge 0$. Then

$$\begin{aligned} \|e^{zu}f\|_\theta &= \|e^{\theta w}f^*\|_\theta \\ &\le C\|f\|_X \end{aligned}$$

by (5.17). Hence

$$\|T(e^{zu}f)\|_\theta \le C\gamma(T)\|f\|_X$$

by (5.18), and hence from the definition of $\| \, \|_\theta$

$$|\int_E e^{-zv} g.T(e^{zu} f) d\mu| \leq C\gamma(t)\|f\|_X \, \|g\|_{X^*}$$

If $-\Re z = \theta > 0$ then we similarly obtain from (5.19) and (5.20),

$$\|T^*(e^{-zv} g)\|_{*,\theta} \leq C\gamma(T)\|g\|_{X^*}$$

and so

$$|\int_E (e^{zu} f).T^*(e^{-zv} g) d\mu| \leq C\gamma(T)\|f\|_X \, \|g\|_{X^*}$$

Thus

$$|F(z)| \leq C\gamma(T)\|f\|_X \, \|g\|_{X^*}$$

for $|\Re z| < \delta$, and by the Cauchy estimates

$$|F'(0)| \leq C\gamma(T)\|f\|_X \, \|g\|_{X^*}.$$

Now

$$F'(0) = \int_E (T(uf).g - (Tf).vg) d\mu$$

i.e.

$$F'(0) = \int_E (\Gamma_\phi(f).T^* g - Tf.\Gamma_\phi(g)) d\mu.$$

We can now apply Theorem 5.4 to both X and X_0^* (since both have finite Boyd indices) to deduce that

$$|\int_E \Gamma_\phi(f.T^* g) - \Gamma_\phi(Tf.g) d\mu| \leq C\gamma(T)\|f\|_X \, \|g\|_{X^*}$$

Finally the quasi-addivity of Γ_ϕ on L_1 (Lemma 4.2) yields

$$|\int_E \Gamma_\phi(B_T(f,g)) d\mu| \leq C\gamma(T)\|f\|_X \, \|g\|_{X^*}$$

as required.

COROLLARY 5.7. *Under the assumptions of the theorem, when X is separable, T commutes with each Γ_ϕ on X.*

The proof of Corollary 5.7 requires the simple observation that $G \subset D(\Gamma_\phi)$ and $T(G) \subset D(\Gamma_\phi)$ for every ϕ when T satisfies conditions of the Theorem.

6. The symmetric Hardy class

We recall that $\mathcal{L}_1^b$ denotes the class of bounded Lipschitz maps $\phi : \mathbf{R} \to \mathbf{R}$ with $\phi(0) = 0$ and $L(\phi) \leq 1$. Let E be an interval or $E = \mathbf{R}$. Then we denote by H_1^{sym} the space of all $f \in L_1(E)$ so that

$$\Lambda_1(f) = \|f\|_1 + \sup_{\phi \in \mathcal{L}_1^b} \left| \int_E \Gamma_\phi(f) d\mu \right| < \infty \tag{6.1}$$

Let us observe that H_1^{sym}, which we refer to as the *symmetric Hardy class*, is a vector space and that Λ_1 is a quasi-norm on H_1^{sym}. In fact it follows from Corollary 3.2 that for $\phi \in \mathcal{L}_1^b$

$$\|\Gamma_\phi(f_1 + f_2) - \Gamma_\phi(f_1) - \Gamma_\phi(f_2)\|_1 \leq C(\|f_1\|_1 + \|f_2\|_1)$$

where C is independent of ϕ and hence that Λ_1 is a quasi-norm and H_1^{sym} is a vector space. We also see that H_1^{sym} is a re-arrangement invariant, i.e. that if $f \in H_1^{sym}$ and $\sigma : E \to E$ is a measure-preserving automorphism then $S_\sigma f \in H_1^{sym}$ and

$$\Lambda_1(S_\sigma f) \leq C\Lambda_1(f)$$

where C is independent of σ. It follows that the definition of H_1^{sym} can be extended unambiguously to any Polish space E and any non-atomic σ-finite measure μ on E.

We will let $H_{1,0}^{sym}$ be the space of mean-zero functions in H_1^{sym}. If $f \in L_1(\mathbf{R})$ is real-valued we shall say that f is *signed-decreasing* if f is non-increasing on $(-\infty, 0)$ and on $(0, \infty)$, is non-positive on $(-\infty, 0)$ and non-negative on $(0, \infty)$. For any real-valued $f \in L_1(E)$ there is a unique signed-decreasing rearrangement $f_d \in L_1(\mathbf{R})$ with

$f \sim f_d$. In fact

$$f_d(t) = (f_+)^*(t) \qquad t > 0$$
$$= -(f_)^*-(-t) \qquad t < 0.$$

We define

$$M(t) = \int_{-t}^{t} f_d(s)ds$$
$$M(\infty) = \int_E f d\mu$$

LEMMA 6.1. *For real-valued functions $f \in L_1(E)$, Λ_1 is equivalent to*

$$\Lambda_0(f) = \int_0^1 \frac{|M(t)|}{t} dt + \int_1^\infty \frac{|M(t) - M(\infty)|}{t} dt + \|f_1\|. \tag{6.2}$$

In particular on $H_{1,0}^{sym}$, Λ_1 is equivalent to

$$\Lambda_0(f) = \int_0^\infty \frac{|M(t)|}{t} dt + \|f\|_1 \tag{6.3}$$

PROOF: If $\phi \in \mathcal{L}_1^b$, $f \in L_1(E)$ with $f \geq 0$,

$$\int_E \Gamma_\phi(f) d\mu = \int_0^\infty \Gamma_\phi(f^*) dt$$

Hence if f is real-valued we deduce that

$$|\int_E \Gamma_\phi(f) d\mu - \int_0^\infty \Gamma_\phi((f_+)^*) dt + \int_0^\infty \Gamma_\phi((f_-)^*) dt| \leq C\|f\|_1$$

where C is dependent of ϕ, f. Now

$$\begin{aligned}\int_0^\infty (\Gamma_\phi((f_+)^*) - \Gamma_\phi((f_-)^*))\,dt &= \int_0^\infty (f_d(t) + f_d(t) + f_d(-t))\phi(\log t)dt \\ &= \int_0^\infty M'(t)\phi(\log t)dt \\ &= -\int_0^1 M(t)\frac{\phi'(\log t)}{t}dt + \int_1^\infty (M(\infty) - M(t))\frac{\phi'(\log t)}{t}dt\end{aligned}$$

so that

$$|\int_E \Gamma_\phi(f)d\mu| \le C\|f\|_1 + \int_0^1 \frac{|M(t)|}{t}dt + \int_1^\infty \frac{|M(t)-M(\infty)|}{t}dt$$

and conversely

$$\int_0^1 \frac{|M(t)|}{t}dt + \int_1^\infty \frac{|M(t)-M(\infty)|}{t}dt \le C\|f\|_1 + \sup_{\phi\in\mathcal{L}_1^b} |\int_E \Gamma_\phi(f)d\mu|$$

so that the lemma follows.

Lemma 6.1 shows immediately that (H_1^{sym}, Λ_1) is complete, i.e. a quasi-Banach space. In fact Λ_0 is a lower-semi-continuous function on $L_1(E)$ and this allows a routine proof of completeness. We state this formally as:

PROPOSITION 6.2. *H_1^{sym} is a quasi-Banach space.*

Lemma 6.1 also relates H_1^{sym} to a theorem of B. Davis [3] which characterizes rearrangements of functions in $\Re H_1$ and justifies our terminology. Let us note that Davis's theorem is actually an immediate consequence of our Theorem 5.6.

