

# Characterizations of strictly singular operators on Banach lattices

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## ABSTRACT

New characterizations of strictly singular operators between Banach lattices are given. It is proved that, for Banach lattices  $X$  and  $Y$  such that  $X$  has finite cotype and  $Y$  satisfies a lower 2-estimate, an operator  $T : X \rightarrow Y$  is strictly singular if and only if it is disjointly strictly singular and  $\ell_2$ -singular. Moreover, if  $T$  is regular then the same equivalence holds provided that  $Y$  is just order continuous. Furthermore, it is shown that these results fail if the conditions on the lattices are relaxed.

## Introduction

Strictly singular operators were introduced by Kato [18] in connection with the perturbation theory of Fredholm operators. Recall that an operator  $T : X \rightarrow Y$  between Banach spaces is *strictly singular* if it is not an isomorphism when restricted to any infinite-dimensional (closed) subspace of  $X$ . Strictly singular operators constitute a closed two-sided operator ideal that contains the ideal of compact operators. Moreover, an operator  $T : X \rightarrow Y$  is strictly singular if and only if, for every infinite-dimensional subspace  $M$  of  $X$ , there exists an infinite-dimensional subspace  $N$  of  $M$  such that the restriction  $T|_N$  is compact.

In the context of Banach lattices, a weaker notion is the following: given a Banach lattice  $X$ , a Banach space  $Y$ , and an operator  $T : X \rightarrow Y$ , we say that  $T$  is *disjointly strictly singular* if it is not an isomorphism when restricted to the closed linear span of any disjoint sequence in  $X$ . This notion is quite a useful tool in the study of strictly singular operators on Banach lattices, for example, in the context of domination problems for positive operators (cf. [10]), and for comparing structures of rearrangement invariant spaces (cf. [13, 14]). Several properties of disjointly strictly singular operators have been studied in [8, 9, 11].

In this paper, we are interested in giving characterizations of the strict singularity of operators acting between Banach lattices. Since strictly singular operators are disjointly strictly singular, we are mainly interested in converse statements.

Our motivation stems from the following facts. First, it is well known that an endomorphism of  $L_p = L_p[0, 1]$ , with  $1 \leq p < \infty$ , is strictly singular if and only if it is  $\ell_p$ -singular and  $\ell_2$ -singular [22, 25]. In other words, an endomorphism  $T$  on  $L_p$  is strictly singular if and only if it is disjointly strictly singular and  $\ell_2$ -singular. Recall that an operator between Banach spaces is called  $\ell_p$ -singular for some  $1 \leq p \leq \infty$  if it is not an isomorphism when restricted to any subspace isomorphic to  $\ell_p$ . For recent results on  $\ell_p$ -singular operators, we refer to [16].

Given an order continuous Banach lattice  $X$ , if an operator  $T : X \rightarrow Y$  is disjointly strictly singular and  $\ell_p$ -singular for every  $1 \leq p \leq 2$ , then  $T$  is strictly singular. This can be seen using the Kadec–Pelczyński disjointification method and Aldous' theorem on subspaces of  $L_1$  (see [2]). Furthermore, in the special case of  $X$  (or  $Y$ ) being a Banach lattice with type 2,

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if  $T : X \rightarrow Y$  is disjointly strictly singular and  $\ell_2$ -singular, then  $T$  is strictly singular. A similar statement also holds for inclusion operators between rearrangement invariant spaces [12].

One of our main results in this direction is the following.

**THEOREM A.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  has finite cotype and  $Y$  satisfies a lower 2-estimate. Then an operator  $T : X \rightarrow Y$  is strictly singular if and only if it is both disjointly strictly singular and  $\ell_2$ -singular.*

Then, we consider the class of regular operators, that is, those that are a difference of positive operators, proving that, for this class, the equivalence given above in Theorem A is also true under much weaker conditions on the lattices.

**THEOREM B.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  has finite cotype and  $Y$  is order continuous. Then a regular operator  $T : X \rightarrow Y$  is strictly singular if and only if it is both disjointly strictly singular and  $\ell_2$ -singular.*

Both Theorems A and B are obtained by means of the following general result. If  $X$  is a Banach lattice with finite cotype,  $Y$  a Banach space, and  $T : X \rightarrow Y$  is disjointly strictly singular and AM-compact, then  $T$  is strictly singular (see Theorem 2.4). Recall that, for a Banach lattice  $X$ , an operator  $T : X \rightarrow Y$  is called AM-compact if the image of every order interval is a relatively compact set. The connection between AM-compact operators and  $\ell_2$ -singular operators is studied in Section 2 (see Propositions 2.5 and 2.6). Let us remark that the motivation of this kind of result dates from Rosenthal [24], where it was proved that, for endomorphisms on  $L_1$  spaces, being AM-compact and  $\ell_2$ -singular are equivalent notions.

As an application of these characterizations, a domination result for positive strictly singular operators is easily obtained, improving a result of [10]. Precisely, given two operators  $0 \leq R \leq T : X \rightarrow Y$ , with  $T$  strictly singular, then we have that  $R$  is also strictly singular provided that  $X$  has finite cotype and  $Y$  is order continuous (Corollary 2.8).

In Section 3, we prove that the hypothesis in Theorem A on the range lattice  $Y$  cannot be weakened, in the sense that the result is no longer true for Banach lattices  $Y$  with a lower  $q$ -estimate for some  $q > 2$ . To this end, we consider the Banach lattice  $L_r(\ell_q)$  that consists of sequences  $x = (x_1, x_2, \dots)$  of elements in  $L_r$  such that

$$\|x\|_{L_r(\ell_q)} = \left\| \left( \sum_{i=1}^{\infty} |x_i|^q \right)^{1/q} \right\|_{L_r} < \infty.$$

**THEOREM C.** *Consider the Banach lattices  $L_p$  and  $L_r(\ell_q)$ , where  $1 < r < p < 2 < q < \infty$ . For each  $p < s < 2$ , there exists an operator  $T : L_p \rightarrow L_r(\ell_q)$  such that it is  $\ell_p$ -singular and  $\ell_2$ -singular, but not  $\ell_s$ -singular.*

*In particular, the operator  $T$  is disjointly strictly singular and  $\ell_2$ -singular, but not strictly singular.*

The proof of this fact requires some preliminary results. First, we present some technical lemmas that make use of known estimates for independent and identically distributed (i.i.d.)  $s$ -stable random variables for  $1 < s < 2$ , given in [15] (see Proposition 3.1). We consider the atomic lattice representation  $H_r$  of the space  $L_r$ , associated to the unconditional Haar basis  $(h_i)$ , in order to define a suitable operator  $R$  from  $H_r$  to  $L_r(\ell_q)$  that, restricted to  $L_p$ , satisfies

the required conditions. More precisely, let  $(w_n)$  denote a block basis of the Haar basis  $(h_i)$  of the form

$$w_n = \sum_{i=q_{n-1}+1}^{q_n} a_i h_i,$$

which is equivalent to a sequence of i.i.d.  $s$ -stable random variables in both  $L_p$  and in  $L_r$  (for  $1 < r < p < s < 2$ ). We can consider the operator  $R : H_r \rightarrow L_r(\ell_q)$ , defined by

$$R((c_i)_{i=1}^\infty) = \left( \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right)_{n=1}^\infty.$$

The operator  $T_s : L_p \rightarrow L_r(\ell_q)$  is now defined as the composition  $RLJ$ , where  $J$  is the canonical inclusion  $L_p \hookrightarrow L_r$  and  $L$  is the isomorphism between  $H_r$  and  $L_r$  mapping each sequence in  $H_r$  to the corresponding expansion as a series with respect to the Haar system.

