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Perturbation and Interpolation Theorems for the H^∞-Calculus with Applications to Differential Operators

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Abstract. We prove comparison theorems for the H^∞-calculus that allow to transfer the property of having a bounded H^∞-calculus from one sectorial operator to another. The basic technical ingredient are suitable square function estimates. These comparison results provide a new approach to perturbation theorems for the H^∞-calculus in a variety of situations suitable for applications. Our square function estimates also give rise to a new interpolation method, the Rademacher interpolation. We show that a bounded H^∞-calculus is characterized by interpolation of the domains of fractional powers with respect to Rademacher interpolation. This leads to comparison and perturbation results for operators defined in interpolation scales such as the L_p-scale. We apply the results to give new proofs on the H^∞-calculus for elliptic differential operators, including Schrödinger operators and perturbed boundary conditions. As new results we prove that elliptic boundary value problems with bounded uniformly coefficients have a bounded H^∞-calculus in certain Sobolev spaces and that the Stokes operator on bounded domains Ω with $\partial\Omega \in C^{1,1}$ has a bounded H^∞-calculus in the Helmholtz scale $L_{p,\sigma}(\Omega)$, $p \in (1, \infty)$.

1. Introduction

It is well established by now that the H^∞-functional calculus of a sectorial operator has important applications in the spectral theory of partial differential operators and the theory of evolution equations, e.g., in determining the domain of fractional powers of a partial differential operator in the solution of Kato's problem (e.g. [55], [16], [26], [8], [4]), in connection with maximal regularity of parabolic evolution equations (e.g. [41], [42], [43], [30], [28], [51]) and certain estimates in control theory ([44], [27]). An essential tool in verifying the boundedness of the H^∞-calculus for a partial differential operator are perturbation theorems. While sectoriality (and R-sectoriality, [40]) are preserved under relatively bounded perturbations, it is well known that this is not true for the boundedness of the

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H^∞-calculus (cf. [47]). In this paper we offer a systematic study of additional conditions, that – together with relative boundedness – lead to useful perturbation theorems.

Our results are based on a characterization of the boundedness of the H^∞-calculus in terms of "square function estimates", which are similar to the ones known for Hilbert- and L_p-spaces (see [15], [45]) but can be formulated in general Banach spaces and refine the results of [30]. They are formulated in terms of Rademacher averages and connect therefore nicely with R-boundedness and related techniques from Banach space theory (see Section 4). For many of our results we need a weaker form of R-sectoriality, which is necessary for the boundedness of the H^∞-calculus without any restrictions on the underlying Banach space. This notion of "almost R-sectoriality" requires that $\{\lambda A R(\lambda, A)^2 : \lambda \in \Sigma\}$ is R-bounded for some sector Σ, it is studied in Section 3.

One main theme of our paper, discussed in Sections 5 and 6, is to show how the boundedness of the H^∞-calculus in "encoded" in the fractional domain spaces of a sectorial operator. For example we show in Section 5 that if $D(A^\alpha)$ and $D(B^\alpha)$ coincide (with equivalent norms) for two different values of α with $0 < |\alpha| \le 3/2$ where A has a bounded H^∞-calculus and B is (almost) R-sectorial then B has a bounded H^∞-calculus, too (Theorem 5.1). This leads to characterizations of the coincidence of $D(A^\beta)$ and $D(B^\beta)$ for certain values of β in terms of R-boundedness conditions on the resolvent of B in the scale defined by $D(A^\alpha)$ (Theorem 5.7) and to perturbation theorems where the perturbation is not just relatively bounded on $D(A)$ but with respect to two norms of the scale $D(A^\alpha)$ (see Theorem 6.1). We also give results modeled after perturbation theorems for forms, i.e. the perturbation maps from $D(A^\alpha)$ to $D(A^{\alpha-1})$. If $\alpha \in (0, 1)$, even one relative bound suffices (see Theorem 6.6). In the Hilbert space case, some of our comparison theorems recover theorems in [56] and [5], which were the starting point of our investigation.

Our second main theme concerns operators defined in a whole interpolation scale, such as L_p-spaces or Helmholtz-spaces. We want to show that, if two operators A and B are "well understood" in one space of the scale (usually a Hilbert space) then one relative boundedness condition suffices to obtain a perturbation result for the remaining spaces of the scale. This is of particular interest when A and B are accretive operators in a Hilbert space extending to an L_p-scale (see Section 8 for details). For this we need a new interpolation method $\langle \cdot, \cdot \rangle_\theta$ — also formulated in terms of Rademacher averages — which interpolates the scale of domains of fractional powers "correctly", i.e., $\widetilde{D(A^\gamma)} = \langle \widetilde{D(A^\alpha)}, \widetilde{D(A^\beta)} \rangle_\theta$, $\gamma = (1-\theta)\alpha + \theta\beta$ if and only if A has an H^∞-calculus ($\widetilde{\ }$ denotes completion for the homogeneous norm, see Section 7 for details and precise assumptions). For Hilbert space and the complex interpolation method, this result is already contained in [5], but for Banach spaces other than Hilbert spaces it is neither true for the complex nor for the the real interpolation method. We also show that, for every (almost) R-sectorial operator, the part of A on the Rademacher interpolation space $\langle X, \widetilde{D(A)}\rangle_\theta$, $\theta \in (0,1)$, has always a bounded H^∞-calculus.

In Section 9 we illustrate our results by giving short proofs for the boundedness of the H^∞-calculus for some classes of partial differential operators. Whereas some of these results are known and shown with different proofs as before (e.g., for partial differential operators on $\mathbb{R}^n$ with Hölder continuous coefficients or for elliptic boundary value problems with Hölder continuous coefficients), we obtain new results for elliptic boundary value problems with bounded uniformly continuous coefficients in Sobolev spaces, for certain operators in divergence form, and for the Stokes operator on bounded domains with a $C^{1,1}$-boundary.

In the following Section 2 we give some necessary definitions and preliminary results concerning fractional powers of sectorial operators and the scales of their domain spaces.

2. Preliminaries on sectorial operators and fractional powers

By X and Y we always denote complex Banach spaces and by $B(X,Y)$ the space of bounded linear operators from X to Y. We write $B(X)$ for $B(X,X)$.

Recall that a closed densely defined operator A on X is a **sectorial operator** of type $0 \le \omega < \pi$ if A is one–to–one with dense range, the spectrum $\sigma(A)$ of A is contained in the closed sector $\overline{\Sigma_\omega}$ where $\Sigma_\omega = \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg(\lambda)| < \omega\}$, and, for all $\sigma > \omega$, the set $\tau_\sigma = \{\lambda R(\lambda, A) : \lambda \neq 0, |\arg(\lambda)| \ge \sigma\}$ is bounded. The infimum over all ω for which A is sectorial of type ω is denoted by $\omega(A)$.

Since the dual operator A^* of a sectorial operator A may not have a norm–dense domain and norm–dense range in X^*, we denote by $A^\sharp$ the part of A^* in $X^\sharp = \overline{D(A^*)}^{X^*} \cap \overline{R(A^*)}^{X^*}$. If we want to emphasize with respect to which sectorial operator A the space $X^\sharp$ is constructed we write $X^\sharp_A$. It is clear that $X^\sharp_A = X^\sharp_{A^{-1}}$, and it is well known (cf. [33, Thm. 3.7], [15]) that $A^\sharp$ is a sectorial operator in $X^\sharp$. In particular, $D(A^\sharp) \cap R(A^\sharp)$ is dense in $X^\sharp$. Note that $D(A^*) \cap R(A^*)$ norms X: Letting $\varphi_n(z) = \frac{n}{n+z} - \frac{1}{1+nz}$ (cf. also the beginning of Section 4), this follows from $\langle \varphi_n(A^*)x^*, x\rangle = \langle x^*, \varphi_n(A)x\rangle \to \langle x^*, x\rangle$ as $n \to \infty$ where $x \in X$, $x^* \in X^*$. Notice that $\varphi_n(A) = n(n+A)^{-1} - n^{-1}(n^{-1}+A)^{-1} = A((n+A)^{-1} - (n^{-1}+A)^{-1})$ is a uniformly bounded sequence in $B(X)$ if A is sectorial.

Since $A^\sharp$ is sectorial in $X^\sharp$, we conclude that also $D(A^\sharp) \cap R(A^\sharp)$ norms X. Recall that $A^\sharp = A^*$ and $X^\sharp = X^*$ if X is reflexive.

For the theory of fractional powers A^α, $\alpha \in \mathbb{C}$, of a sectorial operator A we refer to Komatsu ([Ko1] – [Ko5]).

We now define a scale of Banach spaces related to fractional powers. This scale turns out to be fundamental for our later considerations. For a sectorial operator A on X and $\alpha \in \mathbb{R}$ we define $\dot{X}_\alpha$ as the completion of $D(A^\alpha)$ with respect to the norm $\|x\|_\alpha = \|A^\alpha x\|$, i.e.

$$(\dot{X}_\alpha, \|\cdot\|_{\dot{X}_\alpha}) := (D(A^\alpha), \|A^\alpha \cdot \|_X)^\sim.$$

In particular we have $\dot{X}_0 = X$. Only if $0 \in \varrho(A)$ these spaces do coincide with the usual Sobolev tower as defined, e. g. in [23], where one takes the graph norm on $D(A^\alpha)$ for $\alpha > 0$ (cf. also Remark 5.9 for further details). If $X = L_p(\mathbb{R}^n)$ and $A = -\Delta$ then the $\dot{X}_\alpha$ are Riesz potential spaces, whereas the Sobolev tower defined in [23] consists of Bessel potential spaces.

If we want to emphazise the operator A, whose fractional domains we consider, we write $\dot{X}_{\alpha,A}$.

It is easy to check that $A^\alpha : D(A^\alpha) \to R(A^\alpha)$ extends to an isomorphism $\widetilde{A^\alpha} : \dot{X}_\alpha \to X$ whose inverse $(\widetilde{A^\alpha})^{-1}$ is an extension of the operator $A^{-\alpha} : R(A^\alpha) \to D(A^\alpha)$. Observe that one has to distinguish this extension which acts as an isomorphsim $X \to \dot{X}_\alpha$ from the extension $\widetilde{A^{-\alpha}}$ which acts as an isomorphism $\dot{X}_{-\alpha} \to X$.

It is not hard to see that $X^\sharp_{A^\alpha} = X^\sharp_A$ for any $\alpha \neq 0$. If X is reflexive one also has $(\dot{X}_{\alpha,A})^* = (X^*)^{\cdot}_{-\alpha,A^*}$ and $\dot{X}_{-\alpha,A} = ((X^*)^{\cdot}_{\alpha,A^*})^*$ with respect to the duality $\langle X, X^* \rangle$.

For further considerations we recall the following fundamental property of the fractional powers of a sectorial operator A (cf. [33, Sect.7]): For arbitrary $\alpha, \beta \in \mathbb{C}$ we have

$$A^\alpha A^\beta \subset A^{\alpha+\beta} \quad \text{and} \quad D(A^\alpha A^\beta) = D(A^\beta) \cap D(A^{\alpha+\beta}) \tag{1}$$

is a core for $A^{\alpha+\beta}$. In particular we have for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \geq 0$ that $A^\alpha A^\beta = A^{\alpha+\beta}$. For $\alpha \in \mathbb{R}$ we define the operator $\dot{A}_\alpha$ in $\dot{X}_\alpha$ by $\dot{A}_\alpha := (\widetilde{A^\alpha})^{-1} A \widetilde{A^\alpha}$. Then $\dot{A}_\alpha$ is similar to A and

$$\dot{A}_\alpha \supset A^{-\alpha} A A^\alpha \subset A$$

For $\alpha > 0$ we have $AA^\alpha = A^{1+\alpha} = A^\alpha A$, and for $\alpha < 0$ we have $A^{-\alpha}A = A^{1-\alpha}$, which by (1) leads to

$$D(A^{-\alpha} A A^\alpha) = \begin{cases} D(A^{1+\alpha}) & , \quad \alpha > 0 \\ D(A^\alpha) \cap D(A) \ , & \quad \alpha < 0 \end{cases} .$$

On this set we have coincidence of A and $\dot{A}_\alpha$. Moreover, this set is a core for both operators A and $\dot{A}_\alpha$.

Later on we shall need the following result on “shifting” in the scale $\dot{X}_\alpha$.

Proposition 2.1. *Let A be a sectorial operator in X and $\alpha \in \mathbb{R}$. Define $Y := \dot{X}_{\alpha,A}$ and $B := \dot{A}_\alpha$. Then we have, in a canonical way, $\dot{Y}_{\beta,B} = \dot{X}_{\alpha+\beta,A}$ and $\dot{B}_\beta = \dot{A}_{\alpha+\beta}$ for any $\beta \in \mathbb{R}$.*

Proof. We may assume $\alpha\beta \neq 0$. First we notice

$$B^\beta = (\dot{A}_\alpha)^\beta = (\widetilde{A^\alpha})^{-1} A^\beta \widetilde{A^\alpha}.$$

Using (1) the same argument as above shows that B^β and A^β coincide on

$$D := D(A^{-\alpha}A^\beta A^\alpha) = \begin{cases} D(A^{\alpha+\beta}) & , \quad \alpha\beta > 0 \\ D(A^\alpha) \cap D(A^\beta) \, , & \quad \alpha\beta < 0 \end{cases}$$

which is a core for B^β in Y and for A^β and $A^{\alpha+\beta}$ in X. Thus for $y \in D$ we have $B^\beta y = A^{-\alpha}A^\beta A^\alpha y \in D(A^\alpha)$ and

$$\|B^\beta y\|_Y = \|A^\alpha B^\beta y\|_X = \|A^\beta A^\alpha y\|_X = \|A^{\alpha+\beta} y\|_X$$

which completes the proof. □

Since we shall use interpolation in our scale $(\dot{X}_\alpha)$ we quote the following from [32]: For any $m \in \mathbb{N}$ the pair $(\dot{X}_m \cap \dot{X}_{-m}, \dot{X}_m + \dot{X}_{-m})$ is an interpolation couple. This is obtained by letting $Y_{(-m)} := (X, \|(A(1+A)^{-1})^m \cdot \|_X)^\sim$. It turns out that, in a natural way, $\dot{X}_m + \dot{X}_{-m} = Y_{(-m)}$ and $\dot{X}_m \cap \dot{X}_{-m} = D(A^m) \cap R(A^m)$. For fixed $m \in \mathbb{N}$ all spaces $\dot{X}_\alpha$, $|\alpha| \le m$, are intermediate spaces for the interpolation couple $(\dot{X}_m \cap \dot{X}_{-m}, \dot{X}_m + \dot{X}_{-m})$.

We recall that a sectorial operator A in a Banach space X is said to have **bounded imaginary powers** (or BIP for short) if $A^{it} \in B(X)$, $t \in \mathbb{R}$, and there are constants $c, \gamma > 0$ such that

$$\|A^{it}\| \le ce^{\gamma|t|}, \qquad t \in \mathbb{R}.$$

The infimum of all such γ is denoted $\omega_{BIP}(A)$. If A has BIP then, by (1), $D(A^\alpha) = D(A^{Re\,\alpha})$ and $\|A^\alpha \cdot \| \sim \|A^{Re\,\alpha} \cdot \|$ for all $\alpha \in \mathbb{C}$.

The following is obtained by a reproduction of the proof of [54, Thm.1.15.2].

Proposition 2.2. *Suppose that A is a sectorial operator in a Banach space X and that A has BIP. Then we have for the complex interpolation method*

$$[\dot{X}_\alpha, \dot{X}_\beta]_\theta = \dot{X}_{(1-\theta)\alpha+\theta\beta}$$

with equivalent norms for all $\alpha, \beta \in \mathbb{R}$ and all $\theta \in (0,1)$.

Proof. We may assume $\alpha \neq \beta$ and, by Proposition 2.1, even $\alpha = 0$, $\beta > 0$. We now reproduce the proof of [54, Thm.1.15.2] where we want to draw attention to the fact that Triebel assumes $0 \in \rho(A)$ which makes $(D(A^\beta), \|A^\beta \cdot \|)$ a Banach space and leads to $\dot{X}_\beta = D(A^\beta)$ in our notation. But the proof given there shows that, for $0 \in \sigma(A)$, one has to consider $(D(A^\beta), \|A^\beta \cdot \|)^\sim$, i.e. our space $\dot{X}_\beta$, in place of $D(A^\beta)$. □

A converse statement is known to hold in Hilbert space (cf. [5]), but for general Banach spaces this seems to be open.

3. Almost *R*–sectorial operators

In this section we recall some basic definitions related to R-boundedness and introduce almost R-sectoriality which will be a frequent assumption in some of our main results. We also note some basic duality and interpolation results connected with R-boundedness, which will be useful later.

We recall that a family $\mathcal{F} \subset B(X, Y)$ is called **Rademacher–bounded**, or R–bounded, with R–boundedness constant C if letting $(\varepsilon_k)_{k=1}^{\infty}$ be a sequence of independent Rademachers on some probability space, then for every $n \in \mathbb{N}$ and all choices $x_1, \ldots, x_n \in X$ and $T_1, \ldots, T_n \in \mathcal{F}$ we have

$$\mathbb{E}\left\|\sum \varepsilon_k T_k x_k\right\|_Y^2 \leq C^2 \mathbb{E}\left\|\sum \varepsilon_k x_k\right\|_X^2.$$

The smallest C in this inequality we denote by $R(\mathcal{F})$. For the basic properties of this notion we refer to [12], [30], and [17].

We recall that a sectorial operator A of type ω in a Banach space X is called R**-sectorial** of type $\omega \in [0, \pi)$ if, for any $\sigma > \omega$, the set $\tau_\sigma = \{\lambda R(\lambda, A) : |\arg(\lambda)| \geq \sigma\}$ is R-bounded. The infimum over all ω for which A is R-sectorial of type ω we denote by $\omega_R(A)$.

In connection with the H^∞–calculus, we will find it convenient to consider also a weaker version of R–sectoriality which was introduced in [31], [32]: A sectorial operator A of type ω is **almost *R*–sectorial** of type ω, if the sets $\tau_\sigma = \{\lambda A R(\lambda, A)^2 : |\arg(\lambda)| \geq \sigma\}$, $\sigma > \omega$, are R–bounded. $\omega_r(A)$ is again the infimum over all such ω.

Remark 3.1. Every R–sectorial operator is almost R–sectorial and $\omega_r(A) \leq \omega_R(A)$ (since $\lambda A R(\lambda, A)^2 = [AR(\lambda, A)][\lambda R(\lambda, A)]$). But there are examples of almost R–sectorial operators that are not R–sectorial even on $X = L_p$, $p \neq 2$ (see [32]).

It is known, that BIP implies R–sectoriality if X is a UMD–space (cf. [13]) and that a bounded H^∞–calculus implies R–sectoriality if X has property (Δ) (cf. [30]). For general X we have:

Proposition 3.2. *Let A be a sectorial operator with BIP and $\omega_{BIP}(A) < \pi$. Then A is almost R–sectorial with $\omega_r(A) \leq \omega_{BIP}(A)$.*

Proof. Since $\int_0^\infty s^{z-1} \frac{s}{(1+s)^2}\, ds = \frac{\pi z}{\sin(\pi z)}$ for $0 < \operatorname{Re} z < 1$ we obtain by the Mellin inversion formula (with $0 < c < 1$)

$$\begin{aligned}\frac{\mu}{(1+\mu)^2} &= \frac{1}{2\pi i} \int_{Re\, z = c} \frac{\pi z}{\sin(\pi z)} \mu^{-z}\, dz \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{t}{\sinh(\pi t)} \mu^{it}\, dt,\end{aligned}$$

first for $\mu > 0$ and then for $|\arg \mu| < \pi$ by analytic continuation. Choose $\omega, \nu > 0$ with $\omega + \omega_{BIP}(A) < \nu < \pi$. For λ with $|\arg \lambda| < \omega$ and $x \in D(A) \cap R(A)$ we have

$$\begin{aligned}\left[\frac{\lambda A}{(1+\lambda A)^2}\right] x &= \int_{\partial\Sigma_\nu} \frac{\mu}{(1+\mu)^2} R(\mu, \lambda A) x \, d\mu \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{t}{\sinh(\pi t)} \left(\int_{\partial\Sigma_\nu} \mu^{it} R(\mu, \lambda A) x \, d\mu \right) dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{t}{\sinh(\pi t)} \lambda^{it} A^{it} x \, dt \\ &= \int_{-\infty}^{\infty} h_\lambda(t) N(t) x \, dt,\end{aligned}$$

where $N(t) = e^{-(\nu-\omega)|t|} A^{it}$ has integrable norm on $\mathbb{R}$ and

$$h_\lambda(t) = \frac{1}{2} \frac{t}{\sinh(\pi t)} e^{(\nu-\omega)|t|} \lambda^{it}$$

is uniformly bounded in $t \in \mathbb{R}$ and $\lambda \in \Sigma_\omega$. It follows that $\{-\lambda A R(-\lambda, A)^2 : \lambda \in \Sigma_\omega\}$ is R-bounded and therefore $\omega_r(A) \le \pi - \omega$. □

This proposition shows in particular, that almost R-sectoriality is a necessary condition for the boundedness of the H^∞-calculus in any Banach space.

We denote by $H_0^\infty(\Sigma_\sigma)$ the space of all bounded analytic functions f on Σ_σ which, for some $C, \varepsilon > 0$, satisfy an estimate of the form $|f(z)| \le C \left(\frac{|z|}{1+|z|^2}\right)^\varepsilon$ with $\varepsilon > 0$.

Lemma 3.3. *Let A be an almost R-sectorial operator on X. If $\psi \in H_0^\infty(\Sigma_\sigma)$ with $\sigma > \omega_r(A)$, then $\{\psi(tA) : t > 0\}$ is R-bounded.*

Proof. (cf. [31, Lem.7.6]) Let $\Psi(z) := \int_0^z \frac{\psi(\lambda)}{\lambda} d\lambda$, $z \in \Sigma_\sigma$. Define $\varphi(\lambda) := \Psi(\lambda) - \gamma \frac{\lambda}{1+\lambda}$, where $\gamma = \int_0^\infty \frac{\psi(t)}{t} dt$. Then $\varphi'(\lambda) = \frac{\psi(\lambda)}{\lambda} - \gamma(1+\lambda)^{-2}$ and one can show that $\varphi \in H_0^\infty(\Sigma_\sigma)$. For $\sigma > \nu > \omega_r(A)$ we have

$$\varphi(tA) = \frac{1}{2\pi i} \int_{\partial\Sigma_\nu} \varphi(\lambda) R(\lambda, tA) \, d\lambda$$

and therefore

$$tA\varphi'(tA) = \frac{1}{2\pi i} \int \varphi(\lambda) [tAR(\lambda, tA)^2] \, d\lambda.$$

Hence the set

$$\{tA\varphi'(tA) : t > 0\} = \{\psi(tA) + \gamma tA(1+tA)^{-2} : t > 0\}$$

is R-bounded and the assertion follows. □

With this lemma the proof for the next result given in [33, Sect. 10] for sectoriality extends to (almost) R-sectoriality.

Proposition 3.4. *If A is sectorial (R–sectorial, almost R–sectorial) of type ω and $0 < \alpha < \dfrac{\pi}{\omega}$, then A^α is sectorial (R–sectorial, almost R–sectorial) of type $\alpha\omega$.*

Recall that a Banach space is B–convex (see [18]) if it has non–trivial type, i. e. there is a $p > 1$ and a $C > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$

$$\mathbb{E}\Big\| \sum_k \varepsilon_k x_k \Big\| \le C \left(\sum_k ||x_k||^p \right)^{1/p}. \tag{2}$$

For example L_p-spaces with $1 < p < \infty$ and their closed subspaces and quotient spaces are B–convex. Recall also that, by Kahane's inequality, we may take equivalently any norm $(\mathbb{E}\| \sum \varepsilon_k x_k \|^q)^{1/q}$, $1 \le q < \infty$, on the left hand side of (2).

