JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 46, 228–241 (1974)

Boundary Value Problems for Nonlinear Partial Differential Operators

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Submitted by J. P. LaSalle

1. INTRODUCTION

The present paper is a sequel to our paper [2] on differential games of survival and is also related to the work of Fleming [4] on Cauchy problems for degenerate parabolic equations. Our notation and terminology, again outlined in Section 2, is that of [2].

In our recent paper [3], using a related stochastic game, we proved that for fixed time differential games our value U is the same as the upper value V^+ of Friedman [5] (see also [1, Sect. 4]). In Section 3 of this paper we extend this result to games of survival and prove that if the boundary ∂F of the terminal set F is regular then $U = V = Q^+$, where Q^+ is the upper value introduced in [2].

We next prove that if there are upper and lower solutions of the related Isaacs–Bellman equation then U is Lipschitz continuous and so, almost everywhere, a solution of the equation.

We then consider a nonlinear equation of the form

$$Lv = \frac{\partial v}{\partial t} + G(t, x, \nabla v) = 0$$

together with the boundary condition $v \mid \partial F = g$. Under certain hypotheses on G we show by constructing a related differential game that if there are C^1 functions θ_1 and θ_2 such that $L\theta_1 \geqslant 0 \geqslant L\theta_2$ and $\theta_1 = \theta_2 = g$ on ∂F then the value of the differential is a generalized solution of the above boundary value problem. Finally we prove that the solution thus obtained is independent of how a differential game is associated with the problem, that is, the "differential game solution" is unique.

2. Games of Survival

We consider a differential game of survival with control sets Y and Z which are compact metric spaces. The dynamics of the game are given by

$$dx/dt = \dot{x}(t) = f(t, x, y, z) \tag{1}$$

where $x \in R^m$ and $t \in [t_0, \infty)$. $f: R \times R^{m+1} \times Y \times Z \to R^m$ is continuous and satisfies a constant Lipschitz condition in x:

$$\| f(t, x_1, y, z) - f(t, x_2, y, z) \| \leqslant K \| x_1 - x_2 \| \tag{2}$$

We suppose the trajectory satisfies the initial condition

$$x(t_0) = x_0 . \tag{3}$$

A closed terminal set $F \subset R \times R^m$ such that $F \supset [T, \infty) \times R^m$ for some fixed T is considered and the game ends the first time t_F, that the trajectory enters F. The payoff computed is then

$$P(y, z) = g(t_F, x(t_F)) + \int_{t_0}^{t_F} h(t, x(t), y(t), z(t))\, dt \tag{4}$$

where $g: R \times R^m \to R$ and $h: R \times R^m \times Y \times Z$ are continuous. Player J_1 controlling $y \in Y$ is trying to maximize P whilst player J_2 controlling $z \in Z$ is trying to minimize P. We denote by $G(t_0, x_0)$ the differential game with dynamics (1), initial condition (3) and payoff (4).

$\mathscr{M}_1(t_0)$ denotes the space of measurable functions on $[t_0, \infty)$ with values in Y. (A function $y \in \mathscr{M}_1(t_0)$ is measurable if for every continuous real valued function ψ on Y $\psi \circ y$ is Lebesgue measurable, and functions equal almost everywhere are identified.) $\mathscr{M}_2(t_0)$ denotes the space of measurable functions $[t_0, \infty) \to Z$. For $s \geqslant 0$ an *s-delay strategy* α for J_1 is a map $\alpha: \mathscr{M}_2(t_0) \to \mathscr{M}_1(t_0)$ such that wherever

$$z_1(t) = z_2(t) \qquad \text{a.e.} \qquad t_0 \leqslant t \leqslant t_1 ,$$

then

$$(\alpha z_1)(t) = (\alpha z_2)(t) \qquad \text{a.e.} \qquad t_0 \leqslant t \leqslant t_1 + s.$$