We also show that the characterization for regular operators given in Theorem B fails if the order continuity of the range lattice is missing.

### 1. Preliminaries

Let us start with some notation and definitions. We refer the reader to the monographs [4, 20, 21, 28] for unexplained terminology from Banach lattices and positive operator theory.

Throughout, we will write SS and DSS for strictly singular and disjointly strictly singular operators, respectively. Let us recall that a Banach space  $X$  has *cotype*  $q$  for some  $2 \leq q < \infty$  if there exists a constant  $C < \infty$  so that, for every finite set of vectors  $(x_j)_{j=1}^n$  in  $X$ , we have

$$\left( \sum_{j=1}^n \|x_j\|^q \right)^{1/q} \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\| dt,$$

where  $r_j$  denotes the  $j$ th Rademacher function.

A Banach lattice  $Y$  is  $q$ -concave for some  $1 \leq q < \infty$  if there exists a constant  $C < \infty$  so that, for every choice of vectors  $(y_j)_{j=1}^n$  in  $Y$ , we have

$$\left( \sum_{j=1}^n \|y_j\|^q \right)^{1/q} \leq C \left\| \left( \sum_{j=1}^n |y_j|^q \right)^{1/q} \right\|.$$

Every  $q$ -concave Banach lattice with  $q \geq 2$  is of cotype  $q$ .

A Banach lattice  $Y$  satisfies a *lower  $q$ -estimate* for some  $1 < q < \infty$  if there exists a constant  $C < \infty$  such that, for every choice of pairwise disjoint elements  $(y_j)_{j=1}^n$  in  $Y$ , we have

$$\left( \sum_{j=1}^n \|y_j\|^q \right)^{1/q} \leq C \left\| \sum_{j=1}^n y_j \right\|.$$

Banach lattices with finite cotype are  $q$ -concave for some  $q < \infty$  and have order continuous norm. In what follows, any separable (or with a weak unit) order continuous Banach lattice will be represented as a Köthe function space, that is, it is lattice isomorphic to an (in general, not closed) ideal of  $L_1(\mu)$  for some probability space  $(\Omega, \Sigma, \mu)$  (cf. [20, Theorem 1.b.14]). Let us recall the Kadec–Pełczyński disjointification method for order continuous Banach lattices [7, Theorem 4.1].

PROPOSITION 1.1. *Let  $X$  be an order continuous Banach lattice with a weak unit and  $j : X \hookrightarrow L^1(\mu)$  be the formal inclusion. For any (closed) subspace  $Y \subset X$ , one of the following holds.*

*Either there exists a normalized almost disjoint sequence  $(x_n) \subset Y$  (that is, there exists a disjoint sequence  $(z_n) \subset X$  such that  $\|z_n - x_n\| \rightarrow 0$  when  $n \rightarrow \infty$ ).*

*Or a subspace  $Y$  is isomorphic to a closed subspace of  $L_1(\mu)$  (in fact,  $j : X \hookrightarrow L_1$  is an isomorphism when restricted to  $Y$ ); in this case, we say that  $Y$  is a strongly embedded subspace.*

Note that more can be said if, instead of a subspace, we consider a normalized sequence  $(x_n) \subset X$ ; now the alternatives are as follows:

- (1) either  $(\|x_n\|_{L_1})$  is bounded away from zero;
- (2) or there exist a subsequence  $(x_{n_k})$  and a disjoint sequence  $(z_k) \subset X$  such that  $\|z_k - x_{n_k}\| \rightarrow 0$  when  $k \rightarrow \infty$ .

Recall that a subset  $M$  of an order continuous Banach lattice  $X$  is *equi-integrable* if  $\sup_{f \in M} \|f\chi_A\| \rightarrow 0$  when  $\mu(A) \rightarrow 0$ . This concept has an analogue for general Banach lattices: a bounded subset  $M$  of a Banach lattice  $X$  is  *$L$ -weakly compact* if, for every  $\varepsilon > 0$ , there exists  $x \in X_+^a$  such that  $M \subset [-x, x] + \varepsilon B_X$  (where  $X^a$  denotes the maximal order ideal in  $X$  on which the induced norm is order continuous and  $B_X$  denotes the closed unit ball of  $X$ ). Note that a bounded sequence  $(g_k)$  in an order continuous Köthe function space (over a probability space) is  *$L$ -weakly compact* if and only if it is equi-integrable.

We will make use of the following standard facts (cf., for example, [10, Lemmas 3.2 and 3.3]).

LEMMA 1.2. *Let  $T : X \rightarrow Y$  be a regular operator from a Banach lattice  $X$  into a Banach lattice  $Y$  with order continuous norm. If  $A \subset X$  is  $L$ -weakly compact, then  $T(A)$  is  $L$ -weakly compact.*

LEMMA 1.3. *Let  $X$  be an order continuous Banach lattice with a weak unit, and hence representable as an order ideal in  $L_1(\mu)$  for some probability space. A norm bounded sequence  $(g_n)$  in  $X$  is convergent to zero if and only if  $(g_n)$  is equi-integrable and  $\|\cdot\|_{L_1}$ -convergent to zero.*

A Banach lattice  $X$  with an order continuous norm satisfies the *subsequence splitting property* [15, Chapter 6; 26] if, for every bounded sequence  $(f_n)$  in  $X$ , there exist a subsequence  $(f_{n_k})$  and sequences  $(g_k)$  and  $(h_k)$  in  $X$  with  $|g_k| \wedge |h_k| = 0$  and  $f_{n_k} = g_k + h_k$  for all  $k$ , such that  $(g_k)$  is equi-integrable in  $X$ , and  $(h_k)$  is disjoint. It is known that every Banach lattice with finite cotype has the subsequence splitting property (see [6] and [26, Theorem 2.5]).

We will make use of the following fact.

LEMMA 1.4. *Let  $(f_n)$  be a weakly null normalized sequence in  $L_p(\mu)$  for some finite measure  $\mu$  and  $1 < p < \infty$ . Suppose that  $(f_n)$  is uniformly bounded (that is, there exists  $M < \infty$  such that  $|f_n| \leq M$ ). Then, there is a subsequence  $(f_{n_k})$  equivalent to the unit vector basis of  $\ell_2$ .*

*Proof.* Since the sequence  $(f_n)$  is uniformly bounded, it is, in particular, equi-integrable. Thus, since it is a normalized sequence in  $L_p(\mu)$ , by Lemma 1.3 it follows that  $\inf \|f_n\|_{L_1} > 0$ . Hence,  $(f_n)$  is a seminormalized sequence in every  $L_q(\mu)$ , with  $1 \leq q \leq \infty$ . Moreover, as the sequence  $(f_n)$  is weakly null in  $L_p(\mu)$ , it has to be weakly null in every  $L_q(\mu)$  with  $1 \leq q < \infty$ . Therefore, for every  $1 < q < \infty$ , passing to a subsequence,  $(f_n)$  will be equivalent to a

block basis of the Haar system in  $L_q(\mu)$  (cf. [19, Proposition 1.a.12]). Hence, since the Haar system forms an unconditional basis of  $L_q(\mu)$  (see [20, Theorem 2.c.5]), and the blocks of an unconditional basis are also unconditional, then, passing to a further subsequence,  $(f_{n_k})$  can be assumed to be an unconditional basic sequence in  $L_q(\mu)$ .

