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A Re-Examination of the Roberts Example of a Compact Convex Set Without Extreme Points

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1. Introduction

In [3] (see also [4]) Roberts constructed a compact convex subset of L_p $(0<p<1)$ with no extreme points, resolving a long-standing open problem. An essential ingredient of the construction is the notion of needlepoint. Recently the first author has given an example of a quasi-Banach space with trivial dual which has no needlepoints. In this space, the Krein-Milman Theorem holds [2]. The precise classification of those F-spaces in which the Krein-Milman Theorem fails, or of those which contain compact convex sets with no extreme points, seems still some way off. The aim of this paper is to provide some further information on this problem by a detailed examination of the argument employed by Roberts in [3].

A further aim is to perform the Roberts construction relative to a given convex set. Thus we show that the unit ball of L_1 contains a convex set without extreme points which is compact in the topology of L_p for $0\leqq p<1$, this answers a question put to the authors by H. P. Rosenthal. (Later Bourgain and Rosenthal [1] were able to construct a subspace of L_1 failing the Radon-Nikodym property in which the unit ball is compact in the L_p-topology for $0\leqq p<1$.)

Our notation is fairly standard. All vector spaces will be real. A quasi-norm on a vector space X is a map $x\to\|x\|$ such that

$$\|x\|>0, \quad x\neq 0, \tag{1.0.1}$$

$$\|tx\|=|t|\,\|x\| \quad t\in R, \quad x\in X, \tag{1.0.2}$$

$$\|x+y\|\leqq k(\|x\|+\|y\|) \quad x,y\in X, \tag{1.0.3}$$

where k is independent of x and y. The quasi-norm is p-subadditative if

$$\|x+y\|^p\leqq\|x\|^p+\|y\|^p \quad x,y\in X. \tag{1.0.4}$$

A quasi-norm induces on X a metrizable vector topology; if this topology is complete then X is a quasi-Banach space. X is p-convex $(0<p\leqq 1)$ if it can be

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equivalently re-quasi-normed so that (1.0.4) holds; every quasi-Banach space is p-convex for some $p>0$ ([5]). Throughout this paper we shall deal only with quasi-Banach spaces.

We shall also denote by $\mathscr{B}$ the Borel sets of $[0,1)$ and let λ be Lebesgue measure on $\mathscr{B}$. An Orlicz function is a non-decreasing function $\phi:[0,\infty)\to[0,\infty)$ with $\phi(0)=0$ and ϕ not identically zero. Suppose, in addition for some $0<\alpha,\beta,\gamma<\infty$

$$\alpha\phi(x)\leqq\phi(2x)\leqq\beta\phi(x)\qquad x\geqq\gamma. \tag{1.0.5}$$

Then the Orlicz space L_ϕ of all real Borel functions f on $[0,1)$ (modulo functions almost everywhere zero) satisfying

$$\int_0^1 \phi(|f(t)|)dt<\infty \tag{1.0.6}$$

is a quasi-Banach space if we define the unit ball to be those f satisfying

$$\int_0^1 \phi(|f(t)|)dt\leqq 1. \tag{1.0.7}$$

If X is a quasi-Banach space with a continuous quasi-norm, then by $L_p(X)$ $(0<p<\infty)$ we denote the space of Borel maps $f:[0,1)\to X$ satisfying

$$\|f\|=\{\int\|f(t)\|^p dt\}^{1/p}<\infty.$$

After factoring out functions equal to zero almost everywhere, $L_p(X)$ is a quasi-Banach space and if X is r-convex then $L_p(X)$ is $\min(p,r)$-convex.

2. Needlepoints

From now on we fix X to be a quasi-Banach space. We suppose X is r-convex $(0<r\leqq 1)$ and that the quasi-norm on X is r-subadditive. We shall denote by B the closed unit ball of X.

Definition 2.1. Let K be a closed convex subset of X. Then K is *small* if $\overline{\text{co}}(K\cap\varepsilon B)=K$ for every $\varepsilon>0$ and *strictly small* if $\text{co}(K\cap\varepsilon B)=K$.

The next result is due to Roberts [3].