The set of s-delay strategies for J_1 is denoted by $\Gamma_{t_0}(s)$. s-delay strategies for J_2 are defined similarly and the set denoted by $\Delta_{t_0}(s)$. For $\alpha \in \Gamma_{t_0}(0)$ its value is defined as

$$u(\alpha) = \inf[P(\alpha z, z);\ z \in \mathscr{M}_2(t_0)].$$

Then

$$U(t_0, x_0) = \sup[u(\alpha); \alpha \in \Gamma_{t_0}(0)]. \tag{5}$$

For $\beta \in \Delta_{t_0}(0)$ we define

$$v(\beta) = \sup[P(y, \beta y); y \in \mathscr{M}_1(t_0)]$$

and

$$V(t_0, x_0) = \inf[v(\beta); B \in \Delta_{t_0}(0)]. \tag{6}$$

We also define upper and lower values

$$V^+(t_0, x_0) = \inf \left[v(\beta); \beta \epsilon \bigcup_{s>0} \Delta t_0(s)\right], \tag{7}$$

$$V^-(t_0, x_0) = \sup \left[u(\alpha); \alpha \epsilon \bigcup_{s>0} \Gamma t_0(S)\right]. \tag{8}$$

For β, $\beta' \in \Delta_{t_0}(0)$ we say that $\beta' \in \Delta_{t_0}^s(\beta)$ if for any $y \in \mathscr{M}_2(t_0)$

$$(\beta' y)(t) = (\beta y)(t) \qquad \text{a.e.} \qquad t_0 + s \leqslant t < \infty.$$

The *s-delay value* of $\beta \in \Delta_{t_0}(0)$ is defined as

$$v_s(\beta) = \sup[v(\beta'); \beta' \in \Delta_{t_0}^s(\beta)].$$

Write $\mathscr{M}_2^s(t_0)$ for the space of measurable functions z: $[t_0, t_0 + s] \to Z$ and also define $\Delta_{t_0}[s \mid z]$ as the set of strategies $\beta \in \Delta_{t_0}(s)$ such that $\beta y(t) = z(t)$ a.e. $t_0 \leqslant t \leqslant t_0 + s$, where $z \in \mathscr{M}_2^s[t_0]$ is fixed. We define the value

$$Q_s^+(t_0, x_0) = \sup_{z \in \mathscr{M}_2^s[t_0]} \inf_{\beta \in \Delta_{t_0}[s \mid z]} v(\beta). \tag{9}$$

Clearly we have the relation

$$Q_s^+(t_0, x_0) \leqslant \inf_{\beta \in \Delta_{t_0}(s)} v_s(\beta).$$

Finally

$$\begin{aligned} Q^+(t_0, x_0) &= \lim_{s \to 0} Q_s^+(t_0, x_0) \\ &= \inf_{s>0} Q_s^+(t_0, x_0). \end{aligned} \tag{10}$$

There are similar definitions (see [2]) for the s-delay value $u_s(\alpha)$ of $\alpha \in \Gamma_{t_0}(0)$, $Q_s^-(t_0, x_0)$ and $Q^-(t_0, x_0)$.

From [1, 2, and 3] we collect the following results.

THEOREM 2.1.

$$Q^-(t_0, x_0) \leqslant V^-(t_0, x_0) \leqslant V(t_0, x_0),\ U(t_0, x_0) \leqslant V^+(t_0, x_0) \leqslant Q^+(t_0, x_0). \tag{11}$$

For fixed time games (i.e., $F = [T, \infty) \times R^m$)

$$Q^-(t_0, x_0) = V^-(t_0, x_0) = V(t_0, x_0) \tag{12}$$

and

$$Q^+(t_0, x_0) = V^+(t_0, x_0) = U(t_0, x_0). \tag{13}$$

Consider a map $\tau: \mathscr{M}_1(t_0) \times \mathscr{M}_2(t_0) \to [t_0, \infty)$ which prescribes a stopping time corresponding to any pair of control functions. τ is said to be nonanticipating if whenever

$$y'(t) = y(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau(y, z),$$
$$z'(t) = z(t) \quad \text{a.e.} \quad t_0 \leqslant t \leqslant \tau(y, z),$$

then $\tau(y', z') = \tau(y, z)$. Consider a nonanticipating map τ with $\tau(y, z) \leqslant t_F(y, z)$ for all y, z and a real-valued function θ defined on $R \times R^m$. We define a game $G_\tau(t_0, x_0; \theta)$ as the game with initial condition (3), dynamics (1) and payoff:

$$P_{\tau,\theta}(y, z) = \theta(\tau(y, z), x(\tau(y, z)) + \int_{t_0}^{\tau(y,z)} h(t, x(t), y(t), z(t))\, dt. \tag{14}$$

The various upper and lower values for $G_\tau(t_0, x_0; \theta)$ are defined in analogous ways in [2, Section 3] but the particularly important cases are when θ is itself a value of the game $G(\tau(y, z), x(\tau(y, z)))$. The following dynamic programming results are then obtained in [2].