Let  $1 < q \leq p$  with  $1 < q \leq 2$ , and consider a subsequence  $(f_{n_k})$  that is seminormalized and unconditional in both  $L_q(\mu)$  and  $L_p(\mu)$ . Thus, for every finite sequence of scalars  $(a_i)_{i=1}^n$ , we have

$$\begin{aligned} K_1 \left\| \sum_{i=1}^n a_i f_{n_i} \right\|_{L_q} &\geq \int_0^1 \left\| \sum_{i=1}^n r_i(t) a_i f_{n_i} \right\|_{L_q} dt \\ &\geq C_1 \left( \sum_{i=1}^n \|a_i f_{n_i}\|_{L_q}^2 \right)^{1/2} \\ &\geq C_1 (\inf \|f_j\|_{L_q}) \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2}, \end{aligned}$$

where  $C_1$  is the cotype 2 constant of  $L_q$ , and  $K_1$  is the unconditional constant of  $(f_{n_i})$  in  $L_q(\mu)$ .

On the other hand, we have

$$\begin{aligned} \left\| \sum_{i=1}^n a_i f_{n_i} \right\|_{L_p} &\leq K_2 \int_0^1 \left\| \sum_{i=1}^n r_i(t) a_i f_{n_i} \right\|_{L_p} dt \\ &\leq K_2 C_2 \left\| \left( \sum_{i=1}^n |a_i f_{n_i}|^2 \right)^{1/2} \right\|_{L_p} \\ &\leq 2K_2 C_2 M \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2}, \end{aligned}$$

where  $C_2$  is the constant appearing in [20, Theorem 1.d.6], and  $K_2$  is the unconditional constant of  $(f_{n_i})$  in  $L_p(\mu)$ . This completes the proof.  $\square$

## 2. Proofs of Theorems A and B

In the proofs of Theorems A and B we will make use of the following.

**PROPOSITION 2.1.** *If  $E$  is a Banach lattice with a lower 2-estimate, then every SS operator  $T$  from  $\ell_2$  to  $E$  is compact.*

*Proof.* Since  $E$  satisfies a lower 2-estimate, in particular, it is order continuous and, by [20, Proposition 1.a.9], we can consider an ideal with a weak unit containing  $T(\ell_2)$ . By [20, Theorem 1.b.14], this ideal can be represented as an order dense ideal in  $L_1(\mu)$  for some probability measure  $\mu$ . Thus, let us consider the operator  $T$  as an operator into  $L_1(\mu)$ .

Let us see that  $T : \ell_2 \rightarrow E$  is compact. It clearly suffices to prove that  $\|Te_n\|_E \rightarrow 0$ , where  $(e_n)$  is any weakly null normalized sequence in  $\ell_2$ . Suppose that this is not the case; then, by Proposition 1.1, either  $(\|Te_n\|_{L_1})$  is bounded away from zero or  $(Te_n)$  has a subsequence equivalent to a disjoint sequence.

First assume that the sequence  $(\|Te_n\|_{L_1})$  is bounded away from zero. Then, by [3, Section 6, Theorem], there exists a constant  $\delta > 0$  and a subsequence  $(Te_{n_k})$  such that,

for scalars  $(a_k)_{k=1}^m$ , we have

$$C \|T\| \left( \sum_{k=1}^m |a_k|^2 \right)^{1/2} \geq \left\| \sum_{k=1}^m a_k T e_{n_k} \right\|_E \geq \left\| \sum_{k=1}^m a_k T e_{n_k} \right\|_{L_1} \geq \delta \left( \sum_{k=1}^m |a_k|^2 \right)^{1/2},$$

where  $C > 0$  is the equivalent constant between the sequence  $(e_n)$  and the unit vector basis of  $\ell_2$ .

On the other hand, if  $(T e_{n_j})$  were equivalent to a disjoint sequence, then we would have

$$C \|T\| \left( \sum_{j=1}^k |a_j|^2 \right)^{1/2} \geq \left\| \sum_{j=1}^k a_j T(e_{n_j}) \right\|_E \geq M^{-1} (\inf \|T e_n\|_E) \left( \sum_{j=1}^k |a_j|^2 \right)^{1/2},$$

where  $M$  is the constant appearing in the lower 2-estimate of  $E$ . Thus, in both cases we see that  $T$  is not SS, and we reach a contradiction.  $\square$

The following fact is well known.

LEMMA 2.2. *Let  $X$  be a Banach lattice,  $Y$  a Banach space and  $T : X \rightarrow Y$  an AM-compact operator. For every  $L$ -weakly compact set  $A$  in  $X$ , we have that  $T(A)$  is relatively compact.*

We will make use of the following fact that relates  $\ell_1$ -singular operators with operators that do not preserve a disjoint copy of  $\ell_1$  (compare with [5, 23]).

LEMMA 2.3. *Let  $X$  be a Banach lattice with the subsequence splitting property,  $Y$  a Banach space and  $T : X \rightarrow Y$  an operator. If  $T$  is an isomorphism on a subspace of  $X$  isomorphic to  $\ell_1$ , then there exists a disjoint sequence  $(h_j)$  in  $X$  equivalent to the unit vector basis of  $\ell_1$ , such that  $T$  restricted to the span  $[h_n]$  is an isomorphism.*

*Proof.* Let  $(x_n)$  be a normalized sequence in  $X$  that is equivalent to the unit vector basis of  $\ell_1$  and such that  $T$  restricted to  $[x_n]$  is an isomorphism. By the subsequence splitting property, there exist a subsequence  $(x_{n_k})$  and sequences  $(g_k)$  and  $(h_k)$ , with  $(g_k)$  equi-integrable,  $(h_k)$  disjoint,  $|g_k| \wedge |h_k| = 0$ , and  $x_{n_k} = g_k + h_k$ , for every  $k \in \mathbb{N}$ .

Now, by Rosenthal's lemma, the sequence  $(Th_k)$  has either a weakly Cauchy subsequence or a subsequence equivalent to the unit vector basis of  $\ell_1$ . Suppose that  $(Th_k)$  is weakly Cauchy. Then, since  $(g_k)$  is equi-integrable, it has a weakly convergent subsequence, say  $(g_{k_j})$ . This would imply that  $T x_{n_{k_j}} = T g_{k_j} + T h_{k_j}$  is a weakly Cauchy sequence, in contradiction with the fact that  $T$  is an isomorphism on the span  $[x_n]$ .

Therefore, passing to a subsequence, we can assume that  $(Th_k)$  is equivalent to the unit vector basis of  $\ell_1$ . Thus, for scalars  $(a_k)_{k=1}^n$  and some constant  $C > 0$ , we have

$$C \sum_{k=1}^n |a_k| \leq \left\| \sum_{k=1}^n a_k T h_k \right\| \leq \|T\| \left\| \sum_{k=1}^n a_k h_k \right\| \leq \|T\| \sum_{k=1}^n |a_k|.$$

This completes the proof.  $\square$

THEOREM 2.4. *Let  $X$  be a Banach lattice with finite cotype and  $Y$  be a Banach space. If an operator  $T : X \rightarrow Y$  is DSS and AM-compact, then it is SS.*

*Proof.* Suppose that  $T : X \rightarrow Y$  is DSS, AM-compact, and not SS. Then, using [20, Theorem 1.c.9], we have that  $T$  is an isomorphism when restricted to the span of some

unconditional basic sequence  $(u_n)$  in  $X$ . Now, by Rosenthal’s lemma, we can assume that some subsequence  $(u_{n_k})$  is weakly Cauchy. Otherwise,  $T$  would be an isomorphism on a subspace isomorphic to  $\ell_1$  and, by Lemma 2.3,  $T$  would preserve a disjoint copy of  $\ell_1$ , in contradiction with the fact that  $T$  is DSS. Hence, taking differences, the sequence  $(u_{n_k} - u_{n_{k+1}})$  is weakly null and seminormalized. Thus, there is a subsequence equivalent to a block basis of  $(u_n)$ , which is an unconditional basic sequence (see [19, Proposition 1.a.12, p. 19]). Let us denote by  $(f_n)$  this weakly null, unconditional sequence in  $[u_n]$ .

