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Spaces of Compact Operators

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1. Introduction

In this paper we study the structure of the Banach space $K(E,F)$ of all compact linear operators between two Banach spaces E and F. We study three distinct problems: weak compactness in $K(E,F)$, subspaces isomorphic to l_∞ and complementation of $K(E,F)$ in $L(E,F)$, the space of bounded linear operators.

In § 2 we derive a simple characterization of the weakly compact subsets of $K(E,F)$ using a criterion of Grothendieck. This enables us to study reflexivity and weak sequential convergence. In § 3 a rather different problem is investigated from the same angle. Recent results of Tong [20] indicate that we should consider when $K(E,F)$ may have a subspace isomorphic to l_∞. Although $L(E,F)$ often has this property (e.g. take $E=F=l_2$) it turns out that $K(E,F)$ can only contain a copy of l_∞ if it inherits one from either E^* or F. In § 4 these results are applied to improve the results obtained by Tong and also to approach the problem investigated by Tong and Wilken [21] of whether $K(E,F)$ can be non-trivially complemented in $L(E,F)$ (see also Thorp [19] and Arterburn and Whitley [2]).

It should be pointed out that the general trend of this paper is to indicate that $K(E,F)$ accurately reflects the structure of E and F, in the sense that it has few properties which are not directly inherited from E and F. It is also worth stressing that in general the theorems of the paper do not depend on the approximation property, which is now known to fail in some Banach spaces; the paper is constructed independently of the theory of tensor products.

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2. Weak Compactness in $K(E,F)$

Let E and F be Banach spaces and let $L(E,F)$ denote the space of bounded linear operators between E and F; then $K(E,F)$ is a closed subspace of $L(E,F)$. We shall be interested in two main topologies on $L(E,F)$. The *weak-operator topology* w is defined by the linear functionals $T\to f^*(Te)$ $f^*\in F^*$, $e\in E$; while the *dual weak-operator topology*

w' is defined by the linear functionals $T \to e^{**}(T^* f^*)$ $f^* \in F^*$, $e^{**} \in E^{**}$. We clearly have that $w' \geqq w$, and that if E is reflexive $w' = w$.

Let U denote the unit ball of E^{**} with the weak*-topology $\sigma(E^{**}, E^*)$. Let V be the unit ball of F^* with the weak*-topology $\sigma(F^*, F)$. Then U and V are compact Hausdorff spaces. For $T \in L(E, F)$ we define χ_T a function on $U \times V$ by

$$\chi_T(u, v) = u(T^* v).$$

Lemma 1. *$T \to \chi_T$ defines a linear isometry of $K(E, F)$ onto a closed linear subspace of $C(U \times V)$.*

Proof. Suppose $T \in K(E, F)$ and $u_\alpha \to u$ in U and $v_\alpha \to v$ in V. Then as T^* is compact $\|T^* v_\alpha - T^* v\| \to 0$. Hence

$$\begin{aligned} |\chi_T(u_\alpha, v_\alpha) - \chi_T(u, v)| &= |u_\alpha(T^* v_\alpha) - u(T^* v)| \\ &\leqq |u_\alpha(T^* v_\alpha - T^* v)| + |(u_\alpha - u)\, T^* v| \\ &\to 0, \end{aligned}$$

and therefore $\chi_T \in C(U \times V)$. Clearly $\|\chi_T\| = \|T\|$ and $T \to \chi_T$ is linear.

Theorem 1. *Let A be a subset of $K(E, F)$; then A is weakly compact if and only if A is w'-compact.*

Proof. Suppose A is w'-compact and let $\chi(A) = \{\chi_T; T \in A\}$. Then $\chi(A)$ is compact in the topology of pointwise convergence in $U \times V$ and therefore $\chi(A)$ is weakly compact in $C(U \times V)$ (Grothendieck [8]). Hence A is weakly compact. The converse is immediate since w' is weaker than the weak topology of $K(E, F)$. Theorem 1 is essentially due to Brace and Friend ([5], Theorem 8).