For any Banach space X we define $\mathrm{Rad}(X)$ as the closure of $\{\sum_k r_k x_k : x_k \in X\}$ in $L_2([0,1], X)$ where (r_k) are the Rademacher functions given by $r_k(t) = \operatorname{sign}\sin(2^k \pi t)$, $t \in [0,1]$, $k \in \mathbb{N}$. Then R–boundedness of $\mathcal{F} \subset B(X,Y)$ means that there is a constant $C > 0$ such that, for every sequence $T_k \in \mathcal{F}$, the assignment

$$\sum_k r_k x_k \mapsto \sum_k r_k T_k x_k$$

defines a bounded operator from $\mathrm{Rad}(X)$ to $L_2([0,1], Y)$ with norm $\le C$.

If X is B–convex then, by [50] and [18, Ch.13], the subspace $\mathrm{Rad}(X)$ is complemented in $L_2([0,1], X)$ and $\mathrm{Rad}(X)$ is norming for $\mathrm{Rad}(X^*)$, where duality is given by

$$\left\langle \sum_j r_j x_j, \sum_k r_k x_k^* \rangle = \sum_j \langle x_j, x_j^* \right\rangle.$$

Duality and interpolation results for (almost) R-sectorial operators follow directly from the next two propositions.

Proposition 3.5. *Let $\mathcal{F}$ be a R–bounded subset of $B(X,Y)$. If X is B–convex, then $\{T^* : T \in \mathcal{F}\} \subset B(Y^*, X^*)$ is R–bounded.*

Proof. We consider a sequence (T_k) in $\mathcal{F}$ and the operator

$$\mathrm{Rad}(Y^*) \to \mathrm{Rad}(X^*), \sum_k r_k y_k^* \mapsto \sum_k r_k T_k^* y_k^*,$$

and use the fact that $\mathrm{Rad}(X)$ norms $\mathrm{Rad}(X^*)$ by B–convexity of X. Hence

$$\begin{aligned}\|\textstyle\sum_k r_k T_k^* y_k^*\|_{\mathrm{Rad}(X^*)} &\le C \sup\{|\textstyle\sum_k \langle x_k, T_k^* y_k^*\rangle| : \|\sum_k r_k x_k\|_{\mathrm{Rad}(X)} \le 1\}\\ &\le C' \|\textstyle\sum_k r_k y_k^*\|_{L_2([0,1],Y^*)} \sup\{\|\sum_k r_k T_k x_k\|_{L_2(Y)} : \|\sum_k r_k x_k\|_{\mathrm{Rad}(X)} \le 1\}\\ &\le C' R(\mathcal{F}) \|\textstyle\sum_k r_k y_k^*\|_{L_2([0,1],Y^*)}.\end{aligned}$$

□

Remark 3.6. A consequence of 3.5 is that $A^\sharp$ is R–sectorial (almost R–sectorial) in $X^\sharp$ if A is R–sectorial (almost R–sectorial) in X and X is B–convex.

Proposition 3.7. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples and $\mathcal{F}$ a family of operators from (X_0, X_1) to (Y_0, Y_1) such that $\mathcal{F}$ as a subset of $B(X_0, Y_0)$ and of $B(X_1, Y_1)$ is R–bounded. Let X_θ and Y_θ be interpolation spaces formed by the real or complex interpolation method.*

If X_0 and X_1 are B–convex, then $\mathcal{F}$ is R–bounded as a subset of $B(X_\theta, Y_\theta)$.

Proof. We clearly have $(L_2([0,1], Y_0), L_2([0,1], Y_1))_\theta = L_2([0,1], Y_\theta)$. We consider $\mathrm{Rad}(X_j)$, as complemented subspaces of $L_2([0,1], X_j)$ and use that then $(\mathrm{Rad}(X_0), \mathrm{Rad}(X_1))_\theta = \mathrm{Rad}(X_\theta)$ according to [54, 1.2.4]. The claim follows by considering sequences (T_k) in $\mathcal{F}$. □

We note the following version of the Stein interpolation theorem where we use the notation

$$\Sigma(\theta_0, \theta_1) := \{\lambda \in \mathbb{C} \setminus \{0\} : \theta_0 \le \arg\lambda \le \theta_1\}$$

for two angles $\theta_0 < \theta_1$.

Proposition 3.8. *Assume that (X_0, X_1) is an interpolation couple of B-convex Banach spaces and that $\theta_0 < \theta_1$. Let $N(\lambda)$, $\lambda \in \Sigma(\theta_0, \theta_1)$ be a family of linear maps $N(\lambda) : X_0 \cap X_1 \to X_0 + X_1$ such that, for all $x \in X_0 \cap X_1$, the function $\lambda \mapsto N(\lambda)x \in X_0 + X_1$ is continuous and bounded on $\Sigma(\theta_0, \theta_1)$ and analytic in the interior of $\Sigma(\theta_0, \theta_1)$. Assume, for $j = 0, 1$, that the functions $\mathbb{R}_+ \ni s \to N(se^{i\theta_j})x \in X_j$, $x \in X_0 \cap X_1$, are continuous and that*

$$\{N(te^{i\theta_j}) : t > 0\} \text{ is } R\text{-bounded in } B(X_j). \tag{3}$$

Then for every $\alpha \in (0,1)$ and $\theta = (1-\alpha)\theta_0 + \alpha\theta_1$ the set

$$\{N(te^{i\theta}) : t > 0\} \subset B([X_0, X_1]_\alpha)$$

is R-bounded where $[\cdot,\cdot]_\alpha$ means complex interpolation.

Proof. Put $U = \{\lambda \in \mathbb{C} : \theta_0 \le \operatorname{Re}\lambda \le \theta_1\}$ and let C be larger than $R\{N(te^{i\theta_0}) : t > 0\}$ in $B(X_0)$ and $R\{N(te^{i\theta_1}) : t > 0\}$ in $B(X_1)$. Fix $t_1, \dots, t_n > 0$, $\lambda \in U$, and define

$$\begin{aligned} M(\lambda) : \operatorname{Rad}(X_0 \cap X_1) &\to \operatorname{Rad}(X_0 + X_1) \\ \sum_k \varepsilon_k x_k &\mapsto \sum_{k=1}^n \varepsilon_k N(t_k e^{i\lambda}) x_k. \end{aligned}$$

By (3) we have that, for $j = 0, 1$, $\{M(\theta_j + is) : s > 0\}$ is bounded $\operatorname{Rad}(X_j) \to \operatorname{Rad}(X_j)$ with $\|M(\theta_j + is)\| \le C$. By the abstract Stein interpolation theorem we obtain that $M(\theta)$ is bounded with $\|M(\theta)\| \le C$ in $[\operatorname{Rad}(X_0), \operatorname{Rad}(X_1)]_\alpha$. Since X_0, X_1 are B-convex, this space equals $\operatorname{Rad}[X_0, X_1]_\alpha$. Hence we obtain

$$\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k N(t_k e^{i\theta}) x_k\right\|_{[X_0,X_1]_\alpha} \le C\mathbb{E}\left\|\sum_k \varepsilon_k x_k\right\|_{[X_0,X_1]_\alpha}$$

which implies the assertion. □

This yields the following corollary on (almost) R-sectorial operators in complex interpolation spaces. We assume the following situation, which we shall meet several times: (X_0, X_1) is an interpolation couple, the spaces $X_\theta := [X_0, X_1]_\theta$, $\theta \in (0, 1)$, are obtained by complex interpolation, and there is a family $(A_\theta)_{\theta\in[0,1]}$ of sectorial operators A_θ in X_θ satisfying the consistency condition

$$R(\lambda, A_\theta)x = R(\lambda, A_{\tilde\theta})x, \quad x \in X_\theta \cap X_{\tilde\theta}, \theta, \tilde\theta \in [0, 1].$$

By a connectedness argument the resolvents of A_θ and $A_{\tilde\theta}$ are then consistent on the largest sector contained in $\rho(A_\theta) \cap \rho(A_{\tilde\theta})$.

Corollary 3.9. *Let, in the situation described above,* (X_0, X_1) *be an interpolation couple of* B*-convex Banach spaces. For* $j = 0, 1$, *let* A_j, *be* R*-sectorial (almost* R*-sectorial) in* X_j *of type* ω_j. *Then, for* $\theta \in (0, 1)$, *the operator* A_θ *is* R*-sectorial (almost* R*-sectorial) in* X_θ *of type* $(1-\theta)\omega_0 + \theta\omega_1$.

4. Characterizations of the H^∞–calculus

In this section we recall the definition of the H^∞-calculus and give characterizations of the H^∞-calculus in terms of certain square functions that will be our main tool in proving perturbation theorems. We also remark on the best angle for the H^∞-calculus (e.g., in terms of almost R-sectoriality) and on interpolation of the H^∞-calculus.

Recall that

$$H_0^\infty(\Sigma_\sigma) := \{f \in H^\infty(\Sigma_\sigma) : |f(z)| \le c|z|^s(1+|z|)^{-2s} \text{ for some } c, s > 0\}.$$

For $f \in H_0^\infty(\Sigma_\sigma)$ and a sectorial operator A on X of type $\omega < \sigma$ we define

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\gamma} f(\lambda) R(\lambda, A) d\lambda \tag{4}$$

where $\gamma \in (\omega, \sigma)$. The map $H_0^\infty(\Sigma_\sigma) \ni f \mapsto f(A) \in B(X)$ is linear and multiplicative and $f(A) = AR(\mu_1, A)R(\mu_2, A)$ for $f(\lambda) = \lambda(\mu_1 - \lambda)^{-1}(\mu - \lambda)^{-1}$ with $|\arg \mu_j| > \sigma$, $j = 1, 2$. We say that A admits a **bounded $H^\infty(\Sigma_\sigma)$–calculus**, if this map has a continuous extension from $H^\infty(\Sigma_\sigma)$ to $B(X)$. The infimum over all σ for which A has an $H^\infty(\Sigma_\sigma)$–calculus is denoted by $\omega_H(A)$.

Put $\varphi_n(z) = \dfrac{n}{n+z} - \dfrac{1}{1+nz}$. Then A admits an $H^\infty(\Sigma_\sigma)$–calculus if and only if $\sup\limits_n \|(\varphi_n \cdot f)(A)\|_{B(X)} < \infty$ for all $f \in H^\infty(\Sigma_\sigma)$ and in this case

$$\begin{aligned} f(A)x &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial\Sigma_\sigma} \varphi_n(\lambda) f(\lambda) R(\lambda, A) x d\lambda, \quad \text{for } x \in X \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial\Sigma_\sigma} f(\lambda) R(\lambda, A) x_n d\lambda, \quad \text{for } x_n = \varphi_n(A)x \in D(A) \cap R(A). \end{aligned}$$

For the details of this construction, see [46], [15].

We will need the following characterizations of the H^∞–calculus.

Theorem 4.1. *Let A be an almost R-sectorial operator in X. For $0 \neq \psi \in H_0^\infty(\Sigma_\omega)$ with $\omega > \omega_r(A)$ and $a > 0$ put*

$$\|x\|_{\psi,A} = \sup_{t>0} \sup_N \mathbb{E} \Big\| \sum_{k=-N}^{N} \varepsilon_k \psi(ta^k A) x \Big\|_X$$

and

$$\|x^*\|_{\psi,A}^* = \sup_{t>0} \sup_N \mathbb{E} \Big\| \sum_{k=-N}^{N} \varepsilon_k \psi(ta^k A)^* x^* \Big\|_{X^*}$$

Then each of the following conditions is equivalent to A having an $H^\infty(\Sigma_\sigma)$-calculus for each $\sigma > \omega_r(A)$.

(i) There is a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $t \in [1, 2]$

$$\sup_{\varepsilon_k = \pm 1} \Big\| \sum_{k=-N}^{N} \varepsilon_k \psi(ta^k A) \Big\|_{B(X)} \le C.$$

(ii) There is a constant C such that for all $x \in X$, $x^* \in X^*$

$$\|x\|_{\psi,A} \le C\|x\|, \quad \|x^*\|_{\psi,A}^* \le C\|x^*\|.$$

(iii) There is a constant C such that for all $x \in X$

$$C^{-1}\|x\|_{\psi,A} \le \|x\| \le C\|x\|_{\psi,A}.$$

(iv) There is a constant C such that for all $x \in X$, $x^* \in X^*$

$$\sup_{N\in\mathbb{N}} \sup_{t\in[1,2]} \sum_{k=-N}^{N} |\langle x^*, \psi(ta^k A)x\rangle| \le C\|x\| \cdot \|x^*\|.$$

Remark 4.2. These conditions are motivated by the square function conditions in [46] and [15] in Hilbert and L_p–spaces. It is shown in [32] that, for an almost R-sectorial operator, one obtains a norm equivalent to $\|\cdot\|_{\psi,A}$ if one leaves out the sup over $t > 0$ (i.e., one puts $t = 1$). In [31] and [32] a general method is presented to generalize such square function estimates to the Banach space case. Condition (i) is from [30]. Condition (iv) is a discrete version of conditions that appeared in [15] and [11].

Remark 4.3. Typical functions ψ to which we will apply Theorem 4.1 are, e.g., $\psi^{\nu}_{m,s}(\lambda) = \lambda^s(e^{i\nu} - \lambda)^{-m}$ where $0 < s < m$, $m \in \mathbb{N}$, and $|\arg\nu| > \omega(A)$. Then

$$\|x\|_{\psi,A} = \sup_{t>0} \sup_{N} \mathbb{E}\Big\| \sum_{k=-N}^{N} \varepsilon_k (ta^k)^{m-s} A^s R(e^{i\nu} ta^k, A)^m x \Big\|.$$

It was shown in [30] that an operator with an H^∞–calculus is R–sectorial if X has property (Δ), and that in this case, $\omega_R(A)$ determines the H^∞–type (see [30]). Using the notion of almost R–sectoriality Theorem 4.1 improves this result:

Corollary 4.4. *If A has an H^∞-calculus in an arbitrary Banach space X then A is almost R-sectorial with $\omega_r(A) \le \omega_H(A)$ by Proposition 3.2, and Theorem 4.1 yields $\omega_H(A) = \omega_r(A)$ without additional assumptions on the Banach space X.*

We remark that the answer to the following question in [15]: "Let A be a sectorial operator with an H^∞–calculus. Do we always have $\omega_H(A) = \omega(A)$?" still seems to be open for L_p-spaces, although there are counterexamples in certain subspaces of L_p, $p \neq 2$ (see [32]).

The following estimates use a simple randomization technique but they will be used at various instants, including the proof of Theorem 4.1. We draw attention to the fact that, by the Khintchine–Kahane inequalities, the L_2–norms may be replaced by L_1–norms in a) and b) by the cost of an additional constant on the right hand side. We shall use this fact without further mentioning.

Lemma 4.5. *a) Let $\psi_k, \phi_k, M_k \in B(X)$, $k = 1, \dots, N$. For $x_k \in X$ and $x_k^* \in X^*$ we have*

$$\sum_{k=1}^{N} |\langle \psi_k M_k \phi_k x_k, x_k^*\rangle| \le 2R(M_k)\Big(\mathbb{E}\Big\| \sum_k \varepsilon_k \phi_k x_k \Big\|^2\Big)^{1/2}\Big(\mathbb{E}\Big\| \sum_k \varepsilon_k \psi_k^* x_k^* \Big\|^2\Big)^{1/2}.$$

b) For $a > 0$ and strongly measurable, locally bounded operator valued functions ψ, ϕ and M on $\mathbb{R}_+$ we have for $x \in X$ and $x^* \in X^*$

$$\lim_{R\to\infty} \int_{1/R}^{R} |\langle \psi(t)M(t)\phi(t)x, x^*\rangle| \frac{dt}{t}$$
$$\leq 2R\{M(\cdot)\} \sup_{t>0} \sup_{N} \Big(\mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \phi(a^k t)x \Big\|^2\Big)^{1/2}$$
$$\Big(\mathbb{E}\Big\| \sum_{|k|\leq N} \varepsilon_k \psi(a^k t)^* x^* \Big\|^2\Big)^{1/2}.$$

Proof. For the proof of a) we observe

$$\sum_{k=1}^{N} |\langle \psi_k M_k \phi_k x_k, x_k^*\rangle| \leq \sup_{|a_k|=1} \Big| \sum_{k=1}^{N} a_k \langle M_k \phi_k x_k, \psi_k^* x_k^*\rangle \Big|$$
$$= \sup_{|a_k|=1} \mathbb{E}\Big| \sum_{k=1}^{N} \varepsilon_k^2 \langle a_k M_k \phi_k x_k, \psi_k^* x_k^*\rangle \Big|$$
$$= \sup_{|a_k|=1} \mathbb{E}\Big| \langle \sum_k \varepsilon_k a_k M_k \phi_k x_k, \sum_k \varepsilon_k \psi_k^* x_k^*\rangle \Big|$$
$$\leq \sup_{|a_k|=1} \Big(\mathbb{E}\Big\| \sum_k \varepsilon_k a_k M_k \phi_k x_k \Big\|^2\Big)^{1/2} \Big(\mathbb{E}\Big\| \sum_k \varepsilon_k \psi_k^* x_k^* \Big\|^2\Big)^{1/2}.$$

Now we use the R–boundedness of $\{a_k M_k\}$. For the proof of b) observe

$$\int_{a^{-N}}^{a^{N+1}} |\langle \psi(t)M(t)\phi(t)x, x^*\rangle| \frac{dt}{t} = \sum_{k=-N}^{N} \int_{a^k}^{a^{k+1}} |\cdots| \frac{dt}{t}$$
$$\leq \sup_{t>0} \sum_{k=-N}^{N} |\langle \psi(a^k t)M(a^k t)\phi(a^k t)x, x^*\rangle| \cdot \log a$$

and apply part a). □

In the proof of Theorem 4.1 we shall also use the following proposition which is inspired by similar results for the classical square functions considered in [46] and [45].

Proposition 4.6. *Let A be an almost R-sectorial operator in a Banach space X. Let $\varphi, \psi \in H_0^\infty(\Sigma_\sigma) \setminus \{0\}$ for some $\sigma > \omega_r(A)$. Then there is a constant C such that for all $f \in H^\infty(\Sigma_\sigma)$ and all $x \in X$ we have*

$$\|f(A)x\|_{\psi,A} \leq C\|f\|_\infty \|x\|_{\varphi,A}.$$

In particular, for $f(\lambda) \equiv 1$*, we obtain*

$$C^{-1}\|x\|_{\psi,A} \le \|x\|_{\varphi,A} \le C\|x\|_{\psi,A}.$$

The proof will be based on the following lemma.

Lemma 4.7. *Let* $M, N : \mathbb{R}_+ \to B(X)$ *be strongly measurable, bounded functions and* $h \in L_1(\mathbb{R}_+, \frac{dt}{t})$*. If*

$$M(t) = \int_0^\infty h(ts)N(s)\frac{ds}{s}, \quad t > 0,$$

then for all $x \in X$

$$\sup_{t>0} \mathbb{E}\| \sum_{k\in\mathbb{Z}} \varepsilon_k M(2^k t)x\| \le C \sup_{t>0} \mathbb{E}\| \sum_{k\in\mathbb{Z}} \varepsilon_k N(2^k t)x\|,$$

where $C = 2\int_0^\infty |h(s)| \frac{ds}{s}$*.*

Proof. For every $k \in \mathbb{Z}$ and $t > 0$

$$M(t2^k)x = \sum_{j\in\mathbb{Z}} \int_1^2 h(t2^k s2^j)N(s2^j)x \, \frac{ds}{s}.$$

Hence

$$\begin{aligned}
\| \sum_k \varepsilon_k(\cdot)M(t2^k)x\| &\le \int_1^2 \| \sum_j \sum_k \varepsilon_k(\cdot)h(ts2^{k+j})N(s2^j)x\|\frac{ds}{s} \\
&\le \sum_l \int_1^2 \| \sum_j \varepsilon_{l-j}(\cdot)h(ts2^l)N(s2^j)x\|\frac{ds}{s},
\end{aligned}$$

and therefore by Kahane's contraction principle and Fubini

$$\begin{aligned}
&\mathbb{E}\| \sum_k \varepsilon_k M(t2^k)x\| \\
&\quad\le 2\sum_l \int_1^2 |h(ts2^l)| \frac{ds}{s}\left(\sup_{s\in[1,2]} \mathbb{E}\left\| \sum_j \varepsilon_{l-j} N(s2^j)x \right\| \right) \\
&\quad\le 2\int_0^\infty |h(ts)| \frac{ds}{s} \sup_{s\in[1,2]} \mathbb{E}\| \sum_j \varepsilon_j N(s2^j)x\|.
\end{aligned}$$

□

Proof of Proposition 4.6. We study the case $f \in H_0^\infty(\Sigma_\sigma)$ first. We choose auxiliary functions $g, h \in H_0^\infty(\Sigma_\sigma)$ such that

$$\int_0^\infty g(t)h(t)\varphi(t)\frac{dt}{t} = 1.$$

By analytic continuation we have for all $\lambda \in \Sigma_\sigma$ that

$$\int_0^\infty g(t\lambda)h(t\lambda)\varphi(t\lambda)\frac{dt}{t} = 1,$$

and obtain by the H_0^∞-calculus of A and some $\gamma \in (\omega_r(A), \sigma)$ with Fubini

$$\begin{aligned} f(A) &= \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma}\left(\int_0^\infty g(t\lambda)h(t\lambda)\varphi(t\lambda)\frac{dt}{t}\right) f(\lambda)R(\lambda, A)\,d\lambda \\ &= \int_0^\infty\left(\frac{1}{2\pi i}\int_{\partial\Sigma_\gamma} g(t\lambda)f(\lambda)h(t\lambda)\varphi(t\lambda)R(\lambda, A)\,d\lambda\right)\frac{dt}{t} \\ &= \int_0^\infty g(tA)h(tA)f(A)\varphi(tA)\frac{dt}{t}. \end{aligned}$$

By [30, Prop.4.2] we have

$$\begin{aligned} \psi(sA)g(tA) &= \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma}\psi(s\lambda)g(t\lambda)R(\lambda, A)\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma}\psi(s\lambda)g(t\lambda)\lambda^{1/2}A^{1/2}R(\lambda, A)\frac{d\lambda}{\lambda}. \end{aligned}$$

Hence for $s > 0$ we get by Fubini's theorem

$$\begin{aligned} f(A)\psi(sA)x &= \int_0^\infty[\psi(sA)g(tA)]h(tA)f(A)\varphi(tA)x\frac{dt}{t} \\ &= \int_0^\infty\Big[\frac{1}{2\pi i}\int_{\partial\Sigma_\gamma}\psi(s\lambda)g(t\lambda)\lambda^{1/2}A^{1/2}R(\lambda, A)\frac{d\lambda}{\lambda}\Big]h(tA)f(A)\varphi(tA)x\frac{dt}{t} \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\gamma}\psi(s\lambda)[\lambda^{1/2}A^{1/2}R(\lambda, A)]\left(\int_0^\infty g(t\lambda)f(A)h(tA)\varphi(tA)x\frac{dt}{t}\right)\frac{d\lambda}{\lambda} \end{aligned}$$

$$= \frac{1}{2\pi i} \int_{\partial\Sigma_\gamma} \psi(s\lambda) M(\lambda) x \frac{d\lambda}{\lambda}$$

where $M(\lambda) = \lambda^{1/2} A^{1/2} R(\lambda, A) N(\lambda)$ and $N(\lambda)x = \int_0^\infty g(t\lambda) f(A) h(tA) \varphi(tA) x \frac{dt}{t}$. Since $\{\lambda^{1/2} A^{1/2} R(\lambda, A) : \lambda \in \partial\Sigma_\gamma\}$ is R-bounded by Lemma 3.3, we conclude with Lemma 4.7 that

$$\begin{aligned} \|f(A)x\|_{\psi,A} &\leq C \sup_{\delta=\pm 1} \sup_{s>0} \mathbb{E} \| \sum_j \varepsilon_j M(e^{i\delta\gamma} s 2^j) x \| \\ &\leq C' \sup_{\delta=\pm 1} \sup_{s>0} \mathbb{E} \| \sum_j \varepsilon_j N(e^{i\delta\gamma} s 2^j) x \|. \end{aligned}$$

Now we want to apply Lemma 4.7 to $N(\lambda)$. To this end we have to show that $\{f(A)h(tA) : t > 0\}$ is R-bounded. Using again [30, Prop.4.2] we write

$$\begin{aligned} f(A)h(tA) &= \frac{1}{2\pi i} \int_{\partial\Sigma_\gamma} f(\lambda) h(t\lambda) R(\lambda, A)\, d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_\gamma} f(\lambda) h(t\lambda) \lambda^{1/2} A^{1/2} R(\lambda, A) \frac{d\lambda}{\lambda}. \end{aligned}$$

Since $\{\lambda^{1/2} A^{1/2} R(\lambda, A) : \lambda \in \partial\Sigma_\gamma\}$ is R-bounded by Lemma 3.3, and

$$\int_{\partial\Sigma_\gamma} |f(\lambda) h(t\lambda)| \, |\frac{d\lambda}{\lambda}| \leq \|f\|_{H^\infty(\Sigma_\gamma)} \int_{\partial\Sigma_\gamma} |h(\lambda)| \, |\frac{d\lambda}{\lambda}|,$$

we can apply the convex-hull lemma [12, Lem.3.2] and conclude that, indeed, $\{f(A)h(tA) : t > 0\}$ is R-bounded. Hence we obtain by Lemma 4.7 from the above that

$$\begin{aligned} \|f(A)x\|_{\psi,A} &\leq C' \sup_{\delta=\pm 1} \sup_{s>0} \mathbb{E} \| \sum_j \varepsilon_j N(e^{i\delta\gamma} s 2^j) x \| \\ &\leq C'' \|f\|_{H^\infty(\Sigma_\gamma)} \sup_{s>0} \mathbb{E} \| \sum_j \varphi(s 2^j A) x \| \\ &= C'' \|f\|_{H^\infty(\Sigma_\gamma)} \|x\|_{\varphi,A}. \end{aligned}$$

Of course, we can take $f(\lambda) \equiv 1$ and exchange φ and ψ in these arguments to obtain the equivalence of the square function norms. $\square$

Proof of Theorem 4.1. First we remark that condition (iv) is just a reformulation of (i). It was shown in [30, Lem.4.1] that (i) holds if A has an $H^\infty(\Sigma_\omega)$-calculus. The implication (i) $\Longrightarrow$ (ii) is clear.