Let  $\alpha > 0$  be such that, for every sequence of scalars  $(a_n)$ , we have

$$\left\| T \left( \sum_{n=1}^{\infty} a_n f_n \right) \right\| \geq \alpha \left\| \sum_{n=1}^{\infty} a_n f_n \right\|.$$

Since  $X$  has the subsequence splitting property, we can extract a subsequence (still denoted by  $(f_n)$ ) and sequences  $(g_n)$  and  $(h_n)$  such that  $|g_n|, |h_n| \leq |f_n|$ ,  $f_n = g_n + h_n$ , and  $|g_n| \wedge |h_n| = 0$ , where  $(g_n)$  is equi-integrable in  $X$  and  $(h_n)$  is disjoint.

We first consider the case  $\|h_n\| \rightarrow 0$ ; then, passing to a subsequence if needed, the sequence  $(f_n)$  would inherit the equi-integrability from the sequence  $(g_n)$ . Therefore, since the operator  $T$  is AM-compact, by Lemma 2.2, the sequence  $(Tf_n)$  would have a convergent subsequence, but, since  $T$  is invertible on  $[f_n]$ , this would imply that  $(f_n)$  also has a convergent subsequence. This is a contradiction with the fact that  $(f_n)$  is weakly null and normalized.

Alternatively, let us suppose that  $\|h_n\| \geq \rho > 0$ . We consider the operator  $V : [f_n] \rightarrow X$  defined by

$$V \left( \sum_{n=1}^{\infty} a_n f_n \right) = \sum_{n=1}^{\infty} a_n h_n,$$

which is bounded. Indeed, since  $|h_n| \leq |f_n|$  and  $X$  has finite cotype, for some constant  $C > 0$ , we have (cf. [20, Theorem 1.d.6])

$$\begin{aligned} \left\| V \left( \sum_{i=1}^n a_i f_i \right) \right\| &= \left\| \sum_{i=1}^n a_i h_i \right\| \\ &= \left\| \left( \sum_{i=1}^n |a_i|^2 |h_i|^2 \right)^{1/2} \right\| \\ &\leq \left\| \left( \sum_{i=1}^n |a_i f_i|^2 \right)^{1/2} \right\| \\ &\leq C \left\| \int_0^1 \left| \sum_{i=1}^n a_i f_i r_i(t) \right| dt \right\| \\ &\leq C \int_0^1 \left\| \sum_{i=1}^n a_i f_i r_i(t) \right\| dt. \end{aligned}$$

But now, if  $K$  denotes the unconditional constant of  $(f_n)$ , then we have

$$\int_0^1 \left\| \sum_{i=1}^n a_i f_i r_i(t) \right\| dt = 2^{-n} \sum_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^n \varepsilon_i a_i f_i \right\| \leq K \left\| \sum_{i=1}^n a_i f_i \right\|.$$

Hence, the operator  $V$  is bounded with  $\|V\| \leq KC$ .

Therefore, the restriction operator  $T|_{[f_n]} : [f_n] \rightarrow Y$  can be decomposed as

$$T|_{[f_n]} = TV + T(I_{[f_n]} - V),$$

where  $I_{[f_n]} : [f_n] \hookrightarrow X$  is the identity inclusion. It holds that the operator  $TV : [f_n] \rightarrow Y$  is SS. Indeed, if it were not the case, then  $TV$  would be invertible on some subspace  $M \subset [f_n]$ . Then,  $T$  would also be invertible in the subspace  $V(M) \subset [h_n]$ . Now, arguing as in the first paragraph of the proof, we can find a normalized weakly null sequence in  $V(M)$ , which will be equivalent to a block basis of  $[h_n]$ , which is disjoint. More precisely, by [19, Proposition 1.a.12], there exists a normalized sequence  $(v_n)$  in  $V(M)$  with  $\|v_n - \sum_{i=m_n+1}^{m_{n+1}} c_i h_i\|_X \rightarrow 0$  for some increasing sequence  $(m_n)$  and scalars  $(c_i)$ . If we write  $v'_n = \sum_{i=m_n+1}^{m_{n+1}} c_i h_i$ , then this sequence is clearly disjoint and also satisfies  $\|Tv_n - Tv'_n\|_Y \rightarrow 0$ . So, by the perturbation argument, passing to a further subsequence,  $(Tv'_n)$  is equivalent to  $(Tv_n)$ , and, by the invertibility of  $T$  on  $V(M)$ , this sequence is equivalent to  $(v_n)$ , and hence to  $(v'_n)$ . This is a contradiction with the fact that  $T$  is DSS.

Hence, the operator  $TV$  is SS, and, since  $T|_{[f_n]}$  is an isomorphism, we have, by [19, Proposition 2.c.10], that the operator  $T(I_{[f_n]} - V)$  has a finite-dimensional kernel and a closed range.

Now, by Lemma 2.2, the sequence  $T(I_{[f_n]} - V)(f_n) = T(g_n)$  has a convergent subsequence. Consider the decomposition

$$[f_n] = \text{Ker}(TI_{[f_n]} - V) \oplus Z.$$

Since  $T(I_{[f_n]} - V)$  has a closed range, the operator  $T(I_{[f_n]} - V)$  is invertible on  $Z$ . Hence, the sequence  $(f_n)$  would also have a convergent subsequence. Again, this is a contradiction with the fact that  $(f_n)$  is weakly null and normalized.  $\square$

Recall that an operator  $T : X \rightarrow Y$  is  $\ell_2$ -singular if it is not an isomorphism when restricted to any subspace isomorphic to  $\ell_2$ .

**PROPOSITION 2.5.** *Let  $X$  and  $Y$  be Banach lattices such that  $X$  has finite cotype. If  $T : X \rightarrow Y$  is an  $\ell_2$ -singular operator, then  $T$  is also AM-compact under any of the following conditions.*

- Either the Banach lattice  $Y$  satisfies a lower 2-estimate.*
- Or the Banach lattice  $Y$  is order continuous and  $T$  is regular.*

*Proof.* Let  $x \in X_+$  be fixed and denote by  $E_x$  the closed ideal of  $X$  generated by  $x$ . Since  $X$  is  $q$ -concave for some  $1 < q < \infty$ , we have  $L_q(\mu) \hookrightarrow E_x \hookrightarrow L_1(\mu)$  for a certain probability measure  $\mu$  (see [15, p. 14]). First, let us prove that (in both cases), for every positive constant  $M$ , the set  $T[-M, M]$  is relatively compact. To this end, let  $(f_n)$  be such that  $|f_n| \leq M$ . Since the order intervals in  $X$  are weakly compact, without loss of generality, we can assume that  $f_n \rightarrow 0$  weakly.