Proposition 2.2. *Let K be a strictly small compact convex set and suppose $K\neq\{0\}$. Then $C=\text{co}(K\cup(-K))$ is a compact convex set without any extreme points.*

We now give a modification of a definition due to Roberts, by defining the notion of a needlepoint relative to a given convex set. For this and the succeeding definitions of pinpoints and approachable points, in the special case $K=X$ we shall omit reference to the underlying convex set.

Definition 2.3. Let K be a closed convex subset of X, which contains 0. Then a point $u\in K$ is a *needlepoint* of K if given $\varepsilon>0$ there exist $x_1,\dots,x_n\in K$ with the properties:

$$\|x_i\|\leqq\varepsilon,\qquad i=1,2\dots n \tag{2.3.1}$$

$$\left\|u-\frac{1}{n}(x_1+\dots+x_n)\right\|\leqq\varepsilon. \tag{2.3.2}$$

Wherever $y \in \operatorname{co}\{x_1, \ldots, x_n\}$ there exists α, $0 \leqq \alpha \leqq 1$ (2.3.3)

with

$$\|y - \alpha u\| \leqq \varepsilon .$$

The following Lemma is immediate.

Lemma 2.4. *The set of needlepoints of K is closed.*

We shall say that K is a *needleset* if every point of K is a needlepoint of K.

Suppose $\phi : [0,1) \to X$ is a *simple* Borel function. Then we define a finite rank operator $S_\phi : L_1 \to X$ by

$$S_\phi f = \int_0^1 f(t)\phi(t)dt .$$

in the obvious way.

Lemma 2.5. *Suppose K is a needleset and that $\{A_1, \ldots, A_n\}$ is a partitioning of $[0,1)$ into disjoint sets of positive λ-measure. Let $\mathscr{A}$ be the algebra generated by $\{A_1, \ldots, A_n\}$.*

Suppose $\delta > 0$ and $\phi : [0,1) \to K$ is $\mathscr{A}$-measurable. Then there is a simple $\mathscr{B}$-measurable function $\psi : [0,1) \to K$ and a (non-linear) map $R : L_1 \to L_1(\mathscr{A})$ so that

$$\|S_\phi f - S_\psi f\| \leqq \delta \|f\| \qquad f \in L_1(\mathscr{A}), \tag{2.5.1}$$

$$\|\psi(t)\| \leqq \delta \qquad 0 \leqq t < 1, \tag{2.5.2}$$

$$|Rf| \leqq \mathscr{E}(|f| \,|\mathscr{A}) \qquad f \in L_1, \tag{2.5.3}$$

$$\|S_\phi Rf - S_\psi f\| \leqq \delta \|f\| \qquad f \in L_1 . \tag{2.5.4}$$

Proof. Suppose

$$\phi = \sum_{j=1}^{n} 1_{A_j} u_j ,$$

where $u_j \in K$. Let $\theta = \min \lambda(A_j)$ and choose $\varepsilon > 0$ so that

$$\varepsilon < n^{1-1/r} \theta \delta .$$

For each u_j choose $x_1^j, \ldots, x_{m(j)}^j \in K$ with

$$\left\| u_j - \frac{1}{m(j)} (x_1^j + \ldots + x_{m(j)}^j) \right\| \leqq \varepsilon$$

$$\|x_k^j\| \leqq \varepsilon \qquad k = 1, 2 \ldots n(j)$$

and if $y \in \operatorname{co}(x_k^j : 1 \leqq k \leqq m(j))$ then for some α, $0 \leqq \alpha \leqq 1$,

$$\|y - \alpha u_j\| \leqq \varepsilon .$$

Partition each A_j into disjoint Borel sets A_i^j $(1 \leqq i \leqq m(j))$ of measure $m(j)^{-1} \lambda(A_j)$ and let

$$\psi = \sum_{j=1}^{n} \sum_{i=1}^{m(j)} 1_{A_i^j} x_i^j .$$

Then

$$\|S_\psi 1_{A_j} - u_j\| \leqq \varepsilon .$$

If $f \in L_1(\mathscr{A})$, say

$$f = \sum_{j=1}^{n} \beta_j 1_{A_j},$$

then

$$\begin{aligned}\|S_\psi f - S_\phi f\| &\leqq \varepsilon(\Sigma|\beta_j|^r)^{1/r} \\ &\leqq \varepsilon n^{1/r-1}\Sigma|\beta_j| \\ &\leqq \varepsilon n^{1/r-1}\theta^{-1}\|f\| \\ &\leqq \delta\|f\| .\end{aligned}$$

Since (2.5.2) is immediate, it remains only to establish (2.5.3) and (2.5.4).