(ii) $\Longrightarrow$ (iii): We prove this implication under the additional assumption $\int_0^\infty \psi^2(t)\,\frac{dt}{t} = c \neq 0$. By Proposition 4.6 it is clear that this restriction is not essential. We have $\int_0^\infty \psi^2(zt)\,\frac{dt}{t} = c$ for any $z > 0$, hence by analytic continuation also for any $z \in \Sigma_\omega$. The H_0^∞-calculus for A then yields, for any $x \in X$ and any $x^* \in X^*$,

$$\langle x, x^*\rangle = \int_0^\infty \langle c^{-1}\psi^2(tA)x, x^*\rangle \frac{dt}{t}.$$

By Lemma 4.5 (with $M(t) = I$) and condition (ii) we thus obtain

$$|\langle x, x^*\rangle| \le \frac{2}{|c|}\cdot \|x\|_{\psi,A}\|x^*\|^*_{\psi,A} \le \frac{2C}{|c|}\cdot \|x\|_{\psi,A}\cdot \|x^*\|.$$

Taking the sup over $x^* \in X^*$ with $\|x^*\| \le 1$ we obtain

$$\|x\| \le \frac{2C}{|c|}\cdot \|x\|_{\psi,A}, \quad x \in X,$$

and (iii) is proved.

Finally, Proposition 4.6 shows that (iii) implies boundedness of the $H^\infty(\Sigma_\sigma)$-calculus of A for any $\sigma > \omega_r(A)$. □

Remark 4.8. We remark here that equality $\omega_H(A) = \omega_r(A)$ in Theorem 4.1 may also be shown by an adaption of the arguments that proved [30, Prop.5.1]. Instead of Proposition 4.6 one would use functions $\psi(\lambda) = \psi_\alpha(e^{i\nu}\lambda)$ where $\psi_\alpha(\lambda) = \lambda^\alpha(1-\lambda)^{-1}$, $\alpha \in (0,1)$, $|\nu| > \omega_r(A)$. In order to change the angle ν we resort to the resolvent equation in the form

$$\begin{aligned} t^{1-\beta}A^\beta R(e^{i\nu}t, A) &= t^{1-\beta}A^\beta R(e^{i\sigma}t, A) \\ &\quad + [(e^{i\sigma} - e^{i\nu})t^{1-\alpha}A^\alpha R(e^{i\nu}t, A)]t^{1-\gamma}A^\gamma R(e^{i\sigma}t, A). \end{aligned}$$

where $\alpha, \beta \in (0,1)$ with $\beta > \alpha$ and $\gamma = \beta - \alpha$. By assumption and Lemma 3.3 the set of operators in $[\,\dots\,]$, $t > 0$, is R–bounded. This allows to switch the angle in estimating square function norms.

We end this section with an interpolation result. Again we assume that (X_0, X_1) is an interpolation couple, the spaces $X_\theta := [X_0, X_1]_\theta$, $\theta \in (0,1)$, are obtained by complex interpolation, and there is family $(A_\theta)_{\theta\in[0,1]}$ of sectorial operators A_θ in X_θ satisfying the consistency condition

$$R(\lambda, A_\theta)x = R(\lambda, A_{\tilde\theta})x, \quad x \in X_\theta \cap X_{\tilde\theta}, \theta, \tilde\theta \in [0,1].$$

Proposition 4.9. *Assume in the situation described above that, for $j = 0, 1$, the operator A_j has an $H^\infty(\Sigma_{\sigma_j})$-calculus on X_j. Then, for $\theta \in (0,1)$, A_θ has an $H^\infty(\Sigma_{\sigma_\theta})$-calculus on X_θ where $\sigma_\theta := (1-\theta)\sigma_0 + \theta\sigma_1$.*

Proof. We easily obtain

$$f(A_0)x = f(A_1)x \quad \text{for all } f \in H_0^\infty(\Sigma_{\max(\sigma_0,\sigma_1)}) \text{ and all } x \in X_0 \cap X_1.$$

Let $\sigma_{\max} := \max(\sigma_0, \sigma_1)$. By assumption we have bounded linear maps

$$\Phi_j : H^\infty(\Sigma_{\sigma_{\max}}) \to B(X_j).$$

satisfying $\Phi_0(f) = \Phi_1(f)$ on $X_0 \cap X_1$ for all $f \in H^\infty(\Sigma_{\sigma_{\max}})$. Hence, interpolation gives an $H^\infty(\Sigma_{\sigma_{\max}})$-calculus on X_θ. Moreover, the assumptions imply that we have $\|A^{it}\|_0 \le K_0 e^{i\sigma_0|t|}$ and $\|A^{it}\|_1 \le K_1 e^{i\sigma_1|t|}$. Hence we get $\|A^{it}\|_\theta \le K_\theta e^{i\sigma|t|}$. Now an application of [15, Thm.5.4] yields the result.

In case X_0 and X_1 are B–convex, one can avoid resorting to [15, Thm.5.4] and use Corollary 3.9 for the angle of almost R-sectoriality in X_θ and Remark 4.4. □

5. Comparison theorems

In this section we "compare" the fractional domain spaces of two sectorial operators in order to obtain criteria for the H^∞-functional calculus. We also show that equivalences $\|A^\alpha x\|_X \sim \|B^\alpha x\|_X$ for two R-sectorial operators on a reflexive Banach space X can be characterized by appropriate estimates $\|Ax\|_{\dot{X}_{\beta,A}} \sim \|Bx\|_{\dot{X}_{\beta,A}}$ if A has a bounded H^∞-calculus (see Theorem 5.7 for the precise statement). Note that in this section no "relative smallness" assumptions are involved.

Theorem 5.1. *Suppose that A admits an $H^\infty(\Sigma_\sigma)$-calculus on a Banach space X. Let B be almost R-sectorial with $\omega_r(B) \le \sigma$.*

Assume that for two different u_1 and u_2 with $0 < |u_j| < 3/2$ we have for $j = 1, 2$

$$D(A^{u_j}) = D(B^{u_j}) \text{ and } \frac{1}{C}\|A^{u_j}x\| \le \|B^{u_j}x\| \le C\|A^{u_j}x\| \text{ for } x \in D(A^{u_j}).$$

Then B admits an $H^\infty(\Sigma_\nu)$-calculus for $\nu > \sigma$.

If X is a Hilbert space and $u_1 = 1, u_2 = -1$, then we obtain [5, Thm.3.1] which has motivated the theorem above. We give a different proof using the following lemma which also gives some additional information.

Lemma 5.2. *Suppose that A_1 and A_2 admit an $H^\infty(\Sigma_\sigma)$ calculus on a Banach space X. Let B be almost R–sectorial with $\omega_r(B) \le \sigma$.*

Assume that there are $u_1, u_2, v_1, v_2 \in (0, \frac{3}{2})$ such that for some $C < \infty$

$$\|B^{u_1}x\| \le C\|A_1^{u_1}x\| \quad \text{for } x \in D(A_1^{u_1}) \subset D(B^{u_1}), \tag{5}$$
$$\|B^{-v_1}x\| \le C\|A_1^{-v_1}x\| \quad \text{for } x \in R(A_1^{v_1}) \subset R(B^{v_1}), \tag{6}$$
$$\|A_2^{u_2}x\| \le C\|B^{u_2}x\| \quad \text{for } x \in D(B^{u_2}) \subset D(A_2^{u_2}), \tag{7}$$
$$\|A_2^{-v_2}x\| \le C\|B^{-v_2}x\| \quad \text{for } x \in R(B^{v_2}) \subset R(A_2^{v_2}). \tag{8}$$

Then B admits an $H^\infty(\Sigma_\nu)$–calculus for all $\nu > \sigma$.

Proof. Put $A = A_1, u = u_1, v = v_1$. Denote by M and N the bounded extensions of $B^u A^{-u}$ and $B^{-v} A^v$ to X, respectively. We want to apply Theorem 4.1 and therefore we compare expressions of the form $\lambda^{n-u} T^u R(\lambda, T)^n$ for $T = A$ and $T = B$. We write

$$\begin{aligned} &R(\lambda, B)^3 = R(\lambda, B)^3 (\lambda - A)^2 R(\lambda, A)^2 \\ &= [\lambda^2 R(\lambda, B)^3] R(\lambda, A)^2 - 2[\lambda R(\lambda, B)^3] A R(\lambda, A)^2 + [R(\lambda, B)^3] A^2 R(\lambda, A)^2. \end{aligned}$$

For $\frac{3}{2} > s > u$ we get

$$\begin{aligned} \lambda^{3-s} B^s R(\lambda, B)^3 \; &= [\lambda^{3-s+u} B^{s-u} R(\lambda, B)^3 M] \{\lambda^{2-u} A^u R(\lambda, A)^2\} \\ &\quad -2\, [\lambda^{3-s} B^s R(\lambda, B)^3 \{\lambda A R(\lambda, A)^2\} \\ &\quad +[\lambda^{3-s-v} B^{s+v} R(\lambda, B)^3 N] \{\lambda^v A^{2-v} R(\lambda, A)^2\}. \end{aligned}$$

By almost R–sectoriality of B, Lemma 3.3, and boundedness of M and N, the three sets in $[\ \ldots\ ]$ are R–bounded for $|\arg(\lambda)| \geq \nu, \nu > \sigma$. Observe that, indeed, all exponents are strictly positive. Hence, for $\psi_{s,r}(\lambda) = \lambda^s (e^{i\nu} - \lambda)^{-r}$ we obtain for $x \in X, t > 0$

$$\mathbb{E}\Big\| \sum_k \varepsilon_k \psi_{s,3}(t^{-1} 2^{-k} B) x \Big\| \leq D \sum_{s=u,1,2-v} \mathbb{E}\Big\| \sum_k \varepsilon_k \psi_{s,2}(t^{-1} 2^{-k} A) x \Big\|. \tag{9}$$

By Theorem 4.1, the right hand side of (9) is $\leq D_1 \|x\|$ since $A = A_1$ has an $H^\infty(\Sigma_\sigma)$–calculus.

Taking now $u = u_2$, $v = v_2$ and replacing $(A, B) = (A_1, B)$ by (B, A_2) the arguments used so far show that

$$\mathbb{E}\Big\| \sum_k \varepsilon_k \psi_{s,3}(t^{-1} 2^{-k} A_2) x \Big\| \leq D_2 \sum_{s=u,1,2-v} \mathbb{E}\Big\| \sum_k \varepsilon_k \psi_{s,2}(t^{-1} 2^{-k} B) x \Big\|. \tag{10}$$

By Theorem 4.1, the left hand side of (10) is $\geq D_3 \|x\|$ since A_2 has an $H^\infty(\Sigma_\sigma)$–calculus. By Proposition 4.6 we may reduce the sum on the left hand side to one of its summands by the cost of a constant.

Now the claim follows from Theorem 4.1. □

Remark 5.3. a) If $0 \in \varrho(B) \cap \varrho(A_1)$ then the norm estimate in (5) already follows from the continuous inclusion $D(A_1^{u_1}) \subset D(B^{u_1})$, since $\|A_1^{u_1} x\|$ is equivalent to the graph norm $\|x\|_{A_1^{u_1}}$. The same remark applies to (7).

b) If B and A have commuting resolvents then (7) and (8) follow from (5) and (6) with $A_1 = A_2 = A$.

Proof of Theorem 5.1. We use Lemma 5.2 for $A_1 := A_2 := A$ and study three cases separately assuming $u_1 < u_2$. If $u_1 < 0 < u_2$ then assumption (5) holds for u_2 in place of u_1, (7) holds for u_2, and (6) and (8) hold for $v_1 = v_2 = -u_1$. The assertion follows.

If $0 < u_1 < u_2$ then we shift the scale (cf. Proposition 2.1) taking $Y := \dot{X}_{u_1,A} = \dot{X}_{u_1,B}$ as a new space. Then we use the previous case for $(Y, -u_1, u_2-u_1)$ in place of (X, u_1, u_2). This yields an H^∞-calculus for B in Y. But $\widetilde{B^{u_1}} : Y \to X$ is an isomorphism and we obtain an H^∞–calculus for B in X.

If $u_1 < u_2 < 0$ we apply the previous case to (A^{-1}, B^{-1}) in place of (A, B). □

Before the H^∞-calculus became more widely known, in many papers on differential operators B (e.g. [52]) it was only shown that B has BIP. Then it was observed later, that a refinement of the given argument also yields boundedness of the H^∞-functional calculus. This phenomenon is explained by the following corollary (where, in the situations mentioned above, A would usually be the Laplace operator and $\alpha = 1$).

Corollary 5.4. *Let A be a sectorial operator which has an H^∞-calculus on a Banach space X and let B be a sectorial operator which has BIP on X with $\omega_{BIP} < \pi$. If $D(A^\alpha) = D(B^\alpha)$ and $\|A^\alpha x\| \sim \|B^\alpha x\|$ for* some $\alpha \neq 0$, *then B has an H^∞-calculus.*

Proof. The assumptions imply $\dot{X}_{\alpha,A} = \dot{X}_{\alpha,B}$ with equivalent norms. For $\theta \in (0, 1)$ we then have by complex interpolation

$$\dot{X}_{\theta\alpha,B} = [X, \dot{X}_{\alpha,B}]_\theta = [X, \dot{X}_{\alpha,A}]_\theta = \dot{X}_{\theta\alpha,A}.$$

Hence $D(B^{\theta\alpha}) = \dot{X}_{\theta\alpha,B} \cap X = \dot{X}_{\theta\alpha,A} \cap X = D(A^{\theta\alpha})$ for $\theta \in (0, 1)$. Since B has BIP of angle $< \pi$, B is almost R-sectorial, and we apply Theorem 5.1. □

For reflexive Banach spaces, conditions (6) and (8) in Lemma 5.2 on negative powers may be reformulated as conditions on positive powers of the dual operators. This will follow from the next proposition, which – for later purposes – is formulated in greater generality.

Proposition 5.5. *Let X, Y be reflexive Banach spaces. Let A be an injective linear operator in X with dense domain and range, and let B be an injective linear operator in Y with dense domain and range. Assume that $P : X \to Y$ is a bounded linear operator such that $P^* : Y^* \to X^*$ is injective. Then the following two conditions are equivalent:*

(i) $P(R(A)) \subseteq R(B)$, $\|B^{-1}Px\| \leq C\|A^{-1}x\|$ *for* $x \in R(A)$, *and the continuous extension* $\widetilde{P} : \dot{X}_{-1,A} \to \dot{Y}_{-1,B}$ *of* P *is surjective;*

(ii) $P^*(D(B^*)) \subset D(A^*)$ *and* $C^{-1}\|B^*y^*\| \leq \|A^*P^*y^*\| \leq \|B^*y^*\|$ *for* $y^* \in D(B^*)$.

Proof. $(i) \Rightarrow (ii)$: Let $y^* = (B^*)^{-1}z^* \in D(B^*)$ and let $x = A^{-1}w \in D(A)$. Then we have

$$\begin{aligned} |\langle P^*y^*, Ax\rangle| &= |\langle y^*, Pw\rangle| = |\langle (B^*)^{-1}z^*, Pw\rangle| \\ &= |\langle z^*, B^{-1}Pw\rangle| \le \|z^*\|\|B^{-1}Pw\| \\ &\le C\|B^*y^*\|\|A^{-1}w\| = C\|B^*y^*\|\|x\|. \end{aligned}$$

Hence $P^*(D(B^*)) \subset D(A^*)$ and $\|A^*P^*y^*\| \le C\|B^*y^*\|$ for all $y^* \in D(B^*)$. It rests to show that $A^*P^*(B^*)^{-1}$ is bounded from below. This will follow from surjectivity of the operator $M := (A^*P^*(B^*)^{-1})^* : X \to Y$. By (i) the operator $\widetilde{P} : \dot{X}_{-1,A} \to \dot{Y}_{-1,B}$ is surjective. Since $(\widetilde{A^{-1}})^{-1} : X \to \dot{X}_{-1,A}$ and $\widetilde{B^{-1}} : \dot{Y}_{-1,B} \to Y$ are isomorphisms, the operator $K := \widetilde{B^{-1}}\widetilde{P}(\widetilde{A^{-1}})^{-1} : X \to Y$ is surjective. Now clearly $K \supset B^{-1}PA$ and $M \supset B^{-1}PA$. Moreover, $D(B^{-1}PA) = D(A)$ by (i), and we obtain $M = K$ by denseness of $D(A)$ in X. Hence $M : X \to Y$ is surjective, and $M^* \supset A^*P^*(B^*)^{-1}$ is from below.

$(ii) \Rightarrow (i)$: Let $x = Aw \in R(A)$. We have to show $Px \in R(B) = D(B^{-1})$ and shall use $B^{-1} = ((B^{-1})^*)^*$. So let $y^* = B^*z^* \in D((B^{-1})^*)$. Then by (ii) we have

$$\begin{aligned} |\langle Px, (B^{-1})^*y^*\rangle| &= |\langle x, P^*z^*\rangle| = |\langle Aw, P^*z^*\rangle| = |\langle w, A^*P^*z^*\rangle| \\ &\le \|w\|\|A^*P^*z^*\| \le C\|A^{-1}x\|\|B^*z^*\| = \|A^{-1}x\|\|y^*\|, \end{aligned}$$

which means $Px \in D(B^{-1})$ and $\|B^{-1}Px\| \le C\|A^{-1}x\|$. By (ii), the operator $A^*P^*(B^*)^{-1} : R(B^*) \to X^*$ has a (unique) continuous extension to an operator $M : Y^* \to X^*$ which is an isomorphism $Y^* \to M(Y^*)$. In particular, $M(Y^*)$ is a closed subspace of X^*, and by the closed graph theorem $M^*(X)$ is a closed subspace of Y. On the other hand, $M^*(X)$ is dense in Y since $M = (M^*)^*$ is injective. Hence $M^* : X \to Y$ is surjective. But then also $K := (\widetilde{B^{-1}})^{-1}M^*(\widetilde{A^{-1}}) : \dot{X}_{-1,A} \to \dot{Y}_{-1,B}$ is surjective. Now observe that $M^* \supset B^{-1}PA$ which yields $K \supset BM^*A^{-1} \supset BB^{-1}PAA^{-1} = P|_{R(A)}$. This in turn implies the remaining assertion in (i) with $\widetilde{P} = K$. □

Taking $X = Y$ and $P = I_X$ we obtain

Corollary 5.6. *Let A and B be sectorial operators in a reflexive space X. Then the following conditions are equivalent:*

(i) $D(A^{-1}) \subset D(B^{-1})$ and $\left\|B^{-1}x\right\| \le C\left\|A^{-1}x\right\|$ for $x \in R(A)$;

(ii) $D(B^*) \subset D(A^*)$ and $\|A^*x^*\| \le C\|B^*x^*\|$ for $x^* \in D(B^*)$.

Conditions (6) and (8) of Lemma 5.2 can be dualized in the same way.

The next theorem was motivated by a Hilbert space result of Yagi [56, Thm. 4.1, Lem. 4.2].

Theorem 5.7. *Let A and B be sectorial operators on a reflexive Banach space X and let A have an H^∞–calculus. Fix $-1 < \delta_0 < \delta_1 < 2$ such that $2 > \delta_1 - \delta_0 > 1$. If $1 \in (\delta_0, \delta_1)$, assume in addition that $D(A) = D(B)$ and $\|Ax\| \sim \|Bx\|$ for $x \in D(A)$. Then the following conditions are equivalent:*

(i) B is almost R–sectorial and for all $\delta \in (\delta_0, \delta_1)$ we have

$$D(A^\delta) = D(B^\delta), \quad \|A^\delta x\| \sim \|B^\delta x\| \quad \textit{for } x \in D(A^\delta).$$

(ii) For all $\delta_0 < \sigma < \delta_1 - 1$ we have that $B(D(B^\sigma B)) \subset D(A^\sigma)$, $D(B^\sigma B) \subset D(A^{1+\sigma})$, $B(D(B^\sigma B)) \cap D(B^\sigma B)$ is a dense subset of $D(A^\sigma) \cap D(A^{1+\sigma})$ for the norm $\|A^\sigma \cdot\| + \|A^{1+\sigma} \cdot\|$, and there is an almost R-sectorial operator $\tilde{B}$ in $\dot{X}_{\sigma,A}$ such that $R(\lambda, B)$ and $R(\lambda, \tilde{B})$ are consistent for $\lambda < 0$, $D(\tilde{B}) = D(\dot{A}_\sigma)$ and

$$\|\tilde{B}x\|_{\dot{X}_{\sigma,A}} \sim \|\dot{A}_\sigma x\|_{\dot{X}_{\sigma,A}} \quad \textit{for } x \in D(\dot{A}_\sigma). \tag{11}$$

(iii) For all $\delta_0 + 1 < \sigma < \delta_1$ we have that $D(B^{\sigma-1}B) \subset D(A^\sigma)$, $B(D(B^{\sigma-1}B) \subset D(A^{\sigma-1})$, $D(B^{\sigma-1}B) \cap B(D(B^{\sigma-1}B))$ is a dense subset of $D(A^\sigma) \cap D(A^{\sigma-1})$ for the norm $\|A^\sigma \cdot\| + \|A^{\sigma-1} \cdot\|$, and there is an almost R-sectorial operator $\tilde{B}$ in $\dot{X}_{\sigma,A}$ such that $R(\lambda, B)$ and $R(\lambda, \tilde{B})$ are consistent for $\lambda < 0$, $R(\tilde{B}) = R(\dot{A}_\sigma)$ and

$$\|\tilde{B}^{-1}x\|_{\dot{X}_{\sigma,A}} \sim \|(\dot{A}_\sigma)^{-1}x\|_{\dot{X}_{\sigma,A}} \quad \textit{for } x \in R(\dot{A}_\sigma).$$

Moreover, these conditions imply that also B has an H^∞-calculus in X and $\omega_H(B) = \max(\omega_H(A), \omega_R(B))$.