Corollary 1. *If E is reflexive, a subset A of $K(E, F)$ is weakly compact if and only if it is w-compact.*

Corollary 2. *If E and F are reflexive, and $K(E, F) = L(E, F)$ then $K(E, F)$ is reflexive.*

Proof. By Corollary 1, we need only show that the unit ball of $K(E, F)$ is w-compact. Suppose T_α is a w-Cauchy net with $\|T_\alpha\| \leqq 1$. For each $e \in E$, $T_\alpha e$ is weakly Cauchy in F and hence as F is reflexive $T_\alpha e \to Te$ weakly where $T \in L(E, F)$, and $\|T\| \leqq 1$. By assumption $T \in K(E, F)$ and hence the unit ball of $K(E, F)$ is w-complete; it follows that $K(E, F)$ is reflexive.

Corollary 2 is known when either E or F has the approximation property (cf. Holub [10], Jun [11], Ruckle [18]).

The conditions of Corollary 2 can be fulfilled, for Pitt [15] has shown that $L(l_p, l_q) = K(l_p, l_q)$ if $p > q \geqq 1$. In fact if E or F has the approximation property the converse of Corollary 2 is true.

Corollary 3. *Let T_n be a sequence of compact operators such that $T_n \to T$ in w' where T is compact. Then $T_n \to T$ weakly and there is a sequence S_n of convex combinations of $\{T_n; n = 1, 2, \ldots\}$ with $\|T - S_n\| \to 0$.*

This Corollary is immediate as the set $\{T_n, T\}$ is w'-compact. Note that in the reflexive case we need only assume that $T_n \to T$ in the weak-operator topology. (See Brace and Friend [5].)

The next result pursues further the problems raised by Corollary 2.

Theorem 2. *Let E be an inseparable reflexive space; then $L(E, E)$ is non-reflexive.*

Proof. If E is inseparable and reflexive, Chadwick [6] (cf. Amir and Lindenstrauss [1]) has shown that E has a Schauder decomposition, i.e. a sequence of non-trivial projections Q_n with $I = \sum_{n=1}^{\infty} Q_n$ in the topology w. We have, by an application of the Uniform Boundedness Theorem,

$$\sup_N \left\| \sum_{n=1}^{N} Q_n \right\| < \infty$$

and so, if $L(E, E)$ is reflexive,

$$I = \sum_{n=1}^{\infty} Q_n$$

in the weak topology of $L(E, E)$. Hence there is a sequence

$$S_n = \sum_{i=1}^{n} \lambda_i^{(n)} Q_i$$

with

$$\|I - S_n\| \to 0 .$$

For large enough n

$$\|I - S_n\| < 1$$

and hence S_n is invertible. However for $x \in Q_{n+1}(E)$, $S_n x = 0$, and we have a contradiction.

A Banach space E is called a Grothendieck space if every w^*-convergent sequence in E^* converges weakly in E^* (cf. Grothendieck [9], Theorem 8).

Theorem 3. *The following are equivalent:*

(i) *E is a Grothendieck space.*

(ii) *For any Banach space F, if $T_n \to T$ in the weak-operator topology w on $K(E, F)$, then $T_n \to T$ weakly.*

Proof. (ii)$\Rightarrow$(i) Take $F = \mathbb{R}$.

(i)$\Rightarrow$(ii) Suppose $T_n \to T$ in w; then for $f^* \in F^*$, $T_n^* f^* \to T^* f^*$ weak* in E^*. As E is a Grothendieck space, $T_n^* f^* \to T^* f^*$ weakly in E^* and hence $T_n \to T$ in w'. By Theorem 1, Corollary 3, $T_n \to T$ weakly.

3. Subspaces Isomorphic to l_∞

We first collect together some known results which will be used in this section. The first result is an easy consequence of the Orlicz-Pettis Theorem ([7], p. 318, Pettis [14]). We write $\delta^n = \{\delta_k^n\}$ for the unit vectors in l_∞.

Proposition 1. *Let F be a Banach space and suppose $T: l_\infty \to F$ is weakly compact. Then* $\sum_{n=1}^{\infty} \xi_n T\delta^n$ *converges in norm for each* $\xi = (\xi_n) \in l_\infty$.

The next result is due to Rosenthal [17].

Proposition 2. *Let F be a Banach space containing no isomorphic copy of l_∞. Then every bounded linear map $T: l_\infty \to F$ is weakly compact.*

The other results we require are due to Bessaga and Pelczynski ([3] and [4]). We call a series $\sum_{i=1}^{\infty} x_i$ weakly unconditionally Cauchy (w.u.c.) if

$$\sup_{J \subset N} \left\| \sum_{i \in J} x_i \right\| < \infty$$

where J runs over all finite subsets of the integers N.