For $f \in L_1$

$$S_\psi f = \sum_{j=1}^{n} \sum_{i=1}^{m(j)} \left(\int_{A_i^j} f(t)dt \right) x_i^j .$$

For each $j \leqq n$ there exists α_j with

$$|\alpha_j| \leqq \sum_{i=1}^{m(j)} \left| \int_{A_i^j} f(r)dr \right| \leqq \int_{A_j} |f(r)|dt$$

and

$$\left\| \alpha_j u_j - \sum_{i=1}^{m(j)} \left(\int_{A_i^j} f(t)dt \right) x_i^j \right\| \leqq \varepsilon \int_{A_j} |f(t)|dt .$$

Let

$$Rf = \sum_{j=1}^{n} \alpha_j 1_{A_j} .$$

Then (2.5.3) follows and

$$\begin{aligned}\|S_\phi Rf - S_\psi f\| &\leqq \varepsilon \left(\sum_{j=1}^{n} \left\{ \int_{A_j} |f(r)dt| ^r \right\} \right)^{1/r} \\ &\leqq \varepsilon n^{1/r-1}\|f\| \\ &\leqq \delta\|f\| .\end{aligned}$$

Theorem 2.6. *Let K be a needleset and suppose $K \neq \{0\}$. Then K contains a strictly small compact convex set $C \neq \{0\}$. Thus $\operatorname{co}(K \cup (-K))$ contains a compact convex set with no extreme points.*

Proof. Pick $u \in K$ with $u \neq 0$. Choose $\delta_n > 0$ so that $\sum_{n=1}^{\infty} \delta_n^r < \|u\|^r$. We define $\mathscr{A}_0$ to be the trivial subalgebra of $\mathscr{B}$ and let $\phi_0(t) \equiv u$ $(0 \leqq t < 1)$. Now by induction we choose an increasing sequence $\{\mathscr{A}_n\}$ of finite subalgebras of $\mathscr{B}$ and a sequence $\{\phi_n\}$ of maps from $[0,1)$ to K so that for each n we have

$$\phi_n \text{ is } \mathscr{A}_n\text{-measurable} \quad n \geqq 0, \tag{2.6.1}$$

$$\|\phi_n(t)\| \leqq \delta_n \quad n \geqq 1, \tag{2.6.2}$$

$$\|S_{\phi_{n+1}}f - S_{\phi_n}f\| \leqq \delta_{n+1}\|f\| \quad f \in L_1(\mathscr{A}_n), \quad n \geqq 0. \tag{2.6.3}$$

There exists a (non-linear) map

$$R_{n+1}: L_1 \to L_1(\mathscr{A}_n) \tag{2.6.4}$$

with

$$\left.\begin{aligned} |R_{n+1}f| &\leqq \mathscr{E}(|f|\,|\mathscr{A}_n) \\ \|S_{\phi_n}R_{n+1}f - S_{\phi_{n+1}}f\| &\leqq \delta_{n+1}\|f\| \end{aligned}\right\} n \geqq 0.$$

The existence of such sequences is clear from the preceding lemma.

If $m > n$

$$\|S_{\phi_m}f - S_{\phi_n}R_{n+1}R_{n+2}\dots R_m f\| \leqq \left(\sum_{n+1}^{m} \delta_j^r\right)^{1/r} \|f\| \tag{2.6.5}$$

and hence

$$\|S_{\phi_m}\| \leqq \left(\|u\|^r + \sum_{j=1}^{m} \delta_j^r\right)^{1/r} \leqq 2^{1/r}\|u\|. \tag{2.6.6}$$

If $\mathscr{A}$ is the smallest σ-algebra containing $\bigcup_{n \geqq 0} \mathscr{A}_n$ then we may use (2.6.3) and (2.6.6) to show that if $f \in L_1(\mathscr{A})$

$$Sf = \lim_{n \to \infty} S_{\phi_n}f$$

exists and $\|S\| \leqq 2^{1/r}\|u\|$.