THEOREM 6.3. *(Davis). $H_1(\mathbf{T})$ is contained in $H_1^{sym}(\mathbf{T})$ and hence $\Re H_1(\mathbf{T})$ is also contained in $H_1^{sym}(\mathbf{T})$. If $f \in \Re H_{1,0}(\mathbf{T})$ then*

$$\int_0^\infty \frac{|M(t)|}{t}dt < \infty.$$

PROOF: The Riesz projection $R : L_2 \to H_2$ is of strong type (p,p) for $1 < p < \infty$ and so by Theorem 5.6 if $f, g \in L_2(\mathbf{T})$, $\phi \in \mathcal{L}_1^b$

$$|\int \Gamma_\phi(f.R^*g - Rf.g)d\mu| \le C\|f\|_2\ \|g\|_2$$

where C is independent of ϕ.

Here R^* is the Banach space adjoint of R not the Hilbert-space adjoint, so R^* is the invariant projection of L_2 onto $\bar{H}_2$. Hence if $f \in H_2,\ g \in H_{2,0}$ then

$$|\int \Gamma_\phi(f.g)d\mu| \leq C\|f\|_2\ \|g\|_2$$

Now by factorization if $f \in H_{1,0}$

$$|\int \Gamma_\phi(f)d\mu| \leq C\|f\|_1$$

whence $f \in H_1^{sym}$. The remaining statements are obvious, since if $f \in H_1^{sym}$ then $\Re f$ and $\Im f \in H_1^{sym}$.

Remark. Davis's original proof uses Brownian motion techniques. There is an unpublished proof due to J.L. Lewis using properties of BMO established by Coifman and Rochberg [3]; I am grateful to A. Baernstein for communicating this proof to me. The proof given above is rather inefficient since a considerable amount of machinery has been developed; however, we feel it has some interest. A shorter, direct proof avoiding Theorem 5.6 will be given later, which will also establish an equivalent result of Ceretelli [2].

We now prove a technical lemma which will be useful in the sequel.

LEMMA 6.4. *Let f be a real-valued function in $H_1^{sym}(E)$. Then there is a sequence $f_n \in H_1^{sym}$ with $|f_n| \leq |f|$ for $n \in \mathbf{N}$, $f_n \to f$ a.e., each f_n bounded and of support of finite measure, so that*

$$\Lambda_0(f) = \lim_{n\to\infty} \Lambda_0(f_n).$$

PROOF: It clearly suffices to prove this in the case when $E = \mathbf{R}$ and f is signed-decreasing; the general case follows by rearrangement. For each $n \in \mathbf{N}$ we construct

f_n. Let $a = \frac{1}{n}$, $b = n$. For convenience, let us assume that

$$M(a) = \int_{-a}^{a} f(t)dt \geq a(f(a) + f(-a)) \tag{6.4}$$

$$M(\infty) - M(b) = \int_{|t|\geq b} f(t)dt \geq 0 \tag{6.5}$$

The modifications necessary in the case when (6.4) or (6.5) are violated will be easily seen. Notice that $f(b) = 0$ implies $M(\infty) = M(b)$. Define $d \geq b$ so that $d = b$ if $f(b) = 0$ and otherwise

$$\int_t^d f(t)dt = M(\infty) - M(b).$$

Now define

$$f_n(t) = \begin{cases} \frac{M(a)}{a} - f(-a), & \text{for } 0 \leq t \leq a \\ f(-a), & \text{for } -a \leq t \leq 0 \\ f(t), & \text{for } |a| \leq t \leq |b| \\ f(t), & \text{for } b \leq t \leq d \\ 0, & \text{otherwise} \end{cases}$$

f_n is also signed-decreasing and

$$\|f_n\|_1 = \int_{|a|\leq t\leq|b|} |f|dt + M(a) - 2af(-a) + \int_b^d |f|dt$$

so that

$$\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1.$$

If

$$M_n(t) = \int_{-t}^{t} f_n(s)ds$$

then

$$M_n(t) = \begin{cases} \frac{M(a)}{t}t, & \text{for } 0 \leq t \leq a \\ M(t), & \text{for } a \leq t \leq b \\ M(b) + \int_b^t f(s)ds, & \text{for } b \leq t \leq d \\ M(\infty), & \text{for } t \geq d \end{cases}$$

Thus

$$\int_0^1 \frac{|M_n(t)|}{t} dt \leq |M(a)| + \int_0^1 \frac{|M(t)|}{t} dt,$$

and

$$\int_1^\infty \frac{|M_n(t) - M_n(\infty)|}{t} dt \leq \int_1^b \frac{|M(t) - M(\infty)|}{t} + \int_b^d \frac{|M(\infty) - M(t)|}{t} dt$$

since $M(t) \leq M_n(t) \leq M(\infty)$ for $b \leq t \leq d$.

It follows easily that

$$\limsup_{n\to\infty} \Lambda_0(f_n) \leq \Lambda_0(f)$$

and the lemma follows from the lower-semi-continuity of Λ_0.

We now state our main theorem on H_1^{sym}.

THEOREM 6.5. *There is a constant C so that whenever E is a Polish space, μ is a σ-finite measure on E and Ω is a symmetric homogeneous centralizer on $L_1(E,\mu)$ then*

$$|\int_E \Omega(f) d\mu| \leq C\Delta^*(\Omega)\Lambda_1(f)$$

for $f \in H_{1,0}^{sym} \cap D(\Omega)$.

Before proceeding with the proof of theorem 6.5, we make some initial remarks. First observe that we can reduce the theorem to the case when E is an interval or $E = \mathbf{R}$.

Next suppose Ω is a symmetric homogeneous centralizer on $L_1(E)$. Then $D(\Omega)$ is a dense symmetric order-ideal in $L_1(E)$; in particular, if $\mu(E) < \infty$ then $D(\Omega) \supset L_\infty(E)$. In general $D(\Omega)$ must include all L_∞-functions whose supports have finite measure. Further if f, g are simple functions whose supports have finite measure and

$f \sim g$ there is a measure-preserving automorphism σ of E with $g = S_\sigma f$. Thus

$$|\int_E \Omega(f)d\mu - \int_E \Omega(g)d\mu| \leq \Delta^* \|f\|_1$$

where $\Delta^* = \Delta^*(\Omega)$.

Now we can use Lemma 4.3(b) to deduce that in general if $f, g \in D(\Omega)$ and $f \sim g$ then

$$(6.6) \qquad |\int_E \Omega(f)d\mu - \int_E \Omega(g)d\mu| \leq 13\Delta^* \|f\|_1$$

We shall frequently use the inequality

$$|\int_E \Omega(f_1 + f_2) - \Omega(f_1) - \Omega(f_2)d\mu| \leq 3\Delta(\|f_1\|_1 + \|f_2\|_1)$$

(Lemma 4.2).

We shall now concentrate on the case $E = [-1, 1]$; it will be convenient to naturally identify $L_1[-1, 1]$ as a subspace of $L_1(\mathbf{R})$. We will fix $\alpha = 3/4$ and Σ_0 be the σ-algebra generated by the sets $\pm[\alpha^{n+1}, \alpha^n)$ for $n \geq 0$. We will let $P : L_1[-1, 1] \to L_1(\Sigma_0)$ be the conditional expectation operator. Our proof will require a number of preliminary lemmas.