Let us consider some  $p > \max\{q, 2\}$ . Clearly, the sequence  $(f_n)$  must also be weakly null and seminormalized in  $L_p(\mu)$ . Otherwise, the sequence  $(Tf_n)$  would have a convergent subsequence and that would complete the proof. Moreover, passing to a further subsequence (still denoted by  $(f_n)$ ) and using Lemma 1.4, the sequence  $(f_n)$  spans in  $L_p(\mu)$  a subspace isomorphic to  $\ell_2$ . Therefore, since a normalized disjoint sequence in  $L_p(\mu)$  spans a subspace isomorphic to  $\ell_p$ , by Proposition 1.1, we say that  $[f_n]$  has to be strongly embedded in  $L_p(\mu)$ , which means that on  $[f_n]$  the topology of  $L_p(\mu)$  and  $L_1(\mu)$  coincide.

Thus, for certain constants  $\alpha, \beta > 0$  and any scalars  $(a_i)_{i=1}^n$ , we have

$$\alpha \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^n a_i f_i \right\|_{L_1} \leq \left\| \sum_{i=1}^n a_i f_i \right\|_X \leq \left\| \sum_{i=1}^n a_i f_i \right\|_{L_p} \leq \beta \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2}.$$

Thus,  $(f_n)$  has a subsequence whose span in  $X$  is isomorphic to  $\ell_2$ .

On the one hand, suppose that  $Y$  satisfies a lower 2-estimate. Then  $(T(f_n))$  must have a subsequence that tends to zero in norm. Otherwise, the operator  $T$  restricted to the span  $[f_{n_k}]$  that is isomorphic to  $\ell_2$ , would not be a compact operator, and, by Proposition 2.1,  $T$  would not be  $\ell_2$ -singular. This is a contradiction.

Now consider the case where  $Y$  is order continuous and the operator  $T$  is regular. Then, the sequence  $(T(f_n))$  is equi-integrable by Lemma 1.2. Hence, if  $(Tf_n)$  is not convergent to zero in  $Y$ , then, by Lemma 1.3,  $(\|T(f_n)\|_{L_1})$  is bounded away from zero. This implies that the operator  $R : \ell_2 \rightarrow L_1(\nu)$ , defined by the following diagram, is not compact:

$$\begin{array}{ccc} \ell_2 & \xrightarrow{R} & L_1(\nu) \\ i \downarrow & & \uparrow j \\ [f_{n_k}] & \xrightarrow{T} & Y \end{array}$$

Here  $i$  is just an isomorphism, and  $j$  is the formal inclusion of  $Y$  in some  $L_1(\nu)$  (or of an ideal with weak unit containing  $T(f_n)$ ) (see [20, Theorem 1.b.14]). By Proposition 2.1, we see that  $T$  is not  $\ell_2$ -singular. This is a contradiction.

So far, we have shown that, in both cases,  $T[-M, M]$  is a relatively compact set for every positive constant  $M$ . Now, we use the density of  $L_\infty(\mu)$  in  $X$ , which follows from the representability of  $E_x$  as a function space between  $L_q(\mu)$  and  $L_1(\mu)$ . Hence, given  $\varepsilon > 0$ , we can consider  $M_\varepsilon < \infty$  such that  $x \in [0, M_\varepsilon] + \varepsilon\|T\|^{-1}B_X$ . Therefore,  $T[-x, x] \subset T[-M_\varepsilon, M_\varepsilon] + \varepsilon B_Y$ , and, since  $T[-M_\varepsilon, M_\varepsilon]$  is relatively compact, so is  $T[-x, x]$ . This completes the proof.  $\square$

Proposition 2.5 has a partial converse.

**PROPOSITION 2.6.** *Let  $X$  be a Banach lattice and  $Y$  be a Banach space. Suppose that  $X$  has finite cotype and does not contain any sequence of disjoint elements that span a subspace isomorphic to  $\ell_2$ . If an operator  $T : X \rightarrow Y$  is AM-compact, then  $T$  is also  $\ell_2$ -singular.*

*Proof.* Suppose that  $T : X \rightarrow Y$  is AM-compact, but not  $\ell_2$ -singular. Therefore, there exists a sequence  $(f_n)$  in  $X$  that spans a subspace isomorphic to  $\ell_2$  and such that  $T$  is an isomorphism when restricted to  $[f_n]$ .

Since  $X$  has the subsequence splitting property, passing to a subsequence, we have  $f_n = h_n + g_n$  with  $|h_n| \wedge |g_n| = 0$ ,  $(g_n)$  equi-integrable,  $(h_n)$  disjoint, and  $|h_n|, |g_n| \leq |f_n|$  for all  $n$ . Again, arguing as in the proof of Theorem 2.4, the operator  $V : [f_n] \rightarrow X$  defined by  $V(f_n) = h_n$  is bounded. Hence, since  $[f_n]$  is isomorphic to  $\ell_2$ , all three sequences  $(f_n)$ ,  $(g_n)$ , and  $(h_n)$  must be weakly null.

Thus, by Lemma 2.2, the sequence  $(T(g_n))$  must have a subsequence that goes to zero in norm. Therefore,  $(T(h_n))$  has a subsequence, say  $(T(h_{n_k}))$ , equivalent to  $(T(f_{n_k}))$ , which is equivalent to the unit vector basis of  $\ell_2$ . Then, there exist constants  $\alpha$  and  $\beta$ , such that, for any  $n$  and  $(a_k)_{k=1}^n$ , we have

$$\begin{aligned} \alpha \left( \sum_{k=1}^n |a_k|^2 \right)^{1/2} &\leq \left\| T \left( \sum_{k=1}^n a_k h_{n_k} \right) \right\| \leq \|T\| \left\| \sum_{k=1}^n a_k h_{n_k} \right\| \\ &\leq \|T\| \|V\| \left\| \sum_{k=1}^n a_k f_{n_k} \right\| \leq \beta \|T\| \|V\| \left( \sum_{k=1}^n |a_k|^2 \right)^{1/2}. \end{aligned}$$

This means that  $[h_{n_k}]$  is isomorphic to  $\ell_2$ , and this is impossible according to the hypothesis on  $X$ .  $\square$

*Proof of Theorems A and B.* Clearly, SS operators are DSS and  $\ell_2$ -singular. Conversely, if the operator  $T : X \rightarrow Y$  is  $\ell_2$ -singular, then, by Proposition 2.5,  $T$  must also be AM-compact. The conclusion now follows from Theorem 2.4.  $\square$

REMARK 2.7. Note that Theorem B is still true if  $X$  is just an order continuous Banach lattice with the subsequence splitting property.

As a consequence, we obtain a domination result for positive SS operators, which improves [10, Theorem 3.1].

COROLLARY 2.8. *Let  $X$  be a Banach lattice with finite cotype and  $Y$  be an order continuous Banach lattice. Suppose that  $R \leq T : X \rightarrow Y$  are positive operators. If  $T$  is SS, then  $R$  is also SS.*

*Proof.* Since the operator  $T$  is SS, it is obviously DSS and  $\ell_2$ -singular. Now, since  $Y$  is order continuous, [9, Theorem 1.1] yields that  $R$  is DSS. Moreover, by Proposition 2.5, the operator  $T$  is AM-compact, and, by [21, Proposition 3.7.2], it follows that  $R$  is also AM-compact. Hence, by Theorem 2.4, we conclude that the operator  $R$  is SS.  $\square$

See [17] for related results concerning the domination of complementably  $\ell_p$ -singular operators.