Also

$$\|S1 - u\|^r \leqq \left(\sum_{j=1}^{\infty} \delta_j^r\right)$$

so that $S1 \neq 0$ and $S \neq 0$.

Let W be the closed unit ball of $L_1(\mathscr{A})$, and let $P = \{f \in W : f \geqq 0\}$. Then $S(P) \subset K$, and $S(P) \neq \{0\}$. We show $\overline{S(P)}$ is compact and strictly small.

If $x \in S_{\phi_n}(W)$ then by 2.6.5 if $m > n$

$$\begin{aligned} d(x, S_{\phi_m}(W)) &= \inf_{f \in W} \|x - S_{\phi_m}f\| \\ &\leqq \left(\sum_{n+1}^{m} \delta_j^r\right)^{1/r} \end{aligned}$$

and so if $x \in S(W)$

$$d(x, S_{\phi_n}(W)) \leqq \left(\sum_{n+1}^{\infty} \delta_j^r\right)^{1/r}.$$

Hence S is a compact operator and $C = \overline{S(P)}$ is compact; clearly $0 \in C$.

Suppose $x \in C$ and $\varepsilon > 0$. Choose $f_m \in P$ with $sf_m \to x$ and n so that $\sum_n^\infty \delta_j^r < \varepsilon^r$. We let $\{A_1, \ldots, A_k\}$ be the atoms of $\mathscr{A}_n$, and suppose by selecting a subsequence that

$$\lim_{m\to\infty} \int_{A_i} f_m(t)dt = c_i \quad \text{exists} \quad i = 1, 2, \ldots, k,$$

$$\lim_{m\to\infty} S(f_m 1_{A_i}) = y_i \quad \text{exists} \quad i = 1, 2, \ldots, k.$$

Then if $l \geqq n$

$$\|S_{\phi_l}(f_m 1_{A_i}) - S_{\phi_n} R_{n+1} R_{n+2} \ldots R_l(f_m 1_{A_i})\| \leqq \left(\sum_{n+1}^{\infty} \delta_j^r\right)^{1/r} \|f_m 1_{A_i}\|$$

and

$$|R_{n+1}, R_{n+2} \ldots R_l(f_m 1_{A_i})| \leqq \left(\int_{A_i} f_m(t)dt\right) 1_{A_i}.$$

Hence

$$\|S_{\phi_l}(f_m 1_{A_i})\| \leqq \left(\delta_n^r + \sum_{n+1}^{\infty} \delta_j^r\right)^{1/r} \|f_m 1_{A_i}\|.$$

Thus

$$\|S(f_m 1_{A_i})\| \leqq \varepsilon \|f_m 1_{A_i}\|$$

and

$$\|y_i\| \leqq \varepsilon c_i.$$

Now if $c_i > 0$, $c_i^{-1} y_i C$ and

$$x = \sum_{c_i > 0} c_i(c_i^{-1} y_i) + \left(1 - \sum_{c_i > 0} c_i\right)(0),$$

so that $x \in \mathrm{co}(C \cap \varepsilon B)$. Thus C is strictly small.

3. Approachable Points and Pinpoints

We have seen in Sect. 2 that a key step in producing pathological compact convex sets is to find a needlepoint or a needleset. We now introduce a weaker condition which is necessary for the existence of non-locally convex compact sets (see [2]).

Definition 3.1. Suppose K is a closed convex subset of X, containing 0. Then $u \in K$ is an *approachable point* of K if there is a constant M such that for every $\varepsilon > 0$ there exist $x_1, \ldots, x_n \in K$ with

$$\left\| u - \frac{1}{n}(x_1 + \ldots + x_n) \right\| \leqq \varepsilon, \tag{3.1.1}$$

$$\|x_i\| \leqq \varepsilon \quad i = 1, 2, \ldots, n, \tag{3.1.2}$$

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leqq M \sum_{i=1}^n |a_i|, \tag{3.1.3}$$

whenever

$a_1, \ldots, a_n \in \mathbb{R}$.

It is shown in [2] that if X contains no non-zero approachable points (of X) then every compact convex subset of X is locally convex. An example in [2] shows that X may have trivial dual but no non-zero approachable points.

Every needlepoint is approachable; the argument is elementary. The converse is open at present. It may be related to the following easy result.