THEOREM 2.2. *Suppose τ is non-anticipating and $\tau \leqslant t_F$. Then*

$$U(t_0, x_0) = U(t_0, x_0; U), \tag{15}$$

$$V^+(t_0, x_0) \geqslant V^+(t_0, x_0; V^+), \tag{16}$$

and

$$Q_s^+(t_0, x_0) \leqslant Q_{s,\tau}^+(t_0, x_0; Q_s^+), \tag{17}$$

$$Q_s^+(t_0, x_0) \geqslant Q_{s,\tau}^+(t_0, x_0; U). \tag{18}$$

Similar results hold for the various lower values. Further, if

$$\lim_{s\to 0} Q_s^+(t, x) = Q^+(t, x)$$

uniformly on compacta then

$$Q^+(t_0, x_0) \leqslant Q_\tau^+(t_0, x_0; Q^+). \tag{19}$$

3. Equalities for Value Functions

In this section we extend the identity (13) of Theorem 2.1 to games of survival.

For each $(t, x) \in R^{m+1} - \operatorname{int} F$ define

$$\bar{Q}_s^+(t, x) = \limsup_{(\tau,\xi)\to(t,x)} Q_s^+(t, x).$$

DEFINITION 3.1. A point (t, x) of ∂F is *Q^+-regular* if it satisfies

$$\lim_{s\to 0} \bar{Q}_s^+(t, x) = g(t, x) \tag{20}$$

and U is continuous at (t, x). ∂F is Q^+-regular if every point of ∂F is Q^+-regular.

We can now prove the following result.

THEOREM 3.2. *Suppose f satisfies a Lipschitz condition in both t and x*

$$\|f(t_1, x_1, y, z) - f(t_2, x_2, y, z)\| \leqslant K(|t_1 - t_2| + \|x_1 - x_2\|) \tag{21}$$

and ∂F is Q^+-regular. Then

$$U(t_0, x_0) = V^+(t_0, x_0) = Q^+(t_0, x_0) \qquad \text{for all } (t_0, x_0) \in R^{m+1} - \operatorname{int} F. \tag{22}$$

Proof. Theorem 6.4 of [2] states that under these hypotheses

$$V^+(t_0, x_0) = Q^+(t_0, x_0)$$

so only the identity involving U remains to be proved; the present proof, by contradiction, is similar. As quoted in Theorem 2.1, for fixed time games we know

$$U(t_0, x_0) = V^+(t_0, x_0) = Q^+(t_0, x_0).$$

Suppose $(t_0, x_0) \in \partial F$. Theorem 6.3 of [2] states that if ∂F is Q^+-regular then Q^+ is continuous on $R^{m+1} - \operatorname{int} F$ and $\lim_{s\to 0} Q_s^+(t, x) = Q^+(t, x)$ uniformly on compacta. Therefore, from the definitons, for $(t_0, x_0) \in \partial F$

$$U(t_0, x_0) = V^+(t_0, x_0) = Q^+(t_0, x_0).$$

Suppose now $(t_0, x_0) \in R^{m+1} - F$ is a point where

$$Q^+(t_0, x_0) - U(t_0, x_0) = \epsilon > 0.$$

Then every trajectory with initial point (t_0, x_0) is contained in a sphere

$$B_R = \{(t, x): |t| + \|x\| \leqslant R\}$$

for some $R > 0$. Let

$$\sup_{B_R \times Y \times Z} \|f(t, x, y, z)\| = M.$$

Determine inductively a sequence (t_n, x_n) in B_R for $n \geqslant 1$ with $\rho(t_n, x_n) > 0$ for all n. Here $\rho(t, x)$ denotes the distance from (t, x) to F. Suppose (t_k, x_k) has been determined. Write

$$\tau_k = (2(M+1))^{-1} \rho(t_k, x_k) + t_k \tag{23}$$

and consider $G_{\tau_k}(t_k, x_k; Q^+)$.