LEMMA 6.6. *There is a constant C so that if $f \in L_\infty[-1, 1]$ is signed-decreasing then*

$$(6.7) \qquad \Lambda_1(f - Pf) \leq C\|f\|_1$$

$$\Lambda_1(Pf) \leq C\Lambda_1(f).$$

PROOF: If $\phi \in \mathcal{L}_1^b$

$$\|\Gamma_\phi(f) - \Gamma_\phi(f.1_{[0,1]}) - \Gamma_\phi(f.1_{[-1,0)})\|_1 \leq C\|f\|_1$$

and a similar inequality holds for Pf. Now

$$\Gamma_\phi(f.1_{[0,1]})dt = \int_0^1 f(t)\phi(\log t)dt$$

while

$$\begin{aligned}\Gamma_\phi(Pf.1_{[0,1]})dt &= \int_0^1 Pf(t)\phi(\log t)dt \\ &= \int_0^1 Pf(t)P[\phi(\log t)]dt \\ &= \int_0^1 f(t)P[\phi(\log t)]dt\end{aligned}$$

where for $\alpha^{n+1} \leq t < \alpha^n$

$$P[\phi(\log t)] = \frac{1}{\alpha^n - \alpha^{n+1}} \int_{\alpha^{n+1}}^{\alpha^n} \phi(\log t)dt$$

Thus

$$|P[\phi(\log t)] - \phi(\log t)| \leq \log \frac{4}{3}$$

and so

$$|\int \Gamma_\phi(f.1_{[0,1]}) - \Gamma_\phi(Pf.1_{[0,1]})dt| \leq \log \frac{4}{3} \int_0^1 f(t)dt.$$

If we combine with the similar result on $(-1,0)$ we obtain

$$|\int \Gamma_\phi(f) - \Gamma_\phi(Pf)dt| \leq C\|f\|_1$$

and so

$$|\int \Gamma_\phi(f - Pf)dt| \leq C\|f\|_1$$

so that both (6.7) and (6.8) follow.

LEMMA 6.7. *There is a constant C so that for every symmetric homogeneous centralizer Ω on $L_1[-1,1]$ and every $f \in L_\infty$*

$$|\int \Omega(f')dt - \int \Omega(f)dt| \leq C\Delta^*(\Omega)\|f\|_1$$

where

$$f'(t) = \frac{4}{3}f(\frac{4}{3}t)$$

PROOF: First let $g(t) = f(2t+1) + f(2t-1)$ for $-1 \leq t \leq 1$. Then $g \sim f$ and $g1_{[-1,0]} \sim g1_{[0,1]}$. Hence

$$|2\int \Omega(g1_{[0,1]})dt - \int \Omega(f)dt| \leq C\Delta^*(\Omega)\|f\|_1.$$

Repeating the argument if $h(t) = f(4t+1)$ then

$$|4\int \Omega(h)dt - \int \Omega(f)dt| \leq C\Delta^*(\Omega)\|f\|_1$$

However $f' \sim \frac{4}{3}(g1_{[0,1]} + h)$ so that

$$|\int \Omega(f')dt - \int \Omega(f)dt| \leq C\Delta^*(\Omega)\|f\|_1$$

LEMMA 6.8. *There is a constant C so that if $f \in L_\infty[-1,1]$ is signed-decreasing, with mean zero, and Σ_0-measurable, and if Ω is any symmetric homogeneous centralizer on $L_1[-1,1]$ then*

$$|\int \Omega(f)dt| \leq C\Lambda_1(f)\Delta^*(\Omega).$$

PROOF: Let $g(t) = f(t) + f(-t)$, $-1 \leq t \leq 1$. Then

$$|\int \Omega(g)dt - 2\int \Omega(f)dt| \leq C\Delta^*\|f\|_1$$

Let

$$G(t)=\sum_{n=0}^{\infty}\alpha^{n+1}g(\alpha^{n+1}t) \qquad -\infty<t<\infty.$$

Then

$$g(t)=\alpha^{-1}G(\alpha^{-1}t)-G(t) \tag{6.9}$$

Note that both g and G are even functions. Suppose $g(t)=\gamma_n$ for $\alpha^{n+1}\leq t<\alpha^n$, where $n\geq 0$. If $t\geq 1$ and $\alpha^{-k}\leq t<\alpha^{-(k+1)}$ then

$$\begin{aligned}G(t)&=\sum_{n=0}^{\infty}\gamma_n\alpha^{k+n+1}\\&=\frac{\alpha^{k+1}}{1-\alpha}\int_0^1 g(t)dt=0\end{aligned}$$

since f has mean zero. Thus G is supported on $[-1,1]$ and clearly $G\in L_\infty[-1,1]$.

Now Lemma 6.7 shows with (6.9) that

$$|\int\Omega(g)dt|\leq C\Delta^*(\Omega)\|G\|_1$$

and so

$$|\int\Omega(f)dt|\leq C\Delta^*(\Omega)(\|f\|_1+\|G\|_1).$$

However

$$\int_0^1|G|dt=\sum_{k=0}^{\infty}(\alpha^k-\alpha^{k+1})|\sum_{n=k+1}^{\infty}\gamma_n\alpha^{n-k}|$$

For f we have

$$M(t)=\int_{-t}^{t}f(s)ds=\int_0^t g(s)ds.$$

Thus if $\alpha^{k+1}\leq t<\alpha^k$

$$M(t)=\sum_{n=k+1}^{\infty}(\alpha^n-\alpha^{n+1})\gamma_n+(t-\alpha^{k+1})\gamma_k$$

and so

$$\left|\int_{\alpha^{k+1}}^{\alpha^k} \frac{|M(t)|}{t}dt - \log\frac{1}{\alpha}\left|\sum_{n=k+1}^{\infty}(\alpha^n-\alpha^{n+1})\gamma_n\right|\right| \leq \int_{\alpha^{k+1}}^{\alpha^k}\frac{t-\alpha^{k+1}}{t}|\gamma_k|dt$$
$$\leq |\gamma_k|(\alpha^k-\alpha^{k+1}).$$

Hence

$$\left|\int_0^1 \frac{|M(t)|}{t}dt - \log\frac{1}{\alpha}\int_0^1 |G|dt\right| \leq \int_0^1 |g|dt$$

so that

$$\|G\|_1 \leq C\Lambda_0(f) \leq C\Lambda_1(f)$$

and the lemma follows.

Now we complete the proof of the theorem for $E=[-1,1]$. Suppose first $f\in L_\infty[-1,1]$ is real and has mean zero. We will show that for any symmetric homogeneous centralizer Ω then

$$\left|\int \Omega(f)dt\right| \leq C\Lambda_1(f)\Delta^*(\Omega) \tag{6.10}$$

where C is independent of f,Ω.

