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Hermitian operators on complex Banach lattices and a problem of Garth Dales

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Abstract

Let $E \oplus F$ be a direct sum decomposition of a complex Banach lattice X. Garth Dales asked recently whether the equation $\|x+y\| = \| \, |x| \vee |y| \, \|$ for all $x \in E$ and $y \in F$ implies that E and F are bands. We show that this is the case by using the theory of hermitian operators. We then show that the same result holds if we replace Dales's condition by $\|x+y\| = \|(|x|^p + |y|^p)^{1/p}\|$ for any $p \neq 2$. To do this, we develop a general theory of hermitian operators on a complex Banach lattice, showing in particular that the operators of the form $S + iT$ with S and T hermitian always form a subalgebra of $\mathcal{L}(X)$, and that this subalgebra is (by the Vidav–Palmer theorem) isometrically a C^*-algebra. A particular conclusion is that, if $E \oplus F$ satisfies $\|x+y\| = \|x + e^{i\theta}y\|$ for all $x \in E$, all $y \in F$, and all $\theta \in [0, 2\pi)$, then it also satisfies the equation $\|x+y\| = \|(|x|^2 + |y|^2)^{1/2}\|$ for all $x \in E$ and $y \in F$.

1. *Introduction*

Let us suppose that X is a complex Banach lattice and that $E \oplus F$ is a direct sum decomposition of X. Garth Dales [7] asked whether the condition

$$\|x+y\| = \| \, |x| \vee |y| \, \|, \quad x \in E, \ y \in F, \tag{1.1}$$

implies that $E \oplus F$ is a band decomposition of X, that is,

$$|x| \wedge |y| = 0, \quad x \in E, \ y \in F. \tag{1.2}$$

He was motivated to raise this query by the theory of multi-normed spaces, which was first discussed in [13] (under a different name) and developed in [8]. In this paper, we shall answer Dales's question affirmatively, using the theory of hermitian operators and numerical ranges (for which, see [4, 5]).

The background for Dales's problem is the following. In [8, Chapter 7], there is an extensive theory of 'orthogonality in multi-normed spaces' related to decompositions of Banach spaces with respect to certain multi-norms. This is applied to the example of a 'lattice multi-norm $(\| \cdot \|_n^L : n \in \mathbb{N})$ based on a (complex) Banach lattice'. It is easy to see that a band decomposition of a Banach lattice is 'orthogonal with respect to the lattice multi-norm'; it was hoped in [7] that the converse of this result would be true. Our result shows that this is indeed the case.

Notice that, for real Banach lattices, the question of Dales has a negative answer, as pointed out in [8]. Indeed, let $X = \ell_1^2$ and suppose that $\{e_1, e_2\}$ is the canonical basis of X. We set

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$E = [e_1 + e_2]$ and $F = [e_1 - e_2]$. Then, for each $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\|a(e_1 + e_2) + b(e_1 - e_2)\| &= |a + b| + |a - b| = \max(2|a|, 2|b|)\\ &= \| \, |a| \, |e_1 + e_2| \vee |b| \, |e_1 - e_2| \, \|.\end{aligned}$$

Thus (1.1) holds, but (1.2) fails.

Let X be a complex Banach space, and suppose that $E \oplus F$ is a direct sum decomposition of X. Then $E \oplus F$ is an *hermitian decomposition* if

$$\|x + e^{i\theta} y\| = \|x + y\|, \quad x \in E, \ \ y \in F, \ \ \theta \in \mathbb{R}. \tag{1.3}$$

This is equivalent to the requirement that the induced projection $P : X \to E$ be an hermitian operator (see below for definitions). These hermitian decompositions are discussed in [**8**, Chapter 7]. Now suppose that X is a complex Banach lattice, then of course (1.1) implies (1.3).

The solution that we shall give of Dales's problem also yields the result that (1.1) can be replaced by the related condition that

$$\|x + y\| = \|(|x|^p + |y|^p)^{1/p}\|, \quad x \in E, \ \ y \in F, \tag{1.4}$$

for some $p \geqslant 4$: if (1.4) holds for some $p \geqslant 4$, then $E \oplus F$ is a band decomposition. (For the definition of $(|x|^p + |y|^p)^{1/p}$ in a Banach lattice, see [**12**, pp. 40–42] or § 3.)

This leads us to consider whether (1.4) for some $p \in [1, \infty)$ with $p \neq 2$ is already sufficient for $E \oplus F$ to be a band decomposition. We shall also prove this more general result, but this requires a lot more work. We shall in fact show that, if $E \oplus F$ is an hermitian decomposition of X, then we have

$$\|x + y\| = \|(|x|^2 + |y|^2)^{1/2}\|, \quad x \in E, \ \ y \in F, \tag{1.5}$$

and from this the conclusion follows quickly.

To establish (1.5), we shall prove some very general results about hermitian operators on Banach lattices. If X is a complex Banach space, then let us denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X and by $\mathcal{J}(X)$ the closed subspace of $\mathcal{L}(X)$ consisting of all operators of the form $T = H + iK$, where H and K are hermitian operators. Then the space $\mathcal{J}(X)$ admits a natural involution $\bigstar$, specified by

$$(H + iK)^{\bigstar} = H - iK.$$

If $\mathcal{J}(X)$ is an algebra, then the Vidav–Palmer theorem [**4**, **15**] implies that, with this involution, $\mathcal{J}(X)$ is (isometrically) a C^*-algebra. However, in general, $\mathcal{J}(X)$ need not be an algebra; in particular, if T is hermitian, it does not follow that T^2 is hermitian (see [**4**, § 6, Example 1]). For recent progress on the properties of T^2 when T is hermitian, see, for example, [**6**].

We shall show that, if X is a complex Banach lattice, then the square of an hermitian operator is always hermitian, and hence that $\mathcal{J}(X)$ is always an algebra. These results supplement earlier work of the author with Wood [**11**] for the special case in which the space X has a 1-unconditional basis.

2. *Hermitian operators*

To discuss our problem, we shall use heavily the theory of hermitian operators on a Banach space. We refer to [**4**, **5**] for a full discussion of this elegant theory; see also [**16**, § 2.6].

Suppose that X is a Banach space. Then we denote the closed unit ball of X by B_X and its surface by ∂B_X; if X is a Banach lattice, then $X^+ = \{x \in X : x \geqslant 0\}$.

Now suppose that X is a complex Banach space. The dual space to X is denoted by X^*, and the dual (or adjoint) operator of $T \in \mathcal{L}(X)$ is $T^* \in \mathcal{L}(X^*)$. A *state* on $\mathcal{L}(X)$ is a continuous

linear functional φ on $\mathcal{L}(X)$ such that $\varphi(I_X) = 1 = \|\varphi\|$, where I_X is the identity operator on X. We shall say that φ is a *primary state* if there exist $x \in \partial B_X$ and $x^* \in \partial B_{X^*}$ such that $x^*(x) = 1$ and $\varphi(T) = x^*(Tx)$ for all $T \in \mathcal{L}(X)$; under these circumstances, we refer to (x, x^*) as a *primary state* on X. (The set of primary states on X is denoted by $\Pi(X)$ in [4].)