Then by (19)

$$Q^+(t_k, x_k) \leqslant Q^+_{\tau_k}(t_k, x_k; Q^+).$$

G_{τ_k} is a fixed duration game so $Q^+(t_k, x_k) \leqslant U_{\tau_k}(t_k, x_k; Q^+)$. But by (15)

$$U(t_k, x_k) = U_{\tau_k}(t_k, x_k; U)$$

so

$$Q^+(t_k, x_k) - U(t_k, x_k) \leqslant U_{\tau_k}(t_k, x_k; Q^+) - U_{\tau_k}(t_k, x_k; U).$$

Therefore, there is a trajectory $\xi(t)$ in $G_{\tau_k}(t_k, x_k)$ such that

$$Q^+(t_{k+1}, x_{k+1}) - U(t_{k+1}, x_{k+1}) \geqslant Q^+(t_k, x_k) - U(t_k, x_k) - \epsilon/2^{k+2},$$

where

$$t_{k+1} = \tau_k \qquad \text{and} \qquad \xi(t_{k+1}) = x_{k+1}.$$

Also

$$\rho(t_{k+1}, x_{k+1}) \geqslant \rho(t_k, x_k) - (M+1)(t_{k+1} - t_k) > 0$$

and certainly (t_n, x_n) is a sequence in B_R because each (t_n, x_n) is reached from (t_0, x_0) by a trajectory. By construction

$$Q^+(t_n, x_n) - U(t_n, x_n) \geqslant \epsilon \left(1 - \sum_{k=1}^{n} 2^{-(k+1)}\right) \geqslant \epsilon/2. \tag{24}$$

For each n $(t_n, x_n) \notin F$ so $t_n \leqslant T$ for all n. Thus

$$\lim_{n\to\infty} t_n = t_0 + \sum_{k=1}^{\infty} (t_{k+1} - t_k) = \bar{t}$$

exists and we have

$$\lim_{k\to\infty}(t_{k+1} - t_k) = 0.$$

As $\| x_{k+1} - x_k \| \leqslant M(t_{k+1} - t_k)$ the sequence x_k converges to $\bar{x}$ say and by (23) $\lim_{k\to\infty} \rho(t_k, x_k) = 0$. Therefore $(\bar{t}, \bar{x}) \in F$ so by the continuity of Q^+ and U:

$$\lim_{k\to\infty}(Q^+(t_k, x_k) - U(t_k, x_k)) = 0.$$

However, this contradicts (24) so we must have

$$Q^+(t_0, x_0) = U(t_0, x_0) = V^+(t_0, x_0).$$

4. Lipschitz Continuity

In this section we prove under certain hypotheses that the value functions are Lipschitz continuous and so almost everywhere they are solutions of the Isaac's–Bellman equation. A consequence of Section 3 is that we can confine our attention to the value U, which because it satisfies, for example, identity (15), has several advantages.

Theorem 4.1. *Suppose f satisfies the Lipschitz condition* (21) *in* (t, x), *suppose h is Lipschitz in* (t, x) *and suppose U is uniformly Lipschitz in* (t, x) *on ∂F in any bounded region of R^{m+1}. Then U is uniformly Lipschitz in any region of the form*

$$(R^{m+1} - \operatorname{int} F) \cap ((T_0 \leqslant t \leqslant T) \times R^m).$$