We claim first that for any such f we can find $g\in L_\infty[-1,1]$ also real-valued with mean-zero and such that

$$\|g\|_1 \leq \frac{1}{3}\|f\|_1 \tag{6.11}$$

$$\Lambda_1(g) \leq C\|f\|_1 \tag{6.12}$$

$$\|g\|_\infty \leq \|f\|_\infty + \|f\|_1 \tag{6.13}$$

For any symmetric homogeneous Ω we have

$$\left|\int \Omega(f)dt - \int \Omega(g)dt\right| \leq C\Delta^*(\Omega)\Lambda_1(f) \tag{6.14}$$

To prove this claim, first let f_d be the signed-decreasing rearrangement of f. This is supported on an interval $[-\beta, 2-\beta]$ where $0 \le \beta \le 2$. For convenience we assume $\beta \le 1$ and leave the reader to appropriately modify the argument if $\beta > 1$. Define

$$f'(t) = \begin{cases} f_d(t), & \text{for } -\beta \le t \le 1 \\ f_d(t+2), & \text{for } -1 \le t \le \beta \end{cases}$$

Then $f' \sim f$ and so

$$\left|\int \Omega(f')dt - \int \Omega(f)dt\right| \le C\Delta^*(\Omega)\|f\|_1,$$

for any symmetric homogeneous Ω.

Next let

$$h(t) = \begin{cases} -f_d(t+2), & \text{for } -1 \le t \le -\beta \\ f_d(t+1), & \text{for } 0 \le t \le 1-\beta \\ 0, & \text{otherwise.} \end{cases}$$

Then since h is the difference of two functions with identical distributions we have

$$\left|\int \Omega(h)dt\right| \le C\Delta^*(\Omega)\|f\|_1.$$

Next let $f'' = f' + h$. Then

$$\left|\int \Omega(f'')dt - \int \Omega(f)dt\right| \le C\Delta^*(\Omega)\|f\|_1.$$

In particular by applying this inequality to each Γ_ϕ, $\phi \in \mathcal{L}_1^b$ we obtain

$$\Lambda_1(f'') \le C\Lambda_1(f).$$

f'' is also signed-decreasing, $\|f''\| = \|f\|_1$ and

$$\begin{aligned} \|f''\|_\infty &\le \|f\|_\infty + f_d(1) \\ &\le \|f\|_\infty + \|f\|_1. \end{aligned}$$

Let $g = f'' - Pf''$. Since f'' is signed-decreasing we obtain (6.13) immediately. Lemma 6.6 yields

$$\Lambda_1(g) \leq C\|f''\|_1 = C\|f\|_1$$

i.e. (6.12). For any symmetric homogeneous Ω

$$|\int \Omega(Pf'')dt| \leq C\Lambda_1(Pf'')\Delta^*(\Omega)$$

by Lemma 6.8. Thus by Lemma 6.6

$$\begin{aligned} |\int \Omega(Pf'')dt| &\leq C\Lambda_1(f'')\Delta^*(\Omega) \\ &\leq C\Lambda_1(f)\Delta^*(\Omega). \end{aligned}$$

Now

$$|\int \Omega(g)dt - \int \Omega(f'')dt - \int \Omega(Pf'')dt| \leq C\Delta^*(\Omega)\|f\|_1$$

and (6.14) follows.

Finally for (6.11)

$$\begin{aligned} \int_0^1 |g(t)|dt &= \sum_{n=0}^{\infty} \int_{\alpha^{n+1}}^{\alpha^n} |f''(t) - Pf''(t)|dt \\ &\leq \sum_{n=0}^{\infty} (\alpha^n - \alpha^{n+1})(f''(\alpha^{n+1}) - f''(\alpha^n)) \\ &= (1-\alpha)\sum_{n=0}^{\infty} \alpha^n(f''(\alpha^{n+1}) - f''(\alpha^n)) \\ &\leq (1-\alpha)\sum_{n=1}^{\infty} (\alpha^{n-1} - \alpha^n)f''(\alpha^n) \\ &\leq \frac{1-\alpha}{\alpha}\sum_{n=1}^{\infty} \int_{\alpha^{n+1}}^{\alpha^n} f''(t)dt \\ &= \frac{1-\alpha}{\alpha}\int_0^1 f''(t)dt \end{aligned}$$

so that by a similar argument on $(-1,0)$,

$$\|g\|_1 \leq \frac{1-\alpha}{\alpha}\|f\|_1$$

and $\frac{1-\alpha}{\alpha} = \frac{1}{3}$.

Now use (6.11)-(6.14) to produce a sequence $f_n \in L_\infty[-1,1]$ of mean-zero real-valued functions such that $f_0 = f$ and

$$\|f_{n+1}\|_1 \leq \frac{1}{3}\|f_n\|_1 \qquad n \geq 0 \tag{6.15)}$$

$$\Lambda_1(f_{n+1}) \leq C\|f_n\|_1 \qquad n \geq 0 \tag{6.16}$$

$$\|f_{n+1}\|_\infty \leq \|f_n\|_\infty + \|f_n\|_1 \qquad n \geq 0 \tag{6.17}$$

For any symmetric homogeneous Ω,

$$|\int \Omega(f_{n+1}) - \Omega(f_n)dt| \leq C\Delta^*(\Omega)\Lambda_1(f_n) \tag{6.18}$$

Clearly $\|f_n\|_1 \leq 3^{-n}\|f\|_1$ and

$$\|f_n\|_\infty \leq \|f\|_\infty + \frac{3}{2}\|f\|_1 \qquad n \geq 0.$$

Thus f_n is uniformly bounded and $f_n \to 0$ in measure. By Lemma 4.3,

$$\lim_{n\to\infty} \|\Omega(f_n)\|_1 = 0.$$

Now by (6.14)

$$\begin{aligned}
|\int \Omega(f)dt - \int \Omega(f_n)dt| &\leq C\Delta^*(\Omega)\sum_{k=0}^{n-1}\Lambda_1(f_k) \\
&\leq C\Delta^*(\Omega)(\Lambda_1(f) + \sum_{k=0}^{n-1}\|f_k\|) \\
&\leq C\Delta^*(\Omega)\Lambda_1(f).
\end{aligned}$$

Now letting $n \to \infty$ we obtain (6.10).

Let us now complete the proof when E is a bounded interval (a,b)(and hence for finite nonatomic measure μ). We clearly have for any symmetric homogeneous Ω and $f \in L_\infty(a,b)$ with mean zero

$$|\int_b^a \Omega(f)dt| \leq C\Lambda_1(f)\Delta^*(\Omega).$$

Now suppose $f \in D(\Omega) \cap H_{1,0}^{sym}$. If f is real-valued we can find a sequence $f_n \in L_\infty(a,b)$ with $0 \leq |f_n| \leq |f|$, $f_n \to f$ a.e. so that

$$\Lambda_0(f) = \lim_{n\to\infty} \Lambda_0(f_n)$$

Let

$$c_n = \int_b^a f_n dt.$$

Then $|c_n| \leq \|f\|_1$ and $f_n - c_n \to f$ a.e. Now

$$\begin{aligned} |\int_a^b \Omega(f_n - c_n)dt| &\leq C\Lambda_1(f_n - c_n)\Delta^*(\Omega) \\ &\leq C(\Lambda_1(f_n) + |c_n|)\Delta^*(\Omega) \end{aligned}$$

Now by Lemma 4.2

$$\limsup_{n\to\infty} |\int_a^b \Omega(f_n - c_n)dt - \int_a^b \Omega(f)dt| \leq C\Delta^*(\Omega)\|f\|_1$$

and it follows that

$$|\int_a^b \Omega(f)dt| \leq C\Delta^*(\Omega)\Lambda_1(f).$$

Finally we complete the proof for $E = \mathbf{R}$ or any σ-finite nonatomic measure. First note that if Ω is a symmetric homogeneous centralizer on $L_1(\mathbf{R})$ and $A \subset \mathbf{R}$ is a

Borel set with $\mu A < \infty$ then we can construct a symmetric homogeneous centralizer on $L_1(A)$ by

$$\Omega_A(f) = 1_A\Omega(f) \qquad f \in L_1(A)$$

Indeed

$$\|\Omega_A(f) - \Omega(f)\|_1 \leq \Delta^*(\Omega)\|f\|_1$$

and it quickly follows that

$$\Delta^*(\Omega_A) \leq 3\Delta^*(\Omega)$$

Thus if f is bounded and has support of finite measure we can deduce that if $A = \operatorname{supp} f$

$$|\int \Omega_A(f)dt| \leq C\Delta^*(\Omega)\Lambda_1(f)$$

and hence

$$|\int \Omega(f)dt| \leq C\Delta^*(\Omega)\Lambda_1(f).$$

The proof may now be completed in the same way as the finite case.