We recall that a bounded linear operator $T : X \to X$, where X is a complex Banach space, is *hermitian* if $\varphi(T)$ is real for all states φ on $\mathcal{L}(X)$. This is the requirement that $V(T) \subseteq \mathbb{R}$, where $V(T)$ is the *spatial numerical range* of T. The hermitian operators form a real-linear subspace of $\mathcal{L}(X)$. It turns out that T is hermitian if and only if $\varphi(T)$ is real for all primary states φ.

Note that it follows from [4, § 9, Corollary 6(iii)] that $T^* \in \mathcal{L}(X^*)$ is hermitian if and only if $T \in \mathcal{L}(X)$ is hermitian.

PROPOSITION 2.1. *Suppose that X is a complex Banach space and that $T : X \to X$ is a bounded linear operator. Then the following are equivalent*:

(i) *T is hermitian*;
(ii) *$\exp(itT)$ is an isometry for each $t \in \mathbb{R}$*;
(iii) *there is a dense subset D of ∂B_X such that, for every $x \in D$, there exists $x^* \in \partial B_{X^*}$ with (x, x^*) a primary state and $x^*(Tx) \in \mathbb{R}$.*

Now suppose that $T : X^* \to X^*$ is a bounded linear operator. Then:

(iv) *T is hermitian if and only if, for every primary state (x, x^*) on X, we have $(Tx^*)(x) \in \mathbb{R}$.*

Proof. We refer to [4, § 5, Lemma 2, and § 9, Theorem 3] for the equivalence of (i)–(iii); (iv) follows from (iii) by the Bishop–Phelps theorem [3]. □

For the following basic facts, we also refer to [4, § 5, Lemma 4] or [16, Theorem 2.6.7]. Here $\mathcal{R}(P)$ is the range of P.

PROPOSITION 2.2. *Suppose that X is a complex Banach space, that $S, T : X \to X$ are hermitian operators, and that $P : X \to X$ is a contractive projection. Then*:

(i) *$i[S, T] = i(ST - TS)$ is an hermitian operator*;
(ii) *$PT : \mathcal{R}(P) \to \mathcal{R}(P)$ is an hermitian operator on the Banach space $\mathcal{R}(P)$.* □

LEMMA 2.3. *Suppose that X is a complex Banach space and that $T : X \to X$ is an hermitian operator. Then*:

(i) *if for some $x \in X$, we have $T^2x = 0$, then $Tx = 0$*;
(ii) *if $T^2 = 0$, then $T = 0$.*

Proof. Proposition 2.1(ii) implies that $\|x + it(Tx)\| = \|x\|$ for all $t \in \mathbb{R}$, and so $Tx = 0$, giving (i); clause (ii) is then immediate. □

LEMMA 2.4. *Suppose that X is a complex Banach space, that $T : X \to X$ is an hermitian operator, and that $P : X \to X$ is an hermitian projection. Then*:

(i) *T commutes with P if and only if $(I_X - P)TP = 0$*;
(ii) *if TP is hermitian, then T commutes with P.*

Proof. (i) If T commutes with P, then $(I_X - P)TP = 0$.

Conversely, if $(I_X - P)TP = 0$, then $[P,T] = PT(I_X - P)$. By Proposition 2.2(i), $i[P,T]$ is hermitian. Since $(PT(I_X - P))^2 = 0$, we have $[P,T]^2 = 0$, and so $[P,T] = 0$ by Lemma 2.3(ii).

(ii) If TP is hermitian, then so is $i[P,TP] = i(PTP - TP) = -i(I_X - P)TP$. Since we have $((I_X - P)TP)^2 = 0$, this implies that $(I_X - P)TP = 0$, as before. Now, by (i), T commutes with P. □

Let X be a complex Banach space. To conclude this section, we remark that, as in the introduction, we define $\mathcal{J}(X)$ to be the (complex) linear span of the hermitian operators on X. Then $\mathcal{J}(X)$ is a norm-closed subspace of $\mathcal{L}(X)$, and $\mathcal{J}(X)$ has an involution specified by $(H + iK)^\bigstar = H - iK$ when H and K are hermitian. The space $\mathcal{J}(X)$ is a subalgebra of $\mathcal{L}(X)$ if and only T hermitian implies that T^2 is hermitian, and then $\mathcal{J}(X)$ is isometrically $*$-isomorphic to a C^*-algebra.

3. *Banach lattices*

For background on Banach lattices we refer to [1, 2, 12, 14, 17]. (The original 1985 version of [2] was reprinted in 2006 with changed numbering; we shall give references to the numbers in both versions.) For a systematic and detailed exposition of complex Banach lattices, see [22].

Let us recall (for example, see [1, § 3.2; 12, p. 43], or [17, pp. 134ff.]) that X is a complex Banach lattice if and only if it is the complexification of a real Banach lattice $X_\mathbb{R} \subset X$. Indeed, each $x \in X$ may be uniquely written as $x = u + iv$, where $u, v \in X_\mathbb{R}$, and then

$$\|x\| = \| \, |x| \, \| = \|(|u|^2 + |v|^2)^{1/2}\|;$$

we write $u = \operatorname{Re} x$ and $v = \operatorname{Im} x$, and then set $\bar{x} = u - iv$. If $T : X \to X$ is any operator, then we denote by $\bar{T}$ the linear operator

$$\bar{T} : x \mapsto \overline{T(\bar{x})}, \quad X \longrightarrow X.$$

We shall make free use of the functional or Krivine calculus as explained in [12, pp. 40–42], for example. Thus, if $x_1, \dots, x_n$ are elements in a (real) Banach lattice X and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and homogeneous, we can define $f(x_1, \dots, x_n) \in X$. The map $f \mapsto f(x_1, \dots, x_n)$ is linear and preserves the lattice operations whenever $x_1, \dots, x_n \in X^+$; in particular, if $f(t_1, \dots, t_n) \geqslant 0$ when $t_1, \dots, t_n \geqslant 0$, then $f(x_1, \dots, x_n) \geqslant 0$ whenever $x_1, \dots, x_n \in X^+$.

We can extend the Krivine calculus to complex Banach lattices X. Indeed, take $f : \mathbb{C}^n \to \mathbb{R}$ and regard f as a function from $\mathbb{R}^{2n}$ to $\mathbb{R}$. Suppose that f is now continuous and homogeneous, and take $z_1, \dots, z_n \in X$, say, $u_j = \operatorname{Re} z_j$ and $v_j = \operatorname{Im} z_j$ for $j = 1, \dots, n$. Then we define

$$f(z_1, \dots, z_n) = f(u_1, v_1, \dots, u_n, v_n).$$

In particular, we can use the function given by $f : (s,t) \mapsto (|s|^p + |t|^p)^{1/p}$ on $\mathbb{C}^2$ to define the element $(|x|^p + |y|^p)^{1/p}$ for each x and y in X. Furthermore, [12, Proposition 1.d.2(i)] states that, for every $\theta \in (0,1)$ and $x, y \in X$, we have

$$\left\| \, |x|^\theta \, |y|^{1-\theta} \right\| \leqslant \|x\|^\theta \, \|y\|^{1-\theta}. \tag{3.1}$$

Let X be a complex Banach lattice. An operator $T : X_\mathbb{R} \to X_\mathbb{R}$ is *positive* if $Tx \geqslant 0$ in $X_\mathbb{R}$ whenever $x \in X^+$, and T is *regular* if $T = T_1 - T_2$, where T_1 and T_2 are positive operators; an operator $T : X \to X$ is *regular* if $T = T_1 + iT_2$, where $T_1, T_2 : X_\mathbb{R} \to X_\mathbb{R}$ are regular. Each regular operator is continuous. The space of regular operators on X is denoted by $\mathcal{L}_r(X)$. The book [2] is devoted to a study of positive operators.