Proof. We shall prove $(i) \Longrightarrow (ii) \Longrightarrow (i)$. Having done this we shall prove $(i) \Longleftrightarrow (iii)$.

$(i) \Longrightarrow (ii)$: Assume (i) and let $\delta_0 < \sigma < \delta_1 - 1$. First we recall $D(B^\sigma B) = D(B^{1+\sigma}) \cap D(B)$. Then we observe

$$B(D(B^\sigma B)) \subset D(B^\sigma) = D(A^\sigma) \quad \text{and} \quad D(B^\sigma B) \subset D(B^{1+\sigma}) = D(A^{1+\sigma}).$$

Moreover, $D(B^\sigma B) \cap B(D(B^\sigma B))$ is a dense subset of $D(A^{1+\sigma}) \cap D(A^\sigma)$ for the norm $\|A^{1+\sigma} \cdot\| + \|A^\sigma \cdot\|$ since, by assumption, this norm is equivalent to the norm $\|B^{1+\sigma} \cdot\| + \|B^\sigma \cdot\|$.

By similarity, the operator $\dot{B}_\sigma$ is almost R-sectorial in $\dot{X}_{\sigma,B}$. The assumption implies that $\dot{X}_{\sigma,B} = \dot{X}_{\sigma,A}$ with equivalent norms. Moreover,

$$D(\dot{B}_\sigma) = (\dot{X}_{\sigma,B})^{\cdot}_{1,B} \cap \dot{X}_{\sigma,B} = \dot{X}_{1+\sigma,B} \cap \dot{X}_{\sigma,B} = \dot{X}_{1+\sigma,A} \cap \dot{X}_{\sigma,A} = D(\dot{A}_\sigma),$$

and for $x \in D(B^\sigma B) \cap D(B^\sigma) \subset D(A^{1+\sigma}) \cap D(A^\sigma)$ we have

$$\begin{aligned} \|\dot{B}_\sigma x\|_{\dot{X}_{\sigma,A}} &\sim \|\dot{B}_\sigma x\|_{\dot{X}_{\sigma,B}} = \|B^{\sigma+1}x\| \sim \|A^{\sigma+1}x\| \\ &= \|\dot{A}_\sigma x\|_{\dot{X}_{\sigma,A}} \end{aligned}$$

Hence we have obtained (ii) with $\tilde{B} = \dot{B}_\sigma$.

$(ii) \Longrightarrow (i)$: First we give the proof for $\beta_0 := -\delta_0 > 0, \alpha_0 := \delta_1$ and $\varrho = -\sigma$. Assume (ii) and fix $1 - \alpha_0 < \varrho < \beta_0$. First we observe that

$$\begin{aligned} D(\dot{A}_{-\varrho}) &= (\dot{X}_{-\varrho,A})^{\boldsymbol{\cdot}}_{1,\dot{A}_\varrho} \cap \dot{X}_{-\varrho,A} = \dot{X}_{1-\varrho,A} \cap \dot{X}_{-\varrho,A} \\ &= \dot{X}_{1-\varrho,A} \cap X \cap \dot{X}_{-\varrho,A} = D(A^{1-\varrho}) \cap R(A^\varrho). \end{aligned}$$

We conclude that $D(\tilde{B}) = D(A^{1-\varrho}) \cap R(A^\varrho)$ and that $\tilde{B}$ and B coincide on $D(B) \cap R(B)$ which is a core for both operators. Moreover, we have by (11) that $\tilde{B}(\dot{A}_\varrho)^{-1}$ extends uniquely to an isomorphism $\widetilde{L_\varrho} \in B(\dot{X}_{-\varrho,A})$. Hence $L_\varrho := \widetilde{A^{-\varrho}}\widetilde{L_\varrho}(\widetilde{A^{-\varrho}})^{-1}$ is an isomorphism in X. Clearly, L_ϱ is an extension of $A^{-\varrho}BA^{\varrho-1}$, and we have by assumption

$$\begin{aligned} D(A^{-\varrho}BA^{\varrho-1}) &= A^{1-\varrho}(D(B)), \\ R(A^{-\varrho}BA^{\varrho-1}) &\subset R(A^{-\varrho}) = D(A^\varrho). \end{aligned}$$

Now $D(B)$ is dense in $D(A^{1-\varrho})$ for the graph norm. Hence $D(B)$ is dense in $\dot{X}_{1-\varrho}$ and $A^{1-\varrho}(D(B))$ is dense in X. This implies that the operator L_ϱ is the unique extension of $A^{-\varrho}BA^{\varrho-1}$ to an element of $B(X)$. Moreover, $L_\varrho^{-1} \in B(X)$ is an extension of $A^{1-\varrho}B^{-1}A^\varrho$.

We now recall (cf. [15]) that, for $\phi \in H_0^\infty(\Sigma_\omega)$ with $\omega > \omega(B)$, and $y \in D(B) \cap R(B)$, we have

$$y = c^{-1} \int_0^\infty \phi(tB)y \, \frac{dt}{t}, \tag{12}$$

if $c = \int_0^\infty \phi(t)\, \frac{dt}{t} \neq 0$. In the sequel, we shall only use functions $\phi \in \bigcap_{\omega\in(0,\pi)} H_0^\infty(\Sigma_\omega)$ and write $\phi \in H_0^\infty$ for short. In order to exploit (12) we need suitable representations of expressions $B^s R(\lambda, B)^m$ which we shall use for $\lambda = -t < 0$. Multiplying $B^s R(\lambda, B)^m$ from the left and from the right with

$$I_X = \lambda^2 R(\lambda, A)^2 - 2\lambda A R(\lambda, A)^2 + A^2 R(\lambda, A)^2$$

we obtain

$$B^s R(\lambda, B)^m = \sum_{j,k=}^{2} c_{jk} \lambda^{2-j} R(\lambda, A)^2 B^s R(\lambda, B)^m \lambda^{2-k} A^k R(\lambda, A)^2 \tag{13}$$

for suitable integers c_{jk}. Using (12) we write, for $B^\varrho x \in D(B) \cap R(B) \subset R(A^\varrho)$ and s, m to be specified later,

$$A^{-\varrho} B^\varrho x = c^{-1} \int_0^\infty t^{m-s} A^{-\varrho} B^{\varrho+s} R(-t, B)^m x \, \frac{dt}{t} \tag{14}$$

and use (13) for $B^{\varrho+s}R(\lambda, B)^m$ for $\lambda = -t < 0$. We shall obtain a representation

$$t^{m-s}A^{-\varrho}B^{\varrho+s}R(-t, B)^m x = \sum_{j,k=0}^{2} c_{jk}\varphi_j(t^{-1}A)M_{jk}(t)\psi_k(t^{-1}A)x, \qquad (15)$$

where, for $j, k = 0, 1, 2$, the functions φ_j, ψ_k are in H_0^∞ and the set $\{M_{jk}(t) : t > 0\}$ is R-bounded in $B(X)$. We fix a $\tilde{\varrho} \in (\varrho, \beta_0)$. The idea is to have (15) with

$$M_{jk}(t) = S_{jk}\, \widetilde{A^{-\tilde{\varrho}}}\, t^{m-s_{jk}}\, \tilde{B}^{s_{jk}} R(-t, \tilde{B})^m\, (\widetilde{A^{-\tilde{\varrho}}})^{-1}\, T_{jk} \qquad (16)$$

and $S_{jk}, T_{jk} \in B(X)$, $s_{jk} \in (0, m)$. Then almost R-sectoriality of $\tilde{B}$ in $\dot{X}_{-\varrho,A}$ will imply that the sets $\{M_{jk}(t) : t > 0\} \subset B(X)$ are R-bounded.

In the following we examine separately terms $A^{-\varrho}A^jR(\lambda, A)^2$, which appear on the left, and terms $A^kR(\lambda, A)^2$, which appear on the right of summands in (13). For $j = 0$ we have

$$\lambda^2 A^{-\varrho}R(\lambda, A)^2|_{D(B)} = \lambda^{2+\varrho-\tilde{\varrho}}A^{\tilde{\varrho}-\varrho}R(\lambda, A)^2A^{-\tilde{\varrho}}|_{D(B)}\lambda^{\tilde{\varrho}-\varrho}.$$

For $j = 1$ we have

$$\lambda A^{-\varrho}AR(\lambda, A)^2|_{D(B)} = \lambda^{1+\varrho-\tilde{\varrho}}A^{1+\tilde{\varrho}-\varrho}R(\lambda, A)^2A^{-\tilde{\varrho}}|_{D(B)}\lambda^{\tilde{\varrho}-\varrho}.$$

For $j = 2$ we have

$$\begin{aligned} A^{-\varrho}A^2R(\lambda, A)^2|_{D(B)} &= \lambda^{1+\varrho-\tilde{\varrho}}A^{1+\tilde{\varrho}-\varrho}R(\lambda, A)^2A^{1-\tilde{\varrho}}B^{-1}A^{\tilde{\varrho}}A^{-\tilde{\varrho}}B\lambda^{\tilde{\varrho}-\varrho-1} \\ &= \lambda^{1+\varrho-\tilde{\varrho}}A^{1+\tilde{\varrho}-\varrho}R(\lambda, A)^2L_{\tilde{\varrho}}^{-1}A^{-\tilde{\varrho}}B\lambda^{\tilde{\varrho}-\varrho-1} \end{aligned}$$

For $k = 1, 2$ we obtain

$$\lambda^{2-k}A^kR(\lambda, A)^2 = \lambda^{-\tilde{\varrho}}A^{\tilde{\varrho}}\lambda^{2-k+\tilde{\varrho}}A^{k-\tilde{\varrho}}R(\lambda, A)^2.$$

For $k = 0$ we have

$$I_{D(B)}R(\lambda, A)^2 = B^{-1}A^{\tilde{\varrho}}L_{\tilde{\varrho}}A^{1-\tilde{\varrho}}R(\lambda, A)^2.$$

This means that we have (16) with $S_{jk} = \delta_{j2}(L_{\tilde{\varrho}})^{-1}$, $T_{jk} = \delta_{k0}L_{\tilde{\varrho}}$ and $s_{jk} = s + \varrho + \delta_{j2} - \delta_{k0}$. Thus if we choose s, m such that $s + \varrho \in (1, 2)$ and $m \geq 3$ then the sets $\{M_{jk}(-t) : t > 0\}$ are R-bounded in $B(X)$, and we have (15) with

$$\begin{aligned} &\varphi_0(z) = -z^{\tilde{\varrho}-\varrho}(1+z)^{-2},\ \varphi_1(z) = \varphi_2(z) = z\varphi_0(z), \\ &\psi_0(z) = \psi_1(z) = -z^{1-\tilde{\varrho}}(1+z)^{-2},\ \psi_2(z) = z\psi_1(z). \end{aligned}$$

Now we apply an $x^* \in X^*$ to the integral in (14), use the representation of the integrand above, and obtain by Lemma 4.5 that $|\langle x^*, A^{-\varrho}B^{\varrho}x\rangle|$ can be estimated from above by

$$\begin{aligned} 2\sum_{j,k=0}^{2} |c_{jk}|\, R\{M_{jk}(t) : t < 0\} \quad &\sup_{t>0} \mathbb{E}\left(\left\|\sum_\nu \varepsilon_\nu \psi_k(t^{-1}2^{-\nu}A)x\right\|^2\right)^{1/2} \\ &\sup_{t>0} \mathbb{E}\left(\left\|\sum_\nu \varepsilon_\nu \varphi_j(t^{-1}2^{-\nu}A)^*x^*\right\|^2\right)^{1/2}. \end{aligned}$$

Since A has an H^∞-calculus we can use Theorem 4.1 and conclude that

$$\|A^{-\varrho}x\| \le C\|B^{-\varrho}x\| \text{ for } x \in R(B^\varrho) \subset R(A^\varrho), \varrho \in (1-\alpha_0, \beta_0). \tag{17}$$

Together with the assumption this implies, for $\varrho \in (1-\alpha_0, \beta_0)$, that

$$\|A^{1-\varrho}B^{\varrho-1}x\| = \|[A^{1-\varrho}B^{-1}A^\varrho][A^{-\varrho}B^\varrho]x\| \le C\,\|L_\varrho\|\,\|x\|.$$

Hence we have shown that, for $\varrho \in (1-\alpha_0, \beta_0)$,

$$\|A^{1-\varrho}x\| \le C\|B^{1-\varrho}x\| \text{ for } x \in D(B^{1-\varrho}) \subset D(A^{1-\varrho}). \tag{18}$$

We now estimate, for $(B^*)^{1-\varrho}x^* \in R(B^*) \cap D(B^*)$ and $x \in D(A) \cap R(A)$,

$$\langle (B^*)^{1-\varrho}x^*, A^{\varrho-1}x\rangle = c^{-1}\int_0^\infty t^{m-s}\langle (B^{1-\varrho+s}R(-t,B)^m)x^*, A^{\varrho-1}x\rangle\,\frac{dt}{t}.$$

Again we use (13) and aim at a representation

$$t^{m-s}B^{1-\varrho+s}R(-t,B)^m A^{\varrho-1}x = \sum_{j,k=0}^{2} c_{jk}\varphi_j(t^{-1}A)M_{jk}(t)\psi_k(t^{-1}A)x, \tag{19}$$

where $\varphi_j, \psi_k \in H_0^\infty$ and the sets $\{M_{jk}(t) : t > 0\} \subset B(X)$ are R-bounded. To this end we choose $\tilde\varrho \in (1-\alpha_0, \varrho)$ and check again for a representation with M_{jk} as in (16) with $S_{jk}, T_{jk} \in B(X)$, $s_{jk} \in (0,m)$.

We proceed as before. For $j = 0, 1$ we have

$$\lambda^{2-j}A^jR(\lambda,A)^2|_{D(B)} = \lambda^{2-j-\tilde\varrho}A^{j+\tilde\varrho}R(\lambda,A)^2A^{-\tilde\varrho}|_{D(B)}\lambda^{\tilde\varrho}.$$

For $j = 2$ we have

$$\begin{aligned} A^2R(\lambda,A)^2|_{D(B)} &= \lambda^{1-\tilde\varrho}A^{1+\tilde\varrho}R(\lambda,A)^2A^{1-\tilde\varrho}B^{-1}A^{\tilde\varrho}A^{-\tilde\varrho}B\lambda^{\tilde\varrho-1} \\ &= \lambda^{1-\tilde\varrho}A^{1+\tilde\varrho}R(\lambda,A)^2(L_{\tilde\varrho})^{-1}A^{-\tilde\varrho}B\lambda^{\tilde\varrho-1}. \end{aligned}$$

For $k = 2$ we have

$$A^2R(\lambda,A)^2A^{\varrho-1}x = \lambda^{\varrho-\tilde\varrho-1}A^{\tilde\varrho}\lambda^{1-\varrho+\tilde\varrho}A^{1+\varrho-\tilde\varrho}R(\lambda,A)^2x.$$

For $k = 0, 1$ we have

$$\lambda^{2-k}A^kR(\lambda,A)^2A^{\varrho-1}x = \lambda^{\varrho-\tilde\varrho}B^{-1}A^{\tilde\varrho}L_{\tilde\varrho}\lambda^{2-k-\varrho+\tilde\varrho}A^{k+\varrho-\tilde\varrho}R(\lambda,A)^2x.$$

This leads to (19) for

$$\begin{gathered} \varphi_0 = -z^{\tilde\varrho}(1+z)^{-2},\ \varphi_1(z) = \varphi_2(z) = z\varphi_0(z) \\ \psi_0(z) = -z^{\varrho-\tilde\varrho}(1+z)^{-2},\ \psi_1(z) = \psi_2(z) = z\psi_0(z), \end{gathered}$$

where M_{jk} satisfies (16) with

$$\begin{gathered} S_{jk} = \delta_{j2}(L_{\tilde\varrho})^{-1},\ T_{jk} = (1-\delta_{k2})L_{\tilde\varrho}, \\ s_{jk} = 1-\varrho+s+\delta_{j2}-1-\delta_{k2} = s-\varrho+\delta_{j2}-\delta_{k2}. \end{gathered}$$

We choose s, m such that $s-\varrho \in (1,2)$ and $m \geq 3$, use reflexivity of X, and thus have proved that, for $\varrho \in (1-\alpha_0, \beta_0)$,

$$\|B^{1-\varrho}x\| \leq C\|A^{1-\varrho}x\| \quad \text{for } x \in D(A^{1-\varrho}) \subset D(B^{1-\varrho}). \tag{20}$$

Moreover, this leads to

$$\|B^{-\varrho}A^{\varrho}x\| = \|[B^{1-\varrho}A^{\varrho-1}][A^{1-\varrho}B^{-1}A^{\varrho}]x\| \leq C\,\|(L_{\varrho})^{-1}\|\|x\|,$$

which means that we also have proved that, for $\varrho \in (1-\alpha_0, \beta_0)$,

$$\|B^{-\varrho}x\| \leq C\|A^{-\varrho}x\| \quad \text{for all } x \in R(A^{\varrho}) \subset R(B^{\varrho}). \tag{21}$$

Now (18), (21), (17) and (20) imply that $\|B^{\alpha}x\| \sim \|A^{\alpha}x\|$ for $1-\beta_0 < \alpha < \alpha_0$ and $\|B^{-\beta}x\| \sim \|A^{-\beta}x\|$ for $1-\alpha_0 < \beta < \beta_0$. By

$$BR(\lambda,B)^2 = [B^{-\varrho}A^{+\varrho}][A^{-\varrho}BR(\lambda,B)^2A^{\varrho}][A^{-\varrho}B^{\varrho}].$$

the operator B is almost R-sectorial in X and thus has an H^{∞}–calculus by Lemma 5.2. In particular, both operators A and B have BIP on X.Thus the remaining cases $0 < \alpha \leq 1-\beta_0$ and $0 < \beta \leq 1-\alpha_0$ follow by complex interpolation (cf. Proposition 2.2).

Now we give the proof of $(ii) \implies (i)$ for $\delta_0 \geq 0$. So assume that (ii) holds with $\delta_0 \geq 0$. In this case we have $\delta_1 > 1$ and $Y := \dot{X}_{1,A} = \dot{X}_{1,B}$ with equivalent norms. Hence we use what we have just proved, replacing the space X by Y, and taking $\beta_0 = -\delta_0' := -(\delta_0 - 1)$, $\alpha_0 = \delta_1' := \delta_1 - 1$. We only have to make sure that condition (ii) holds for Y and the operators $\dot{A}_1$ and $\dot{B}_1$ and (δ_0', δ_1') in place of X, A, B, and (δ_0, δ_1). So let $\sigma' \in (\delta_0', \delta_1' - 1)$. Then $\sigma' < 0$, $\varrho := -\sigma' > 0$, and $1-\varrho, 2-\varrho \in (\delta_0, \delta_1)$. We also let $\sigma := 1+\sigma' = 1-\varrho$. We have

$$D((\dot{B}_1)^{\sigma}\dot{B}_1) = D(\dot{B}_1) = (\dot{X}_{1,B})^{\cdot}_{1,\dot{B}_1} \cap \dot{X}_{1,B} = \dot{X}_{2,B} \cap \dot{X}_{1,B}$$

and

$$R(\dot{B}_1) = (\dot{X}_{1,B})^{\cdot}_{-1,\dot{B}_1} \cap \dot{X}_{1,B} = X \cap \dot{X}_{1,B} = D(B).$$

This yields $D(\dot{B}_1) \cap R(\dot{B}_1) = \dot{X}_{2,B} \cap \dot{X}_{1,B} \cap X = D(B^2)$. We also have

$$D((\dot{A}_1)^{1-\varrho}) = \dot{X}_{2-\varrho,A} \cap \dot{X}_{1,A}, \quad D((\dot{A}_1)^{-\varrho}) = \dot{X}_{1-\varrho,A} \cap \dot{X}_{1,A},$$

and on the natural domain the norm $\|(\dot{A}_1)^{1-\varrho}\cdot\|_{\dot{X}_{1,A}} + \|(\dot{A}_1)^{-\varrho}\cdot\|_{\dot{X}_{1,A}}$ is the same as $\|A^{1+\sigma}\cdot\| + \|A^{\sigma}\cdot\|$. By assumption we have

$$\begin{aligned} &R(\dot{B}_1) = D(B) \subset \dot{X}_{1,B} = \dot{X}_{1,A} \quad \text{and} \\ &R(\dot{B}_1) = D(B) = D(A) \subset D(A^{\sigma}) \subset \dot{X}_{1-\varrho,A}, \end{aligned}$$

which yields $R(\dot B_1) \subset D((\dot A_1)^{-\varrho})$. We turn to the inlcusion $D(\dot B_1) \subset D((\dot A_1)^{1-\varrho})$, and observe that by assumption

$$D(\dot B_1) \cap X = D(B^2) \subset D(B) \subset \dot X_{1,B} = \dot X_{1,A} \quad \text{and}$$
$$D(\dot B_1) \cap X = D(B^2) \subset D(B^{\sigma+1}) \subset D(A^{\sigma+1}) \subset \dot X_{2-\varrho,A}.$$

To conclude we need a norm estimate. We start from

$$D(B) = D(A) \subset D(A^\sigma) \subset \dot X_{1-\varrho,A},$$

which implies $\|A^\sigma \cdot\| \le C(\|B\cdot\| + \|\cdot\|)$ and thus $\|A^\sigma B\cdot\| \le C(\|B^2\cdot\| + \|B\cdot\|)$. But by assumption we have $\|A^{\sigma+1}\cdot\| \sim \|A^\sigma B\cdot\|$. Hence we conclude that $\|A^{\sigma+1}\cdot\| \le C'(\|B^2\cdot\| + \|B\cdot\|)$ and

$$D(\dot B_1) = \dot X_{2,B} \cap \dot X_{1,B} \subset \dot X_{2-\varrho,A}.$$

Since $D(\dot B_1) \subset \dot X_{1,B} = \dot X_{1,A}$, we have proved $D(\dot B_1) \subset \dot X_{2-\varrho,A} \cap \dot X_{1,A} = D((\dot A_1)^{1-\varrho})$. The assumptions imply that $D(\dot B_1) \cap R(\dot B_1) = D(B^2)$ is dense in $D(A^\sigma) \cap D(A^{1+\sigma})$ for the norm $\|A^\sigma\cdot\| + \|A^{\sigma+1}\cdot\|$. Thus $D(B^2)$ is also dense in

$$D((\dot A_1)^{-\varrho}) \cap D((\dot A_1)^{1-\varrho}) = \dot X_{1-\varrho,A} \cap \dot X_{2-\varrho,A},$$

which means that we have checked the first part of (ii) for Y, $\dot A_1$, $\dot B_1$, δ_0', δ_1' in place of X, A, B, δ_0, δ_1.

Now $\tilde B$ is by assumption an almost R-sectorial operator in $\dot X_{1-\varrho,A} = (\dot X_{1,A})^{\cdot}_{-\varrho,\dot A_1}$, and $D(\tilde B) = D(\dot A_{1-\varrho}) = \dot X_{2-\varrho,A} \cap \dot X_{1-\varrho,A} = D((\dot A_1)^{\cdot}_{-\varrho})$. Finally we obtain (11) by

$$\|\tilde B x\|_{\dot X_{1-\varrho,A}} = \|\tilde B x\|_{\dot X_{1-\varrho,B}} = \|B^{2-\varrho}x\| \sim \|A^{2-\varrho}x\| = \|\dot A_{1-\varrho}x\|_{\dot X_{1-\varrho,A}}.$$

The proof given above thus shows (i) for (δ_0', δ_1'), $(\dot A_1, \dot B_1)$ and Y in place of (δ_0, δ_1), (A, B) and X, and $\dot B_1$ has an H^∞-calculus in Y. Using again $Y = \dot X_{1,A} = \dot X_{1,B}$ we obtain (i) for (δ_0, δ_1) in the space X. By similarity B is almost R-sectorial in X and has an H^∞-calculus in X.

$(i) \Longleftrightarrow (iii)$: Again we distinguish the cases $0 \in (\delta_0, \delta_1)$ and $1 \in (\delta_0, \delta_1)$. If $0 \in (\delta_0, \delta_1)$ then 1) remains true for $(A^{-1}, B^{-1}, -\delta_1, -\delta_0)$ in place of $(A, B, \delta_0, \delta_1)$, and (iii) is just a reformulation of (ii) for that situation. Since equivalence of (i) and (ii) is already proved, equivalence of (i) and (iii) follows.