Proposition 3. (i) *Let E be a Banach space containing no copy of c_0; then every w.u.c. series converges.*

(ii) *Let E be a Banach space such that E^* contains a copy of c_0; then E^* contains a copy of l_∞ and E contains a complemented copy of l_1.*

(iii) *Let E be a Banach space containing no complemented copy of l_1; then if* $\sum_{i=1}^{\infty} e_i^*$ *is weak*-unconditionally convergent in E^*, then Σe_i^* converges in norm.*

The next proposition is very closely related to results of Rosenthal [17], and could be used to prove the results of Rosenthal in the countable case. However, it is not clear to the author that it can be derived from Rosenthal's results easily, and in any case the proof given here is quite simple. We denote the nth coordinate map $\xi \to \xi_n$ in l_∞ by $\xi \to \pi_n(\xi)$, and for M a subset of the positive integers N, $l_\infty(M)$ is defined to be the set of $\xi \in l_\infty$ with $\xi_k = 0$ for $k \notin M$.

Proposition 4. *Let $A: l_\infty \to l_\infty$ be a bounded linear map. Suppose $A\delta^n = 0$ for all n; then there exists an infinite subset M of N such that $A(\xi) = 0$ for $\xi \in l_\infty(M)$.*

Proof (cf. Whitley [21], Lindenstrauss [12]).

We may choose (see Whitley [21] for a quick proof) an uncountable collection $(N_\alpha; \alpha \in \mathscr{A})$ of infinite subsets of N such that for $\alpha \neq \beta$ $N_\alpha \cap N_\beta$

is finite. Suppose the proposition is false: then for each α, there exists $\xi^{(\alpha)} \in l_\infty(N_\alpha)$ such that $\|\xi^\alpha\| = 1$ and $A\xi^{(\alpha)} \neq 0$. Let $\mathscr{I}$ be a finite subset of $\mathscr{A}$; then

$$\sum_{\alpha \in \mathscr{I}} \xi^{(\alpha)} = \eta + \zeta$$

where $\|\eta\| \leqq 1$ and $\zeta \in \operatorname{lin}(\delta^{(1)}, \delta^{(2)}, \ldots,)$. Hence

$$\left\| \sum_{\alpha \in \mathscr{I}} A(\xi^{(v)}) \right\| \leqq \|A\| \,.$$

Hence

$$\left| \sum_{\alpha \in \mathscr{I}} \pi_m(A(\xi^{(\alpha)})) \right| \leqq \|A\| \qquad \mathscr{I} \subset \mathscr{A} \,.$$

Therefore the set $\mathscr{A}_m = \{\alpha; \pi_m(A(\xi^{(\alpha)})) \neq 0\}$ is countable and $\cup \mathscr{A}_m$ is also countable. If $\alpha \notin \cup \mathscr{A}_m$ then $A\xi^{(\alpha)} = 0$; since $\mathscr{A}$ is uncountable we have a contradiction.

We next apply Proposition 4 to obtain the form required for studying spaces of bounded linear operators.

Proposition 5. *Let E be a separable Banach space and suppose $\Phi : l_\infty \to L(E, l_\infty)$ is a bounded linear operator with $\Phi(\delta^n) = 0$ for all n. Then there is an infinite subset M of N such that for $\xi \in l_\infty(M)$, $\Phi(\xi) = 0$.*

Proof. Let $\{e_n\}$ be a countable dense subset of the unit ball of E and define

$$\theta : L(E, l_\infty) \to l_\infty \,,$$

$$\pi_m(\theta(T)) = \pi_{\alpha_1(m)}(Te_{\alpha_2(m)})$$

where $\alpha : N \to N \times N$ is some bijection. θ is an isometric embedding and so the result follows from Proposition 4.

We now come to the main theorem of this section.