Further, we shall say that an operator $M \in \mathcal{L}(X)$ is *a multiplier* or a *central operator* (see [1, p. 112]) or [14, § 3.1]) if, for some $C > 0$, it obeys an estimate

$$|Mx| \leqslant C|x|, \ x \in X.$$

We shall use the fact [1, Theorem 3.29] that $M \in \mathcal{L}(X)$ is a multiplier if and only if M is an *orthomorphism*, in the sense that $|x| \wedge |My| = 0$ whenever $x, y \in X$ with $|x| \wedge |y| = 0$. The collection of all multipliers is called the *centre* of X and denoted by $\mathcal{Z}(X)$ (see [1, p. 112]). We shall say that a multiplier M is *real* if, additionally, $M(X_{\mathbb{R}}) \subset X_{\mathbb{R}}$; the real multipliers form the real-linear space $\mathcal{Z}_{\mathbb{R}}(X)$. Every multiplier is a regular operator, and indeed, if M is a multiplier, then we can define the modulus $|M|$ of M; by [1, Theorem 3.30], $|M|$ is also a (real) multiplier and

$$|M|(|x|) = |M(|x|)| = |Mx|, \quad x \in X.$$

The centre $\mathcal{Z}(X)$ of X is a commutative, unital subalgebra of $\mathcal{L}(X)$, and $\mathcal{Z}_{\mathbb{R}}(X)$ is a real lattice, where, for example, $(M_1 \vee M_2)(x) = M_1 x \vee M_2 x$ for $M_1, M_2 \in \mathcal{Z}_{\mathbb{R}}(X)$ and $x \in X^+$; $\mathcal{Z}(X)$ is isometrically algebra- and lattice-isomorphic to the space $\mathcal{C}(\Omega)$ of all complex-valued, continuous functions on some compact Hausdorff space Ω (which is uniquely determined and which we call the *carrier space* of X). Furthermore,

$$\|M\| = \| \, |M| \, \| = \inf\{C > 0 : \ |Mx| \leqslant C|x|, \ x \in X\} \tag{3.2}$$

for all $M \in \mathcal{Z}(X)$.

Note that, if M is a real multiplier, then M is an hermitian operator, and so $\mathcal{Z}(X) \subset \mathcal{J}(X)$; the involution restricted to $\mathcal{Z}(X)$ is given by $M^{\bigstar} = \bar{M}$ for $M \in \mathcal{Z}(X)$.

A complex Banach lattice X is *order-complete*, or *Dedekind complete*, if every non-empty subset of $X_{\mathbb{R}}$ which is bounded above has a supremum. In the case where X is an order-complete Banach lattice, $\mathcal{L}_r(X)$ is also an order-complete Banach lattice, and $\mathcal{Z}(X)$ is a band in $\mathcal{L}_r(X)$ (see [1, Theorem 3.31], taken from [19]), and so $\mathcal{Z}(X)$ is also order-complete. In particular, the dual space X^* is the *dual Banach lattice*; this lattice is always order-complete. Finally, we recall that a Banach lattice of the form $\mathcal{C}(\Omega)$ for a compact space Ω is order-complete if and only if Ω is extremally disconnected. These facts are contained in [1, 2, 19].

The order-ideal generated by an element x of a Banach lattice X is

$$X_x = \{y \in X : |y| \leqslant C|x| \text{ for some } C \geqslant 0\}.$$

A real Banach lattice X has a *quasi-interior point* e if $e \in X^+$ and the order-ideal X_e is norm-dense in X. In particular, notice that, if X is separable, then it always has a quasi-interior point. For this definition and result, see [1, § 4.2; 2, p. 259/266].

Let X be a real Banach lattice with a quasi-interior point e. Then, using the Kakutani representation theorem, there is a compact Hausdorff space Ω such that X_e is lattice-isomorphic to the space $\mathcal{C}_{\mathbb{R}}(\Omega)$ of all real-valued continuous functions on Ω in such a way that e corresponds to the constant function 1 on Ω. For complex Banach lattices, one obtains a similar statement for the complex space $\mathcal{C}(\Omega)$. Thus X is the completion of $\mathcal{C}(\Omega)$ under a lattice norm, and then $\mathcal{C}(\Omega)$ is a lattice-ideal in X. In this case, the multipliers can be identified with (the extensions of) multiplication operators of the form

$$M_f : x \mapsto fx, \quad \mathcal{C}(\Omega) \longrightarrow \mathcal{C}(\Omega),$$

for some $f \in \mathcal{C}(\Omega)$. Hence $\mathcal{Z}(X)$ can be identified with $\mathcal{C}(\Omega)$ and Ω can be identified with the carrier space of X. It is clear that, if $|M_f x| \leqslant C|x|$, then $\|M_f\| \leqslant C$, and indeed that $\|M_f\| = \|f\|_{\mathcal{C}(\Omega)}$. For details of the above discussion, see [1, §§ 3.3, 4.2].

The above considerations lead to the following proposition.

PROPOSITION 3.1. *Let X be a complex Banach lattice with a quasi-interior point, e. Suppose that $x \in X$ with $|x| \leqslant e$. Then there is a unique multiplier $M \in \mathcal{Z}(X)$ such that $Me = x$. Furthermore, $\|M\| \leqslant 1$, and, in the case where $x \in X^+$, the operator M is positive.*

In general, however, $\mathcal{Z}(X)$ can reduce to the set $\{\lambda I : \lambda \in \mathbb{C}\}$ (see [10] or [20]), so that the carrier space can reduce to one point. However, if X is order-complete, we must have plenty of multipliers.

PROPOSITION 3.2. *Let X be an order-complete, complex Banach lattice. Suppose that $x \in X$ and $y \in X^+$ with $|x| \leqslant y$. Then there is a multiplier $M : X \to X$ with $My = x$ and such that $|Mw| \leqslant |w|$ for all $w \in X$.*

Proof. This is basically proved in [2, Theorem 8.15/2.49], but for real scalars. Let $Y = \bar{X}_y$ be the closed order-ideal generated by y. We use Proposition 3.1 to produce a multiplier $M_0 : Y \to Y$ with $\|M_0\| \leqslant 1$ and $M_0 y = x$. Consider the real-linear map $w \mapsto \mathrm{Re}\,(M_0 w)$ from Y to $X_{\mathbb{R}}$. Then

$$|\mathrm{Re}\,(M_0 w)| \leqslant |w|, \quad w \in Y,$$

and, using the order-completeness of X and a Hahn–Banach argument, we can find a real-linear extension $L : X \to X_{\mathbb{R}}$ of this map such that $|Lw| \leqslant |w|$ for $w \in X$.