Proposition 3.2. *The approachable points of K form a convex set. If $K=X$ they form a linear subspace.*

Remark. Contrast with Lemma 2.4. We do not know if the approachable points of X need be closed or the needlepoints be a linear subspace.

Note in Definition 3.1, that if K is bounded then (3.1.3) is superfluous and then the approachable points of K are closed. An easy example to illustrate the notion is to take $u=1$ in L_p $(0<p<1)$ and $K=P$, the positive part of the unit ball of L_1.

We now introduce yet another notion to bridge the gap between approachable points and needlepoints.

Definition 3.3. Suppose K is a closed convex subset of X. Then $u \in K$ is a *pinpoint* of K if there is a constant M, so that for every $\varepsilon>0$, there exist $p=p(\varepsilon)>1$, $0<R(\varepsilon)<\infty$ and sequence $\{x_n\}$ in K with

$$\|x_n\| \leqq \varepsilon \qquad n \in \mathbb{N}, \tag{3.3.1}$$

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leqq M \sum_{n=1}^{\infty} |a_n|, \tag{3.3.2}$$

for every finitely non-zero sequence $\{a_n\}$ in $\mathbb{R}$,

$$\left\| \sum_{n=1}^{\infty} a_n (u - x_n) \right\| \leqq R(\varepsilon) \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} + \varepsilon \sum_{n=1}^{\infty} |a_n| \tag{3.3.3}$$

for every finitely non-zero sequence $\{a_n\}$ in $\mathbb{R}$.

Again (3.3.2) is superfluous if K is bounded. The following properties of pinpoints are straightforward.

Proposition 3.4. *The sets of pinpoints of K form a convex subset; if $K=X$ they form a linear subspace.*

Our main result of this section is a simple reworking of an argument of Roberts [3].

Theorem 3.5. *Suppose either (a) X is r-convex for some $r>\frac{1}{2}$ or (b) K is bounded. Then every pinpoint of K is a needlepoint of K.*

Proof. [We shall suppose the quasi-norm of X is r-subadditive where in case (a) $\frac{1}{2}<r\leqq 1$.] In either case fix $0<\delta<1$ and choose $\{\varepsilon_k : k=1,2\ldots\}$ to be positive and such that

$$\sum_{k=1}^{\infty} \varepsilon_k^r < \tfrac{1}{3}\delta^r. \tag{3.5.1}$$

Then we choose by induction sequences $\{x^k : n=1,2\ldots\}$ and a decreasing sequence $\{a_k\}$ with $0<a_k<1$ so that if $c_n \geqq 0$ and $\Sigma c_n \leqq 1$ with $\{c_n\}$ finitely non-zero,

$$\Big\| \sum_{c_n \geqq a_k} c_n x_n^k \Big\| \leqq \varepsilon_k, \tag{3.5.2}$$

$$\Big\| \sum_{c_n \leqq a_{k+1}} c_n (x_n^k - u) \Big\| \leqq \varepsilon_k. \tag{3.5.3}$$

Indeed we take $a_1 = 1/2$ to start the induction. Now suppose a_k has been chosen. Then choose $(x_n^k : n \in \mathbb{N})$ to satisfy Definition 3.3 with $\varepsilon = a_k^{1/r}\varepsilon_k$, and let $R=R(\varepsilon)$ and $p=p(\varepsilon)$. If $c_n \geqq 0$ and $\Sigma c_n \leqq 1$ then

$$\begin{aligned}\Big\| \sum_{c_n \geqq a_k} c_n x_n^k \Big\| &\leqq \left(\frac{1}{a_k}\right)^{1/r} \max \|x_n^k\| \\ &\leqq \varepsilon_k.\end{aligned}$$

Now choose $a_{k+1} < a_k$ so that

$$R a_{k+1}^{1-1/p} < (1 - a_k^{1/r})\varepsilon_k.$$

Then

$$\begin{aligned}\Big\| \sum_{c_n \leqq a_{k+1}} c_n (x_n^k - u) \Big\| &\leqq R\Big(\sum_{c_n \leqq a_{k+1}} |c_n|^p \Big)^{1/p} + a_k^{1/r}\varepsilon_k \\ &\leqq R(\Sigma a_{k+1}^{p-1} c_n)^{1/p} + a_k^{1/r}\varepsilon_k \\ &\leqq R a_{k+1}^{1-1/p} + a_k^{1/r}\varepsilon_k \\ &\leqq \varepsilon_k.\end{aligned}$$

Thus (3.5.3) is fulfilled.