Theorem 4. *Let E and F be Banach spaces; the following are equivalent:*

(i) *$K(E, F)$ contains a copy of l_∞.*

(ii) *Either F contains a copy of l_∞ or E contains a complemented copy of l_1.*

Proof. (ii)$\Rightarrow$(i) If F contains a copy of l_∞ let $S : l_\infty \to F$ be a linear embedding. Then fix $e^* \in E^*$ with $\|e^*\| = 1$ and define $\Phi : l_\infty \to K(E, F)$ by

$$\Phi(\xi)\, e = e^*(e)\, S(\xi) \qquad e \in E \,.$$

Then $\|\Phi(\xi)\| = \|S(\xi)\|$ so that Φ is a linear embedding.

If E contains a complemented copy of l_1 E^* contains a copy of l_∞. Let $S : l_\infty \to E^*$ be a linear embedding; fixing $f \in F$ with $\|f\| = 1$ we define $\Phi : l_\infty \to K(E, F)$ by

$$\Phi(\xi)\, e = S(\xi)\,(e) f \qquad e \in E$$

and then $\|\Phi(\xi)\| = \|S(\xi)\|$, as before.

(i)$\Rightarrow$(ii) Suppose (ii) is false and that $\Phi : l_\infty \to K(E,F)$ is a linear embedding. Let $\Phi(\delta^n) = T_n$ in $K(E,F)$. For $e \in E$ we define $\Lambda_e : l_\infty \to F$

$$\Lambda_e \xi = \Phi(\xi)\, e .$$

Then Λ_e is linear and continuous. As F contains no copy of l_∞, Propositions 1 and 2 together yield that $\sum_{n=1}^{\infty} \xi_n \Lambda_e \delta^n$ converges for $\xi \in l_\infty$. Hence the series $\Sigma\, \xi_n T_n$ converges in the strong-operator topology in $L(E,F)$ and therefore, a fortiori, in the topology w. We define $\Psi : l_\infty \to L(E,F)$ by

$$\Psi(\xi) = \sum_{n=1}^{\infty} \xi_n T_n .$$

It is easy to see that Ψ is a bounded linear operator; for example, this may be proved by showing Ψ continuous for the topologies $\sigma(l_\infty, l_1)$ and w and then applying the Closed Graph Theorem.

For $f^* \in F^*$

$$\Psi(\xi)^* f^* = \sum_{n=1}^{\infty} \xi_n T_n^* f^*$$

in the weak*-topology on E^*, and convergence is unconditional. Now we apply Proposition 3 to give that

$$\Psi(\xi)^* f^* = \sum_{n=1}^{\infty} \xi_n T_n^* f^*$$

in the weak topology on E^*.

Choose $e_n \in E$ and $f_n^* \in F^*$ such that

$$|f_n^*(T_n e_n)| \geqq \tfrac{1}{2} \|T_n\|$$

and $\|e_n\| = \|f_n^*\| = 1$. We let G be the closed linear span of $\{e_1, e_2, \ldots,\}$ and define a map $S : F \to l_\infty$ by

$$Sf = \{f_n^*(f)\} .$$

We can then construct $\Gamma : l_\infty \to K(G, l_\infty)$ and $\Delta : l_\infty \to L(G, l_\infty)$ by

$$\Gamma(\xi) = S\Phi(\xi)\, J ,$$

$$\Delta(\xi) = S\Psi(\xi)\, J$$

where $J : G \to E$ is the inclusion map. Then $\Gamma(\delta^n) = \Delta(\delta^n) = S T_n J = Q_n$ say. Then

$$\begin{aligned} \|Q_n\| &\geqq |\pi_n(Q_n e_n)| \\ &= |f_n^*(T_n e_n)| \qquad (1) \\ &\geqq \tfrac{1}{2} \|T_n\| . \end{aligned}$$

Furthermore, for $\gamma \in l_\infty^*$

$$\Delta(\xi)^*\gamma = J^*\Psi(\xi)^*S^*\gamma$$
$$= J^*\left(\sum_{n=1}^{\infty} \xi_n T_n^* S^*\gamma\right)$$

where the series converges weakly in E^*. Since $J^*: E^* \to G^*$ is continuous for the weak topologies

$$\Delta(\xi)^*\gamma = \sum_{n=1}^{\infty} \xi_n J^* T_n^* S^*\gamma \qquad \text{weakly}$$
$$= \sum_{n=1}^{\infty} \xi_n Q_n^*\gamma \qquad \text{weakly}$$

i.e.

$$\Delta(\xi) = \sum_{n=1}^{\infty} \xi_n Q_n \quad (w'). \tag{2}$$

We now quote Proposition 5. As G is separable and $\Delta(\delta^n) = \Gamma(\delta^n) = Q_n$ we may conclude the existence of an infinite subset M of N such that $\Delta(\xi) = \Gamma(\xi)$ for $\xi \in l_\infty(M)$. Thus for $\xi \in l_\infty(M)$

$$\Gamma(\xi) = \sum_{n=1}^{\infty} \xi_n Q_n \quad (w')$$

by (2) and the convergence is subseries in $(K(G, l_\infty), w')$.