Proof. The initial part of the proof is similar to that of Theorem 6.3 of [2]. Consider the set $C = B_r \cap (R^{m+1} - \operatorname{int} F)$ for some $r > 0$. Then any trajectory of (1) with initial point in C is contained for $r \leqslant T$ in some set B_R. Suppose (t_1, x_1) and $(t_2, x_2) \in C$ and write:

$$\sigma = t_2 - t_1 \tag{25}$$

$$\delta = | t_2 - t_1 | + \| x_2 - x_1 \|. \tag{26}$$

For $y = y(t) \in \mathscr{M}_1(t_1)$ and $z = z(t) \in \mathscr{M}_2(t_1)$ we associate controls

$$y^* = y^*(t) \in \mathscr{M}_1(t_2) \qquad \text{and} \qquad z^* = z^*(t) \in \mathscr{M}_2(t_2)$$

by

$$y^*(t) = y(t - \sigma), \qquad z^*(t) = z(t - \sigma).$$

The map $(y, z) \to (y^*, z^*)$ is a nonanticipating bijection and it induces a map between the spaces of strategies in $G(t_1, x_1)$ and $G(t_2, x_2)$. Suppose $\xi_1(t)$ and $\xi_2(t)$ are trajectories corresponding to (y, z) and (y^*, z^*), respectively, in $G(t_1, x_1)$ and $G(t_2, x_2)$. Then

$$\frac{d}{dt}(\xi_1(t) - \xi_2(t + \sigma)) = f(t, \xi_1(t), y(t), z(t)) - f(t + \sigma, \xi_2(t + \sigma), y(t), z(t))$$

from which we deduce

$$\|\xi_1(t) - \xi_2(t + \sigma)\| \leqslant \delta e^{K(t-t_1)} - |\sigma| \leqslant \delta e^{K(T+r)} - |\sigma| \qquad \text{for } t_1 \leqslant t \leqslant T. \tag{27}$$

A stopping time $\tau: \mathscr{M}_1(t_1) \times \mathscr{M}_2(t_1) \to R$ is defined by $\tau = \tau(y, z)$ where

$$\min\{\rho(\tau, \xi_1(\tau)); \rho(\tau + \sigma, \xi_2(\tau + \sigma))\} = 0$$

but

$$\min\{\rho(t, \xi_1(t)); \rho(t + \sigma, \xi_2(t + \sigma))\} > 0 \qquad \text{for } t_1 \leqslant t < \tau.$$

Then τ is non-anticipating and $\tau \leqslant T$.

Suppose, without loss of generality, that $(\tau, \xi_1(\tau)) \in \partial F$. Then by (27)

$$\rho(\tau + \sigma, \xi_2(\tau + \sigma)) \leqslant \delta e^{(T+r)} - |\sigma|.$$

By hypothesis there is a constant K' such that

$$|h(t_1, x_1, y, z) - h(t_2, z_2, y, z)| \leqslant K'(|t_1 - t_2| + \|x_1 - x_2\|) \tag{28}$$

and a constant L such that

$$|U(\tau, \xi_1(\tau)) - U(\tau + \sigma, \xi_2(\tau + \sigma))| \leqslant L(|\sigma| + \|\xi_1(\tau) - \xi_2(\tau + \sigma)\|),$$

that is:

$$|g(\tau, \xi_1(\tau)) - U(\tau + \sigma, \xi_2(\tau + \sigma))| \leqslant L\delta e^{K(T+r)}. \tag{29}$$

Therefore,

$$\left| \int_{t_1}^{\tau} h(t, \xi_1(t), y(t), z(t)) - \int_{t_2}^{\tau+\sigma} h(t, \xi_2(t), y^*(t), z^*(t))\, dt \right|$$
$$\leqslant K' \int_{t_1}^{\tau} (|\sigma| + \delta e^{K(T+r)} - |\sigma|)\, dt$$
$$\leqslant K'(r + T)\, \delta e^{K(T+r}$$

So,

$$| P_{\tau,u}(t_1, x_1; y, z) - P_{\tau+\sigma,u}(t_2, x_2; y^*, z^*)| \leqslant \delta e^{K(T+r)}(L + K'(r + T)). \tag{30}$$

The same inequality is true if $(\tau + \sigma, \xi_2(\theta + \sigma)) \in \partial F$ so by the remarks above on strategies in $G(t_1, x_1)$ and $G(t_2, x_2)$:

$$| U_{\tau,u}(t_1, x_1) - U_{\tau+\sigma,u}(t_2, x_2)| \leqslant M\delta \tag{31}$$

where

$$M = e^{K(T+r)}(L + K'(r + T)).$$

By (15)

$$| U(t_1, x_1) - U(t_2, x_2)| \leqslant M\delta$$

and so U is uniformly Lipschitz continuous in regions of the form

$$(R^{m+1} - \operatorname{int} F) \cap ((T_0 \leqslant t \leqslant T) \times R^m).$$