If $1 \in (\delta_0, \delta_1)$, then we let again $Y := \dot X_{1,A} = \dot X_{1,B}$ and take a detour via condition (ii) and condition (iii) for $(Y, \dot A_1, \dot B_1, \delta_0 - 1, \delta_1 - 1)$ in place of $(X, A, B, \delta_0, \delta_1)$. □

Remark 5.8. a) Another version of Theorem 5.7 is obtained when we replace “almost R-sectoriality” of B or $\tilde B$ in conditions (i), (ii), and (iii) above by “R-sectoriality” of B and $\tilde B$, respectively, thus obtaining conditions (i'), (ii'), and (iii'). Then (i'), (ii'), (iii') are equivalent. Moreover, for $0 \in (\delta_0, \delta_1)$ and $\beta_0 := -\delta_0$, $\alpha_0 := \delta_1$, these conditions are also equivalent to the following condition (iv').

(iv') For all $1-\alpha_0 < \varrho < \beta_0$ we have $R(B) \subset R(A^\varrho)$, $D(B)$ is a dense subset of $D(A^{1-\varrho})$ with respect to the graph norm, and the operators

$$L := A^{-\varrho} B A^{\varrho-1}, \qquad N(\lambda) := A^{1-\varrho} R(\lambda, B) A^\varrho, \ \lambda < 0,$$

extend to bounded operators $\tilde{L}, \tilde{N}(\lambda) \in B(X)$ such that the set $\{\tilde{N}(\lambda) : \lambda < 0\}$ is R–bounded.

The proof can be done $(i') \Longrightarrow (ii') \Longrightarrow (iv') \Longrightarrow (i')$, the arguments for the equivalence of (iii') being the same as before. The longest part is again the proof of $(iv') \Longrightarrow (i')$, but one may use simpler representations of expressions such as, e.g., $A^\varrho B^{-\varrho} x$.

b) Yagi's original Hilbert space result [56, Thm. 4.1] has $-\beta_0 = \delta_0 < 0 < \delta_1 = \alpha_0$ and states that, in a Hibert space X, condition (iv') above is equivalent to

(v) B is sectorial and for all $0 < \alpha < \alpha_0$ and all $0 < \beta < \beta_0$ we have

$$\begin{aligned} D(A^\alpha) = D(B^\alpha), &\quad \|A^\alpha x\| \sim \|B^\alpha x\| \quad \text{for } x \in D(A^\alpha) \\ R(A^\beta) = R(B^\beta), &\quad \|A^{-\beta} x\| \sim \|B^\beta x\| \quad \text{for } x \in R(A^\beta). \end{aligned}$$

Since X is reflexive, the condition on the ranges in (v) is by Corollary 5.6 equivalent to: For all $0 < \beta < \beta_0$ we have $D((A^*)^\beta) = D((B^*)^\beta)$ and $\|(A^*)^\beta x\| \sim \|(B^*)^\beta x\|$ for $x \in D((A^*)^\beta)$.

c) Notice that, in contrast to Yagi's result, we do not assume that the operators are boundedly invertible, we also allow for an interval around $\delta = 1$, and that conditions (ii) and (iii) in Theorem 5.7 are new.

Remark 5.9. a) Let $\rho \in [0, 1]$. Let $X_{1-\rho} = X_{1-\rho,A}$ denote $D(A^{1-\rho})$ equipped with the graph norm and $X_{-\rho}$ denote the completion of X with respect to the norm $\|R(\lambda, A)^\rho \cdot \|_X$ where $\lambda \in \rho(A)$. If X is reflexive, then $X_{-\rho} = ((X^*)_{\rho,A^*})^*$. In any space X we have $X_{1-\rho} \hookrightarrow \dot{X}_{1-\rho}$ and $\dot{X}_{-\rho} \hookrightarrow X_{-\rho}$, with equality if $0 \in \rho(A)$.

Assume the situation we are given in (iv') in the remark above, i.e., X is reflexive, $R(B) \subset R(A^\rho)$ and $D(B)$ is a dense subset of $D(A^{1-\rho})$ with respect to the graph norm. Then the following conditions are equivalent:

$(*)$ $L := A^{-\rho} B A^{\rho-1}$ has a bounded extension $\tilde{L} : X \to X$.
$(**)$ B has a bounded extension $\tilde{B} : \dot{X}_{1-\rho} \to \dot{X}_{-\rho}$.
$(***)$ B has a bounded extension $\widehat{B} : X_{1-\rho} \to X_{-\rho}$ and there is a constant C such that

$$|\langle \widehat{B} x, x^* \rangle| \le C \|A^{1-\rho} x\| \|(A^*)^\rho x^*\|, \quad x \in D(A^{1-\rho}), x^* \in D((A^*)^\rho).$$

By the denseness assumption on $D(B)$ all extensions of B above are unique. The relation of $\tilde{L}$ and $\tilde{B}$ is given by $\tilde{B} = (\widetilde{A^{-\rho}})^{-1} \tilde{L} \widetilde{A^{1-\rho}}$.

b) The condition $(***)$ in a) above has the form of the assumption used in [40, Thm.14]. In this context, we want to remark that a linear operator B in a reflexive space X satisfying that $D(B)$ is dense in $D(A^{1-\rho})$ for the graph norm and, for some $C > 0$,

$$|\langle Bx, x^*\rangle| \le C\|A^{1-\rho}x\|\|(A^*)^\rho x^*\|, \quad x \in D(B), x^* \in D((A^*)^\rho),$$

actually satisfies $R(B) \subset \dot{X}_{-\rho} \cap X = D(A^{-\rho}) = R(A^\rho)$.

6. Perturbation theorems for the H^∞-calculus

It is known ([47]) that $\|Bx\| \le C\|Ax\|$ for small enough C does not imply that $A + B$ has an H^∞-calculus if A has an H^∞-calculus. What kind of conditions can we add so that the perturbation theorem holds?

First we show that it is enough to add a second relative estimate $\|A^{\alpha-1}Bx\| \le C\|A^\alpha x\|$ for $\alpha \ne 1$. Then we consider perturbation theorems where only such an estimate with $\alpha \in (0, 1)$ and small C is assumed.

Theorem 6.1. *Let A be an R-sectorial operator in a Banach space X with $\omega_R(A) < \sigma$ and assume that A has an $H^\infty(\Sigma_\sigma)$-calculus. Let $\delta \in (0, 1)$ and suppose that B is a linear operator in X satisfying $D(B) \supset D(A)$, $R(B) \subset R(A^\delta)$ and*

$$\|Bx\| \le C_0\|Ax\|, \quad x \in D(A),$$
$$\|A^{-\delta}Bx\| \le C_1\|A^{1-\delta}x\|, \quad x \in D(A),$$

where $C_0, C_1 < R_0^{-1}$ where $R_0 := R\big(\{AR(\lambda, A) : |\arg\lambda| \ge \sigma\}\big)$.

Then $A+B$ is R–sectorial with $\omega_R(A+B) \le \sigma$ and $A+B$ has an H^∞-calculus with $\omega_H(A + B) \le \sigma$.

Proof. Using the smallness of C_0 we obtain by [40, Thm.1] that $A + B$ is R–sectorial in X with $\omega_R(A + B) \le \sigma$ where

$$R(\lambda, A + B) = R(\lambda, A)\sum_{k=0}^{\infty}(BR(\lambda, A))^k, \quad |\arg\lambda| > \sigma.$$

We shall apply Theorem 4.1 and check condition (iv). Observe that $A^{-\delta}BA^{\delta-1}$ extends to a bounded operator L on X with norm $\|L\| < R_0^{-1}$. For $|\arg\lambda| > \sigma$, we now write

$$\begin{aligned} R(\lambda, A)BR(\lambda, A) &= A^\delta R(\lambda, A)A^{-\delta}BA^{\delta-1}A^{1-\delta}R(\lambda, A) \\ &= A^\delta R(\lambda, A)LA^{1-\delta}R(\lambda, A), \end{aligned}$$

and define $M(\lambda) := \sum_{k=0}^{\infty}(LAR(\lambda, A))^k$. The series converges absolutely, and $\{M(\lambda) : |\arg\lambda| > \sigma\}$ is R-bounded. We obtain

$$R(\lambda, A + B) = R(\lambda, A) + A^\delta R(\lambda, A)M(\lambda)LA^{1-\delta}R(\lambda, A), \quad |\arg\lambda| > \sigma.$$

We now use the following lemma for $A + B$ in place of B to finish the proof. $\square$

Lemma 6.2. *Let A be a linear operator in a Banach space X and assume that A has an $H^\infty(\Sigma_\sigma)$-calculus. Suppose that B is another linear operator in X satisfying, for some $\varrho \in (0,1)$ and some $\omega > \sigma$, a representation of the form*

$$R(\lambda, B) = R(\lambda, A) + A^\varrho R(\lambda, A) M(\lambda) L A^{1-\varrho} R(\lambda, A), \quad |\arg \lambda| \geq \omega, \tag{22}$$

where the set $\{M(\lambda) : |\arg \lambda| \geq \omega\}$ is R-bounded. Then B has an H^∞-calculus in X and $\omega_H(B) \leq \omega$.

Proof. For $|\nu| > \omega$ we define $\psi_\nu \in H_0^\infty(\Sigma_\sigma)$ by $\psi_\nu^2(z) := (e^{i\nu} - z)^{-1} - 2(2e^{i\nu} - z)^{-1} = z(e^{i\nu} - z)^{-1}(2e^{i\nu} - z)^{-1}$. With $\varphi_\varrho(z) := z^\varrho (e^{i\nu} - z)^{-1}$ we then have by Lemma 4.5

$$\begin{aligned}
&\sup_{\varepsilon_k = \pm 1} \sum_k |\langle x^*, \psi_\nu^2(t2^k(B))x\rangle| \\
&\leq \sup_{\varepsilon_k = \pm 1} \sum_k |\langle \psi_\nu(t2^k A)^* x^*, \psi_\nu(t2^k A)x\rangle| \\
&\quad + \sup_{\varepsilon_k = \pm 1} \sum_k |\langle \varphi_\delta(t2^k A)^* x^*, M(t2^k) L \varphi_{1-\delta}(t2^k A)x\rangle| \\
&\quad + \sup_{\varepsilon_k = \pm 1} \sum_k |\langle \varphi_\delta(t2^{k+1} A)^* x^*, M(t2^{k+1}) L \varphi_{1-\delta}(t2^{k+1} A)x\rangle| \\
&\leq C\|x\|_{\psi_\nu, A} \|x^*\|_{\psi_\nu, A}^* + C\|x\|_{\varphi_{1-\delta}, A} \|x^*\|_{\varphi_\delta, A}^*.
\end{aligned}$$

Now the claim follows from Theorem 4.1 since A has an H^∞-calculus. □

Using interpolation we may weaken the assumption on smallness of the constant C_1 in the previous theorem.

Corollary 6.3. *Assume that the assumptions of Theorem 6.1 hold where C_0 is sufficiently small and $C_1 < \infty$ is arbitrary. Then $A + B$ has an H^∞-calculus in X.*

Proof. The operator A has BIP in X. We use Proposition 2.2. If C_0 is sufficiently small and δ_1 is sufficiently close to 0, the norm of $B : \dot{X}_{1-\delta_1} \to \dot{X}_{-\delta_1}$ (or, equivalently, the norm in $B(X)$ of the bounded extension of $A^{-\delta_1} B A^{\delta_1 - 1}$) is less than the constant R_0^{-1} in Theorem 6.1. Thus we may apply Theorem 6.1 for δ_1 in place of δ. □

The usual corollary on relative A-small perturbations reads as follows.

Corollary 6.4. *Let A be an R-sectorial operator in a Banach space X which has an H^∞-calculus. Let $\delta \in (0,1)$ and assume that B is a linear operator in X satisfying $D(B) \supset D(A)$, $R(B) \subset R(A^\delta)$ and*

$$\begin{aligned}
\|Bx\| &\leq a\|Ax\| + b\|x\|, \quad x \in D(A), \\
\|Bx\|_{X_{-\delta}} &\leq C_1 \|x\|_{X_{1-\delta}}, \quad x \in D(A),
\end{aligned}$$

where $a > 0$ is sufficiently small and $b, C_1 > 0$ are arbitrary. Then $A + B + \nu$ is R-sectorial and has an H^∞-calculus for $\nu > 0$ sufficiently large.

Proof. We have, for R_0 as in Theorem 6.1 and any $\nu > 0$,

$$\|B(\nu + A)^{-1}\| \le (a + b/\nu)(R_0 + 1)\|x\|, \quad x \in X.$$

If a is sufficiently small, then choosing $\nu > 0$ sufficiently large, the first estimate in Theorem 6.1 holds for B and $\nu + A$ in place of A with a constant $C_0 < R_0^{-1}$, i.e. a constant having the required smallness. Since $0 \in \rho(\nu + A))$ we have $X_{\beta,A} = \dot{X}_{\beta,\nu+A}$ for any $\beta \in \mathbb{R}$. This means that the second estimate in Theorem 6.1 holds with a modified constant $\tilde{C}_1$. Now we can apply Corollary 6.3 for $\nu + A$ in place of A. □

As a consequence of Corollary 6.3 we obtain also the perturbation theorem in [16] which we had obtained independently with our method.

Corollary 6.5. *Let A be an R-sectorial operator in a Banach space X and assume that A has an $H^\infty(\Sigma_\sigma)$-calculus. Let $\delta \in (0, 1)$ and suppose that B is a linear operator in X satisfying $D(B) \supset D(A)$, $B(D(A^{1+\delta})) \subset D(A^\delta)$ and*

$$\|Bx\| \le C_0\|Ax\|, \quad x \in D(A),$$
$$\|A^\delta Bx\| \le C_1\|A^{1+\delta}x\|, \quad x \in D(A^{1+\delta})$$

where $C_0, C_1 > 0$ and C_0 is sufficiently small. Then $A + B$ is R–sectorial and $A + B$ has an H^∞-calculus with $\omega_H(A + B) \le \sigma$.

Proof. By changing δ if necessary (cf. the proof of Corollary 6.3) we may assume that also C_1 is sufficiently small. In the following the spaces with no index are contructed with respect to the operator A. We apply Theorem 6.1 for $\dot{A}_\delta$ and B in $\dot{X}_\delta$. Then $\dot{A}_\delta + B$ is R-sectorial and has an H^∞-calculus in $\dot{X}_\delta$. We observe

$$(\dot{X}_\delta)^{\cdot}_{1,\dot{A}_\delta+B} = (\dot{X}_\delta)^{\cdot}_{1,\dot{A}_\delta} = \dot{X}_{1+\delta}$$

with equivalent norms where we have used Proposition 2.1. By similarity, $\dot{A}_\delta + B$ thus has an H^∞-calculus in $\dot{X}_{1+\delta}$. By interpolation we obtain an H^∞-calculus for $A + B$ in $\dot{X}_1$. Since we have $\dot{X}_{1,A+B} = \dot{X}_1$ with equivalent norms, we obtain again by similarity an H^∞-calculus for $A + B$ in X. □

We now present a result for perturbations $B : \dot{X}_\alpha \to \dot{X}_{\alpha-1}$ in the fractional scale. As explained in [40] (where perturbation theorems for R-sectorial operators were considered) such perturbation theorems can be considered as abstract versions of form perturbation theorems in Hilbert space such as the KLMN-Theorem. As already mentioned in Remark 5.9, if X is reflexive, the boundedness of

$B : \dot{X}_\alpha \to \dot{X}_{\alpha-1}$ is equivalent to the boundedness of $\widehat{B} := B|_{D(A^\alpha)} : X_\alpha \to X_{\alpha-1}$ and the existence of a constant $C > 0$ such that

$$|\langle \widehat{B}x, x^*\rangle| \le C\|A^\alpha x\|\|(A^*)^{1-\alpha}x^*\|, \quad x \in D(A^\alpha), x^* \in D((A^*)^{1-\alpha}).$$

In the reflexive situation, this type of assumptions has been used in [40]. Observe that, in contrast to the situation in 5.9 c), B takes its values not in X but in the larger space $X_{\alpha-1}$ which means that the part of B in X may have trivial domain $\{0\}$. A case where this actually happens is the perturbation of boundary conditions (cf. Section 9).

It is somehow surprising that, in the fractional scale, a *single* smallness condition is sufficient to obtain the H^∞-calculus for the perturbed operator.

Theorem 6.6. *Let A be an R-sectorial operator in a Banach space X that has an H^∞-calculus. Assume that $0 < \alpha < 1$ and that $B : X_\alpha \to X_{\alpha-1}$ is a linear operator satisfying,*

$$\|B\|_{\dot{X}_\alpha \to \dot{X}_{\alpha-1}} \le \eta$$

where $\eta < (R_0)^{-1}$ and R_0 is as in Theorem 6.1. Then there is a unique sectorial operator C in X whose resolvents are consistent with those of $\dot{A}_{\alpha-1} + B$ in $\dot{X}_{\alpha-1}$. Moreover, the operator C is R-sectorial and has an H^∞-calculus with $\omega_R(C) = \omega_H(C) \le \omega_H(A) = \omega_R(A)$.

Proof. We first derive a representation of the resolvent of the perturbed operator which is different from the one in the proof of [40, Thm.14].

Let $\omega > \omega_R(A) = \omega_H(A)$. For $\lambda \in \Sigma_{\pi-\omega}$ we let $S_\lambda := (\lambda + A)^{-1}$ and $T_\lambda := (\lambda + \dot{A}_{\alpha-1})^{-1}$. We recall the isometries $\widetilde{A^\alpha} : \dot{X}_\alpha \to X$ and $(\widetilde{A^{\alpha-1}})^{-1} : X \to \dot{X}_{\alpha-1}$. Using these and R-sectoriality of A we see that the sets $\{\lambda^\alpha T_\lambda : \lambda \in \Sigma_{\pi-\omega}\}$, $\{\lambda^{1-\alpha} S_\lambda : \lambda \in \Sigma_{\pi-\omega}\}$, and $\{T_\lambda : \lambda \in \Sigma_{\pi-\omega}\}$ are R-bounded in $B(\dot{X}_{\alpha-1}, X)$, $B(X, \dot{X}_\alpha)$, and $B(\dot{X}_{\alpha-1}, \dot{X}_\alpha)$, respectively. Moreover, the R-bound of the last set is $\le R_0$.

Now we define, for $\lambda \in \Sigma_{\pi-\omega}$,

$$R_\lambda := S_\lambda - T_\lambda \sum_{k=0}^{\infty} (-BT_\lambda)^k BS_\lambda,$$

and put $V_\lambda := \sum_{k=0}^{\infty} (-BT_\lambda)^k$. The series is absolutely convergent, since the R-bound of $\{BT_\lambda : \lambda \in \Sigma_{\pi-\omega}\}$ in $B(\dot{X}_{\alpha-1})$ is < 1 by assumption. This also implies that the set $\{V_\lambda : \lambda \in \Sigma_{\pi-\omega}\}$ is R-bounded in $B(\dot{X}_{\alpha-1})$. Writing, for $\lambda \in \Sigma_{\pi-\omega}$,

$$\lambda R_\lambda = \lambda S_\lambda - \lambda^\alpha T_\lambda V_\lambda B \lambda^{1-\alpha} S_\lambda,$$

we conclude that $\{\lambda R_\lambda : \lambda \in \Sigma_{\pi-\omega}\}$ is R-bounded in $B(X)$. Clearly, we have consistency of $R_\lambda \in B(X)$ with $\widetilde{R_\lambda} \in B(\dot{X}_{\alpha-1})$ given by

$$\widetilde{R_\lambda} = T_\lambda - T_\lambda V_\lambda B T_\lambda, \quad \lambda \in \Sigma_{\pi-\omega}.$$

Now it is not hard to check that we have, for $\lambda \in \Sigma_{\pi-\omega}$,

$$\widetilde{R_\lambda} = (\lambda + \dot{A}_{\alpha-1} + B)^{-1}.$$

We conclude that R_λ is a pseudo-resolvent in X. If $x \in X$ satisfies $R_\lambda x = 0$ then $S_\lambda x = T_\lambda V_\lambda B S_\lambda x$, and $V_\lambda^{-1} x = B S_\lambda x$, which in turn implies $x = 0$. Hence R_λ is the resolvent of a closed linear operator C in X. Observe that we have $\lambda^{1-\alpha} S_\lambda x \to 0$ $(\lambda \to \infty)$ in $\dot{X}_\alpha$ for any $x \in X$, since $\lambda^{1-\alpha} A^\alpha (\lambda + A)^{-1} x \to 0$ in X for $x \in D(A^\alpha)$ and $D(A^\alpha)$ is dense in X. We conclude that $\lambda R_\lambda x \to x$ $(\lambda \to \infty)$ for any $x \in X$ which implies that C is densely defined in X.

Observe now that, as in the usual perturbation situation, we have $D(\dot{A}_{\alpha-1} + B) = D(\dot{A}_{\alpha-1}) = R(A^{1-\alpha}) \cap D(A^\alpha)$. Hence the range of $(\dot{A}_{\alpha-1} + B)\widetilde{R_\lambda}^2$ is dense in $R(A^{1-\alpha}) \cap D(A^\alpha)$ for the norm $\|A^{\alpha-1} \cdot \| + \|A^\alpha \cdot \|$. We conclude that the superset $R(C R_\lambda)$ is dense in X for the original norm. Thus C has dense range.

We have shown that C is a sectorial operator in X. The uniqueness assertion is clear, since $\dot{X}_{\alpha-1} \cap X = R(A^{1-\alpha})$ is dense in X. Again, we recall the isometries $\widetilde{A^\alpha} : \dot{X}_\alpha \to X$ and $\widetilde{A^{\alpha-1}} : \dot{X}_{\alpha-1} \to X$. By assumption the operator $L := \widetilde{A^{\alpha-1}} B (\widetilde{A^\alpha})^{-1}$ belongs to $B(X)$ and has norm $\le \eta$. We observe that $T_\lambda (\widetilde{A^{\alpha-1}})^{-1} = A^{1-\alpha} S_\lambda$ and $\widetilde{A^{\alpha-1}} B T_\lambda (\widetilde{A^{\alpha-1}})^{-1} = LAS_\lambda$ and obtain

$$R_\lambda = S_\lambda - A^{1-\alpha} S_\lambda \sum_{k=0}^{\infty} (-LAS_\lambda)^k LA^\alpha S_\lambda, \quad \lambda \in \Sigma_{\pi-\omega}.$$

Since $\{LAS_\lambda : \lambda \in \Sigma_{\pi-\omega}\}$ is R-bounded with R-bound < 1, we can apply Lemma 6.2 and conclude that C has an H^∞-calculus in X. □

We note that, for any $\alpha \le 1$, the operator A can be extended in a canonical way to an operator $A_{\alpha-1}$ in $X_{\alpha-1}$ with domain $D(A_{\alpha-1}) = X_\alpha$.

Corollary 6.7. *Let A be a sectorial operator in a Banach space X which has an H^∞-calculus. Assume that $0 < \alpha < 1$ and that $B : X_\alpha \to X_{\alpha-1}$ is a linear operator such that*

$$\liminf_{\nu \to \infty} \|(\nu + A_{-1})^{\alpha-1} B (\nu + A)^{-\alpha}\|_{B(X)}$$

is sufficiently small. Then the conclusion of Theorem 6.6 holds for $\nu + A$ in place of A and ν sufficiently large.

Remark 6.8. If X is reflexive, the condition on B is satisfied if

$$|\langle Bx, x^*\rangle| \le a(\|A^\alpha x\| + b\|x\|)(\|(A^*)^{1-\alpha} x^*\| + b\|x^*\|),$$
$$x \in D(A^\alpha),\ x^* \in D((A^*)^{1-\alpha}),$$

where $a > 0$ is sufficiently small and $b \ge 0$ is arbitrary.

The following yields in particular the persistence of the H^∞-calculus under perturbations of "lower order" in the fractional scale. The case $\beta = 1$ is, of course, well known (see, e.g., [2]).