### 3. Proof of Theorem C

In this section, we will construct operators in order to show that the conditions on Theorems A and B cannot be relaxed.

First, we need some preliminary lemmas. The first of them will be deduced from the following result for i.i.d.  $p$ -stable random variables given in [15, Lemma 3.10].

PROPOSITION 3.1. *Let  $X$  be a finite-dimensional Banach space with a 1-unconditional basis  $(x_i)_{i=1}^n$ , and let  $Y$  be a  $q$ -concave Banach lattice for some  $1 < q < \infty$ . Let  $T$  be an isomorphism from  $X$  into  $Y$  and let  $(g_i)_{i=1}^n$  be a sequence of i.i.d.  $p$ -stable random variables over a probability space  $(\Omega, \Sigma, \mu)$  for some  $1 < p < 2$ . Then, for every scalar sequence  $(a_i)_{i=1}^n$  for which*

$$\int_{\Omega} \left\| \sum_{i=1}^n a_i g_i(\omega) x_i \right\|_X d\mu(\omega) \leq \left\| \sum_{i=1}^n a_i x_i \right\|_X,$$

the following inequality holds:

$$K \left\| \sum_{i=1}^n a_i x_i \right\|_X \leq \left\| \max_{1 \leq i \leq n} |a_i T x_i| \right\|_Y \leq \left\| \left( \sum_{i=1}^n |a_i T x_i|^p \right)^{1/p} \right\|_Y \leq \|T\| \|g_1\|_{L_1}^{-1} \left\| \sum_{i=1}^n a_i x_i \right\|_X$$

for certain constant  $K$  (depending only on the  $q$ -concavity constant of  $Y$ ).

In what follows, the expression  $\| \sum_{n=1}^k a_n f_n \| \sim \| \sum_{n=1}^k a_n g_n \|$  will mean, as usual, that there exist constants  $c, C > 0$  such that, for any  $k \in \mathbb{N}$  and any  $(a_n)_{n=1}^k$ , we have

$$c \left\| \sum_{n=1}^k a_n f_n \right\| \leq \left\| \sum_{n=1}^k a_n g_n \right\| \leq C \left\| \sum_{n=1}^k a_n f_n \right\|.$$

LEMMA 3.2. *Let  $1 < q < s < 2$  and let  $T : \ell_s \rightarrow L_q(\mu)$  be an isomorphic embedding. If  $f_n = T(e_n)$  denotes the image under  $T$  of  $(e_n)$ , the canonical basis of  $\ell_s$ , then, for every  $2 \leq r < \infty$ , we have*

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \left\| \left( \sum_{n=1}^k |a_n f_n|^r \right)^{1/r} \right\|_{L_q} \sim \left\| \max_{1 \leq n \leq k} |a_n f_n| \right\|_{L_q}$$

for any scalar sequence  $(a_n)_{n=1}^k$ .

*Proof.* Take  $n \in \mathbb{N}$  and let  $X = \ell_s^n$ ,  $Y = L_q(\mu)$ , and  $T = T|_X$  in Proposition 3.1. If  $(g_i)_{i=1}^n$  is a sequence of i.i.d.  $p$ -stable random variables with  $s < p < 2$ , then

$$\begin{aligned} \int_{\Omega} \left\| \sum_{i=1}^n a_i g_i(\omega) e_i \right\|_{\ell_s} d\mu(\omega) &= \int_{\Omega} \left( \sum_{i=1}^n |a_i g_i(\omega)|^s \right)^{1/s} d\mu(\omega) \\ &\leq \left( \int_{\Omega} \sum_{i=1}^n |a_i g_i(\omega)|^s d\mu(\omega) \right)^{1/s} \\ &= \left( \sum_{i=1}^n |a_i|^s \int_{\Omega} |g_i(\omega)|^s d\mu(\omega) \right)^{1/s} \\ &= \|g_1\|_{L_s} \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_s}. \end{aligned}$$

Since  $\|g_1\|_{L_s} < \infty$ , using Proposition 3.1, we obtain

$$\begin{aligned} K \|g_1\|_{L_s}^{1/2} \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_s} &\leq \left\| \max_{1 \leq i \leq n} |a_i f_i| \right\|_{L_q} \\ &\leq \left\| \left( \sum_{i=1}^n |a_i f_i|^p \right)^{1/p} \right\|_{L_q} \\ &\leq \|T\| \|g_1\|_{L_1}^{-1} \|g_1\|_{L_s}^{1/2} \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_s} \end{aligned}$$

for certain constant  $K$  independent of  $n$ .

Since

$$\left\| \max_{1 \leq i \leq n} |a_i f_i| \right\|_{L_q} \leq \left\| \left( \sum_{i=1}^n |a_i f_i|^2 \right)^{1/2} \right\|_{L_q} \leq \left\| \left( \sum_{i=1}^n |a_i f_i|^p \right)^{1/p} \right\|_{L_q}$$

by the previous inequality, we immediately obtain

$$\left\| \max_{1 \leq i \leq n} |a_i f_i| \right\|_{L_q} \sim \left\| \left( \sum_{i=1}^n |a_i f_i|^2 \right)^{1/2} \right\|_{L_q} \sim \left\| \left( \sum_{i=1}^n |a_i f_i|^p \right)^{1/p} \right\|_{L_q}.$$

On the other hand, since  $(f_n)$  is an unconditional basic sequence, we have

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \int_0^1 \left\| \sum_{n=1}^k r_n(t) a_n f_n \right\|_{L_q} dt.$$

Thus, [20, Theorem 1.d.6(i)] yields

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \left\| \left( \sum_{n=1}^k |a_n f_n|^2 \right)^{1/2} \right\|_{L_q}.$$

Therefore, for every  $2 \leq r \leq \infty$ , we have

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \left\| \left( \sum_{n=1}^k |a_n f_n|^r \right)^{1/r} \right\|_{L_q}. \quad \square$$

The next result shows why Lemma 3.2 cannot be extended to the case  $s = 2$ . Given functions  $f, g : \mathbb{N} \rightarrow \mathbb{R}$ , by  $f = o(g)$  we mean, as usual, that  $f(m)/g(m) \rightarrow 0$  when  $m \rightarrow \infty$ .

LEMMA 3.3. *Let  $1 < p < 2$ . If  $T : \ell_2 \rightarrow L_p(\mu)$  is a bounded operator and  $f_n = T(e_n)$ , where  $(e_n)$  denotes the canonical basis of  $\ell_2$ , then, for each natural number  $m$ , we have*

$$\inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p} = o(\sqrt{m}).$$

Moreover, for any  $2 < q < \infty$ , we have

$$\inf_{|A|=m} \left\| \left( \sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} = o(\sqrt{m}).$$

*Proof.* By Krivine’s theorem (cf. [20, Theorem 1.f.14]), given any finite family  $(x_i)_{i=1}^n$  of vectors in  $\ell_2$ , we have

$$\left\| \left( \sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\|_{L_p} \leq K_G \|T\| \left\| \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_{\ell_2} = K_G \|T\| \left( \sum_{i=1}^n \|x_i\|_{\ell_2}^2 \right)^{1/2},$$

where  $K_G$  is Grothendieck’s constant. Now, by Maurey’s factorization theorem (cf. [1, Theorem 7.1.2]), there exists a density function  $h$  on  $\Omega$  (that is,  $h > 0$  and  $\int h d\mu = 1$ ) such that

$$\begin{array}{ccc} \ell_2 & \xrightarrow{T} & L_p(\mu) \\ \tilde{T} \downarrow & & \uparrow J \\ L_2(h d\mu) & \xrightarrow{i} & L_p(h d\mu) \end{array}$$

where  $\tilde{T}(x) = h^{-1/p} T(x)$  for every  $x \in \ell_2$ ,  $i$  denotes the canonical inclusion, and  $J$  is the isometry mapping each  $f \in L_p(h d\mu)$  to  $J(f) = fh^{1/p}$ .