Now suppose $N \in \mathbb{N}$ is chosen in case (a) so that

$$N^{1-2r} M^r \leqq \sum_{k=1}^{\infty} \varepsilon_k^r \tag{3.5.4}$$

or in case (b) so that

$$N^{-1} \sup_{x \in K} \|x\| \leqq \sum_{k=1}^{\infty} \varepsilon_k^r \tag{3.5.5}$$

and let

$$v_n = \frac{1}{N} \sum_{k=1}^{N} x_n^k \qquad n \in \mathbb{N}.$$

Choose m so that $m > a_{N+1}^{-1}$, and consider $\{v_1, \ldots, v_m\} \in K$. If $c_n \geqq 0$ and $\sum_{n=1}^{m} c_n \leqq 1$, then

$$\begin{aligned}&\Big\| \sum_{n=1}^{m} c_n v_n - \Big(\frac{1}{N} \sum_{k=1}^{N} \sum_{c_n \leqq a_{k+1}} c_n \Big) u \Big\|^r \\ &\quad \leqq \Big\| \frac{1}{N} \sum_{k=1}^{N} \sum_{c_n \geqq a_k} c_n x_n^k \Big\|^r \\ &\qquad + \Big\| \frac{1}{N} \sum_{k=1}^{N} \sum_{c_n \leqq a_{k+1}} (c_n x_n^k - u) \Big\|^r + \|w\|^r,\end{aligned}$$

where

$$w=\frac{1}{N}\sum_{k=1}^{N}\sum_{a_{k+1}<c_n<a_k} c_n x_n^k.$$

Now in case (a)

$$\begin{aligned}\|w\|^r &\leqq N^{-r}\sum_{k=1}^{N}\Big\|\sum_{a_{k+1}<c_n<a_k} c_n x_n^k\Big\|^r\\ &\leqq N^{-r}M^r\sum_{k=1}^{N}\Big(\sum_{a_{k+1}<c_n<a_k} c_n\Big)^r\\ &\leqq N^{1-2r}M^r \leqq \sum_{k=1}^{\infty}\varepsilon_k^r.\end{aligned}$$

In case (b)

$$\|w\| \leqq N^{-1}\sup_{x\in K}\|x\| \leqq \sum_{k=1}^{\infty}\varepsilon_k^r.$$

In either case

$$\Big\|\sum_{n=1}^{m} c_n v_n - \frac{1}{N}\Big(\sum_{k=1}^{N}\sum_{c_n\leqq a_{k+1}} c_n\Big)u\Big\|^r \leqq 3\sum_{k=1}^{\infty}\varepsilon_k^r \leqq \delta.$$

In particular since $m^{-1}<a_{N+1}$,

$$\Big\|u-\frac{1}{m}(v_1+\ldots+v_m)\Big\| \leqq \delta$$

and so u is a needlepoint of K.

Remark. If K is compact it has no non-zero pinpoints (although it may have needlepoints). To see this suppose $u\in K$ is a pinpoint of K and suppose $\{x_n\}$ satisfies (3.3.1)–(3.3.3). Let v be a cluster point of $\{x_n\}$. In fact we may suppose

$$\|x_n - v\| \leqq 2^{-n/r}\varepsilon.$$

Then

$$\Big\|\frac{1}{n}(x_1+\ldots+x_n)-v\Big\| \leqq \varepsilon \qquad n=1,2,\ldots$$

and

$$\Big\|\frac{1}{n}(x_1+\ldots+x_n)-u\Big\|^r \leqq R(\varepsilon)n^{(1-p)r/p}+\varepsilon^r \qquad n=1,2\ldots.$$

Hence

$$\|u-v\| \leqq 2^{1/r}\varepsilon.$$

However $\|v\|\leqq\varepsilon$ and so $\|u\|\leqq 3^{1/r}\varepsilon$. Since ε is arbitrary, $u=0$.