Corollary 6.9. *Let A be a sectorial operator in a Banach space X which has an H^∞-calculus. Assume that $0 \le \alpha < \beta \le 1$ and that $B : X_\alpha \to X_{\beta-1}$ is a bounded linear operator. Then, for $\alpha \le \gamma \le \beta$, and $\lambda > 0$ sufficiently large, the part of $(\lambda + A)_{\gamma-1} + B$ in X has an H^∞-calculus.*

Proof. We consider the norms $\|(1+A)^\alpha \cdot \|$ and $\|(1+A_{-1})^{\beta-1} \cdot \|$ on X_α and $X_{\beta-1}$, respectively. The assumption means that $K := (1+A_{-1})^{\beta-1}B(1+A)^{-\alpha} \in B(X)$. For $\nu > 0$ and $\gamma \in [\alpha, \beta]$ we then have

$$(\nu + A_{-1})^{\gamma-1}B(\nu + A)^{-\gamma} = [(1 + A_{-1})^{1-\beta}(\nu+A_{-1})^{\gamma-1}]K[(1+A)^\alpha(\nu+A)^{-\gamma}].$$

Both terms in square brackets are uniformly bounded in $B(X)$ for $\nu > 0$. Since $\alpha < \gamma$ or $\gamma < \beta$, one term is square brackets tends to 0 in $B(X)$ as $\nu \to \infty$. Hence we apply Theorem 6.6 for $\nu + A$ in place of A and ν sufficiently large. □

We close this section with the following proposition on H^∞–calculi for translated operators.

Proposition 6.10. *Suppose A is sectorial on X. If A has an $H^\infty(\Sigma_\sigma)$-calculus then $A + cI$ has an $H^\infty(\Sigma_\sigma)$-calculus for all $c > 0$. Conversely, if $A + c_0 I$ has an H^∞-calculus for some $c_0 > 0$ then $A + cI$ has an H^∞-calculus for all $c > 0$.*

Proof. For $f \in H_0^\infty(\Sigma_\sigma)$:

$$\begin{aligned}
&\int_\Gamma f(\lambda)R(\lambda, A + c)\, d\lambda - \int_\Gamma f(\lambda)R(\lambda, A)\, d\lambda \\
&\quad = c \int f(\lambda)R(\lambda, A)R(\lambda, A + c)\, d\lambda \\
&\quad = c \int f(\lambda)R(\lambda, A)R(0, A + c)\, d\lambda \\
&\qquad + c \int f(\lambda)[R(\lambda, A + c) - R(0, A + c)]R(\lambda, A)\, d\lambda \\
&\quad = cR(-c, A) \int f(\lambda)R(\lambda, A)\, d\lambda \\
&\qquad + c \int f(\lambda)\lambda R(\lambda, A)R(\lambda, A + c)\, d\lambda\, R(0, A + c).
\end{aligned}$$

Call the last integral $I(f)$ and note that $\|I(f)\|_{B(X)} \le D_c\|f\|_{H^\infty}$. Hence

$$\begin{aligned}
\int f(\lambda)R(\lambda, A + c)\, d\lambda &= [I - (-c)R(-c, A)] \int f(\lambda)R(\lambda, A)\, d\lambda + I(f) \\
&= [-AR(-c, A)] \int f(\lambda)R(\lambda, A)\, d\lambda + I(f).
\end{aligned}$$

We obtain $\|f(A+c)\| \le D\|f(A)\| + \|I(f)\|$. For the converse we can assume that A and then $L = -AR(-c, A)$ are invertible so that

$$\|f(A)\| \le \|L^{-1}\|(\|f(A+c)\| + \|I(f)\|)$$

which ends the proof. □

7. Characterization of the H^∞-calculus by interpolation

It is shown in [5] that, in a Hilbert space, a sectorial operator A has a bounded H^∞-calculus if and only if $[\dot{X}_\alpha, \dot{X}_\beta]_\theta = \dot{X}_\gamma$ with $\gamma = (1-\theta)\alpha + \theta\beta$ for $\alpha \neq \beta$, $\theta \in (0,1)$. On a Banach space X which is not isomorphic to a Hilbert space, boundedness of the H^∞-functional calculus cannot be characterized by the complex or the real interpolation method in such a way. Complex interpolation is linked to the weaker condition BIP (cf. [54, 1.15.2] and Proposition 2.2 above), and A has always a bounded H^∞-functional calculus on real interpolation spaces $(X, X_1)_{\alpha,q}$, $\alpha \in (0,1)$ (cf. [19], [20]). Real interpolation spaces, however, do usually not coincide with the scale X_α; if $X = L_p(\mathbb{R}^n)$ and $A = -\Delta$ then the real interpolation spaces of X and $D(A)$ are Besov spaces, not Bessel potential spaces $H^s_p(\mathbb{R}^n)$. Therefore we introduce here the **Rademacher interpolation method**. For Banach spaces with finite cotype, this method is isomorphic to a special case of the interpolation methods introduced in [32] in connection with Euclidean structures (see [32] for further details). With this method it will be possible to extend the above mentioned Hilbert space result and give criteria for the H^∞-calculus.

As before let (ε_k) be a sequence of independent symmetric $\{-1,1\}$–valued random variables on a probability space. One can take the Rademacher functions $r_k(t) = \operatorname{sign}\sin(2^k\pi t)$, $k \in \mathbb{N}$, on $[0,1]$.

Definition 7.1. *Let (Y_0, Y_1) be an interpolation couple. For every $\theta \in (0,1)$, the* ***Rademacher interpolation space*** *$\langle Y_0, Y_1\rangle_\theta$ consists of all $y \in Y_0 + Y_1$ which can be represented as a sum*

$$y = \sum_{k=-\infty}^{\infty} y_k, \quad y_k \in Y_0 \cap Y_1 \tag{23}$$

convergent in $Y_0 + Y_1$, such that

$$C_0(y_k) = \sup_N \mathbb{E}\Big\| \sum_{k=-N}^{N} \varepsilon_k 2^{-k\theta} y_k \Big\|_{Y_0} < \infty$$

$$C_1(y_k) = \sup_N \mathbb{E}\Big\| \sum_{k=-N}^{N} \varepsilon_k 2^{k(1-\theta)} y_k \Big\|_{Y_1} < \infty.$$

The norm on $\langle Y_0, Y_1\rangle_\theta$ *is given by*

$$\|y\|_\theta = \inf\{\max(C_0(y_k), C_1(y_k)) : \textit{ all representations (23) of } y\}.$$

Remark 7.2. Although we shall not use this fact, we note that $\langle L_{p_0}, L_{p_1}\rangle_\theta = L_p$ for $1 \le p_0 < p_1 < \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ (cf. [32]).

It will be important to us that, for a large class of Banach spaces including uniformly convex spaces, the Rademacher method is self dual. The following result is shown in [32] and will not be proved here:

Proposition 7.3. *If X is B–convex then, for an arbitrary interpolation couple (X, Y), we have $\langle X, Y\rangle_\theta^* = \langle X^*, Y^*\rangle_\theta$.*

Applying the Rademacher interpolation method to the completions $\dot{X}_\alpha$ of the fractional domains $D(A^\alpha)$ (see Section 2) we obtain

Theorem 7.4. *Let A admit an H^∞–calculus. For $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ and $\gamma = (1-\theta)\alpha + \theta\beta$ with $\theta \in (0,1)$ we have for the Rademacher interpolation method*

$$\dot{X}_\gamma = \langle \dot{X}_\alpha, \dot{X}_\beta\rangle_\theta.$$

In particular: $X = \left\langle \widetilde{D(A)}, \widetilde{R(A)}\right\rangle_{1/2}$, where $\widetilde{R(A)}$ is the completion of $(R(A), \|A^{-1}x\|)$ and $\widetilde{D(A)}$ is the completion of $(D(A), \|Ax\|)$.

Proof. First we show that $\dot{X}_\gamma \subset \langle \dot{X}_\alpha, \dot{X}_\beta\rangle_\theta$. To this end put $d = \beta - \alpha$, $a = 2^{1/d}$ so that $\alpha = \gamma - \theta d$ and $\beta = \gamma + (1-\theta)d$. For some $l > 1 + d$ we define

$$g_0(z) = \frac{2l}{\pi}\int_1^a \frac{(tz)^l}{1+(tz)^{2l}}\frac{dt}{t}$$

and $g_s(z) = z^s g_0(z)$ for $-l < s < l$. Then for every $0 < \sigma < \pi$ we have $g_s \in H_0^\infty(\Sigma_\sigma)$ for $-l < s < l$. We also have for all $z \in \Sigma_\sigma$

$$\sum_{k\in\mathbb{Z}} g_0(a^k z) = \frac{2l}{\pi}\int_0^\infty \frac{t^l}{1+t^{2l}}\frac{dt}{t} = 1.$$

Then we can write $x = \sum_{k\in\mathbb{Z}} g_0(a^k A)x$ (unconditionally) for all $x \in D(A^\alpha) \cap D(A^\beta)$, and for all $\sigma_k = \pm 1$ and $N \in \mathbb{N}$ we have by Theorem 4.1 (i)

$$\begin{aligned}
&\Big\| \sum_{|k|\le N} \sigma_k 2^{-k\theta} g_0(a^k A)x \Big\|_{D(A^\alpha)} \\
&= \Big\| \sum_{|k|\le N} \sigma_k a^{-kd\theta} g_0(a^k A) A^{-d\theta}(A^\gamma x)\Big\|_X \\
&= \Big\| \sum_{|k|\le N} \sigma_k g_{-d\theta}(a^k A)(A^\gamma x)\Big\| \le C\|A^\gamma x\|,
\end{aligned}$$

and also

$$\begin{aligned}&\Big\|\sum_{|k|\le N}\sigma_k 2^{k(1-\theta)}g_0(a^kA)x\Big\|_{D(A^\beta)}\\&\quad=\Big\|\sum_{|k|\le N}\sigma_k g_{(1-\theta)d}(a^kA)(A^\gamma x)\Big\|_X\le C\|A^\gamma x\|\end{aligned}$$

Hence

$$\|x\|_{\langle \dot X_\alpha,\dot X_\beta\rangle_\theta}\le C\|x\|_{\dot X_\gamma}\ \text{for}\ x\in D(A^\alpha)\cap D(A^\beta).$$

Now we prove the converse inequality. Since $X^\sharp$ norms X, there is a constant C_1 such that for every $x\in D(A^\alpha)\cap D(A^\beta)$ there is an $x^*\in D((A^\sharp)^{\theta d})\cap D((A^\sharp)^{-(1-\theta)d})$ such that $\|A^\gamma x\|\le C_1\langle A^\gamma x,x^*\rangle$ and $\|x^*\|=1$. By definition of the interpolation method there is a representation

$$x=\sum_{k\in\mathbb{Z}}x_k,\quad x_k\in D(A^\alpha)\cap D(A^\beta),$$

such that for $L=\|x\|_{\langle \dot X_\alpha,\dot X_\beta\rangle_\theta}$ we have

$$\mathbb{E}\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k 2^{-k\theta}A^\alpha x_k\Big\|\le 2L \tag{24}$$

$$\mathbb{E}\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k 2^{k(1-\theta)}A^\beta x_k\Big\|\le 2L. \tag{25}$$

The operator $A^\sharp$ on $X^\sharp$ has also an H^∞–calculus and so we can apply the first part of the proof to $(X^\sharp)_{\alpha'}$, $(X^\sharp)_{\beta'}$ with $\alpha'=(\theta-1)d$, $\beta'=\theta d$. Since $\theta\alpha'+(1-\theta)\beta'=0$ there is a decomposition $x^*=\sum_{k\in\mathbb{Z}}x_k^*$, $x_k^*\in(X^\sharp)_{\alpha'}\cap(X^\sharp)_{\beta'}$, such that

$$\mathbb{E}\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k 2^{-k(1-\theta)}(A^\sharp)^{(\theta-1)d}x_k^*\Big\|\le C\|x^*\| \tag{26}$$

$$\mathbb{E}\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k 2^{k\theta}(A^\sharp)^{d\theta}x_k^*\Big\|\le C\|x^*\|. \tag{27}$$

Now we have

$$\langle A^\gamma x,x^*\rangle=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle A^\gamma x_j,x_k^*\rangle=\sum_{r\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\langle A^\gamma x_j,x_{j+r}^*\rangle.$$

For a fixed $r \geq 0$ we estimate using Lemma 4.5 (for $M = I$) and (24), (27):

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \langle A^\gamma x_j, x^*_{j+r}\rangle &= 2^{-\theta r} \sum_j \langle 2^{-j\theta} A^\alpha x_j, 2^{\theta(j+r)} (A^\sharp)^{\theta d} x^*_{j+r}\rangle \\ &\leq C 2^{-\theta r} \mathbb{E}\Big\| \sum_j \varepsilon_j 2^{-j\theta} A^\alpha x_j \Big\| \cdot \mathbb{E} \Big\| \sum_j \varepsilon_j 2^{\theta j} (A^\sharp)^{\theta d} x^*_j \Big\| \\ &\leq 2^{-\theta|r|} 2C' L.\end{aligned}$$

For a fixed $r < 0$ we use Lemma 4.5 (for $M = I$) together with (25) and (26):

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \langle A^\gamma x_j, x^*_{j+r}\rangle &= 2^{(1-\theta)r} \sum_j \langle 2^{(1-\theta)j} A^\beta x_j, 2^{(\theta-1)(j+r)} (A^\sharp)^{(\theta-1)d} x^*_{j+r}\rangle \\ &\leq C 2^{(1-\theta)r} \mathbb{E}\Big\| \sum_j \varepsilon_j 2^{(1-\theta)j} A^\beta x_j \Big\| \\ &\cdot \mathbb{E} \Big\| \sum_j \varepsilon_j 2^{(\theta-1)j} (A^\sharp)^{(\theta-1)d} x^*_j \Big\| \leq 2^{-(1-\theta)|r|} 2C' L.\end{aligned}$$

Combining these inequalities we get

$$\|A^\gamma x\| \leq C_1 \langle A^\gamma x, x^*\rangle \leq 2C'' \Big(\sum_{r\in\mathbb{Z}} 2^{-\min\{\theta, 1-\theta\}|r|} \Big) L$$

and the claim follows. □

If A has an H^∞–calculus on X, then $A^\sharp$ has an H^∞–calculus on $X^\sharp$ and the theorem implies

$$(X^\sharp)^{\dot{}}_\gamma = \langle (X^\sharp)^{\dot{}}_\alpha, (X^\sharp)^{\dot{}}_\beta \rangle_\theta.$$

This leads to the following characterization of the H^∞–calculus.

Theorem 7.5. *Let A and $A^\sharp$ be almost R–sectorial on X and $X^\sharp$, respectively. Then A has an H^∞–calculus if and only if we have continuous embeddings*

$$\dot{X}_{\gamma_0} \subset \langle \dot{X}_{\alpha_0}, \dot{X}_{\beta_0}\rangle_{\theta_0}, \qquad (X^\sharp)^{\dot{}}_{\gamma_1} \subset \langle (X^\sharp)^{\dot{}}_{\alpha_1}, (X^\sharp)^{\dot{}}_{\beta_1}\rangle_{\theta_1}$$

for some $\alpha_i < \beta_i$, $\gamma_i = (1-\theta_i)\alpha_i + \theta_i \beta_i$ with $\theta_i \in (0,1)$ for $i = 0, 1$.

The main estimate of the proof is contained in the following lemma.

Lemma 7.6. *Let A be almost R–sectorial in X. For $\alpha < \gamma < \beta$ with $\gamma = (1-\theta)\alpha + \theta\beta$ choose $l > d := \beta - \alpha$ and $a = 2^{1/d}$. For $g(z) = z^l(1-z)^{-2l}$ and $|\omega| > \omega_r(A)$ there is a constant C such that*

$$\sup_N \mathbb{E} \| \sum_{|j|\leq N} \varepsilon_j g(e^{i\omega} t a^j A) A^\gamma x \| \leq C \|x\|_{\langle \dot{X}_\alpha, \dot{X}_\beta\rangle_\theta}$$

for all $x \in \dot{X}_\alpha \cap \dot{X}_\beta$ and $t > 0$.

Proof. Put $Y = \langle \dot{X}_\alpha, \dot{X}_\beta \rangle_\theta$ and

$$g_+(z) = z^{\theta d} g(z), \quad g_-(z) = z^{(\theta-1)d} g(z).$$

Since g, g_+, g_- are in $H_0^\infty(\Sigma_\sigma)$ for all $0 < \sigma < \pi$, the set $\{g_\pm(sA) : s > 0\}$ is R–bounded with R–bound M. For every $x \in D(A^\alpha) \cap D(B^\alpha)$ there is a representation $x = \sum_{k\in\mathbb{Z}} x_k$, $x_k \in D(A^\alpha) \cap D(A^\beta)$, such that

$$\sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k 2^{-k\theta} A^\alpha x_k \Big\| \le C\|x\|_Y \tag{28}$$

$$\sup_N \mathbb{E}\Big\| \sum_{|k|\le N} \varepsilon_k 2^{k(1-\theta)} A^\beta x_k \Big\| \le C\|x\|_Y. \tag{29}$$

With x satisfying (28) and (29) we have for $\sigma_j = \pm 1$ and $s = e^{i\omega} t$ with $|\omega| > \omega_Y(A)$ and $t \in [1, a]$,

$$\sum_{|j|\le N} \sigma_j g(sa^j A) A^\gamma x = \sum_{r\in\mathbb{Z}} \sum_{|j|\le N} \sigma_j g(sa^j A) A^\gamma x_{j+r}.$$

For a fixed $r \ge 0$ we can write

$$\begin{aligned}&\sum_{|j|\le N} \sigma_j g(sa^j A) A^\gamma x_{j+r}\\ &\quad = 2^{-(1-\theta)r} s^{(1-\theta)d} \sum_{|j|\le N} \sigma_j (sa^j A)^{-(1-\theta)d} g(sa^j A)\Big[2^{(j+r)(1-\theta)} A^\beta x_{r+j}\Big]\end{aligned}$$

and obtain from (29) and the R–boundedness of $g_-(sA)$ the estimate

$$\begin{aligned}&\mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g(sa^j A) A^\gamma x \Big\|\\ &\quad \le 2 \cdot 2^{-(1-\theta)|r|} \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g_-(sa^j A)\Big[2^{(j+r)(1-\theta)} A^\beta x_{r+j}\Big]\Big\|\\ &\quad \le 2 \cdot 2^{-(1-\theta)|r|} MC\|x\|_Y.\end{aligned}$$

Similarly we argue for a fixed $r < 0$:

$$\begin{aligned}&\sum_{|j|\le N} \sigma_j g(sa^j A) A^\gamma x\\ &\quad = s^{-\theta d} 2^{\theta r} \sum_j \sigma_j (sa^j A)^{\theta d} g(sa^j A)\Big[2^{-\theta(j+r)} A^\alpha x_{r+j}\Big]\end{aligned}$$

and therefore by (28)

$$\begin{aligned}\mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g(sa^j A)A^\gamma x\Big\| &\\ \le 2^{-\theta|r|}\mathbb{E}\Big\| \sum_{|j|\le N} g_+(sa^j A)[2^{-\theta(j+r)}A^\alpha x_{r+j}]\Big\| &\\ \le 2^{-\theta|r|}MC\|x\|_Y. &\end{aligned}$$

Combining the last two estimates we get

$$\sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g(e^{i\omega}ta^j A)A^\gamma x\Big\| \le C_1\|x\|_Y$$

which ends the proof. □

Proof of Theorem 7.5. Clearly, the condition is necessary by the last theorem. We prove the converse.

By the lemma and $\dot X_{\gamma_0} \subset \langle \dot X_{\alpha_0}, \dot X_{\beta_0}\rangle_{\theta_0}$ we get for all $y = A^{\gamma_0}x \in (A^{\gamma_0})$ that

$$\sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g(ta^j A)y\Big\| \le C_1\|y\|.$$

If l was chosen such that $l > \beta_1 - \alpha_1$ we can repeat this argument for $A^\sharp$ and obtain for $y^\sharp \in R((A^\sharp)^{\gamma_1})$

$$\mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g(ta^j A^\sharp)y^\sharp\Big\| \le C_2\|y^\sharp\|.$$

Since $R((A^\sharp)^{\gamma_1})$ is a dense subset of $X^\sharp$ the claim follows now from Theorem 4.1. □

Corollary 7.7. *Let A be almost R-sectorial on the B–convex Banach space X. If $\dot X_\gamma = \langle \dot X_\alpha, \dot X_\beta\rangle_\theta$ for some $\gamma = (1-\theta)\alpha + \theta\beta$ with $\alpha < \beta$ and $\theta \in (0,1)$, then A has an H^∞-calculus.*

This extends the characterization of the H^∞-calculus given in [5, Thm.4.2] for Hilbert spaces, since in Hilbert spaces the complex and the Rademacher interpolation method coincide. Of course, Corollary 7.7 gives an alternative proof of Theorem 5.1 but the argument given there is direct and more general.

In the next section we will need

Corollary 7.8. *Assume that A is almost R-sectorial on X and that X is B–convex and reflexive. For $\alpha < \beta$ and $\theta \in (0,1)$ we have that A has an $H^\infty(\Sigma_\sigma)$-calculus on $\langle \dot X_\alpha, \dot X_\beta\rangle_\theta$ for $\sigma > \omega_r(A)$.*

Proof. Applying Lemma 7.6 we obtain for some $|\omega| > \omega_r(A)$

$$N(x) := \sup_{t>0} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g(e^{i\omega} t a^j A) A^\gamma x \Big\| \le C\|x\|_{\langle \dot{X}_\alpha, \dot{X}_\beta\rangle_\theta}$$

for $x \in D(A^\alpha) \cap D(A^\beta)$ and $t > 0$.

Since X is B–convex and reflexive, A^* is also almost R–sectorial, $(\dot{X}_\nu)^* = (X^*)^{\boldsymbol{\cdot}}_{-\nu,A^*}$ for $\nu = \alpha, \beta, \gamma$. Moreover, $Y = \langle \dot{X}_\alpha, \dot{X}_\beta\rangle_\theta$ has the dual space $Y^* = \langle (X^*)^{\boldsymbol{\cdot}}_{-\alpha}, (X^*)^{\boldsymbol{\cdot}}_{-\beta}\rangle_\theta$. Applying Lemma 7.6 to A^* gives

$$N^*(x^*) := \sup_{t>0} \sup_N \mathbb{E}\Big\| \sum_{|j|\le N} \varepsilon_j g(e^{i\omega} t a^j A^*)(A^*)^{-\gamma} x^* \Big\| \le C\|x^*\|_{Y^*}$$

for all $x^* \in D((A^*)^{-\alpha}) \cap D((B^*)^{-\beta})$ and $t > 0$. By Theorem 4.1 (ii) A has an $H^\infty(\Sigma_\sigma)$–calculus on Y for $\sigma > \omega_r(A)$. ◻

This should be compared with [19], [20] where it is shown that, for $\alpha = 0$, $\beta = 1$, any *sectorial* operator A in an *arbitrary* Banach space X has an H^∞-calculus on the *real* interpolation spaces $(X, X_1)_{\theta,q}$, $q \in [1, \infty]$ for all $\sigma > \omega(A)$.

Corollary 7.7 leads to the following useful criteria for the H^∞-calculus.

Theorem 7.9. *Let Y be a complemented subspace of a B–convex Banach space X. Let A have an H^∞-calculus on X and let B be almost R–sectorial on Y.*

If $P(\dot{X}_{\beta_j,A}) = \dot{Y}_{\beta_j,B}$ for two different $\beta_1, \beta_2 \neq 0$ then B has an H^∞-calculus on Y.

Here the equality $P(\dot{X}_{\gamma,A}) = \dot{Y}_{\gamma,B}$ is meant in the following sense: the projection $P : X \to Y$, restricted to $X \cap \dot{X}_{\gamma,A} = D(A^\gamma)$, has a continuous extension $\tilde{P} : \dot{X}_{\gamma,A} \to \dot{Y}_{\gamma,B}$ which is surjective. This also implies that P is compatible with the interpolation couples $(X, \dot{X}_{\gamma,A})$ and $(Y, \dot{Y}_{\beta,A})$.