Let us write  $f_n = \tilde{T}(e_n) \in L_2(h d\mu)$ . Since  $L_2(h d\mu)$  satisfies the subsequence splitting property, there exist a subsequence  $(f_{n_k})$  and sequences  $(g_k)$  and  $(h_k)$ , with  $(h_k)$  disjoint and  $(g_k)$  equi-integrable in  $L_2(h d\mu)$ , such that  $f_{n_k} = g_k + h_k$  and  $|g_k| \wedge |h_k| = 0$ . Therefore,

the sequence  $(J(h_k))$  is disjoint in  $L_p(\mu)$ , and, for scalars  $(a_k)_{k=1}^n$ , we have

$$\begin{aligned} \left\| \sum_{k=1}^n a_k J(h_k) \right\|_{L_p} &\leq \left\| \sum_{k=1}^n a_k h_k \right\|_{L_2(h)} \\ &= \left\| \left( \sum_{k=1}^n |a_k h_k|^2 \right)^{1/2} \right\|_{L_2(h)} \\ &\leq \left\| \left( \sum_{k=1}^n |a_k \widetilde{f_{n_k}}|^2 \right)^{1/2} \right\|_{L_2(h)} \\ &\leq K_G \|\widetilde{T}\| \left\| \left( \sum_{k=1}^n |a_k e_{n_k}|^2 \right)^{1/2} \right\|_{\ell_2} \\ &= C \left( \sum_{k=1}^n |a_k|^2 \right)^{1/2}, \end{aligned}$$

with a constant  $C$  independent of  $n$  (cf. [20, Theorem 1.f.14]). Now, if  $\inf_k \|J(h_k)\|_{L_p} > 0$ , then, for some constant  $c$  and for all  $(a_k)_{k=1}^n$ , we have

$$c \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} \leq \left\| \sum_{k=1}^n a_k J(h_k) \right\|_{L_p} \leq C \left( \sum_{k=1}^n |a_k|^2 \right)^{1/2},$$

which is impossible (since  $1 < p < 2$ ).

Thus, passing to a subsequence, we can assume that  $\|J(h_k)\|_{L_p} \rightarrow 0$  as  $k \rightarrow \infty$ . Now, for each  $m \in \mathbb{N}$ , let us write

$$\phi(m) = \left\| \max_{1 \leq k \leq m} |g_k| \right\|_{L_2(h)},$$

and let us take disjoint measurable sets  $A_1^m, A_2^m, \dots, A_m^m$  in  $\Omega$  such that

$$\max_{k \leq m} |g_k| = \sum_{k=1}^m |g_k| \chi_{A_k^m}.$$

CLAIM 3.4. We have  $\phi(m)/\sqrt{m} \rightarrow 0$  when  $m \rightarrow \infty$ .

*Proof of Claim 3.4.* Assuming the contrary, then there exist  $\varepsilon > 0$  and an increasing sequence  $m_n \rightarrow \infty$  such that

$$\frac{\phi(m_n)}{\sqrt{m_n}} \geq \varepsilon;$$

that is, for all  $n \in \mathbb{N}$ , we can choose an integer  $m_n$  and disjoint sets  $A_1^{m_n}, A_2^{m_n}, \dots, A_{m_n}^{m_n}$  such that

$$\frac{1}{m_n} \sum_{k=1}^{m_n} \int_{A_k^{m_n}} |g_k|^2 h \, d\mu = \frac{1}{m_n} \left\| \sum_{k=1}^{m_n} |g_k| \chi_{A_k^{m_n}} \right\|_{L_2(h)}^2 = \left( \frac{\phi(m_n)}{\sqrt{m_n}} \right)^2 \geq \varepsilon^2 \tag{3.1}$$

for every natural  $n$ . From this fact we conclude that, for every  $N \in \mathbb{N}$ , we can find  $B_1, \dots, B_N$  disjoint sets in  $\Omega$  such that

$$\sup_k \left\{ \int_{B_n} |g_k|^2 h \, d\mu \right\} \geq \frac{\varepsilon^2}{2}$$

for all  $n = 1, \dots, N$ .

Otherwise, suppose that there exists  $N$  such that, for  $n$  large enough, the set

$$S_n = \left\{ k \leq m_n : \int_{A_k^{m_n}} |g_k|^2 h \, d\mu \geq \frac{\varepsilon^2}{2} \right\}$$

always has cardinality less than  $N$ . Then, for  $n$  large enough so that  $(m_n - N)/m_n < 1$  and  $N \sup_k \|g_k\|_{L_2}/m_n < \varepsilon^2/2$ , we have

$$\frac{1}{m_n} \sum_{k=1}^{m_n} \int_{A_k^{m_n}} |g_k|^2 h \, d\mu < \frac{1}{m_n} \left[ (m_n - N) \frac{\varepsilon^2}{2} + N \sup_k \|g_k\|_{L_2} \right] < \varepsilon^2,$$

which is a contradiction with (3.1).

Hence, by [27, Theorem III.C.12], we reach a contradiction with the fact that  $(g_k)$  is equi-integrable in  $L_2(h \, d\mu)$ . Therefore,  $\phi(m)/\sqrt{m} \rightarrow 0$  when  $m \rightarrow \infty$ , and the claim is proved.  $\square$

Now, since  $\|J(h_k)\|_{L_p} \rightarrow_{k \rightarrow \infty} 0$ , we have that, for every  $m \in \mathbb{N}$  and for every  $\varepsilon > 0$ , there exists a set  $A_\varepsilon = \{k_1, k_2, \dots, k_m\}$  of natural numbers such that  $\|J(h_k)\|_{L_p} < \varepsilon/m$  for all  $k \in A_\varepsilon$ . Therefore,

$$\begin{aligned} \inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p} &\leq \inf_{|A|=m} \left\| \max_{j \in A} J(|g_j| + |h_j|) \right\|_{L_p} \\ &\leq \phi(m) + \left\| \max_{j \in A_\varepsilon} J(|h_j|) \right\|_{L_p} \\ &\leq \phi(m) + \varepsilon, \end{aligned}$$

and, since this inequality holds for all  $\varepsilon > 0$ , we obtain that  $\inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p} \leq \phi(m)$ , which implies that

$$\inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p} = o(\sqrt{m}).$$

Now, the second assertion of the lemma is obtained by a Hölder-type inequality [20, Proposition 1.d.2]. Indeed, given  $m \in \mathbb{N}$ , for any  $A \subset \mathbb{N}$  with  $|A| = m$ , we have

$$\begin{aligned} \left\| \left( \sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} &\leq \left\| \left( \left( \sum_{j \in A} |f_j|^2 \right)^{1/2} \right)^\theta \left( \max_{j \in A} |f_j| \right)^{1-\theta} \right\|_{L_p} \\ &\leq \left\| \left( \sum_{j \in A} |f_j|^2 \right)^{1/2} \right\|_{L_p}^\theta \left\| \max_{j \in A} |f_j| \right\|_{L_p}^{1-\theta} \end{aligned}$$

for  $\theta = 2/q \in (0, 1)$ . Now, by the first part of the lemma, the function

$$\varphi(m) = \inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p}$$

satisfies  $\varphi(m)/\sqrt{m} \rightarrow 0$  as  $m \rightarrow \infty$ . Moreover, by passing to a subsequence,  $(f_j)$  can be assumed to be unconditional, and so we have

$$\left\| \left( \sum_{j \in A} |f_j|^2 \right)^{1/2} \right\|_{L_p} \sim \left\| \sum_{j \in A} f_j \right\|_{L_p} \leq \|T\| \sqrt{m}.$$