PROPOSITION 6.9. *Let X be a separable r.i. function space on (E,μ) and let Ω be a symmetric homogeneous centralizer on X. If $f \in X$ satisfies $f^*(t)|\log t| \in X$ then $f \in D(\Omega)$.*

PROOF: If we let, as usual, G consist of all L_∞-functions whose supports have finite measure then it is easy to see that $G \subset D(\Omega)$. If f satisfies the conditions of the proposition, then, since X is separable, we can find a $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ so that ψ is monotonic, $\lim_{x\to 0}\psi(x) = 0$, $\lim_{x\to 0}\frac{\psi(x)}{x} = \lim_{x\to\infty}\frac{\psi(x)}{x} = \infty$ and $h = \psi(|f|)$ satisfies $h^*(t)|\log t| \in X$.

Now fix a function $a \in L_1$ with $a \in D(\Omega^{[1]})$ and

$$\int_E a\,d\mu = 1.$$

For example, if $\mu E < \infty$ let a be a constant function.

In the following argument K is a constant which may vary from line to line and depend on f, h, Ω but not on g or ϕ. We argue first that the conditions on h imply that if $\phi : \mathbf{R} \to \mathbf{R}$ is Lipschitz with $L(\phi) \leq 1$ and $\phi(0) = 0$ then

$$\|\Gamma_\phi(h)\|_X \leq K$$

Hence for $g \in X^*$

$$\|\Gamma_\phi(h).g\|_1 \leq K\|g\|_{X^*}$$

and so (Theorem 5.4)

$$\|\Gamma_\phi(hg)\|_1 \leq K\|g\|_{X^*}$$

Hence

$$\Lambda_1(gh) \leq K\|g\|_{X^*}$$

and if

$$\gamma = \int_E gh\,d\mu$$

$$\Lambda_1(gh - \gamma a) \leq K\|g\|_{X^*}$$

Now by Theorem 6.5,

$$\left|\int \Omega^{[1]}(gh - \gamma a)\,d\mu\right| \leq K\|g\|_{X^*}$$

and hence by Lemma 4.2

$$\left|\int \Omega^{[1]}(gh)\,d\mu\right| \leq K\|g\|_{X^*}$$

Now by Theorem 5.1,

$$|\int g.\Omega(h)d\mu| \leq K\|g\|_{X^*}$$

and we conclude that $\Omega(h) \in X_{max}$, and so $\Omega(f) \in X_{max}$.

Now for $n \in \mathbf{N}$ let $A = \{t : n^{-1} < |f(t)| \leq n\}$. Then $\Omega(f)-\Omega(f.1_A)-\Omega(f.1_B) \in X$ where $B = E \setminus A$. However $f.1_A \in D(\Omega)$ so that $\Omega(f) - \Omega(f.1_B) \in X$.

Thus $d(\Omega(f), X) = d(\Omega(f.1_B), X)$. However

$$f.1_B = uh$$

where

$$\begin{aligned} \|u\|_\infty &= \max\{\sup_{|t|\geq n} \frac{t}{\psi(t)}, \sup_{|t|\leq n^{-1}} \frac{t}{\psi(t)}\} \\ &= \theta_n \quad \text{say.} \end{aligned}$$

Then

$$\begin{aligned} d(\Omega(f.1_B), X) &= d(u.\Omega(h), X) \\ &\leq \theta_n d(\Omega(h), X). \end{aligned}$$

Therefore

$$d(\Omega(f), X) \leq \theta_n d(\Omega(h), X)$$

where $\theta_n \to 0$ and so $\Omega(f) \in X$ i.e. $f \in D(\Omega)$.

Now the main theorem of the paper is immediate.

THEOREM 6.10. *Let E be a Polish space and let μ be a finite or σ-finite Borel measure on E. Let X be a separable r.i. function space on (E,μ) whose Boyd indices satisfy $1 \leq p_0 < p_X < q_X < p_1 < \infty$. Let $T : X \to X$ be a bounded linear operator of strong types (p_0, p_0) and (p_1, p_1).*

Then for any symmetric centralizer Ω on X, T commutes with Ω i.e.

$$T(D(\Omega)) \subset D(\Omega)$$

and

$$\|[T,\Omega](f)\|_X \leq \delta(\|f\|_X) \qquad f \in D(\Omega) \tag{6.19}$$

where $\delta : \mathbf{R}_+ \to \mathbf{R}_+$ is an increasing function.

PROOF: By Proposition 6.9, $L_{p_0} \cap L_{p_1} \subset D(\Omega)$ and T maps $L_{p_0} \cap L_{p_1}$ into itself. We now appeal to Theorem 5.5 with $I = J = G$, the space of bounded functions with supports of finite measure. If

$$B_T(f,g) = Tf.g - f.T^*g \qquad f,g \in G$$

then by Theorem 5.6 we have

$$\Lambda_1(B_T(f,g)) \leq K\|f\|_X \; \|g\|_{X^*}$$

for some constant K. Now Theorem 6.5 implies, since

$$\int_E B_T(f,g)d\mu = 0$$

that

$$\Omega^{[1]}(B_T(f,g))d\mu \leq K\|f\|_X \; \|g\|_{X^*}$$

for some K. Now by Theorem 5.5, T and Ω commute.

We refer the reader to Section 8 for the proof that when X is super-reflexive equation (6.19) is valid for all $f \in X$ (so that in particular $T\Omega(f)$ is well-defined). Probably this conclusion holds in general, but we have not pursued this technical point.

7. Some remarks on Hardy spaces

In this section we give an alternative treatment, avoiding using the main Theorems 5.6 and 6.10, to derive the conclusions of Theorem 6.10 for the Riesz projection and the Hilbert transform. First, we give a direct and rather simple proof of the B. Davis theorem 6.3 via an alternative description of H_1^{sym}.

PROPOSITION 7.1. *There is a constant C so that if*

$$\Lambda_2(f) = \|f\|_1 + \sup_{\phi \in \mathcal{L}_1^b} \left| \int_E f\phi(\log|f|)d\mu \right|$$

then for $f \in L_1$ with mean zero

$$C^{-1}\Lambda_2(f) \leq \Lambda_1(f) \leq C\Lambda_2(f)$$

(and $\Lambda_2(f) < \infty$ if and only if $\Lambda_1(f) < \infty$).

Remark. For mean zero functions, Λ_2 is a homogeneous function (i.e. $\Lambda_2(\alpha f) = |\alpha|\Lambda_2(f)$) and is hence an equivalent quasi-norm to Λ_1.