By Corollary 3 to Theorem 1

$$\Gamma(\xi) = \sum_{n=1}^{\infty} \xi_n Q_n \quad \text{(weakly)}$$

and by the Orlicz-Pettis theorem

$$\Gamma(\xi) = \sum_{n=1}^{\infty} \xi_n Q_n \quad \text{(norm)}.$$

Hence

$$\inf_{n \in M} \|Q_n\| = 0$$

and by (1)

$$\inf_{n \in N} \|T_n\| = 0$$

contrary to the assumption that Γ was an isomorphism.

Corollary. *If E contains no complemented copy of l_1 and F no copy of l_∞, then every bounded linear operator $\Phi: l_\infty \to K(E, F)$ is weakly compact.*

Proof. Proposition 2.

Let us now indicate a result on operator-valued measures, which perhaps clarifies some ideas concerning spectral measures for compact operators.

Theorem 5. *Let E be a Banach space containing no complemented copy of l_1, and let F be any Banach space. Let μ be a $K(E, F)$-valued measure defined on a σ-algebra $\mathcal{S}$ which is σ-additive for the weak-operator topology. Then μ is norm σ-additive. Equivalently if $\sum_{n=1}^{\infty} T_n$ is w-subseries convergent in $K(E, F)$ the ΣT_n converges in norm.*

Proof. Let $A_n \in \mathcal{S}$ be a sequence of disjoint sets. Then

$$\sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

in the weak-operator topology, unconditionally. For $f^* \in F^*$

$$\sum_{n=1}^{\infty} (\mu(A_n))^* f^* = \left[\mu\left(\bigcup_{n=1}^{\infty} A_n\right)\right]^* f^*$$

in the weak*-topology, unconditionally. By Proposition 3, the series converges weakly in E^*, i.e.

$$\sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \quad (w').$$

By Theorem 1, Corollary 3, and the Orlicz-Pettis Theorem, $\Sigma\,\mu(A_n)$ converges in norm to $\mu\left(\bigcup_{n=1}^{\infty} A_n\right)$.

4. Unconditional Bases

Tong [20] and Tong and Wilken [21] have obtained some interesting results concerning $K(E, F)$ when either E or F has an unconditional basis. Theorem 6 below improves and complements their results; in [19] it is established that (i), (ii), and (iii) are equivalent provided F is a dual space, while in [21] (iv) and (i) are shown equivalent when F has an unconditional basis (but not necessarily E). We assume a slightly less restrictive assumption that of an unconditional basis of E, namely that E has an unconditional finite-dimensional expansion of the identity, i.e. a sequence of bounded finite-dimensional operators $A_n: E \to E$ such that for $x \in E$

$$x = \sum_{n=1}^{\infty} A_n x$$

unconditionally. It is easy to show that for $\xi \in l_\infty$, the series $\sum_{n=1}^{\infty} \xi_n A_n$ converges in the strong operator topology of $L(E, F)$ and that the induced map $l_\infty \to L(E, F)$ is norm continuous. In fact this assumption is not very much more general than that of an unconditional basis, for Pelcyzński and Wojtaszczyzk [13] show that if E has an unconditional finite-dimensional expansion of the identity then E is isomorphic to a complemented subspace of a space with an unconditional finite-dimensional Schauder decomposition.

We first prove some preparatory results concerning the complementation of $K(E,F)$ in $L(E,F)$.