Proof. We use 0 as the third point with $P(\dot{X}_{0,A}) = Y_{0,B}$ and let $\{\beta_1, \beta_2, 0\} =: \{\alpha_1, \alpha, \alpha_2\}$ where $\alpha_1 < \alpha < \alpha_2$ and choose θ such that $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$. Then we apply Corollary 7.7 to B since we have

$$\langle \dot{Y}_{\alpha_0,B}, \dot{Y}_{\alpha_2,B}\rangle = \langle P(\dot{X}_{\alpha_0,A}), P(\dot{X}_{\alpha_1,A})\rangle_\theta = P(\dot{X}_{\alpha,A}) = \dot{Y}_{\alpha,B},$$

where we used [54, 1.2.4] for the third equality. ◻

Corollary 7.10. *If, in the last theorem, B has even BIP, then $P(\dot{X}_{\alpha,A}) = \dot{Y}_{\alpha,B}$ for one $\alpha \neq 0$ is sufficient for B to have an H^∞-calculus in Y.*

Proof. Here, we use [54, 1.2.4] for Rademacher interpolation and for complex interpolation

$$\langle Y, \dot{Y}_{\alpha,B}\rangle_\theta = P(\langle X, \dot{X}_{\alpha,A}\rangle_\theta) = P([X, \dot{X}_{\alpha,A}]_\theta) = [Y, \dot{Y}_{\alpha,B}]_\theta = \dot{Y}_{\theta\alpha,B},$$

and apply Corollary 7.7. ◻

8. H^∞-calculus on interpolation scales

In Section 5 we have shown that, if A and B are almost R-sectorial operators on X which satisfy two domain conditions $D(A^{\alpha_j}) = D(B^{\alpha_j})$ with $\alpha_1 < 0 < \alpha_0$, then B has a bounded H^∞-calculus provided A has a bounded H^∞-calculus. Now we will see that, if A_j and B_j are consistent operators on an interpolation couple (X_0, X_1) and we can check **one** of the domain conditions on X_0 and X_1, respectively, then we can obtain a bounded H^∞-calculus for B on the complex interpolation spaces X_θ, $\theta \in (0, 1)$. This is of particular interest if A_0 and B_0 are accretive operators on a Hilbert space. In the next section we will apply this result to differential operators on the L_p-scale or the scale of Helmholtz spaces.

Lemma 8.1. *Suppose that X is B–convex and that A has an H^∞-calculus on X. If B is almost R-sectorial on X and $D(A^\alpha) \sim D(B^\alpha)$ then B has an H^∞-calculus on $\dot{X}_{\beta,A}$ for $0 < \beta < \alpha$.*

Proof. By Lemma 7.8, B has an H^∞-calculus on

$$\langle X, \dot{X}_{\alpha,B}\rangle_\theta = \langle X, \dot{X}_{\alpha,A}\rangle_\theta = \dot{X}_{\theta\alpha,A}$$

for $\theta \in (0, 1)$, where we used Theorem 7.4 for the last equality. $\square$

In the following, we assume that (X_0, X_1) is an interpolation couple, the spaces $X_\theta := [X_0, X_1]_\theta$, $\theta \in (0, 1)$, are obtained by complex interpolation, and there is a family $(A_\theta)_{\theta\in[0,1]}$ of sectorial operators A_θ in X_θ satisfying the consistency condition

$$R(\lambda, A_\theta)x = R(\lambda, A_{\tilde{\theta}})x, \quad x \in X_\theta \cap X_{\tilde{\theta}}, \theta, \tilde{\theta} \in [0, 1].$$

Moreover we assume that (Y_0, Y_1) is another interpolation couple with scale (Y_θ) of complex interpolation spaces and a family $(B_\theta)_{\theta\in[0,1]}$ of sectorial operators satisfying a similar consistency condition.

Theorem 8.2. *Let, in the situation described above, (X_0, X_1) be an interpolation couple of reflexive and B-convex spaces and assume that, for $j = 0, 1$, Y_j is a complemented subspace of X_j with associated and compatible projections P_0, P_1. Assume, for $j = 0, 1$, that A_j has an H^∞-calculus on X_j and that B_j is almost R-sectorial on Y_j. Assume moreover that there are $\alpha < 0 < \beta$ such that*

$$P_0((X_0)^{\cdot}_{\alpha,A_0}) = (Y_0)^{\cdot}_{\alpha,B_0}, \quad P_1((X_1)^{\cdot}_{\beta,A_1}) = (Y_1)^{\cdot}_{\beta,B_1}. \tag{30}$$

Then, for $\theta \in (0, 1)$, the operator B_θ has an H^∞-calculus on the complex interpolation space $Y_\theta = [Y_0, Y_1]_\theta$.

Concerning the meaning of (30) see the remark after Theorem 7.9.

Proof. We first assume $Y_j = X_j$ and $P_j = I$ for $j = 0, 1$. By Lemma 8.1 the operator B_0 has an H^∞-calculus on $(X_0)^\cdot_{\theta\alpha,A_0}$ and B_1 has an H^∞-calculus on $(X_1)^\cdot_{\theta\beta,A_1}$ for all $\theta \in (0, 1)$. We omit subscripts for the operators.

Let $\alpha' \in (\alpha, 0)$ and $\beta' \in (0, \beta)$, and define $\theta \in (0, 1)$ by $(1-\theta)\alpha' + \theta\beta' = 0$. By complex interpolation we obtain an H^∞-calculus for B in the space $[(X_0)^\cdot_{\alpha'}, (X_1)^\cdot_{\beta'}]_\theta$. We now show that this space equals $[X_0, X_1]_\theta$. For the complex method we refer to [54, 1.9]. Observe first that, for $j = 0, 1$, A_j has BIP in X_j, and we can choose $M, \gamma \geq 0$ such that $\|A_j^{it}\|_{B(X_j)} \leq Me^{\gamma|t|}$ for $j = 0, 1$ and all t. Let $\gamma' := \gamma(|\alpha'| + |\beta'|)$. If $f \in \mathcal{F}(X_0, X_1, 0)$ with $f(\theta) = x$ then $g : z \mapsto A^{(1-z)\alpha'+z\beta'} f(z)$ satisfies $g(\theta) = x$ as well and belongs to $\mathcal{F}((X_0)^\cdot_{\alpha'}, (X_1)^\cdot_{\beta'}, \gamma')$ and the norm of g in this space is $\leq C\|f\|_{\mathcal{F}(X_0,X_1,0)}$. If, on the other hand, $g \in \mathcal{F}((X_0)^\cdot_{\alpha'}, (X_1)^\cdot_{\beta'}, 0)$ with $g(\theta) = x$ then $f : z \mapsto A^{(z-1)\alpha'-z\beta'} g(z)$ satisfies $f(\theta) = x$ and belongs to $\mathcal{F}(X_0, X_1, \gamma')$ with norm $\leq C\|g\|_{\mathcal{F}((X_0)^\cdot_{\alpha'},(X_1)^\cdot_{\beta'},0)}$.

This proves the claim, and by choosing α' and β' accordingly we can obtain an H^∞-calculus for B_θ in any space $X_\theta = [X_0, X_1]_\theta$ for any $\theta \in (0, 1)$.

For the general case we shall use again [54, 1.2.4]. By Corollary 7.8, B has an H^∞-calculus in $\langle Y_0, (Y_0)^\cdot_{\alpha,B}\rangle_\theta$ and in $\langle Y_1, (Y_1)^\cdot_{\beta,B}\rangle_\theta$ for all $\theta \in (0, 1)$. By [54, 1.2.4] these spaces coincide with

$$P(\langle X_0, (X_0)^\cdot_{\alpha,A}\rangle_\theta) = P((X_0)^\cdot_{\theta\alpha,A}) \text{ and } P(\langle X_1, (X_1)^\cdot_{\beta,A}\rangle_\theta) = P((X_1)^\cdot_{\theta\beta,A}).$$

Now taking α', β', and θ as before, i.e. satisfying $(1-\theta)\alpha' + \theta\beta' = 0$ we obtain an H^∞-calculus for B in the space

$$\begin{aligned}[P((X_0)^\cdot_{\alpha',A}), P((X_1)^\cdot_{\beta',A})]_\theta &= P([(X_0)^\cdot_{\alpha',A}, (X_1)^\cdot_{\beta',A}]_\theta) \\ &= P([X_0, X_1]_\theta) = [Y_0, Y_1]_\theta.\end{aligned}$$

Here we have used [54, 1.2.4] in the first and in the last equality. □

As an application we prove the following for a σ-finite measure space (Ω, μ).

Corollary 8.3. *Let A_2, B_2 with $D(A_2) = D(B_2)$ and $0 \in \rho(A_2) \cap \rho(B_2)$ be accretive operators in $L_2(\Omega)$. Assume that, for some $p_0 \in (1, \infty)$, there are operators A_p, B_p in $L_p(\Omega)$ for p between p_0 and 2 whose resolvents are consistent with those of A_2 and B_2, respectively, on a sector $-\Sigma_\theta \cup \{0\}$. Suppose that $D(A_{p_0}) = D(B_{p_0})$, A_{p_0} has an H^∞-calculus in $L_{p_0}(\Omega)$ and that B_{p_0} is almost R-sectorial in $L_{p_0}(\Omega)$. Then B_p has an H^∞-calculus in $L_p(\Omega)$ for all p strictly between 2 and p_0.*

Proof. Since A_2 and B_2 are accretive, they have BIP and we obtain $D(A_2^s) = D(B_2^s)$ for all $s \in (0, 1)$. By accretiveness, we also have $D(A_2^s) = D((A_2^*)^s)$ and $D(B_2^s) = D((B_2^*)^s)$ for all $s \in (0, 1/2)$. Hence $D((A_2^*)^s) = D((B_2^*)^s)$ for all $s \in (0, 1/2)$. By Corollary 5.6 we obtain $D(A_2^{-s}) = D(B_2^{-s})$ for all $s \in (0, 1/2)$.

We apply Theorem 8.2 with $Y_0 = X_0 = L_2(\Omega)$, $Y_1 = X_1 = L_{p_0}(\Omega)$ and $P_j = I_{X_j}$, taking $\alpha \in (-1/2, 0)$ and $\beta = 1$. Since, in $L_{p_0}(\Omega)$, A_{p_0} has an

H^∞-calculus and B_{p_0} is almost R-sectorial, the assumptions of the theorem are satisfied. □

Combining Theorem 8.2 with Theorem 5.7 we obtain a result which is of particular interest when the scale $(X_0)_{\alpha,A}$ consists of classical Sobolev spaces.

Corollary 8.4. *Let (X_0, X_1) be an interpolation couple where X_0 is a Hilbert space and X_1 is B-convex and reflexive. Let A have a bounded H^∞-calculus on X_0 and X_1 and assume*

(i) B is almost R-sectorial on X_1 and for some $\alpha \neq 0$ (e.g., $\alpha = 1$) we have $\|A^\alpha x\| \sim \|B^\alpha x\|$ for $x \in D(A^\alpha) = D(B^\alpha)$ in X_1,

(ii) there is an interval $\emptyset \neq I \subset \mathbb{R}_+$ such that, for any $\rho \in I$, there is a sectorial operator $\tilde{B}$ consistent with B on X_0 satisfying

$$\|\tilde{B}x\|_{(X_0)^{\cdot}_{-\rho,A}} \sim \|\dot{A}_{-\rho}\|_{(X_0)^{\cdot}_{-\rho,A}} \qquad x \in D(\dot{A}_{-\rho}).$$

Then B has a bounded H^∞-calculus on $X_\theta = [X_0, X_1]_\theta$ for $0 < \theta < 1$.

9. Applications to differential operators

We illustrate our results by applying them to several classes of elliptic differential operators. In certain cases this leads to new results, in other cases the results are known, sometimes even stronger results have been proved before. In all applications listed below our methods provide new proofs.

(a) Elliptic operators in non-divergence form

We start with operators on $\mathbb{R}^n$ where we use the usual multiindex notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and write $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ with $D_j := -i\frac{\partial}{\partial x_j}$. Recall (cf., e.g., [2]) that, for $M > 0$, $\omega_0 \in [0, \pi)$, an operator of the form $Au(x) := \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x)$ with $a_\alpha \in L_\infty(\mathbb{R}^n, \mathbb{C})$, $|\alpha| \leq 2m$, is called (M, ω_0)–elliptic if $\sum_{|\alpha|=2m} \|a_\alpha\|_\infty \leq M$ and the principal symbol

$$A_\pi(x, \xi) := \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha, \qquad x, \xi \in \mathbb{R}^n,$$

of A satisfies, for all $x, \xi \in \mathbb{R}^n$,

$$A_\pi(x, \xi) \subset \overline{\Sigma_{\omega_0}}, \qquad |A_\pi(x, \xi)| \geq M^{-1}|\xi|^{2m}.$$

Here, we confine ourselves to scalar equations but the arguments will also apply to systems where the coefficients satisfy $a_\alpha \in L_\infty(\mathbb{R}^n, \mathbb{C}^{N\times N})$ for some integer $N > 1$. For an (M, ω_0)–elliptic operator A of order $2m$ and $p \in (1, \infty)$ we consider its realization A_p in $L_p(\mathbb{R}^n)$ with domain $D(A_p) := W_p^{2m}(\mathbb{R}^n)$. Recall that the Sobolev space $W_p^{2m}(\mathbb{R}^n)$ equals the Bessel potential space $H_p^{2m}(\mathbb{R}^n)$.

Due to the perturbation result for lower order perturbations we may concentrate on operators of the form $Au(x) = \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha$. For operators of this form boundedness of H^∞-calculi has been established in [2] for Hölder continuous coefficients, in [21] for BUC (bounded uniformly continuous continuous) coefficients, and in [22] for VMO-coefficients in case $m = 1$. For UMD state space, R-sectoriality has been shown in [17] for BUC-coefficients on $\mathbb{R}^n$, and boundedness of the H^∞-calculus has been shown in [16] for Hölder continuous coefficients and general boundary conditions on the half space and on bounded domains.

Remark 9.1. Let A be an (M, ω_0)–elliptic operator of order $2m$ with **constant coefficients** on $\mathbb{R}^n$. We consider the realization A_p of A in $L_p(\mathbb{R}^n)$, $1 < p < \infty$, with domain $D(A_p) = H_p^{2m}(\mathbb{R}^n)$. By [17] or [40] the operator A_p is R–sectorial. Since $(A_p)^*$ is an operator of the same form, also $(A_p)^*$ is R–sectorial in $L_{p'}(\mathbb{R}^n)$. Defining the spaces $\dot{X}_\alpha := \dot{X}_{\alpha,A_p}$ where $X = L_p(\mathbb{R}^n)$ we find that $\dot{X}_1 = \dot{H}_p^{2m}(\mathbb{R}^n)$ and $\dot{X}_{-1} = \dot{H}_p^{-2m}(\mathbb{R}^n)$ are Riesz potential spaces. These spaces are independent of the particular coefficients. Hence it suffices to know that $(-\Delta)^m$ has an H^∞-calculus in $L_p(\mathbb{R}^n)$ and to use Theorem 5.1 to obtain an H^∞-calculus for A_p. Note that the coefficents of A may be complex.

For later use we note that by Mihlin's theorem we obtain, for $X = L_p(\mathbb{R}^n)$ and all $\beta \in \mathbb{R}$, that $\dot{X}_{\beta,A} = \dot{H}_p^{2m\beta}$ with equivalent norms. Indeed, the symbols $\xi \to (|\xi|^{\pm 2m} A_\pi(\xi)^{\mp 1})^\beta$ are C^∞ on $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree 0. Hence they induce L_p–bounded operators.

We turn to operators with **variable coefficients**. If the coefficients are sufficiently regular we may use comparison as in the case of constant coefficients.

Proposition 9.2. *Let $p \in (1, \infty)$, and assume that $Au := \sum_{|\alpha|=2m} a_\alpha D^\alpha$ is an (M, ω_0)–elliptic operator. Suppose that $a_\alpha \in BUC^{2m}(\mathbb{R}^n)$ for $|\alpha| = 2m$, i.e. all partial derivatives of order $\le 2m$ of the coefficients a_α are bounded and uniformly continuous on $\mathbb{R}^n$. Then, for $\nu > 0$ sufficiently large, the operator $\nu + A_p$ has an $H^\infty(\Sigma_\theta)$–calculus for each $\theta > \omega_0$.*

Proof. Let $X := L_p(\mathbb{R}^n)$. For some $\nu > 0$, the operator $\nu + A_p$ is R–sectorial with $\omega_R(\nu + A_p) \le \omega_0$ by, e.g., [40]. The dual operator $(A_p)^*$ of A_p is given in $L_{p'}(\mathbb{R}^n)$ by $(A_p)^* v = \sum_{|\alpha|=2m} \partial^\alpha (a_\alpha v) =: A^* v$. By $a_\alpha \in BUC^{2m}$ the highest order coefficients of A^* are in $BUC(\mathbb{R}^n)$. By [40] again, the operator $(A_p)^*$ has domain $W_{p'}^{2m}(\mathbb{R}^n) = H_{p'}^{2m}(\mathbb{R}^n)$ and $(\nu + A_p)^*$ is R–sectorial in $L_{p'}(\mathbb{R}^n)$. Now we apply Theorem 5.1 again and compare $\nu + A_p$ with $\nu + (-\Delta)^m$. This yields an $H^\infty(\Sigma_\theta)$–calculus for $\nu + A_p$ and every $\theta > \omega_0$. □

If the coefficients are merely **bounded and uniformly continuous** we obtain the following result:

Proposition 9.3. *Let $p \in (1,\infty)$, and assume that $Au := \sum_{|\alpha|=2m} a_\alpha D^\alpha$ is an (M,ω_0)–elliptic operator such that $a_\alpha \in BUC(\mathbb{R}^n)$ for $|\alpha| = 2m$, i.e., all coefficients a_α are bounded and uniformly continuous on $\mathbb{R}^n$. Then, for $\nu > 0$ sufficiently large, the operator $\nu + A_p$ has an $H^\infty(\Sigma_\theta)$–calculus in $H_p^\beta(\mathbb{R}^n)$ for $\theta > \omega_0$ and $\beta \in (0,2m)$.*

Proof. Since by, e.g., [40] the operator $\nu + A$ is R-sectorial in $L^p(\mathbb{R}^n)$ and $D(A) = H_p^{2m}(\mathbb{R}^n)$ the assertion follows from Corollary 7.8 and the following lemma. □

Lemma 9.4. *Let $p \in (1,\infty)$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq \beta$. For any $\theta \in (0,1)$ we have*

$$\langle \dot{H}_p^\alpha(\mathbb{R}^n), \dot{H}_p^\beta(\mathbb{R}^n)\rangle_\theta = \dot{H}_p^{(1-\theta)\alpha+\theta\beta}(\mathbb{R}^n) \text{ and}$$
$$\langle H_p^\alpha(\mathbb{R}^n), H_p^\beta(\mathbb{R}^n)\rangle_\theta = H_p^{(1-\theta)\alpha+\theta\beta}(\mathbb{R}^n).$$

Proof. The operator $A = -\Delta$ has an H^∞-calculus in $X = L^p(\mathbb{R}^n)$ and $\dot{X}_{\gamma/2} = \dot{H}_p^\gamma(\mathbb{R}^n)$ for all $\gamma \in \mathbb{R}$. Hence, Rademacher interpolation and complex interpolation coincide, and this yields the first assertion. The second assertion is proved by the same arguments applied to the operator $A = 1 - \Delta$ observing that this time $\dot{X}_{\gamma/2} = H_p^\gamma(\mathbb{R}^n)$, $\gamma \in \mathbb{R}$. □

For the case of **Hölder continuous coefficients** we propose the following proof (which is quite different from known ones): Use the same approach as above but in $X = H_p^{-\beta}(\mathbb{R}^n)$ for small $\beta > 0$ instead of $X = L^p(\mathbb{R}^n)$. If the domain X_1 of the operator A, considered as an operator in $X = H_p^{-\beta}(\mathbb{R}^n)$, equals $H_p^{2m-\beta}(\mathbb{R}^n)$ and $\nu + A$ is R-sectorial in X, then Lemma 9.4 and Corollary 7.8 yield an H^∞-calculus for $\nu + A$ in $L^p(\mathbb{R}^n) = \langle X, X_1\rangle_{\beta/(2m)}$. Precisely, we have

Proposition 9.5. *Let $p \in (1,\infty)$, and assume that $Au := \sum_{|\alpha|=2m} a_\alpha D^\alpha$ is an (M,ω_0)–elliptic operator such that $a_\alpha \in C^\gamma(\mathbb{R}^n)$ for some $\gamma > 0$ and all $|\alpha| = 2m$. Then, for $\nu > 0$ sufficiently large, the operator $\nu + A_p$ has an $H^\infty(\Sigma_\sigma)$–calculus in $H_p^\beta(\mathbb{R}^n)$ for $\sigma > \omega_0$ and $\beta \in [0,2m]$.*

For the proof we need R-sectoriality of $\nu + A$ in $H_p^{-\beta}(\mathbb{R}^n)$ for some small $\beta > 0$. This could be done directly by repeating the proof for L^p-spaces in negative Sobolev spaces, i.e., by localization and small perturbations. But it can also be achieved by Sneiberg's Lemma from interpolation theory (cf. [8, Lem. 23, p. 53], or [29, Thm.2.7] for a more general context).

Lemma 9.6 (Sneiberg's Lemma). *Let $(\mathcal{X}_\theta)$ and $(\mathcal{Y}_\theta)$ be complex interpolation scales of Banach spaces where $\theta \in (0,1)$. If S is a linear operator which is bounded $\mathcal{X}_\theta \to \mathcal{Y}_\theta$ for each $\theta \in (0,1)$, then the following subsets of $(0,1)$ are open:*

$$\{\theta : \exists\eta > 0 : \|Sx\|_{\mathcal{Y}_\theta} \geq \eta\|x\|_{\mathcal{X}_\theta}\}, \quad \{\theta : S : \mathcal{X}_\theta \to \mathcal{Y}_\theta \text{ is an isomorphism}\}.$$

Proof of Proposition 9.5. The use of Lemma 9.6 is similar to the way it has been used in [39]. For convenience we change the range of θ. The scales $(\mathcal{X}_\theta)$ and $(\mathcal{Y}_\theta)$ will consist of Rad-like spaces. We choose $\nu \geq 0$ such that $\nu + A$ is R-sectorial in $L^p(\mathbb{R}^n)$ and a dense sequence (λ_j) in $\nu + \Sigma_\sigma$ and let $\mathcal{X}_\theta$ denote the space of all sequences (u_j) of distributions on $\mathbb{R}^n$ such that

$$\sum_{k=0}^{2m} \mathbb{E}\| \sum_j r_j \lambda_j^{1-k/(2m)} \nabla^k u_j \|_{H_p^\theta} < \infty,$$

where ∇^k denotes the vector of all partial derivatives of order k and (r_j) is the Rademacher sequence on $[0, 1]$. We also let $\mathcal{Y}_\theta := \mathrm{Rad}(H_p^\theta)$,i.e. the space of all sequences (u_j) such that $\mathbb{E}\| \sum_j r_j u_j \|_{H_p^\theta} < \infty$. Then $(\mathcal{X}_\theta)$ and $(\mathcal{Y}_\theta)$ are complex interpolation scales. We consider the operator $S : (u_j) \mapsto ((\lambda_j + A)u_j)$ where A is as in the assumption. Since the coefficients of A are in C^γ, we obtain that $S : \mathcal{X}_\theta \to \mathcal{Y}_\theta$ is bounded for $|\theta| < \gamma$. By choice of ν and definition the operator S is an isomorphism for $\theta = 0$. An application of Lemma 9.6 yields that S is an isomorphism for $|\theta| < \gamma_0$ for some $\gamma_0 \in (0, \gamma)$. In particular we have

$$\sum_{k=0}^{2m} \mathbb{E}\| \sum_j r_j \lambda_j^{1-k/(2m)} \nabla^k u_j \|_{H_p^{-\beta}} \leq C_\beta \mathbb{E}\| \sum_j r_j (\lambda_j + A) u_j \|_{H_p^{-\beta}}$$

for $\beta \in (0, \gamma_0)$. Applying this to $(u_j) = ((\lambda_j + A)^{-1} f_j)$ we see that, for $\beta \in (0, \gamma_0)$, the operator $\nu + A$ with domain $D(A) = H_p^{2m-\beta}$ is R-sectorial in $H_p^{-\beta}$ (recall that the sequence (λ_j) is dense in $\nu + \Sigma_\sigma$). As announced, we obtain by Lemma 9.4 and Corollary 7.8 an H^∞-calculus for A in any space $H_p^\sigma(\mathbb{R}^n)$, $\sigma \in (-\beta, 2m - \beta)$, hence in particular in $L_p(\mathbb{R}^n) = H_p^0(\mathbb{R}^n)$. □

The same method of proof may be used for elliptic **boundary value problems** on domains $\Omega \subset \mathbb{R}^n$. First we present a substitute for Lemma 9.4.