PROOF: It will suffice to consider $E = \mathbf{R}$ and to restrict to real-valued functions f. We may further assume f signed-decreasing and non-zero. Let $g(t) = |f(t)| + |f(-t)|$, $-\infty < t < \infty$ and set

$$h(t) = \sum_{n=1}^{\infty} b^{-n} g(2^{-n}t) + \sum_{n=0}^{\infty} a^n g(2^n t)$$

where a, b are fixed with $1 < a < 2 < b$. Then h is even, positive, monotone decreasing on $(0, \alpha)$ and

$$\|h\|_1 \leq C\|f\|_1$$

where $C = C(a, b)$.

We also have $|f| \leq h$ and

$$ah(2t) \leq h(t) \leq bh(2t) \qquad t > 0$$

Now if ϕ is any bounded Lipschitz function, we can find a bounded Lipschitz ψ with

$$\psi(k \log 2) = \phi(\log h(2^k)) \qquad k \in \mathbf{Z}$$

and

$$L(\psi) \leq L(\phi) \frac{\log b}{\log 2}$$

Then

$$\begin{aligned} |\psi(\log t) - \phi(\log h(t))| &\leq L(\phi) \log b + L(\psi) \log 2 \\ &\leq CL(\phi). \end{aligned}$$

Thus

$$\left| \int_0^\infty (f(t) + f(-t))\psi(\log t) dt - \int f\phi(\log |h|) dt \right| \leq CL(\phi) \|f\|_1$$

Now if we set

$$\Omega_\phi(f) = f\phi(\log |f|)$$

then Ω_{phi} is a strong centralizer on L_1 with $\Delta^*(\Omega_\phi) \leq CL(\phi)$. Thus

$$\|\Omega_\phi(f) - fh^{-1}\Omega_\phi(h)\|_1 \leq CL(\phi)\|f\|_1$$

and

$$\left| \int f\phi(\log |f|) dt - \int f\phi(\log |h|) dt \right| \leq CL(\phi)\|f\|_1$$

Also

$$\left| \int_0^\infty (f(t) + f(-t))\psi(\log t) dt - \int \Gamma_\psi(f) dt \right| \leq CL(\psi)\|f\|_1$$

and so finally

$$|\int f\phi(\log|f|)dt - \int \Gamma_\psi(f)dt| \le CL(\phi)\|f\|_1$$

or

$$|\int f\phi(\log|f|)dt| \le CL(\phi)\Lambda_1(f)$$

This yields one half of the desired inequality.

The converse is very similar. If ϕ is bounded and Lipschitz we can construct a bounded Lipschitz ψ with

$$\phi(k\log 2) = \psi(\log(h(2^k)) \qquad k \in \mathbf{Z}$$

and

$$L(\psi) \le \frac{\log 2}{\log a} L(\phi).$$

Then

$$|\phi(\log t) - \psi(\log h(t))| \le CL(\phi)$$

and as before

$$|\int \Gamma_\phi(f)dt - \int f\psi(\log|f|)dt| \le CL(\phi)\|f\|_1$$

whence

$$\Lambda_1(f) \le C\Lambda_2(f).$$

The following lemma extends a similar lemma in [11] (cf.[12]).

LEMMA 7.2. *If $f \in H_{1,0}(\mathbf{T})$ and $\phi \in \mathcal{L}_1^b$ then*

$$|\int_0^{2\pi} f(e^{i\theta})\phi(|\log f(e^{i\theta})|)\frac{d\theta}{2\pi}| \le 4\sqrt{6}\|f\|_1$$

PROOF: By approximating a triangular function we can find for $\delta > 0$ a C^∞-function of compact support $\rho : \mathbf{R} \to \mathbf{R}$ with $\rho \geq 0$

$$\int_{-\infty}^{\infty} \rho(t)dt = 1$$
$$\int_{-\infty}^{\infty} |\rho'(t)|dt \leq 4 + \delta$$
$$\int_{-\infty}^{\infty} |t|\rho(t)dt \leq \frac{2}{3} + \delta$$

For $a > 0$, let

$$\psi(t) = \frac{1}{a}\int_{-\infty}^{\infty} \rho(\frac{t-s}{a})\phi(s)ds.$$

Then $\psi \in \mathcal{L}_1^b$ and

$$|\psi(t) - \phi(t)| \leq \frac{1}{a}\int_{-\infty}^{\infty}(t-s)\rho(\frac{t-s}{a})ds \leq (\frac{2}{3} + \delta)a$$

and

$$|\psi''(t)| = |\frac{1}{a^2}\int_{-\infty}^{\infty} \rho'(\frac{t-s}{a})\phi'(s)ds| \leq \frac{4+\delta}{a}.$$

Now consider the function $x\psi(\log|x|)$ on $\mathbf{C}$. For $z \neq 0$

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(x\psi(\log|z|) = \frac{x}{|z|^2}(\psi''(\log|z|) + 2\psi'(\log|z|)).$$

Thus if we set $\lambda = \frac{1}{a}(4+\delta) + 2a$ then $\lambda|z| - x\psi(\log|z|)$ is subharmonic on $\mathbf{C}$. (We define $x\psi(\log|z|) = 0$ at the origin).

Now

$$\Phi(f) = \int_0^{2\pi} \lambda|f| - (Ref)\psi(\log|f|)\frac{d\theta}{2\pi}$$

is plurisubharmonic and continuous on H_1. If $f \in H_{1,0}$ then $F : \bar{\Delta} \to H_1$ is defined by

$$F(z)(e^{i\theta}) = F(ze^{i\theta})$$

is analytic on Δ and so $\Phi \circ F$ is subharmonic. Since $\Phi \circ F$ is constant on circles $|z| = r$ we have

$$\begin{aligned}\Phi \circ F(1) &= \lim_{r \to 1} \Phi \circ F(r) \\ &\geq \Phi \circ F(0) = 0.\end{aligned}$$

i.e.

$$\Re\Big(\int_0^{2\pi} f\psi(\log|f|)\frac{d\theta}{2\pi}\Big) \leq \lambda\|f\|_1.$$

Arguing with αf in place of f with $|\alpha| = 1$ we obtain

$$\Big|\int_0^{2\pi} f\psi(\log|f|)\frac{d\theta}{2\pi}\Big| \leq \lambda\|f\|_1$$

and so

$$\Big|\int_0^{2\pi} f\phi(\log|f|)\frac{d\theta}{2\pi}\Big| \leq (\lambda + (\frac{2}{3} + \delta)a)\|f\|_1$$

As $\delta > 0$ is arbitrary and taking $a = \frac{2}{3}$ we obtain the lemma.

Alternative proof of Theorem 6.3. By Lemma 7.2 if $f \in H_{1,0}$ then $\Lambda_2(f) \leq 4\sqrt{6}\|f\|_1$ and so by Proposition 7.1, $\Lambda_1(f) \leq C\|f\|_1$ i.e. $f \in H_{1,0}^{sym}$.