Lemma 2. *Suppose E is separable and $K(E,F)$ is complemented in $L(E,F)$ and $\Phi: l_\infty \to L(E,F)$ has the following properties*

(i) $\Phi(\delta^n) \in K(E,F)$ $\quad n=1,2,\dots,$

(ii) $\{\Phi(\xi)\,e;\ \xi \in l_\infty, e \in E\}$ *is separable,*

then for every infinite subset M of N, there exists an infinite subset M_0 of M with $\Phi(\xi) \in K(E,F)$ for $\xi \in M_0$.

Proof. Let $F_0 = \operatorname{lin}\{\Phi(\xi)\,e;\ \xi \in l_\infty, e \in E\}$ so that F_0 is a separable subspace of F. Hence there is an isometry $J: F_0 \to l_\infty$ and this may be extended to a linear operator $S: F \to l_\infty$ with $\|S\| \leqq 1$. Let $\Gamma: L(E,F) \to K(E,F)$ be a bounded projection.

It is clearly sufficient to establish the result when $M=N$. We define maps

$$\Psi: l_\infty \to L(E, l_\infty)$$

$$\Psi_1: l_\infty \to K(E, l_\infty)$$

by

$$\Psi(\xi)\,e = S\Phi(\xi)\,e$$

$$\Psi_1(\xi)\,e = S\Gamma[\Phi(\xi)]\,e\,.$$

Then both Ψ and Ψ_1 are bounded and linear and $\Psi(\delta^n) = \Psi_1(\delta^n)$, $n=1,2,\dots$. Hence by Proposition 5 there exists an infinite subset M_0 of N with $\Psi(\xi) = \Psi_1(\xi)$, $\xi \in l_\infty(M_0)$. In particular $\Psi(\xi) \in K(E, l_\infty)$, $\xi \in l_\infty(M_0)$, and as S is an isometry on F_0, $\Phi(\xi) \in K(E,F)$, $\xi \in l_\infty(M_0)$.

Lemma 3. *Suppose E contains a complemented subspace isomorphic to l_1 and that F is infinite-dimensional. Then $K(E,F)$ is uncomplemented in $L(E,F)$.*

Proof. Suppose $\Gamma: L(E,F) \to K(E,F)$ is a bounded projection and that $H \subset E$ is a subspace isomorphic to l_1 with a bounded projection $P: E \to H$. We define $\Delta: L(H,F) \to K(H,F)$ by

$$\Delta(T)\,h = \Gamma(TP)\,h, \qquad h \in H,$$

and then Δ is a bounded projection. Hence it suffices to prove the result if $E = l_1$.

Let $\{f_n\}$ be any sequence in F such that $\|f_n\| \leqq 1$ and

$$\inf_{n \neq m} \|f_n - f_m\| = \varepsilon > 0\,.$$

Such a sequence exists since F is infinite-dimensional. We define the map

$$\Phi: l_\infty \to L(l_1, F)$$

by

$$\Phi(\xi)\,x = \sum_{n=1}^{\infty} \xi_n x_n f_n, \qquad x = \{x_n\} \in l_1\,.$$

By Lemma 2 if $K(l_1, F)$ is complemented in $L(l_1, F)$, there exists an infinite subset M of N with $\Phi(\chi) \in K(l_1, F)$ where $\chi_i = 1$ $i \in M$, $\chi_i = 0$ otherwise.

For $m \in M$

$$\Phi(\chi)\,\delta^m = f_m$$

and so the sequence $\{\Phi(\chi)\,\delta^m; m \in M\}$ has no clusterpoint, and we have a contradiction.

Theorem 6. *Let E be a Banach space with an unconditional finite-dimensional expansion of the identity $\{A_n\}$. If F is any infinite-dimensional Banach space the following are equivalent.*

(i) $K(E, F) = L(E, F)$.

(ii) *$K(E, F)$ contains no copy of c_0.*

(iii) *$L(E, F)$ contains no copy of l_∞.*

(iv) *$K(E, F)$ is complemented in $L(E, F)$.*

(v) *For $T \in L(E, F)$ the series $\sum_{n=1}^{\infty} TA_n$ converges in norm.*

Proof. We show (i)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(v)$\Rightarrow$(i) and (i)$\Rightarrow$(iv)$\Rightarrow$(v).

(i)$\Rightarrow$(iii). Suppose (iii) is false; then by Theorem 4 either (a) E contains a complemented subspace isomorphic to l_1 or (b) F contains a subspace isomorphic to l_∞. In case (a) there is a surjection of E onto any closed separable subspace of F and hence $K(E, F) \neq L(E, F)$; in case (b) there is an embedding of the separable space E into F and hence $K(E, F) \neq L(E, F)$.