Lemma 9.7. *Let $p \in (1, \infty)$, $m \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^m$. For $\theta \in (0, 1)$ and $\alpha, \beta \in (-1 + 1/p, m)$ we have*

$$\langle H_{p,0}^\alpha(\Omega), H_{p,0}^\beta(\Omega)\rangle_\theta = H_{p,0}^{(1-\theta)\alpha+\theta\beta}(\Omega)$$
$$\langle H_p^\alpha(\Omega), H_p^\beta(\Omega)\rangle_\theta = H_p^{(1-\theta)\alpha+\theta\beta}(\Omega).$$

Proof. We first assume $\partial\Omega \in C^{2m}$ and consider the operators $A_j := (-\Delta)^m$, $j = 0, 1$, in $L_p(\Omega)$ with domains $D(A_0) = H_p^{2m}(\Omega) \cap H_{p,0}^m(\Omega)$ and $D(A_1) := \{u \in H^{2m} : (\partial/\partial\nu)^k u|_{\partial\Omega} = 0, k = m, \ldots, 2m-1\}$. Both operators are induced by quadratic forms in $L_2(\Omega)$ with form domain $H_{2,0}^m(\Omega)$ for A_0 and form domain $H_2^m(\Omega)$ for A_1. Moreover, we have an H^∞-calculus for A_j, $j = 0, 1$, in all $L_p(\Omega)$, $p \in (1, \infty)$. Moreover, the operators are self-dual. Hence Rademacher and complex interpolation coincide and the assertion follows from Seeley's result ([53]) if we recall that $H_p^{-\beta}(\Omega) = (H_{p',0}^\beta(\Omega))' = (H_{p'}^\beta(\Omega))'$ for $\beta \in (0, 1 - 1/p)$.

For the general case $\partial\Omega \in C^m$ we forget about the operators and observe that the asserted equalities are preserved under localization and a C^m-change of coordinates. □

We now prepare the setting for our result on boundary value problems: Let $p \in (1, \infty)$, $m \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{2m}$. Let $Au := \sum_{|\alpha|=2m} a_\alpha D^\alpha$ be an (M, ω_0)-elliptic operator with complex-valued coefficients $a_\alpha \in C(\overline{\Omega})$, $|\alpha| = 2m$. Let $B_j := \sum_{|\beta|\leq m_j} b_{j\beta} D^\beta$, $j = 1, \dots, m$, be differential operators on the boundary of orders $m_j \leq 2m-1$ and with coefficients $b_{j\beta} \in C^{2m-m_j}(\overline{\Omega})$ (the continuation of the coefficients from $\partial\Omega$ into Ω is possible without loss of generality). We assume that (A, B) satisfies the Lopatinskij-Shapiro conditions (see, e.g., [17]) at every boundary point $x \in \partial\Omega$. In $L_p(\Omega)$ we then consider the operator A_B given by

$$\begin{aligned} A_B u &= Au \text{ for} \\ u \in D(A_B) &:= \{v \in H_p^{2m}(\Omega) : B_j v = 0 \text{ on } \partial\Omega \text{ for all } j = 1, \dots, m\}. \end{aligned}$$

Our result on boundary value problems then reads:

Proposition 9.8. *The operator A_B has an H^∞-calculus in $H_p^\beta(\Omega)$ for $\beta \in (0, 1/p)$. If, for some $\gamma \in (0, 1)$, the coefficients a_α of A satisfy $a_\alpha \in C^\gamma(\Omega)$, $|\alpha| = 2m$, and the coefficients $b_{j\beta}$ of B_j satisfy $b_{j\beta} \in C^{2m-m_j+\gamma}(\Omega)$, $j = 1, \dots, m$, then $\nu + A_B$ has an H^∞-calculus in $L_p(\Omega)$ for $\nu > 0$ sufficiently large.*

Proof. By [17] the operator $\nu + A_B$ is R-sectorial in $L_p(\Omega)$ for a suitable $\nu > 0$. By $H_{p,0}^{2m}(\Omega) \subset D(A_B) \subset H^{2m}(\Omega)$ we clearly have, for $\theta \in (0, 1)$,

$$\langle L_p(\Omega), H_{p,0}^{2m}(\Omega)\rangle_\theta \subset \langle L_p(\Omega), D(A_B)\rangle_\theta \subset \langle L_p(\Omega), H_p^{2m}(\Omega)\rangle_\theta. \tag{31}$$

We use Lemma 9.7 and obtain that all spaces in (31) equal $H_p^{2m\theta}(\Omega)$ for $2m\theta < 1/p$. As before, Corollary 7.8 proves the first assertion.

For the proof of the second assertion we may use the same arguments as soon as we have proven that $\nu + A_B$ is R-sectorial in $H_p^{-\beta}(\Omega)$ for some small $\beta \in (0, 1 - 1/p)$ (of course, the domain of A_B in this space is $\{u \in H_p^{2m-\beta}(\Omega) : B_j u = 0 \text{ on } \partial\Omega \text{ for } j = 1, \dots, m\}$). Again, we shall use Sneiberg's Lemma, but now we apply it to estimates for the full inhomogeneous boundary value problem

$$\begin{aligned} (\lambda + A)u &= f \quad \text{in } \Omega \\ B_j u &= g_j \quad \text{on } \partial\Omega \text{ for } j = 1, \dots, m, \end{aligned}$$

where $f \in H_p^\theta(\Omega)$ and $g_j \in H^{\theta+m_j}(\Omega)$ for $j = 1, \dots, m$ (cf. the setting in [17]). For notational reasons we give details only for the case $m = 1$, $m_1 = 1$, $B_1 = B = \sum_{|\beta|\leq 1} b_\beta D^\beta$. Then the assumption reads $b_\beta \in C^{1+\gamma}(\Omega)$, $|\beta| \leq 1$.

Similar to the case of the whole space $\mathbb{R}^n$ we choose a dense sequence (λ_j) in $\nu + \Sigma_\sigma$ and let $\mathcal{X}_\theta$ denote the space of all sequences (u_j) such that

$$\sum_{k=0}^{2} \mathbb{E}\| \sum_j r_j \lambda_j^{1-k/2} \nabla^k u_j \|_{H_p^\theta(\Omega)} < \infty,$$

where $\theta \in I_p := (-1 + 1/p, 1/p)$. For $\mathcal{Y}_\theta$ we take all sequences $((f_j, g_j))$ of pairs such that

$$\mathbb{E}\| \sum_j r_j f_j \|_{H_p^\theta(\Omega)} + \mathbb{E}\| \sum_j r_j \nabla g_j \|_{H_p^\theta(\Omega)} + \mathbb{E}\| \sum_j r_j \lambda_j^{1/2} g_j \|_{H_p^\theta(\Omega)} < \infty.$$

The operator S is now given by $S((u_j)_j) := (((\lambda_j + A)u_j, Bu_j)_j)$. It is straight forward to check that S is bounded $\mathcal{X}_\theta \to \mathcal{Y}_\theta$ for $\theta \in I_p$ satisfying $|\theta| < \gamma$. But for $\theta = 0$ the operator S is an isomorphism (cf. [17, Thms.6.10, 7.3, 8.2]). Lemma 9.6 yields that S is an isomorphism $\mathcal{X}_{-\beta} \to \mathcal{Y}_{-\beta}$ for some small $\beta > 0$, in particular, there is a constant $C > 0$ such that (taking $g_j = 0$ and $u_j = (\lambda_j + A_B)^{-1} f_j$)

$$\mathbb{E}\| \sum_j r_j \lambda_j (\lambda_j + A_B)^{-1} f_j \|_{H_p^{-\beta}} \le C\mathbb{E}\| \sum_j r_j f_j \|_{H_p^{-\beta}}$$

for all choices $f_j \in H_p^{-\beta}(\Omega)$. We conclude that $\nu + A_B$ is R-sectorial in $H_p^{-\beta}(\Omega)$, and the proof is finished. □

Remark 9.9. The first assertion in Proposition 9.8 on the H^∞-calculus of A_B in $H_p^{-\beta}(\Omega)$ for $\beta \in (0, 1/p)$ under the sole assumption $a_\alpha \in C(\overline{\Omega})$ is new. The second assertion is slightly weaker than the main result of [16] for the case of scalar equations. By a totally different proof it is shown there that the assumption $b_{j\beta} \in C^{2m-m_j}(\overline{\Omega})$ is sufficient. If we consider, e.g., for the case $m = 1$ the Neumann type condition $(\partial/\partial\nu)u = 0$ on $\partial\Omega$ where $\nu : \partial\Omega \to \mathbb{R}^n$ denotes the outer normal unit vector, then we have $B_1 u(x) = \nu(x) \cdot \nabla u(x)$, $x \in \partial\Omega$, which means that we need $\nu \in C^{1+\gamma}(\partial\Omega)$, i.e., $\partial\Omega \in C^{2+\gamma}$, for our result. For an application of [16] it would be sufficient to have $\partial\Omega \in C^2$.

(b) Elliptic operators in divergence form

We consider operators A on $\mathbb{R}^n$ given by forms

$$\mathfrak{a}(u, v) = \int \overline{\nabla v}^t a \nabla u \, dx = \langle Au, v\rangle,$$

where $a \in L_\infty(\mathbb{R}^n, \mathbb{C}^{n\times n})$ satisfies

$$Re \left(\sum_{jk=1}^{n} a_{jk}(x) \xi_j \overline{\xi_k} \right) \ge \delta |\xi|^2, \quad x \in \mathbb{R}^n, \xi \in \mathbb{C}^n$$

for some $\delta > 0$. Then the adjoint operator A^* is given by the form

$$\mathfrak{a}^*(u, v) = \int \overline{\nabla v}^t a^* \nabla u \, dx = \langle A^* u, v \rangle.$$

The following relates the H^∞-calculus to boundedness of Riesz transforms. For more details and results on Riesz transforms $\nabla A^{-1/2}$, $\nabla (A^*)^{-1/2}$ we refer to [8], [10] and [3].

Proposition 9.10. *Let A be R-sectorial in L_p where $p \in (1, \infty)$. If $\|\nabla u\|_p \sim \|A^{1/2} u\|_p$ and $\|\nabla v\|_{p'} \sim \|(A^*)^{1/2} v\|_{p'}$ then A has an H^∞-calculus in L_p.*

Proof. We have $X = L_p$ and the assumptions lead to $\dot{X}_{1/2} = \dot{H}^1_p$ and $\dot{X}_{-1/2} = ((X^*)^{\cdot}_{1/2, A^*})^* = (\dot{H}^1_{p'})^* = \dot{H}^{-1}_p$. Since $-\Delta$ has an H^∞-calculus on X we obtain by Corollary 7.7 that A has an H^∞-calculus. $\square$

We now consider these operators in Bessel potential spaces. In [48], sectoriality of these operators has been proved in $H^{-1}_p(\mathbb{R}^n)$ under the assumption that the matrix a is bounded and uniformly continuous. Our methods enable us to prove the following to our knowledge new result on the H^∞-calculus.

Proposition 9.11. *Suppose that a is bounded uniformly continuous and satisfies the assumptions above. Then the operator $1 + A$ has an H^∞-calculus in $H^{-\beta}_p(\mathbb{R}^n)$ for any $\beta \in (-1, 1)$.*

Proof. We first study the case that A with coefficients $a_{jk} \in L_\infty$ is a small perturbation of an operator A_0 with constant coefficients a^0_{jk} that satisfies the same assumptions. Here smallness means that

$$\max_{jk} \sup_{x \in \mathbb{R}^n} |a_{jk}(x) - a^0_{jk}|$$

is sufficiently small. Observe that $\dot{X}_{\alpha, A_0} = \dot{H}^{2\alpha}_p(\mathbb{R}^n)$ for any $\alpha \in \mathbb{R}$. We let $B = A - A_0$. Then B has small norm $\dot{X}_{1/2, A_0} \to \dot{X}_{-1/2, A_0}$. Hence we may apply Theorem 6.6 in any space $\dot{H}^{2\beta}_p = \dot{X}_{\beta, A_0}$ for $\beta \in (-1/2, 1/2)$ to obtain an H^∞-calculus for A in this space.

For the general case of BUC-coefficients we may localize the operator in a way similar to what is usually done for operators in non-divergence form in $L_p(\mathbb{R}^n)$ (cf. [2] and also [48]). $\square$

Remark 9.12. If the matrix a is assumed to be Hölder continuous then we can show (via Corollary 6.3) that A has an H^∞-calculus in $H^{-1}_p(\mathbb{R}^n)$. We also refer to [8], [7], and [6] for related results.

(c) Schrödinger type operators

The following is an application of Corollary 6.7 to operators $-\Delta + V$ with potentials V from the Kato class. For $\beta = 0$ the assertion may be shown by other means, but this is not clear for $\beta \neq 0$.

Proposition 9.13. *Let $V : \mathbb{R}^n \to \mathbb{C}$ belong to the Kato class. Then, for $1 < p < \infty$, the Schrödinger type operator $-\Delta + V + \nu$ has an H^∞-calculus in $H_p^\beta(\mathbb{R}^n)$ for all $\beta \in (-2/(p'), 2/p)$ and ν sufficiently large.*

Proof. By definition of the Kato class we have $\| \, |V|(\lambda - \Delta)^{-1}\|_{L_1 \to L_1} \to 0$ as $\lambda \to \infty$. By duality and interpolation we obtain

$$\|(\lambda - \Delta)^{-1/(p')} V (\lambda - \Delta)^{-1/p}\|_{L_p \to L_p} \to 0 \quad (\lambda \to \infty),$$

which menas that, for $X = L_p(\mathbb{R}^n)$ and $A = -\Delta$, V acts boundedly $X_{1/p} \to X_{-1/(p')}$. Then we apply Corollary 6.7 in any space X_α, $\alpha \in (-1/(p'), 1/p)$ and recall $X_\alpha = H_p^{2\alpha}(\mathbb{R}^n)$. □

(d) Stokes operators

Let Ω be a bounded domain in $\mathbb{R}^n$ and $\partial\Omega \in C^{1,1}$. We let $p \in (1, \infty)$ and write $\mathbb{L}_p := \mathbb{L}_p(\Omega) := L_p(\Omega)^n$ and $\mathbb{P}_p$ for the Helmholtz projection in $\mathbb{L}_p$. We denote $\mathbb{H}_p^k := H_p^k(\Omega)^n$, and $\mathbb{H}_{p,0}^k$ is the closure in $\mathbb{H}_p^k$ of the C^∞-functions with compact support in Ω.

We let A denote the negative Dirichlet Laplacian on Ω, $A = -\Delta_D$, considered as an operator in $\mathbb{L}_2 = L_2(\Omega)^n$, and we let $B := \mathbb{P}A$ denote the Stokes operator in $\mathbb{L}_{2,\sigma}$. Moreover, for $1 < p < \infty$, A_p denotes the negative Dirichlet Laplacian in $\mathbb{L}_p$ and B_p denotes the Stokes operator in $\mathbb{L}_{p,\sigma}$. Observe that $0 \in \rho(A_p) \cap \rho(B_p)$ for $1 < p < \infty$. The proof of the following proposition employs essentially the arguments used in the proof of [26, Lem.6] and Proposition 5.5 to verify the assumptions of Theorem 8.2.

Proposition 9.14. *Let $1 < p < \infty$, $X := \mathbb{L}_p$ and $Y := \mathbb{L}_{p,\sigma}$. Then the Helmholtz projection $\mathbb{P}_p : X \to Y$ has a continuous and surjective extension $\widetilde{\mathbb{P}_p} : X_{-1,A_p} \to Y_{-1,B_p}$.*

Proof. We first observe that $(A_p)^* = A_{p'}$, $(B_p)^* = B_{p'}$, and $(\mathbb{P}_p)^* : \mathbb{L}_{p',\sigma} \to \mathbb{L}_{p'}$, $g \mapsto g$. Hence $D((B_p)^*) = \mathbb{H}_{p',0}^1 \cap \mathbb{H}_{p'}^2 \cap \mathbb{L}_{p',\sigma} \subset \mathbb{H}_{p',0}^1 \cap \mathbb{H}_{p'}^2 = D((A_p)^*)$. By continuity of $\mathbb{P}_{p'}$ we have for $g \in D((B_p)^*)$

$$\|(B_p)^* g\|_{p'} = \|B_{p'} g\|_{p'} = \|\mathbb{P}_{p'} A_{p'} g\|_{p'} \leq C \|A_{p'} g\|_{p'} = \|(A_p)^* g\|_{p'}.$$

By $0 \in \rho(A_{p'}) \cap \rho(B_{p'})$ we also have

$$\|(A_p)^* g\|_{p'} = \|A_{p'} g\|_{p'} \leq C' \|g\|_{\mathbb{H}_{p'}^2} \leq C'' \|B_{p'} g\|_{p'} = C'' \|(B_p)^* g\|_{p'}.$$

The assertion follows from $(ii) \Rightarrow (i)$ of the Proposition 5.5. □

The previous proposition verifies part of the hypothesis of Theorem 8.2 (for the operators A^{-1} and B^{-1}). We now look at the situation in $\mathbb{L}_2$.

Proposition 9.15. *For all* $s \in (0, 1/4)$ *we have that* $\mathbb{P}$ *maps* $D(A^s)$ *onto* $D(B^s)$.

Proof. By [14, Rem.1.10] we have $\mathbb{P}(\mathbb{H}^1_{2,0}) \subset \mathbb{H}^1_2$. Since $\mathbb{P}\mathbb{L}_2 \subset \mathbb{L}_2$ we obtain by complex interpolation $\mathbb{P}([\mathbb{L}_2, \mathbb{H}^1_{2,0}]_\theta) \subset [\mathbb{L}_2, \mathbb{H}^1_2]_\theta$ for all $\theta \in (0, 1)$. It is well known that $[\mathbb{L}_2, \mathbb{H}^1_{2,0}]_\theta = [\mathbb{L}_2, \mathbb{H}^1_2]_\theta = \mathbb{H}^\theta_2$ for $\theta \in (0, 1/2)$. Hence $\mathbb{P}(\mathbb{H}^\theta_2) \subset \mathbb{H}^\theta_2$ for $\theta \in (0, 1/2)$ which immediately implies $\mathbb{P}(\mathbb{H}^\theta_2) \subset \mathbb{H}^\theta_2 \cap \mathbb{L}_{2,\sigma}$ for $\theta \in (0, 1/2)$. On the other hand $\mathbb{H}^\theta_2 \cap \mathbb{L}_{2,\sigma} = \mathbb{P}(\mathbb{H}^\theta_2 \cap \mathbb{L}_{2,\sigma}) \subset \mathbb{P}(\mathbb{H}^\theta_2)$ for all θ. Thus we have shown $\mathbb{P}(\mathbb{H}^\theta_2) = \mathbb{H}^\theta_2 \cap \mathbb{L}_{2,\sigma}$ for $\theta \in (0, 1/2)$.

Next we recall $D(A^{1/2}) = \mathbb{H}^1_{2,0}$ and observe that, for $\theta \in (0, 1/2)$, $\mathbb{H}^\theta_2 = [\mathbb{L}_2, \mathbb{H}^1_{2,0}]_\theta = [\mathbb{L}_2, D(A^{1/2})]_\theta = D(A^{\theta/2})$ since the self-adjoint operator A has BIP in $\mathbb{L}_2$.

Finally we use the arguments (in the proof of) [26, Lem.6]: since B is self-adjoint in $\mathbb{L}_{2,\sigma}$ it has BIP and this leads to $D(B^s) = D(A^s) \cap \mathbb{L}_{2,\sigma}$ for all $s \in (0, 1)$. We conclude that, for all $s \in (0, 1/4)$,

$$D(B^s) = D(A^s) \cap \mathbb{L}_{2,\sigma} = \mathbb{H}^{2s}_2 \cap \mathbb{L}_{2,\sigma} = \mathbb{P}(\mathbb{H}^{2s}_2) = \mathbb{P}(D(A^s))$$

as asserted. □

In order to apply Theorem 8.2 we cite the following result due to Andreas Fröhlich ([25]).

Proposition 9.16. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded with* $\partial\Omega \in C^{1,1}$. *Then, for* $1 < p < \infty$, *the operator* B_p *is* R*-sectorial in* $\mathbb{L}_{p,\sigma}$ *and* $\omega_R(B_p) = 0$.

Combining these results we can prove

Theorem 9.17. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded with* $\partial\Omega \in C^{1,1}$. *Then, for* $1 < p < \infty$, *the operator* B_p *has an* $H^\infty(\Sigma_\nu)$*-calculus for all* $\nu > 0$.

Proof. We apply Theorem 8.2 with $X_0 = \mathbb{L}_2$, $Y_0 = \mathbb{L}_{2,\sigma}$, $X_1 = \mathbb{L}_p$, $Y_1 = \mathbb{L}_{p,\sigma}$, $P = \mathbb{P}_p$ for the operators A_p^{-1} and B_p^{-1} where $\alpha \in (-1/4, 0)$ and $\beta = 1$. □

Remark 9.18. It seems that Theorem 9.17 can also be obtained by the methods in [1]. However, the methods introduced there are totally different. They rely on a parameter-dependent version of Boutet de Monvel's pseudo-differential calculus for operator-valued symbols with low regularity.

For bounded domains Ω with $\partial\Omega \in C^3$, boundedness of the H^∞-calculus for the Stokes operator in $\mathbb{L}_{p,\sigma}(\Omega)$ was shown in [49, Thm.16]. There the proof is done via localization by using the perturbation result for the H^∞-calculus from [16] (cf. Corollary 6.5 above).

(e) Perturbation of boundary conditions

We take up [40, 5.3] and let $p \in (1, \infty)$, $X = L_p(-1, 1)$. Assume $\delta \in (1-1/p, 1)$ and $\phi_\delta := (1-x^2)^{-\delta}$. We want to show that a suitable translate of $-\Delta$ with boundary conditions $u(\pm 1) = \int_{-1}^{1} \phi_\delta u\, dx$ has an H^∞-calculus in X.

To this end we let $A = -\Delta$ in X with Dirichlet boundary conditions, i.e., with $D(A) = \{u \in H_p^2(-1, 1) : u(\pm 1) = 0\}$. The A has an H^∞-calculus, and $\dot{X}_\alpha = X_\alpha = H_p^{2\alpha}(-1, 1)$ for $2\alpha < 1/p$. For β, s satisfying $1/p > 2\beta > s > 1/p + \delta - 1$ we obtain that the operator

$$B : u \mapsto Bu := -\langle \phi_\delta, u\rangle A_{\beta-1} 1$$

is bounded $H_p^s(-1, 1) \to X_{\beta-1}$, i.e. $X_\alpha \to X_{\beta-1}$ if we let $\alpha := s/2$. By Corollary 6.9 we obtain an H^∞-calculus for a suitable translate of the operator $A_B := (A_{\beta-1} + B)|_X$. We refer to [40, 5.3] where it is shown that $A_B u = -\Delta u$ with domain $D(A_B) = \{u \in H_p^2(-1, 1) : u(\pm 1) = \int_{-1}^{1} \varphi_\delta u\, dx\}$.

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