Let X be a separable r.i. function space on $\mathbf{T}$. Define H_X to be the subspace of $f \in X$ such that $\hat{f}(n) = 0$ for $n < 0$ where

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta.$$

THEOREM 7.3. *Let X be a separable r.i. function space on $\mathbf{T}$ and suppose Ω is a symmetric homogeneous centralizer on X. Then there is a constant C so that if $f \in H_X$ and $\Omega(f) \in X$ then*

$$d(\Omega(f), H_X) \leq C\Delta^*(\Omega)\|f\|_X.$$

PROOF: By the Hahn-Banach theorem there exists $g \in X^*$ such that $\|g\|_{X^*} = 1$, $\hat{g}(n) = 0$ for $n \leq 0$ and

$$\int_0^{2\pi} \Omega(f).g\frac{d\theta}{2\pi} = d(\Omega(f), H_X).$$

Then $fg \in H_{1,0} \cap D(\Omega^{[1]})$ and so by Theorem 6.5

$$|\int_0^{2\pi} \Omega^{[1]}(fg)\frac{d\theta}{2\pi}| \leq C\Delta^*(\Omega)\|fg\|_1$$

which implies, by Theorem 6.3, that

$$|\int_0^{2\pi} \Omega(f).g\frac{d\theta}{2\pi}| \leq C\Delta^*(\Omega)\|f\|_X$$

i.e.

$$d(\Omega(f), H_X) \leq C\Delta^*(\Omega)\|f\|_X.$$

Remark 1. If Ω is not assumed homogeneous we obtain

$$d(\Omega(f), H_X) \leq \delta(\|f\|_X)$$

for some function $\delta : \mathbf{R}_+ \to \mathbf{R}_+$.

Remark 2. If X is super-reflexive the Riesz projection R is bounded on X and we obtain that R commutes with Ω very easily from Theorem 7.3 (cf.[11]). In fact, if $f \in X$ and $f \in L_\infty$

$$f = f_1 + f_2$$

where $f_1 \in H_X$, $f_2 \in \bar{H}_{X,0}$ with $\|f_i\|_X \leq C\|f\|_X$. Also $\Omega(f_1)$, $\Omega(f_2) \in X$. Then $\Omega(Rf) = \Omega(f_1)$ while

$$\|\Omega(f) - \Omega(f_1) - \Omega(f_2)\|_X \leq C\|f\|_X$$

and

$$\begin{aligned}\|R\Omega(f_2)\| &\leq Cd(\Omega(f_2), \bar{H}_{X,0}) \\ &\leq C\|f\|_X\end{aligned}$$

Thus

$$\|R\Omega(f) - R\Omega(f_1)\|_X \leq C\|f\|_X$$

and

$$\begin{aligned}\|R\Omega(f_1) - \Omega(f_1)\|_X &\leq Cd(\Omega(f_1), H_X) \\ &\leq C\|f\|_X.\end{aligned}$$

We conclude

$$\|R\Omega(f) - \Omega(Rf)\|_X \leq C\|f\|_X.$$

8. Final remarks

Let us start by proving the promised converse for Theorem 5.1.

THEOREM 8.1. *Let X be a separable Kothe function space on (E, μ) and suppose X satisfies a q-concavity condition for some $q < \infty$ (or X has finite cotype). Suppose Ω is a homogeneous centralizer on $L_1(\mu)$. Then there is a homogeneous centralizer Ω_X on X such that $\Omega_X^{[1]}$ is equivalent to Ω.*

PROOF: We first prove this under the additional hypothesis that X has non-trivial type (or is super-reflexive). In this case X is a B-convex space and it follows from [9] that there is a constant K so that if Z_0 is a dense subspace of X^* and $\Phi : Z_0 \to \mathbf{C}$ is a quasilinear map satisfying

$$(8.1) \qquad |\Phi(g_1 + g_2) - \Phi(g_1) - \Phi(g_2)| \leq \alpha(\|g_1\|_{X^*} + \|g_2\|_{X^*})$$

then there is a linear map $\psi : Z_0 \to \mathbf{C}$ with

$$(8.2) \qquad |\Phi(g) - \psi(g)| \leq K\alpha\|g\|_X \qquad f \in Z_0$$

Now suppose $f \in X$. Pick $u = u_f > 0$ a.e. so that $|u| \leq 1$ a.e. and $u\Omega(f) \in L_1$. Then $\Omega(uf) \in L_1$. Let $Z_f = \{g \in X^* : |g| \leq m|u| \text{ for some } m \in \mathbf{N}\}$. For $g \in Z_f$, set

$$\Phi_f(g) = \int_E \Omega(fg) d\mu$$

and Φ_f satisfies (8.1) with $\alpha \leq 3\Delta\|f\|_X$, so that there exists a linear map $\psi_f : Z_f \to \mathbf{C}$ with

$$|\Phi_f(g) - \psi_f(g)| \leq 3K\Delta\|f\|_X \ \|g\|_{X^*}$$

for $g \in Z_f$. It may be supposed in the above argument that if $\lambda \in \mathbf{C} \setminus \{0\}$ then $Z_{\lambda f} = Z_f$ and $\psi_{\lambda f} = \lambda \psi_f$.

Note that if $|g| \leq |u|$

$$|\Phi_f(g)| \leq \|\Omega(fg)\|_1$$

and

$$\|\Omega(fg) - gu^{-1}\Omega(fu)\|_1 \leq \Delta \|f\|_X$$

Thus

$$\sup_{|g| \leq |u|} |\Phi_f(g)| < \infty$$

Also if $|g_n| \leq |u|$ and $g_n \to 0$ a.e. then

$$\lim_{n \to \infty} \|\Omega(fg_n)\|_1 = 0$$

and so

$$\lim_{n \to \infty} |\Phi_f(g_n)| = 0.$$

It follows that

$$\sup_{|g| \leq |u|} |\psi_f(g)| < \infty$$

and if $|g_n| \leq |u|$, with $g_n \to 0$ a.e.

$$\lim_{n \to \infty} |\psi_f(g_n)| = 0.$$

Hence there exists $h \in L_1(u.\mu)$ such that

$$\psi_f(g) = \int ghd\mu.$$

We shall define $\Omega_X(f) = h$. We then have

$$|\int_E \Omega(fg)d\mu - \int_E g\Omega_X(f)d\mu| \leq 3K\Delta \|f\|_X \ \|g\|_{X^*}$$

for $f \in X,\ g \in Z_f$. By our method of selection Ω_X is homogeneous.

Now suppose $|v| \leq 1$ a.e. Then for $g \in Z_f \cap Z_{vf}$

$$|\int_E \Omega(fgv)d\mu - \int_E gv\Omega_X(f)d\mu| \leq 3K\Delta\|f\|_X\ \|g\|_{X^*}$$

and

$$|\int_E \Omega(fgv)d\mu - \int_E g\Omega_X(vf)d\mu| \leq 3K\Delta\|f\|_X\ \|g\|_{X^*}$$

Thus

$$|\int_E g(v\Omega_X(f) - \Omega_X(vf))d\mu| \leq 6K\Delta\|f\|_X\ \|g\|_{X^*}.$$

As $Z_f \cap Z_{vf}$ is a dense subspace of X^* we conclude

$$\|v\Omega_X(f) - \Omega_X(vf)\|_X \leq 6K\Delta\|f\|_X$$

so that Ω_X is a centralizer.

Finally if $|v| \leq 1$ a.e.

$$|\int_E \Omega(fgv)d\mu - \int_E gv\Omega_X(f)d\mu| \leq 3K\Delta\|f\|_X\ \|g\|_{X^*}$$

and so

$$|\int_E \Omega(fg)vd\mu - \int_E gv\Omega_X(f)d\mu| \leq 4K\Delta\|f\|_X\ \|g\|_{X^*}$$

so that

$$\|\Omega(fg) - g.\Omega_X(f)\|_1 \leq 4K\Delta\|f\|_X\ \|g\|_{X^*}$$

so that $\Omega_X^{[1]}$ is equivalent to Ω.