Finally, define $Mw = Lw - iL(iw)$ for $w \in X$, so that M is complex-linear and

$$\begin{aligned} |M(w)| &= (L(w)^2 + L(iw)^2)^{1/2} \\ &= \sup_{-\pi \leqslant \theta \leqslant \pi} |(\cos\theta) L(w) + (\sin\theta) L(iw)| \\ &= \sup_{-\pi \leqslant \theta \leqslant \pi} |L((\cos\theta) w + i(\sin\theta) w)| \leqslant |w|, \end{aligned}$$

as required. □

COROLLARY 3.3. *Let X be an order-complete, complex Banach lattice. For each $x \in X$, there exists $M \in \mathcal{Z}(X)$ with $M(|x|) = x$ and $|M| = I_X$.*

Proof. Take $x \in X$. By Proposition 3.2, there exists $M \in \mathcal{Z}(X)$ with $M(|x|) = x$. Since $|M|\,(|x|) = |M(|x|)| = |x|$, it follows that $|M|$ is the identity operator on $X_{|x|}$. Now re-define M to be the identity on the disjoint complement of $X_{|x|}$ for the result. □

The following proposition will be required later. An account of essentially this result is given in [21].

PROPOSITION 3.4. *Let X be a complex Banach lattice.*

(i) *Suppose that X has a quasi-interior point and that $T : X \to X$ is a bounded operator which commutes with every multiplier. Then T is a multiplier.*

(ii) *Suppose that X is order-complete and that $T : X \to X$ is a bounded operator which commutes with every band projection on X. Then T is a multiplier.*

Proof. (i) We may suppose that $\|T\| \leqslant 1$. Let the quasi-interior point of X be e.

We argue first that, if $|Te| \leqslant \lambda e$ for some $\lambda > 0$, then T is a multiplier. Indeed in this case, if $|x| \leqslant e$, we can find a multiplier $M_x : X \to X$ with $M_x e = x$. Then

$$|Tx| = |TM_x e| = |M_x Te| = |M_x|(|Te|) \leqslant |M_x|(\lambda e) = \lambda |M_x e| = \lambda |x|.$$

By scaling and the density of X_e, this estimate extends to all $x \in X$.

Now suppose that we do not have an estimate $|Te| \leqslant \lambda |e|$. For each $\varepsilon > 0$, let $v = |Te| + \varepsilon e$. Then v is also a quasi-interior point, and so, by Proposition 3.1, there is a positive multiplier $M : X \to X$ such that $Mv = |Te| \wedge \varepsilon^{-1} e$ and $\|M\| \leqslant 1$. Since $|Te| \leqslant v$, we see that

$$|MTe| = M(|Te|) \leqslant Mv \leqslant \varepsilon^{-1} e,$$

so that, by the earlier reasoning, MT is a multiplier. Certainly, $\|M\| \leqslant 1$ and $\|MT\| \leqslant 1$ and so $Me \leqslant e$ and $|MTe| \leqslant e$ by (3.2). Thus

$$|Te| \wedge \varepsilon^{-1} e = Mv = M(|Te| + \varepsilon e) = |MTe| + \varepsilon Me \leqslant (1 + \varepsilon) e.$$

Since e is a quasi-interior point, we know that $\lim_{n\to\infty} \|x - x \wedge ne\| = 0$ for each $x \in X^+$ (see [**2**, Theorem 15.13/4.85]), and so this implies that $|Te| \leqslant |e|$. Thus, T is a multiplier.

The above result also follows from [**2**, Theorems 8.3/2.37 and 15.4/4.76].

(ii) Since X is order-complete, there is a band projection onto each band [**2**, Theorem 3.8/1.42]. Since T commutes with every band projection, T is order-bounded [**2**, Theorem 15.4/4.76], and so T is an orthomorphism [**2**, Definition 8.7/2.41], and hence a multiplier [**2**, Theorem 15.5/4.77]. □

We shall say that X is a *dual Banach lattice* if X is the dual of some Banach lattice. Note that every dual Banach lattice is order-complete. Let X be a dual Banach lattice. We shall need the existence of a projection from X^{**} to X which respects multipliers; this is given in the following result.

PROPOSITION 3.5. *Let X be an order-complete, real or complex Banach lattice, and suppose that there is a positive, contractive projection from X^{**} onto X. Then there is a positive, contractive projection $\Pi : X^{**} \to X$ such that $\Pi M^{**} = M\Pi$ for every $M \in \mathcal{Z}(X)$.*

In particular, in the case where X is a dual Banach lattice, there is a positive, contractive projection $\Pi : X^{**} \to X$ such that $\Pi M^{**} = M\Pi$ for every $M \in \mathcal{Z}(X)$.

Proof. Consider the non-empty set $\mathcal{P}$ of positive, contractive projections $R : X^{**} \to X$. We order this set by the canonical ordering: $R_1 \geqslant R_2$ if $R_1 x^{**} \geqslant R_2 x^{**}$ for every $x^{**} \geqslant 0$ in X^{**}. Using the order-completeness of X, we see that there is a minimal projection $\Pi \in \mathcal{P}$.

Now let P be a band projection in X, and set $Q = I_X - P$. We consider the map Π' defined by $\Pi' = P\Pi P^{**} + Q\Pi Q^{**}$. Then Π' is a projection of X^{**} onto X, and

$$\Pi' = \tfrac{1}{2}(\Pi + (P - Q)\Pi(P - Q)^{**}),$$

so that $\|\Pi'\| = 1$. Further, we have

$$(P - Q)\Pi(P - Q)^{**} x^{**} \leqslant |\Pi(P - Q)^{**} x^{**}| \leqslant \Pi x^{**}$$

for each $x^{**} \geqslant 0$ because $\Pi \geqslant 0$, so that $\Pi' \in \mathcal{P}$ and $\Pi' \leqslant \Pi$. Hence, $\Pi' = \Pi$ and so

$$\Pi P^{**} = \Pi' P^{**} = P\Pi' P^{**} = P\Pi.$$

Since the carrier space of X is extremally disconnected, this implies a similar result for all multipliers.

Suppose that $X = Y^*$ is a dual Banach lattice. Then X is order-complete and the canonical projection of $X^{**} = Y^{***}$ onto $X = Y^*$ is contractive and positive. □

We next prove a simple technical lemma which we shall need later.

LEMMA 3.6. *Let X be an order-complete, real or complex Banach lattice, and suppose that $T : X \to X$ is a bounded operator. Then the set*

$$\{x \in X : |Tx| \leqslant c|x| \text{ for some } c \geqslant 0\}$$

is dense in X.

Proof. We may suppose that $\|T\| \leqslant 1$. It will suffice to show that, for each $x_0 \in X$ with $\|x_0\| = 1$ and for each $\varepsilon \in (0,1)$, there exists $x \in X$ with $\|x - x_0\| < \varepsilon$ and $|Tx| \leqslant (2/\varepsilon)|x|$.

By Corollary 3.3, there exists $M \in \mathcal{Z}(X)$ with $M(|x_0|) = x_0$ and $|M| = I_X$. Define a sequence (x_n) in X inductively by setting $x_n = M(|Tx_{n-1}|)$ for $n \in \mathbb{N}$ (with x_0 as specified). Now take $n \in \mathbb{N}$. Then we see that $\|x_n\| \leqslant 1$ and

$$|x_n| = |M(|Tx_{n-1}|)| = I_X(|Tx_{n-1}|) = |Tx_{n-1}|,$$

and hence that $M(|x_n|) = M(|Tx_{n-1}|) = x_n$.