THEOREM 4.2. *Suppose f and h are Lipschitz in (t, x) and suppose that there exist functions θ_1, θ_2, both C^1 on $R^{m+1} - \operatorname{int} F$, such that $\theta_1 = \theta_2 = g$ on ∂F and $L^+\theta_1 \leqslant 0 \leqslant L^+\theta_2$ on $R^{m+1} - \operatorname{int} F$, where*

$$L^+\theta = \frac{\partial \theta}{\partial t} + \min_z \max_y (\nabla\theta f + h).$$

Then the value U of the game is almost everywhere a solution of the equation $L^+U = 0$ satisfying the boundary condition $U|_{\partial F} = g$.

Remark. It is proved in Theorem 7.1 of [2] that the existence of functions θ_1 and θ_2 as above implies ∂F is Q^+-regular. Therefore, by Theorem 3.2 of the present paper $U = V^+ = Q^+$, so that V^+ and Q^+ also satisfy the equation and boundary condition.

Proof of 4.2. By Theorems 5.3 and 5.4 of [2], for $(t, x) \in R^{m+1} - \operatorname{int} F$:

$$\theta_2(t, x) \leqslant U(t, x) \leqslant \theta_1(t, x).$$

Certainly

$$\theta_2(\tau, \xi) = U(\tau, \xi) = \theta_1(\tau, \xi) = g(\tau, \xi) \qquad \text{for } (\tau, \xi) \in \partial F.$$

and so

$$\theta_2(t, x) - \theta_2(\tau, \xi) \leqslant U(t, x) - U(\tau, \xi) \leqslant \theta_1(t, x) - \theta_1(\tau, \xi).$$

By the mean value theorem for θ_1 and θ_2 there is a constant L such that

$$| U(t, x) - U(\tau, \xi)| \leqslant L(| t - \tau | + \| x - \xi \|) \qquad \text{for } (\tau, \xi) \in \partial F$$

L is just a bound for the derivative of θ_1 and θ_2, and so is bounded in compact sets of R^{m+1}. U is, therefore, uniformly Lipschitz in (t, x) on ∂F in any bounded region of R^{m+1} and so by Theorem 4.1, U is uniformly Lipschitz in bounded regions of $R^{m+1} - \operatorname{int} F$.

By Rademacher's Theorem (see Section 4.1 of [5]) $U(t, x)$ is thus differentiable at almost all points (t, x) and so by Theorem 4.4 of [2] $L^+U = 0$ almost everywhere.

We state without proof the analogous theorem for lower values:

THEOREM 4.3. *Suppose f and h are Lipschitz in (t, x) and suppose there exist functions θ_1, θ_2 both C^1 on $R^{m+1} - \operatorname{int} F$, such that $\theta_1 = \theta_2 = g$ on ∂F and $L^-\theta_1 \geqslant 0 \geqslant L^-\theta_2$ on $R^{m+1} - \operatorname{int} F$ where*

$$L^-\theta = \partial\theta/\partial t + \max_y \min_z (\nabla\theta \cdot f + h).$$

Then the value V of the game is almost everywhere a solution of the equation $L^-V = 0$ satisfying the boundary condition $V|_{\partial F} = g$.

5. NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

We now apply Theorem 4.3 to obtain results for boundary value problems for certain non-linear partial differential equations. This extends work of Fleming [4] on the Cauchy problem for such equations. We construct a generalized solution, that is a function differentiable almost everywhere which at points of differentiability satisfies the equation.