Now let us relax the type assumption. Suppose merely that X verifies a q-concavity condition. Let us denote the 2-convexification of X (cf. [18]) by

$$Y = \{f \in L_0 : |f|^2 \in X\}$$

We norm Y by

$$\|f\|_Y = \||f|^2\|_X^{\frac{1}{2}}$$

Then Y is a 2-convex and $2q$-concave lattice and is therefore of non-trivial type ([18]). Then by the first part there is a homogeneous centralizer Ω_Y on Y so that $\Omega_Y^{[1]}$ is equivalent to Ω.

Let us define for $f \in X$

$$\Omega_X(f) = (\text{sgn } f)|f|^{1/2}\Omega_Y(|f|^{1/2})$$

If $|u| \leq 1$ a.e.

$$\Omega_X(uf) = \text{sgn } u.\text{sgn } f|uf|^{1/2}\Omega_Y(|u|^{1/2}|f|^{1/2})$$

and so

$$\Omega_X(uf) - u\Omega_X(f) = \text{sgn } (uf)|uf|^{1/2}(\Omega_Y(|u|^{1/2}|f|^{1/2}) - |u|^{1/2}\Omega_Y(|f|^{1/2}))$$

If we let

$$w_1 = |uf|$$

$$w_2 = (\Omega_Y(|u|^{1/2}|f|^{1/2}) - |u|^{1/2}\Omega_Y(|f|^{1/2}))^2$$

Then

$$\begin{aligned}\|\Omega_X(uf) - u\Omega_X(f)\|_X &= \|w_1^{1/2}\ w_2^{1/2}\|_X \\ &\leq \|w_1\|_X^{1/2}\ \|w_2\|_X^{1/2} \\ &\leq \|f\|_X^{1/2}\ \|w_2\|_X^{1/2}.\end{aligned}$$

However

$$\begin{aligned}\|w_2\|_X^{1/2} &= \|\Omega_Y(|u|^{1/2}|f|^{1/2}) - |u|^{1/2}\Omega_Y(|f|^{1/2}\|_Y \\ &\leq \Delta(\Omega_Y)\||f|^{1/2}\|_Y \\ &= \Delta(\Omega_Y)\|f\|_X^{1/2}.\end{aligned}$$

Thus Ω_X is a centralizer and $\Delta(\Omega_X) \leq \Delta(\Omega_Y)$.

Next suppose $g \in X^*$ and $f \in X$. Then if $h \in Y$, $|f|^{1/2}\, h \in X$ and

$$\begin{aligned}\||f|^{1/2}h\|_X &\leq \|f\|_X^{1/2}\ \|h^2\|_X^{1/2}\\ &= \|f\|_X^{1/2}\ \|h\|_Y.\end{aligned}$$

Hence

$$\|g|f|^{1/2}h\|_1 \leq \|f\|_X^{1/2}\ \|g\|_{X^*}\|h\|_Y$$

so that $g|f|^{1/2} \in Y^*$ and

$$\|g|f|^{1/2}\|_{Y^*} \leq \|f\|_X^{1/2}\ \|g\|_{X^*}$$

Now

$$g.\Omega_X(f) = (sgn f)g|f|^{1/2}\Omega_Y(|f|^{1/2})$$

and so

$$\begin{aligned}\|\Omega(fg) - g\Omega_X(f)\|_1 &\leq C\||f|^{1/2}g\|_{Y^*}\ \||f|^{1/2}\|_Y\\ &\leq C\|f\|_X\ \|g\|_{X^*}\end{aligned}$$

and the result is proved.

If X is super-reflexive and Ω is a homogeneous centralizer, then $X \oplus_\Omega X$ is isomorphic to a Banach space (cf.[9]). Now applying Theorem 8.1 to $\Omega^{[1]}$ and X^* yields a *dual centralizer* $\Omega^* : X^* \to L_0$ with

$$\|f.\Omega^* g - \Omega f.g\|_1 \leq C\|f\|_X\ \|g\|_{X^*} \tag{8.3}$$

Now for $(f_1, f_2) \in X^2$, $(g_1, g_2) \in (X^*)^2$ define

$$\langle (f_1, f_2), (g_1, g_2)\rangle = \int_E (f_1 g_2 - g_2 f_1)d\mu \tag{8.4}$$

Then (8.3) yields that

$$|\langle (f_1, f_2),\ (g_1, g_2)\rangle| \leq C\|(f_1, f_2)\|_\Omega \|(g_1, g_2)\|_{\Omega^*}$$

provided $(f_1, f_2) \in X \oplus_\Omega X$, $(g_1, g_2) \in X^* \oplus_{\Omega^*} X^*$. It is then not difficult to identify $(X \oplus_\Omega X)^*$ with $X^* \oplus_{\Omega^*} X^*$ via the duality (8.4). We omit the details. Special cases of this calculation are established in [10] and [14].

Let us use the idea of the dual centralizer to show that if X is super-reflexive the conclusion of Theorem 6.10 can be improved to read in place of (6.19).

$$\|T\Omega(f) - \Omega T(f)\|_X \leq \delta(\|f\|_X) \qquad f \in X \tag{8.5}$$

It suffices to consider the case when Ω is homogeneous. In this case if $g \in L_{q_0} \cap L_{q_1}$ then by Proposition 6.9, $g \in D(\Omega^*)$ and hence if $f \in X$, $\Omega(f).g \in L_1$. Thus $\Omega(f) \in L_{p_0} + L_{p_1}$ whenever $f \in X$ and in particular $T\Omega(f)$ is well-defined.

Now let f_n be a sequence in G so that $\|f_n - f\|_X \leq 2^{-n}$, and hence $\|Tf_n - Tf\|_X \leq 2^{-n}$. We may find $h \in X$ with $\|h\|_X \leq C\|f\|_X$ and

$$|h| \geq |f_n| \qquad n \in \mathbf{N}$$

$$|h| \geq |Tf_n| \qquad n \in \mathbf{N}$$

Now

$$\|T\Omega(f_n) - \Omega T(f_n)\|_X \leq C\|f\|_X$$

so that

$$\|T(f_n h^{-1}\Omega(h)) - (Tf_n)h^{-1}\Omega(h)\|_X \leq C\|f\|_X.$$

Letting $n \to \infty$ and using the fact that the X-unit ball is closed in $L_{p_0} + L_{p_1}$ we obtain

$$\|T(fh^{-1}\Omega(h)) - (Tf)h^{-1}\Omega(h)\|_X \leq C\|f\|_X$$

and it follows that

$$\|[T,\Omega](f)\|_X \le C\|f\|_X$$

We state this conclusion formally:

THEOREM 8.2. *Under the hypotheses of Theorem 6.10, if X is super-reflexive then we may replace (6.19) by*

$$\|[T,\Omega](f)\|_X \le \delta(\|f\|_X) \qquad f \in X \tag{8.6}$$

Let us conclude with the observation that Theorems 5.1 and 8.1 together indicate a correspondence between centralizers on X and centralizers on L_1 at least when X has finite cotype. If X is not locally convex (e.g. $X = L_p$ for $0 < p < 1$) it is unclear whether such a correspondence can in general be established. To make this precise we ask whether if $\Omega : L_p \to L_0$ is a homogeneous centralizer one can always find a homogeneous centralizer $\Omega^{[1]} : L_1 \to L_0$ so that Ω is equivalent to

$$\Omega'(f) = (\mathrm{sgn}\ f)|f|^{1-p}\Omega^{[1]}(|f|^p).$$

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