We now define

$$x = \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2}\right)^n x_n \in X,$$

so that $\|x - x_0\| \leqslant \varepsilon/(2-\varepsilon) < \varepsilon$ and

$$x = \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2}\right)^n M(|x_n|) = M\left(\sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2}\right)^n |x_n|\right).$$

Again using the fact that $|M| = I_X$, we see that

$$|x| = \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2}\right)^n |x_n|.$$

Now

$$|Tx| \leqslant \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2}\right)^n |Tx_n| = \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2}\right)^n |x_{n+1}| \leqslant \frac{2}{\varepsilon} \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2}\right)^n |x_n| = \frac{2}{\varepsilon}|x|,$$

giving the result. □

The following example shows that Lemma 3.6 cannot hold for general Banach lattices.

Let Ω be a compact space of topological dimension at least 2 (see [18]), and set $X = \mathcal{C}(\Omega)$. Then there exists $f_0 \in X$ and $\varepsilon > 0$ such that, for every $f \in X$ with $\|f_0 - f\| < \varepsilon$, there exists $t \in \Omega$ such that $f(t) = 0$ (or otherwise the topological dimension of Ω is at most 1). Since $f_0 \neq 0$, we may suppose that there exists $t_0 \in \Omega$ with $f_0(t_0) = 1$. Define

$$T : f \mapsto f(t_0)1 - f, \quad X \longrightarrow X,$$

where 1 is the constant 1 function.

We *claim* that, for every $f \in X$ such that $\|f_0 - f\| < \varepsilon < 1$, there is no constant c with $|Tf| \leqslant c|f|$. Indeed, suppose that $f \in X$ is such an element. Then there exists $t \in \Omega$ with $f(t) = 0$. We have $|f_0(t)| < \varepsilon$, and so $t \neq t_0$. We also see that $|1 - f(t_0)| = |f_0(t_0) - f(t_0)| < \varepsilon$, and this implies that

$$|(Tf)(t)| = |f(t_0) - f(t)| = |f(t_0)| > 1 - \varepsilon > 0,$$

whereas $f(t) = 0$. So there is no constant $c > 0$ such that $|Tf| \leqslant c|f|$.

4. *The problem of Dales*

We first consider a specific function defined as follows:

$$\psi(z,w) = \int_0^{2\pi} \left|z + e^{i\theta} w\right| \frac{\mathrm{d}\theta}{2\pi}, \quad z, w \in \mathbb{C}.$$

We note that, in fact $\psi(z,w) = \psi(|z|,|w|)$ for $z, w \in \mathbb{C}$, and that the function ψ is continuous and homogeneous on $\mathbb{C}^2$, identified with $\mathbb{R}^4$.

We also note that $\psi(1,t) > 1$ for $t \in (0,1]$, and this inequality is all that is required for our first main result, Theorem 4.2. In fact, we shall show for later use that

$$\psi(1,t) \geqslant 1 + \frac{t^2}{5} \quad \text{for } 0 \leqslant t \leqslant 1. \tag{4.1}$$

Indeed, first suppose that $t \in [0,1)$, and set $s = 2t/(1+t^2)$, so that $s \in [0,1)$. Then

$$|1 + t\, e^{i\theta}| = \sqrt{1+t^2}\sqrt{1+s\cos\theta} = \sqrt{1+t^2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} s^n \cos^n\theta,$$

where the series is uniformly convergent on $[0,k]$ for each $k < 1$. Thus

$$\psi(1,t) = \frac{1}{2\pi}\sqrt{1+t^2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} s^n \int_0^{2\pi} \cos^n\theta \, d\theta.$$

The integral takes the value 2π when $n = 0$, is 0 when n is odd, and belongs to $[0,\pi]$ when n is even. Also, for $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{2k} s^{2k} &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} s^n + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-s)^n\right) \\ &= \frac{1}{2}\left(\sqrt{1+s} + \sqrt{1-s}\right) = \frac{1}{\sqrt{1+t^2}}. \end{aligned}$$

Since each coefficient $\binom{\frac{1}{2}}{2k}$ is negative for $k \in \mathbb{N}$, we have

$$\psi(1,t) \geqslant \frac{1}{2\pi}\sqrt{1+t^2}\left(2\pi + \pi\left(\frac{1}{\sqrt{1+t^2}} - 1\right)\right) = \frac{1}{2}(1 + \sqrt{1+t^2}).$$

Since $\sqrt{1+t^2} \geqslant 1 + 2t^2/5$, inequality (4.1) for $t \in [0,1)$ follows; the inequality also holds for $t = 1$ by continuity.

In fact, $\psi(1,t) \geqslant 1 + t^2/4$ for $0 \leqslant t \leqslant 1$; this is proved in [**9**], where it is attributed to Peter Goddard.

We make further remarks about the function ψ. First, it is clear that $\psi(s,t) \geqslant s \vee t$ when $s, t \geqslant 0$. Also, for each $\varepsilon > 0$, we have

$$s + t \leqslant (1+\varepsilon)(s \vee t) + \frac{5}{\varepsilon}(\psi(s,t) - s \vee t), \quad s, t \geqslant 0. \tag{4.2}$$

To see this, we may suppose that $s \geqslant t$, so that $\psi(s,t) \geqslant s(1 + t^2/5s^2)$ by (4.1), and so it suffices to show that $t \leqslant \varepsilon s + t^2/\varepsilon s$. But this is immediate by considering separately the cases where $0 \leqslant t \leqslant \varepsilon s$ and $\varepsilon s \leqslant t \leqslant s$.

Throughout the remainder of this section, we take X to be a complex Banach lattice.

We now extend the definition of ψ to have domain X^2, rather than $\mathbb{C}^2$. Indeed, we set

$$\psi(u,v) = \int_0^{2\pi} |u + e^{i\theta} v| \frac{\mathrm{d}\theta}{2\pi}, \quad u, v \in X. \tag{4.3}$$

In fact, $\psi(u,v)=\psi(|u|,|v|)$ for all $u,v\in X$ by the uniqueness of the Krivine calculus. Thus, by the Krivine calculus, $\psi(|u|,|v|)\geqslant |u|\vee|v|$ and

$$|u|+|v|\leqslant (1+\varepsilon)(|u|\vee|v|)+\frac{5}{\varepsilon}(\psi(|u|,|v|)-|u|\vee|v|),\quad u,v\in X. \tag{4.4}$$

LEMMA 4.1. *Let X be a complex Banach lattice, and take $x\in X$ with $\|x\|=1$. Suppose that $x=u+v$, where $u,v\in X$ and, for every $-\pi\leqslant\theta\leqslant\pi$, we have*

$$\|u+e^{i\theta}v\|=\|\,|u|\vee|v|\,\|=1.$$

Then there exists $x^*\in X^*$ such that (x,x^*) is a primary state on X and $x^*(Mu)$ is real for every $M\in\mathcal{Z}_{\mathbb{R}}(X)$.

Proof. We may suppose that $e=|u|+|v|$ is a quasi-interior point of X.