4. Main Results

Now we fix p, $1 \leqq p < \infty$ and define $K^* \subset L_p(X)$ to be the set of $f \in L_p(X)$ such that $f(t) \in K$ almost everywhere.

Theorem 4.1. *Suppose $f \in K^*$ is simple and $f(t)$ is approachable in K for almost every t, $0 < t < 1$. Then f is a pinpoint of K^*.*

Proof. Suppose

$$f = \sum_{i=1}^{m} u_i 1_{A_i},$$

where $\{A_1, \ldots, A_m\}$ are disjoint Borel sets of positive measure which partition $[0, 1)$, and suppose u_i are approachable in K.

There is an $M < \infty$ (independent of $i = 1, 2 \ldots m$) so that for fixed $\varepsilon > 0$ we may find $N \in \mathbb{N}$ and $(x_{ij} : 1 \leqq i \leqq m,\ 1 \leqq j \leqq N)$ so that

$$\|x_{ij}\| < (\tfrac{1}{2})^{1/r} \varepsilon \quad 1 \leqq i \leqq m, \quad 1 \leqq j \leqq N,$$

$$\left\| \sum_{j=1}^{N} a_j x_{ij} \right\| \leqq M \sum_{j=1}^{N} |a_j| \quad 1 \leqq i \leqq m,$$

$$\|u_i - v_i\| \leqq (\tfrac{1}{2})^{1/r} \varepsilon \quad 1 \leqq i \leqq m,$$

where

$$v_i = \frac{1}{N} \sum_{j=1}^{N} x_{ij}.$$

For each n let $(B_{nj} : 1 \leqq j \leqq N)$ be a Borel partitioning of $[0, 1)$ each of measure $\frac{1}{N}$ so that the finite algebras $\mathscr{B}_n$ generated by (B_{nj}) are mutually independent and also independent of $(A_1, \ldots, A_m)$.

Let

$$g_{ni} = \sum_{j=1}^{N} x_{ij} 1_{B_{nj}}.$$

Then if X_i is the linear span of $\{x_{ij} : 1 \leqq j \leqq N\}$ we have $g_{ni} - v_i \in L_p(X_i)$. Now X_i is finite-dimensional and $g_{ni} - v_i$ are independent, identically distributed with mean zero and uniformly bounded. Hence for some $C_i < \infty$,

$$\left\| \sum a_n (g_{ni} - v_i) \right\| \leqq C_i \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2}$$

for every finitely non-zero $\{a_n\}$. If $C = \max_{1 \leqq i \leqq m} C_i$, then

$$\left\| \sum a_n (g_{ni} - u_i) \right\| \leqq 2^{1/r} C \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} + \varepsilon \sum_{n=1} |a_n|.$$

Now let

$$h_n = \sum_{i=1}^{m} g_{ni} 1_{A_i}.$$

Then

$$\|h_n\| \leqq \max_{0 \leqq t < 1} \|h_n(t)\| \leqq \varepsilon$$

and $h_n \in K^*$. If $t \in A_i$, $\sum a_n h_n(t)$ belongs to $(\sum |a_n|)V_i$ where V_i is the absolutely convex hull of $\{x_{ij} : 1 \leqq j \leqq N\}$. Hence

$$\left\| \sum_{n=1}^{\infty} a_n h_n \right\| \leqq M \sum_{n=1}^{\infty} |a_n| .$$

Also

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} a_n(h_n - f) \right\|^p &= \int_0^1 \left\| \sum_{n=1}^{\infty} a_n(h_n(t) - f(t)) \right\|^p dt \\ &= \sum_{i=1}^{m} \int_{A_i} \left\| \sum_{n=1}^{\infty} a_n(g_{ni}(t) - u_i) \right\|^p dt \\ &= \sum_{i=1}^{m} \lambda(A_i) \int_0^1 \left\| \sum^{\infty} a_n(g_{ni}(t) - u_i) \right\|^p dt \\ &\leqq \left(\sum^{m} \lambda(A_i) \right) (2^{1/r} C (\sum |a_n|^2)^{1/2} + \varepsilon \sum |a_n|)^p . \end{aligned}$$

Hence

$$\left\| \sum_{n=1}^{\infty} a_n(h_n - f) \right\| \leqq 2^{1/r} C (\sum |a_n|^2)^{1/2} + \varepsilon \sum |a_n| .$$

and so f is a pinpoint of K^*.