As in Section 1 $F \subset R \times R^m$ denotes a closed set such that $F \supset [T, \infty) \times R^m$ for some fixed T. Also $t \in [t_0, \infty)$ and $x \times R^m$. Consider the nonlinear partial differential operator

$$Lv = \partial v/\partial t + G(t, x, \nabla v). \tag{32}$$

and $G(t, x, p)$ is uniformly Lipschitz in all variables. Further, we suppose there is a constant B_1 such that

$$| G(t, x, 0)| \leqslant B_1 . \tag{33}$$

For suitable bounds α, β we introduce the control sets

$$Y = \{y \in R^m : | y | \leqslant \alpha\} \tag{34}$$

$$Z = \{z \in R^m : | z | \leqslant \beta\} \tag{35}$$

and the functions

$$f(t, x, y, z) = \frac{G(t, x, y)\, y}{1 + |y|^2} + z \tag{36}$$

$$h(t, x, y, z) = \frac{G(t, x, y)}{1 + |y|^2} - y \cdot z. \tag{37}$$

Writing

$$G_0(y) = G(t, x, y)/1 + |y|^2$$

and, for $i = 1, \ldots, m$,

$$G_i(y) = G(t, x, y)\, y_i/1 + |y|^2$$

it is proved in Fleming [4, p. 998] that there is a constant B_2 such that

$$\left| G_0(y) - G_0(p) + \sum_{i=1}^{m} (G_i(y) - G_i(p))\, p_i \right| \leqslant B_2 \,|\, y - p \,|\,. \tag{38}$$

From this Fleming deduces that if $|\, p \,| \leqslant \alpha$ and $B_2 = \beta$ then

$$\begin{aligned} &\max_Y \min_Z (p \cdot f + h) \\ &\quad = \max_Y \min_Z (G(t, x, y)\,(1 + p \cdot y)\,(1 + |\, y \,|^2)^{-1} + z \cdot (p - y)) \qquad (39) \\ &\quad = G(t, x, p). \end{aligned}$$

THEOREM 5.1. *Consider the nonlinear partial differential equation*

$$Lv = \partial v/\partial t + G(t, x, \nabla v) = 0 \tag{40}$$

together with the boundary condition $v\,|_{\partial F} = g$.

Here G is uniformly Lipschiltz in (t, x, p) *and satisfies inequality* (33). *Suppose there are functions θ_1 and θ_2 both C^1 on $R^{m+1} - \operatorname{int} F$ such that the derivatives of θ_1 and θ_2 are uniformly bounded on $R^{m+1} - \operatorname{int} F$, $\theta_1 = \theta_2 = g$ on ∂F and*

$$L(\theta_1) \geqslant 0 \geqslant L(\theta_2) \qquad \text{on} \qquad R^{m+1} - \operatorname{int} F.$$

Then there is a generalized solution $V(t, x)$ of the boundary value problem (40) *in any region of the form*

$$(R^m - \operatorname{int} F) \cap ((T_0 \leqslant t \leqslant T) \times R^m).$$

Proof. Consider the differential game with dynamics

$$\dot{x}(t) = f(t, x, y, z), \tag{40}$$

initial condition $x(\tau) = \xi$ and payoff

$$P(y, z) = g(t_F, x(t_F)) + \int_\tau^{t_F} h(t, x(t), y(t), z(t))\, dt \tag{41}$$

where f and h are the functions defined in (36) and (37). The control sets Y and Z are subsets of Euclidean space as in (34) and (35). The bound β is taken to be the constant B_2 given by inequality (38). Because the derivatives of θ_1 and θ_2 are bounded on $R^{m+1} - \operatorname{int} F$ by L' say, we see as in Theorem 4.2 that the value $V(t, x)$ of this game is uniformly Lipschitz continuous on ∂F and so on all $R^{m+1} - \operatorname{int} F$. This Lipschitz constant L'' for $V(t, x)$ is, therefore, a bound for the derivatives of V at points of differentiability. Taking $\alpha = \max(L', L'')$ we see from (39) that

$$\max_Y \min_Z (\nabla\theta_i \cdot f + h) = G(t, x, \nabla\theta_i), \qquad i = 1, 2$$

and at points of differentiability:

$$\max_Y \min_Z (\nabla V \cdot f + h) = G(t, x, \nabla V).$$

Therefore, for the differential game (40), (41) we have that

$$\begin{aligned} L^-\theta_1 &= \partial\theta_1/\partial t + G(t, x, \nabla\theta_1) \\ &= L\theta_1 \geqslant 0, \\ L^-\theta_2 &= L\theta_2 \leqslant 0 \end{aligned}$$

and so by Theorem 4.3 the value V of this game is a generalized solution of $L^-V = LV = 0$ satisfying $V|_{\partial F} = g$.