Since $\|\,|u|\vee|v|\,\|=1$, there exists a positive linear functional w^* on X such that $\|w^*\|=1$ and $w^*(|u|\vee|v|)=1$. It now follows from (4.3) that

$$\|\psi(|u|,|v|)\|=\|\psi(u,v)\|=\left\|\int_0^{2\pi}|u+e^{i\theta}v|\frac{\mathrm{d}\theta}{2\pi}\right\|\leqslant\int_0^{2\pi}\|u+e^{i\theta}v\|\,\frac{\mathrm{d}\theta}{2\pi}\leqslant 1,$$

and so $w^*(\psi(|u|,|v|))\leqslant 1$. However, $\psi(|u|,|v|)\geqslant|u|\vee|v|$ and w^* is positive, and so $w^*(\psi(|u|,|v|))\geqslant w^*(|u|\vee|v|)=1$. Thus $w^*(\psi(|u|,|v|))=1$. We now have

$$w^*(\psi(|u|,|v|)-|u|\vee|v|)=0.$$

Note that, for each $\varepsilon>0$, it follows from (4.4) that

$$e\leqslant(1+\varepsilon)(|u|\vee|v|)+\frac{5}{\varepsilon}(\psi(|u|,|v|)-|u|\vee|v|).$$

Since $e=|u|\vee|v|+|u|\wedge|v|$, we have

$$1=w^*(|u|\vee|v|)\leqslant w^*(e)\leqslant(1+\varepsilon)w^*(|u|\vee|v|)=1+\varepsilon$$

for each $\varepsilon>0$, and so $w^*(e)=1$. From this, we have $w^*(|u|\wedge|v|)=0$.

Now let L_u and L_v be the unique multipliers given by Proposition 3.1 such that $L_ue=u$ and $L_ve=v$. Note that $|L_u|+|L_v|=I_X$. We define $x^*=(\bar{L}_u^*+\bar{L}_v^*)w^*$. Then $\|x^*\|\leqslant 1$ and

$$x^*(x)=w^*((|L_u|^2+|L_v|^2+\bar{L}_uL_v+\bar{L}_vL_u)(e)).$$

We have $|w^*(\bar{L}_uL_ve)|\leqslant w^*(|L_u|\,|L_v|e)\leqslant w^*(|u|\wedge|v|)=0$, and so $w^*(\bar{L}_uL_ve)=0$. Similarly, $w^*(\bar{L}_vL_ue)=0$, and so

$$x^*(x)=w^*(|L_u|^2e+|L_v|^2e)=w^*(e)-2w^*(|L_u|\,|L_v|e)=1$$

because $w^*(|L_u|\,|L_v|e)=0$. This shows that (x,x^*) is a primary state on X.

Suppose that $M\in\mathcal{Z}_{\mathbb{R}}(X)$. Then

$$x^*(Mu)=x^*(ML_ue)=w^*(M|L_u|^2e)+w^*(M\bar{L}_vL_ue).$$

Here the first term is real and the second term can be estimated by

$$|w^*(M\bar{L}_vL_ue)|\leqslant w^*(|M|\,|L_v|\,|L_u|e)\leqslant\|M\|w^*(|u|\wedge|v|)=0.$$

The result follows. □

The following theorem answers the original question of Dales.

THEOREM 4.2. *Let X be a complex Banach lattice, and suppose that $X = E \oplus F$ is a decomposition such that*

$$\|x + y\| = \| \, |x| \vee |y| \, \|, \quad x \in E, \ \ y \in F.$$

Then $E \oplus F$ is a band decomposition, that is,

$$|x| \wedge |y| = 0, \quad x \in E, \ \ y \in F.$$

Proof. We denote by P the projection of X onto E with kernel F. Then

$$\|x\| = \|\max(|Px|, |x - Px|)\|, \quad x \in X,$$

so that P is hermitian.

Let $M \in \mathcal{Z}_{\mathbb{R}}(X)$. By Lemma 4.1 we have that, for every $x \in X$ with $\|x\| = 1$, we can find x^* such that (x, x^*) is a primary state on X and $x^*(MPx)$ is real. Thus, using Proposition 2.1, MP is hermitian. By Lemma 2.4(ii), M and P commute.

Now, in the case where X has a quasi-interior point, we can conclude from Proposition 3.4(i) that $P \in \mathcal{Z}(X)$, and hence that P is a band projection.

For X, a general Banach lattice, choose $u \in E$ and $v \in F$, and let X_0 be the smallest closed sublattice containing u and v and invariant under P. Then X_0 is a separable Banach lattice, and so has a quasi-interior point. Thus, $P \mid X_0 : X_0 \to X_0$ is a band projection and $|u| \wedge |v| = 0$. This implies that P is a band projection on X. □

REMARK. In fact, the above argument can be easily modified to show that it is also true that $E \oplus F$ is a band decomposition whenever

$$\|x + y\| = \|(|x|^p + |y|^p)^{1/p}\|, \quad x \in E, \ \ y \in F,$$

for some $p \geqslant 4$. This follows from the fact that

$$\psi(1, t) > 1 + \frac{t^2}{5} > (1 + t^4)^{1/4}, \quad 0 < t \leqslant 1.$$

We are therefore naturally led to the problem of whether a similar statement is true whenever $2 < p < 4$ or $1 \leqslant p < 2$. It is fairly easy to see that it is enough to show that

$$\|x + y\| = \|(|x|^2 + |y|^2)^{1/2}\|, \quad x \in E, \ \ y \in F,$$

whenever $E \oplus F$ is an hermitian decomposition of X. We shall prove this below.

5. *Hermitian operators on Banach lattices*

In this section, we shall first show that $\mathcal{J}(X)$ is a C^*-algebra for each Banach lattice X, and then deduce the above result.

THEOREM 5.1. *Let X be a dual Banach lattice. Suppose that $T : X \to X$ is an hermitian operator and that M is a real multiplier on X. Then MTM is also hermitian.*

Proof. First, we recall from Proposition 3.5 that there is a positive, contractive projection $\Pi : X^{**} \to X$ such that $\Pi M^{**} = M\Pi$ for every multiplier M on X.

We shall refer to $(P_1, \ldots, P_n)$ as a *band decomposition* of X if each P_j is a band projection, if $\sum_{j=1}^n P_j = I_X$, and if $P_j P_k = 0$ for $j \neq k$. We start with the observation that, since the carrier space of the order-complete Banach lattice X is extremally disconnected, we can suppose that $M = \sum_{j=1}^m a_j Q_j$, where $(Q_1, \ldots, Q_m)$ a fixed band decomposition of X and $-1 \leqslant a_j \leqslant 1$ for

$j = 1, \dots, m$. We then choose $(b_j)_{j=1}^m$ such that $b_j \geqslant 0$ and $a_j^2 + b_j^2 = 1$ for $j = 1, \dots, m$, and we define $M' = \sum_{j=1}^m b_j Q_j$.

Let $\mathcal{P}$ be the collection of all band decompositions of X. We say that $(P_1, \dots, P_n)$ *refines* $(R_1, \dots, R_l)$ in $\mathcal{P}$ if $P_j R_k = 0$ or $P_j R_k = P_j$ for every $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant l$. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{P}$ containing, for each $(R_1, \dots, R_l)$, the set of all $(P_1, \dots, P_n)$ which refine $(R_1, \dots, R_l)$.

Given a band decomposition $(P_1, \dots, P_n)$ of X, we define an hermitian operator $V(P_1, \dots, P_n) \in \mathcal{L}(X)$ by

$$V(P_1, \dots, P_n)x = \sum_{j=1}^n P_j T P_j x, \quad x \in X. \tag{5.1}$$

Let us first check that $V(P_1, \dots, P_n)$ is hermitian and that $\|V(P_1, \dots, P_n)\| \leqslant \|T\|$. This follows because we can express $V(P_1, \dots, P_n)$ as an average over choices of signs, namely

$$V(P_1, \dots, P_n) = \operatorname*{Ave}_{\varepsilon_j = \pm 1} \left(\sum_{j=1}^n \varepsilon_j P_j \right) T \left(\sum_{j=1}^n \varepsilon_j P_j \right),$$

and each term is hermitian since $\sum_{j=1}^n \varepsilon_j P_j$ is an isometry whose square is the identity.