Theorem 4.2. *Suppose K is a bounded small set. Then K^* is a needleset.*

Proof. Each $x \in K$ is approachable. Hence each simple $f \in K^*$ is a pinpoint of K^* and hence also a needlepoint (since K^* is bounded in $L_p(X)$). As the needlepoints of K^* are closed in K^* we conclude by a simple approximation argument that K^* is a needleset.

Example 4.3. Suppose ϕ is an Orlicz function satisfying (1.0.5), and such that

$$\liminf_{t \to \infty} \frac{\phi(t)}{t} = 0 , \tag{4.3.1}$$

$$\limsup_{t \to \infty} \frac{\phi(t)}{t} < \infty . \tag{4.3.2}$$

Then the unit ball of L_1 contains a compact convex set with no extreme points.

First we observe from (1.0.5) that for some $p < \infty$ we have

$$\phi(st) \leqq A(s^p + 1)\phi(t) + B \qquad 0 \leqq s, t < \infty$$

for some constants A and B. Hence the map defined initially for simple functions by

$$Tf(s, t) = f(s)(t)$$

is a continuous linear operator of $L_p(L_\phi)$ into $L_\phi([0,1)\times[0,1))$. Indeed if $f\in L_p(L_\phi)$ is simple then, with

$$f(s)=\|f(s)\|g(s),$$

$$\int_0^1 \phi(|g(s)(t)|)dt \leqq 1 \qquad 0\leqq s<1.$$

Hence

$$\int_0^1\int_0^1 \phi(|Tf(s,t)|)dtds \leqq \int_0^1 A(\|f(s)\|^p+1)+Bds.$$

In particular if

$$\int_0^1 \|f(s)\|^p ds \leqq 1$$

then

$$\int_0^1\int_0^1 \phi(|Tf(s,t)|)dtds \leqq 2A+B.$$

Now if K is the unit ball of L_1, then $K\subset L_\phi$ is small and bounded and so K^* is a needleset of $L_p(L_\phi)$. Hence $T(K^*)$ is a needleset of $L_\phi([0,1)\times[0,1))$ and $T(K^*)$ is contained in the unit ball of $L_1([0,1)\times[0,1))$.

Clearly this implies that $K\subset L_\phi$ contains a needleset and the result is proved.

More generally Theorem 4.2 implies:

Theorem 4.3. *Suppose X is a quasi-Banach space containing a non-trivial bounded small set. Then for any p, $1\leqq p<\infty$, $L_p(X)$ contains a needleset and hence a compact convex set without extreme points.*

We also have

Theorem 4.4. *Suppose X is an r-convex quasi-Banach space where $\frac{1}{2}<r\leqq 1$, and that $1\leqq p<\infty$.*

(a) *If X contains a non-zero approachable point then $L_p(X)$ contains a non-zero needlepoint.*

(b) *If the set of approachable points in X is dense then every $f\in L_p(X)$ is a needlepoint.*

(c) *Every $f\in L_p(X)$ is a needlepoint if and only if every $f\in L_p(X)$ is approachable.*

Proof. (a) This follows from 3.5 and 4.1.

(b) This again follows from 3.5 and 4.1 and the fact that the needlepoints of $L_p(X)$ form a closed set.

(c) Observe that $L_p(X)\cong L_p(L_p(X))$ and apply (b).

5. Concluding Remarks

(1) The fact (Example 4.3) that in L_p $(0<p<1)$ the unit ball of L_1 contains a compact convex set without extreme points may be contrasted with a result of the second author that every compact convex set in the positive cone of L_p $(0<p<1)$

has an extreme point (although it may be strictly small and have only one extreme point.

Even more striking is the fact that every L_p-closed subset of the unit ball of L_1 is dentable (this may be proved by establishing a Martingale convergence theorem). Results of Roberts show that L_p does contain non-dentable compact convex sets.

2) Except for the partial results of Theorem 4.4 we do not know whether every approachable point of a quasi-Banach space need be a needlepoint. This problem is related to the question of whether of the needlepoints of a given space form a linear subspace.

3) The condition $\frac{1}{2}<r<1$ in Theorem 3.5 and 4.4 seems superfluous, but we have not been able to eliminate it.

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