Remark 5.2. If the complement Ω of the terminal set F is bounded then we need only require that $G(t, x, 0)$ and the derivatives of θ_1 and θ_2 are bounded in Ω.

In Equations (36) and (37) we have, following Fleming, given one way of constructing a differential game associated with a nonlinear equation. However, the solution obtained is independent of how the differential game is constructed.

THEOREM 5.3. *The differential game solution of the boundary value problem of Theorem* 5.1 *is unique.*

Proof. We first prove that for a nonlinear equation of the form (40) with a fixed final time Cauchy condition

$$v(T, x) = g(x) \tag{42}$$

the differential game solution is unique. Suppose Λ_1 and Λ_2 are two differential games of fixed duration associated with this equation. That is, if Λ_i has dynamics $\dot{x}_i(t) = f(t, x_i, y_i, z_i)$, initial condition $x_i(\tau) = \xi \in R^m$ and payoff $\int_\tau^T h_i(t, x_i, y_i, z_i)\, dt$ then for p,

$$x \in R^m \max_{y_i} \min_{z_i}(p \cdot f + h) = G(t, x, p)$$

for $i = 1$ and 2.

The value functions $V_1(\tau, \xi)$, $V_2(\tau, \xi)$ obtained from Λ_1 and Λ_2 respectively are then generalized solutions of the Equation (40) satisfying $V_i(T, x) = g(x)$, $i = 1, 2$. However, as proved in [3] or [4], each V_i is equal almost everywhere to $\lim_{\epsilon \to 0} \phi_\epsilon$ where ϕ_ϵ is the unique solution of the non-linear parabolic equation $\epsilon \nabla^2 v + L_v = 0$ with boundary condition (42). By continuity $V_1 = V_2$ everywhere.

Consider now the situation of Theorem 5.1. Suppose again that Λ_1 and Λ_2 are differential games associated with (40), with value functions $V_1(t, x)$, $V_2(t, x)$, respectively, both satisfying the boundary value problem. Suppose there is some point $(t_0, x_0) \in R^{m+1} - F$ where

$$V_1(t_0, x_0) - V_2(t_0, x_0) = \epsilon > 0.$$

Now suppose τ is a constant such that $\tau \leqslant t_F$ for any trajectory from (t_0, x_0) in both Λ_1 and Λ_2. Consider fixed time games with the same dynamics and integral payoff as Λ_1 and Λ_2, starting at (t_0, x_0) and ending at time τ. By the identity for V corresponding to (15) we have:

$$V_1(t_0, x_0) = V_{1,\tau}(t_0, x_0; V_1), \tag{43}$$

$$V_2(t_0, x_0) = V_{2,\tau}(t_0, x_0; V_2). \tag{44}$$

However. we could consider Λ_1 with a terminal payoff V_2 at time τ, so by uniqueness for fixed time games we have:

$$V_2(t_0, x_0) = V_{1,\tau}(t_0, x_0; V_2). \tag{45}$$

Therefore

$$V_1(t_0, x_0) - V_2(t_0, x_0) = V_{1,\tau}(t_0, x_0; V_1) - V_{1,\tau}(t_0, x_0; V_2)$$

and we can apply the inductive construction of Theorem 3.2 to deduce a contradiction. Therefore, $V_1(t_0, x_0) = V_2(t_0, x_0)$ for all $(t_0, x_0) \in R^{m+1} - \operatorname{int} F$ and the differential game solution is unique.

THEOREM 5.4. *Under the hypotheses of Theorem* 5.1 *a unique differential game solution of the boundary value problem* (40) *exists in* $R^{m+1} - \operatorname{int} F$.

Proof. We proved that a differential game solution exists in

$$(R^{m+1} - \operatorname{int} F) \cap ((T_0 \leqslant t \leqslant T) \times R^m).$$

If $T_0' \leqslant T_0$ a differential game solution also exists in

$$(R^{m+1} - \operatorname{int} F) \times ((T_0' \leqslant t \leqslant T) \times R^m).$$

By Theorem 5.3 the two solutions are equal on the intersection of their domains.

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