It follows that we can define $V : X \to X$ by

$$Vx = \Pi \left(\lim_{\mathcal{U}} V(P_1, \dots, P_n)x \right), \quad x \in X, \tag{5.2}$$

where the limit is taken in X^{**} with respect to the weak-$*$ topology. Clearly, $\|V\| \leqslant \|T\|$.

We claim that V is also hermitian. Indeed, V is the compression via Π of the operator $\tilde{V} : X^{**} \to X^{**}$ given by

$$\tilde{V}x^{**} = \lim_{\mathcal{U}} V(P_1, \dots, P_n)^{**} x^{**}, \quad x^{**} \in X^{**},$$

with the limit again in the weak-$*$ topology. We shall show that $\tilde{V}$ is hermitian on X^{**}. By Proposition 2.1(iv), we need to check only that $(\tilde{V}x^{**})(x^*)$ is real when (x^*, x^{**}) is a primary state on X^*. But this is clear since

$$(\tilde{V}x^{**})(x^*) = \lim_{\mathcal{U}} V(P_1, \dots, P_n)^{**} x^{**}(x^*)$$

and each $V(P_1, \dots, P_n)^{**}$ is hermitian.

Now suppose that R is any band projection on X. Then

$$\begin{aligned} VRx &= \Pi \left(\lim_{\mathcal{U}} V(P_1, \dots, P_n)Rx \right) = \Pi \left(\lim_{\mathcal{U}} RV(P_1, \dots, P_n)x \right) \\ &= \Pi R^{**} \lim_{\mathcal{U}} V(P_1, \dots, P_n)x = RVx, \quad x \in X. \end{aligned}$$

It now follows from Proposition 3.4(ii) that $V \in \mathcal{Z}(X)$, and hence that $V \in \mathcal{Z}_{\mathbb{R}}(X)$. It follows that $M'VM'$ is also a real multiplier, and thus $M'VM'$ is hermitian.

Let us consider the operator $H = MTM + M'VM'$. Clearly, we have

$$Hx = MTMx + \Pi \lim_{\mathcal{U}} V(P_1, \dots, P_n)(M')^2 x, \quad x \in X.$$

If $(P_1, \dots, P_n)$ refines the fixed band decomposition $(Q_1, \dots, Q_m)$, then we can write

$$M = \sum_{j=1}^n \alpha_j P_j \quad \text{and} \quad M' = \sum_{j=1}^n \beta_j P_j,$$

where α_j and β_j are real and $\alpha_j^2 + \beta_j^2 = 1$ for $j = 1, \dots, n$. Then

$$MTM + V(P_1, \dots, P_n)(M')^2 = \operatorname*{Ave}_{\varepsilon_j = \pm 1} \left(\sum_{j=1}^n (\alpha_j - i\varepsilon_j \beta_j) P_j \right) T \left(\sum_{j=1}^n (\alpha_j + i\varepsilon_j \beta_j) P_j \right).$$

Each term on the right-hand side is hermitian since it is of the form $U^{-1}TU$, with U an invertible isometry. Thus, $MTM + V(P_1, \dots, P_n)(M')^2$ is hermitian whenever $(P_1, \dots, P_n)$ refines $(Q_1, \dots, Q_m)$. Arguing as above, we see easily that H is hermitian. Thus the operator $H - M'VM' = MTM$ is also hermitian. □

THEOREM 5.2. *Let X be a complex Banach lattice. If T is an hermitian operator on X, then T^2 is also hermitian. Hence $\mathcal{J}(X)$ is isometrically $*$-isomorphic to a C^*-algebra.*

Proof. It will suffice to consider the case where X is a dual Banach lattice (for otherwise consider $T^* \in \mathcal{L}(X^*)$, and note that, if $(T^*)^2$ is hermitian, then so is T^2 because $(T^*)^2 = (T^2)^*$). Then, for all real multipliers M on X, we see that

$$\tfrac{1}{2}((M + I_X)T(M + I_X) - (M - I_X)T(M - I_X)) = MT + TM$$

is hermitian by Theorem 5.1. On the other hand, $i(MT - TM)$ is hermitian by Proposition 2.2. It now follows that, if φ is any state on $\mathcal{L}(X)$, then we have

$$\varphi(TM) = \overline{\varphi(MT)}. \tag{5.3}$$

If M is possibly complex-valued, we can use (5.3) on both the real and imaginary parts to obtain

$$\varphi(\bar{M}T) = \overline{\varphi(TM)}. \tag{5.4}$$

Since X is a dual Banach lattice, it is order-complete, and so Proposition 3.2 and Lemma 3.6 apply to X.

Now let us show that T^2 is hermitian. By Proposition 2.1(iv) and Lemma 3.6, we need to consider only a primary state (x, x^*) such that $|Tx| \leqslant c|x|$ for some constant c. By Proposition 3.2, there is a multiplier M such that $Tx = Mx$. Then, using (5.4), we see that

$$x^*(T^2x) = x^*(TMx) = \overline{x^*(\bar{M}Tx)} = \overline{x^*(|M|^2x)},$$

which is real since $|M|^2$ is hermitian. Thus T^2 is hermitian. □

THEOREM 5.3. *Let X be a complex Banach lattice, and suppose that $S_1, \dots, S_n \in \mathcal{J}(X)$. Then, for each $x_1, \dots, x_n \in X$, we have*

$$\left\| \left(\sum_{j=1}^{n} |S_j x_j|^2 \right)^{1/2} \right\| \leqslant \left(\max_{1 \leqslant j \leqslant n} \|S_j\| \right) \left\| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} \right\|.$$

Proof. We consider the Banach lattice $Y = X^n$ with the norm $\|\cdot\|_Y$ given by

$$\|(x_1, \dots, x_n)\|_Y = \left\| \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} \right\|, \quad x_1, \dots, x_n \in X.$$

We shall show that, if $T_1, \dots, T_n$ are hermitian on X, then

$$\mathcal{T}(x_1, \dots, x_n) = (T_1x_1, \dots, T_nx_n), \quad x_1, \dots, x_n \in X,$$

defines an hermitian operator $\mathcal{T}$ on Y.

To do this, it suffices to consider the case where X is separable (since, given $x \in X$, we can pass to a separable sublattice X_0 containing x and invariant for each T_j). Hence, we may assume the existence of a quasi-interior point e with $\|e\| = 1$.