(iii)$\Rightarrow$(ii). First suppose F contains a copy of c_0. Then since E is separable there is a sequence e_n^* in E^* such that $\|e_n^*\| = 1$ and $e_n^* \to 0$, $\sigma(E^*, E)$. Now define a map

$$\Phi : l_\infty \to L(E, c_0) \subsetneq L(E, F),$$

by

$$\Phi(\xi)\,e = \{\xi_n e_n^*(e)\}.$$

Then Φ is an embedding of l_∞ into $L(E, F)$.

Hence we assume F contains no copy of c_0 and that $\Psi : c_0 \to K(E, F)$ is an embedding. For $e \in E$, the series $\sum_{n=1}^{\infty} \xi_n \Psi(\delta^n)e$ converges unconditionally in F (Proposition 3). We therefore define $\Phi : l_\infty \to L(E, F)$ by

$$\Phi(\xi) = \sum_{n=1}^{\infty} \xi_n \Psi(\delta^n) \qquad \text{(strong operator topology)}.$$

It is easy to verify that Φ is indeed a bounded linear map (cf. the same construction in Theorem 4).

Now by Proposition 2 Φ is weakly compact and by Proposition 1 we obtain

$$\begin{aligned}\lim_{n\to\infty} \|\Phi(\delta^n)\| &= \lim_{n\to\infty} \|\Psi(\delta^n)\| \\ &= 0,\end{aligned}$$

contrary to the initial assumption that Ψ was an embedding.

(ii)$\Rightarrow$(v). Since $\Sigma A_n = I$ unconditionally in the strong operator topology we have

$$\sup_{J \subset N} \left\| \sum_{n \in J} A_n \right\| < \infty$$

where J runs over all finite subsets of N, by the Uniform Boundedness Principle. Therefore for $T \in L(E,F)$ $\sum_{n=1}^{\infty} TA_n$ is w.u.c. in $K(E,F)$. By Proposition 3, $\sum_{n=1}^{\infty} TA_n$ converges in norm.

(v)$\Rightarrow$(i). Since $T = \sum_{n=1}^{\infty} TA_n$ in the strong-operator topology, we have that $\left\| T - \sum_{n=1}^{m} TA_n \right\| \to 0$. Therefore T is the uniform limit of finite-dimensional operators and is compact.

(i)$\Rightarrow$(iv). Trivial.

(iv)$\Rightarrow$(v). By Lemma 3 we may assume E contains no complemented copy of l_1. Suppose (v) fails; then for some $T \in L(E,F)$ there is an $\varepsilon > 0$ and an increasing sequence n_k with $n_0 = 0$ such that if

$$C_k = \sum_{n_{k-1}+1}^{n_k} A_i \qquad k = 1, 2, \ldots$$

then

$$\|TC_k\| \geqq \varepsilon .$$

We define

$$\Phi : l_\infty \to L(E,F)$$

by

$$\Phi(\xi)\, e = T\left(\sum_{k=1}^{\infty} \xi_k C_k e \right)$$

and it is easy to check that Φ is bounded and linear. Further $\{\Phi(\xi)\, e, \xi \in l_\infty, e \in E\} \subset T(E)$ and is therefore separable, and

$$\Phi(\delta^k) = TC_k$$

is finite-dimensional and therefore compact. Hence for some infinite subset M of N, $\Phi(\xi) \in K(E,F)$ for $\xi \in l_\infty(M)$. Then $\sum_{m \in M} \Phi(\delta^m)$ is weak-operator subseries convergent in $K(E,F)$ for if $M_1 \subset M$,

$$\sum_{m \in M_1} \Phi(\delta^m)\, e = T\left(\sum_{m \in M_1} C_m e \right)$$
$$= \Phi(\chi)\, e$$

where χ is the characteristic function of M_1 and $\chi \in l_\infty(M)$. Hence by Theorem 5, $\sum_{m \in M} \Phi(\delta^m)$ converges in norm and so

$$\inf_{m \in M} \|\Phi(\delta^m)\| = \inf_{m \in M} \|TC_m\| = 0$$

contrary to assumption.

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