Let us suppose that $y = (x_1, \dots, x_n) \in X^n$ with $\|y\|_Y = 1$; set

$$x = \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2}.$$

Now, for each $\delta > 0$, we define

$$w_\delta = (\delta^2 e^2 + x^2)^{1/2} \in X.$$

Then, for each $j = 1, \dots, n$, we can find unique multipliers $M_j = M_j(\delta)$ such that $M_j w_\delta = x_j$. Let $L = L_\delta$ be the positive multiplier such that $L w_\delta = \delta e$. Using a concrete representation, it is clear that

$$\sum_{j=1}^{n} |M_j|^2 + L^2 = I_X$$

and that, for any $v_1, \dots, v_n \in X$, we have

$$\sum_{j=1}^{n} |M_j|\,|v_j| \leqslant \left(\sum_{j=1}^{n} |v_j|^2 \right)^{1/2}.$$

Let w_δ^* be a positive, norm-one functional on X such that $w_\delta^*(w_\delta) = \|w_\delta\|$. We consider the functional $y_\delta^* \in Y^*$ given by

$$y_\delta^*(v_1, \dots, v_n) = \sum_{j=1}^{n} w_\delta^*(\bar{M}_j v_j), \quad v_1, \dots, v_n \in X.$$

It follows from the remarks stated above that $\|y_\delta^*\| \leqslant 1$.

Now

$$\begin{aligned} 1 \geqslant y_\delta^*(y) = w_\delta^* \left(\sum_{j=1}^{n} \bar{M}_j M_j w_\delta \right) &= w_\delta^*(w_\delta) - w_\delta^*(L^2 w_\delta) \\ &\geqslant \|w_\delta\| - w_\delta^*(L w_\delta) \\ &\geqslant 1 - \delta w_\delta^*(e) \geqslant 1 - \delta. \end{aligned}$$

It follows that, if y^* is any weak-$*$ cluster point of the net (y_δ^*) as $\delta \searrow 0$, then $y^*(y) = 1$ and so (y, y^*) is a primary state on Y.

We see that

$$y_\delta^*(\mathcal{T} y) = \sum_{j=1}^{n} w_\delta^*(\bar{M}_j T_j M_j w_\delta).$$

Thus, since $\mathcal{J}(X)$ is a C^*-algebra, the operators $\bar{M}_j T_j M_j = M_j^{\bigstar} T_j M_j$ are all hermitian. Hence, since $(w_\delta / \|w_\delta\|, w_\delta^*)$ is a primary state, each term in the sum is real and so $y_\delta^*(\mathcal{T} y)$ is real. Thus, $\mathcal{T}$ is an hermitian element of a C^*-algebra, and so $\|\mathcal{T}\|$ coincides with its spectral radius, which again coincides with $\max_{1 \leqslant j \leqslant n} \|T_j\|$.

Returning to the general case, let

$$\mathcal{S}(x_1, \dots, x_n) = (S_1 x_1, \dots, S_n x_n), \quad x_1, \dots, x_n \in X,$$

where $S_1, \dots, S_n \in \mathcal{J}(X)$. Then $\mathcal{S} \in \mathcal{J}(Y)$ and

$$\|\mathcal{S}\|^2 = \|\mathcal{S}^{\bigstar} S\| = \max_{1 \leqslant j \leqslant n} \|S_j^{\bigstar} S_j\| = \max_{1 \leqslant j \leqslant n} \|S_j\|^2.$$

This completes the proof. □

THEOREM 5.4. *Let X be a complex Banach lattice, and suppose that $E \oplus F$ is an hermitian decomposition of X. Then*

$$\|x+y\| = \|(|x|^2+|y|^2)^{1/2}\|, \quad x \in E,\ y \in F.$$

Proof. Suppose that P is the induced projection onto E, and set $Q = I_X - P$. Then the operator $P-Q$ is an hermitian operator whose square is the identity. We apply Theorem 5.3 with $S_1 = P+Q = I_X$ and $S_2 = P-Q$, so that $S_1, S_2 \in \mathcal{J}(X)$ and $\|S_1\| = \|S_2\| = 1$. Take $x \in E$ and $y \in F$. With $x_1 = x_2 = x+y$, we obtain

$$\|(|x+y|^2+|x-y|^2)^{1/2}\| \leqslant \sqrt{2}\|x+y\|.$$

With $x_1 = x+y$ and $x_2 = x-y$, we obtain

$$\sqrt{2}\|x+y\| \leqslant \|(|x+y|^2+|x-y|^2)^{1/2}\|.$$

Since $\|(|x+y|^2+|x-y|^2)^{1/2}\| = \sqrt{2}\|(|x|^2+|y|^2)^{1/2}\|$ by the Krivine calculus, the result follows. □

THEOREM 5.5. *Let X be a complex Banach lattice. Suppose that $E \oplus F$ is a decomposition of X such that, for some $1 \leqslant p < \infty$ with $p \neq 2$, we have*

$$\|x+y\| = \|(|x|^p+|y|^p)^{1/p}\|, \quad x \in E,\ y \in F.$$

Then $E \oplus F$ is a band decomposition.

Proof. The cases where $1 \leqslant p < 2$ and $2 < p < \infty$ are distinct, but the proofs are very similar, and so we shall prove only the case where $p > 2$. Thus we suppose that $2 < p < \infty$.

Since $t^p \leqslant t^2$ for $t \in [0,1]$, we have $(|x|^p+|y|^p)^{1/p} \leqslant (|x|^2+|y|^2)^{1/p}\,|x|^{1-2/p}$ for each $x, y \in \mathbb{R}$ with $|x| \geqslant |y|$, and so

$$(|x|^p+|y|^p)^{1/p} \leqslant ((|x|^2+|y|^2)^{1/2})^{2/p}(|x| \vee |y|)^{1-2/p}, \quad x, y \in \mathbb{R}.$$

By the Krivine calculus, this inequality also holds for $x, y \in X$. It now follows from (3.1) (applied with $\theta = 2/p \in (0,1)$) that

$$\|(|x|^p+|y|^p)^{1/p}\| \leqslant \|(|x|^2+|y|^2)^{1/2}\|^{2/p}\|\,|x| \vee |y|\,\|^{1-2/p}, \quad x, y \in X.$$

Now take $x \in E$ and $y \in F$ with $\|x+y\| = 1$. Since $E \oplus F$ is an hermitian decomposition of X, it follows from Theorem 5.4 that $\|x+y\| = \|(|x|^2+|y|^2)^{1/2}\|$, and so

$$\|x+y\| = \|(|x|^p+|y|^p)^{1/p}\| \leqslant \|(|x|^2+|y|^2)^{1/2}\|^{2/p}\|\,|x| \vee |y|\,\|^{1-2/p} = \|\,|x| \vee |y|\,\|^{1-2/p}.$$

This shows that $\|\,|x| \vee |y|\,\| \geqslant 1$. However, we know that $|x| \vee |y| \leqslant (|x|^2+|y|^2)^{1/2}$, and so

$$\|\,|x| \vee |y|\,\| \leqslant \left\|(|x|^2+|y|^2)^{1/2}\right\| = 1.$$

Hence $\|\,|x| \vee |y|\,\| = \|x+y\| = 1$.

We can now use Theorem 4.2 to show that $E \oplus F$ is a band decomposition. □

Of course the constraint that $p \neq 2$ in the above theorem is necessary. For let $X = \ell_2^2$, and set $E = \{(z,z) : z \in \mathbb{C}\}$ and $F = \{(w,-w) : w \in \mathbb{C}\}$, so that $X = E \oplus F$. For $x = (z,z) \in E$ and $y = (w,-w) \in F$, we have

$$\left\|x + e^{i\theta}y\right\|^2 = 2(|z|^2+|w|^2), \quad \theta \in [0, 2\pi),$$

and so the decomposition is hermitian. However it is not a band decomposition.

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