Thus, for some constant  $C < \infty$  and for any  $q > 2$ , we have

$$\inf_{|A|=m} \left\| \left( \sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} \leq (C \|T\| \sqrt{m})^\theta (\varphi(m))^{1-\theta},$$

and clearly

$$\frac{(C \|T\| \sqrt{m})^\theta \varphi(m)^{1-\theta}}{\sqrt{m}} \rightarrow 0$$

as  $m \rightarrow \infty$ . □

LEMMA 3.5. *Let  $1 < p < 2$ . If a sequence  $(f_n) \subset L_p(0, 1)$  satisfies*

$$\left\| \left( \sum_{n=1}^\infty |a_n|^2 |f_n|^2 \right)^{1/2} \right\|_{L_p} \leq C \left( \sum_{n=1}^\infty |a_n|^2 \right)^{1/2}$$

for some constant  $C > 0$  and every finitely non-zero scalar sequence  $(a_n)$ , then, for  $2 < q < \infty$ , we have

$$\inf_{|A|=m} \left\| \left( \sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} = o(\sqrt{m}).$$

*Proof.* Let us consider the operator  $T : \ell_2 \rightarrow L_p([0, 1] \times [0, 1])$  defined by  $T(e_n) = f_n \otimes r_n$ , where  $(e_n)$  denotes the canonical basis of  $\ell_2$ ,  $(r_n)$  are the Rademacher functions on  $[0, 1]$ , and  $f_n \otimes r_n(s, t) = f_n(s)r_n(t)$ . By Khintchine’s inequality, for scalars  $(a_n)$ , we have

$$\begin{aligned} \left\| \sum_{n=1}^k a_n f_n \otimes r_n \right\|_{L_p} &= \left( \int_0^1 \int_0^1 \left| \sum_{n=1}^k a_n f_n(s)r_n(t) \right|^p dt \right)^{1/p} \\ &\leq C_p \left\| \left( \sum_{n=1}^\infty |a_n|^2 |f_n|^2 \right)^{1/2} \right\|_{L_p} \end{aligned}$$

for certain constant  $C_p$ . Therefore, by the hypothesis of the lemma, this operator is bounded, and hence, by Lemma 3.3, we have

$$\inf_{|A|=m} \left\| \left( \sum_{j \in A} |f_j \otimes r_j|^q \right)^{1/q} \right\|_{L_p} = o(\sqrt{m}).$$

But, since  $|f_j| = |f_j \otimes r_j|$ , the proof is completed. □

We are now in a position to state and prove the main result of this section. We shall consider the Banach lattice  $L_r(\ell_q)$  for  $1 \leq r, q < \infty$ , defined as the set of all sequences  $x = (x_1, x_2, \dots)$  of elements of  $L_r$  such that

$$\|x\|_{L_r(\ell_q)} = \sup_n \left\| \left( \sum_{i=1}^n |x_i|^q \right)^{1/q} \right\|_{L_r} < \infty$$

(cf. [20, pp. 46–47]).

*Proof of Theorem C.* Since  $1 < r < p < 2$ , we can consider the formal inclusion  $J : L_p[0, 1] \hookrightarrow L_r[0, 1]$ . Let us denote by  $H_r$  the atomic Banach lattice whose lattice structure comes from the unconditional Haar basis  $(h_n)$  in  $L_r[0, 1]$ , and which is isomorphic to  $L_r[0, 1]$ . Let  $L : L_r[0, 1] \rightarrow H_r$  be this isomorphism.

Now, given  $p < s < 2$ , consider a sequence  $(f_n)$  of i.i.d.  $s$ -stable random variables in  $L_p[0, 1]$ . Hence the span  $[f_n]$  is isometrically isomorphic to  $\ell_s$  both in  $L_p[0, 1]$  and  $L_r[0, 1]$ . Now,

since  $f_n \rightarrow 0$  weakly in  $L_p[0, 1]$ , there exists a block basis  $(w_n)$  of  $(h_n)$  of the form

$$w_n = \sum_{i=q_{n-1}+1}^{q_n} a_i h_i,$$

with  $(q_n)$  being an increasing sequence of natural numbers, such that  $(w_n)$  is equivalent to a subsequence of  $(f_n)$  (still denoted by  $(f_n)$ ). In fact, we have that  $\|f_n - w_n\|_{L_p} \rightarrow 0$  and, since  $r < p$ , we also have  $\|f_n - w_n\|_{L_r} \rightarrow 0$ . Hence, passing to a further subsequence, we have that  $(f_n)$  and  $(w_n)$  are equivalent both in  $L_p[0, 1]$  and  $L_r[0, 1]$ .

Let us now consider the operator  $R$  defined by

$$R : H_r \longrightarrow L_r(\ell_q)$$

$$(c_i)_{i=1}^\infty \longmapsto \left( \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right)_{n=1}^\infty,$$

which is clearly bounded. Indeed, since  $q > 2$  and the Haar basis  $(h_n)$  is unconditional in  $L_r[0, 1]$ , we have

$$\begin{aligned} \|R(c_i)\|_{L_r(\ell_q)} &= \sup_k \left\| \left( \sum_{n=1}^k \left| \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right|^q \right)^{1/q} \right\|_{L_r} \\ &\leq \sup_k \left\| \left( \sum_{n=1}^k \left| \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right|^2 \right)^{1/2} \right\|_{L_r} \\ &\leq C \left\| \sum_{i=1}^\infty c_i h_i \right\|_{L_r} = C \|(c_i)\|_{H_r} \end{aligned}$$

for certain constant  $C > 0$ .

Let us now consider the operator  $T : L_p[0, 1] \rightarrow L_r(\ell_q)$  defined as follows.

$$\begin{array}{ccc} L_p & \xrightarrow{T} & L_r(\ell_q) \\ \downarrow J & & \uparrow R \\ L_r & \xrightarrow{L} & H_r \end{array}$$

The operator  $T = RLJ$  is not  $\ell_s$ -singular. Indeed,  $T$  is an isomorphism when restricted to the subspace  $[w_n]$  in  $L_p$ , which is isomorphic to  $\ell_s$ , since, by Lemma 3.2, we have

$$\begin{aligned} \left\| T \left( \sum_{n=1}^\infty b_n w_n \right) \right\|_{L_r(\ell_q)} &= \sup_k \left\| \left( \sum_{n=1}^k \left| b_n \sum_{i=q_{n-1}+1}^{q_n} a_i h_i \right|^q \right)^{1/q} \right\|_{L_r} \\ &\sim \left\| \sum_{n=1}^\infty b_n w_n \right\|_{L_r} \sim \left\| \sum_{n=1}^\infty b_n w_n \right\|_{L_p} \\ &\sim \left( \sum_{n=1}^\infty |b_n|^s \right)^{1/s}. \end{aligned}$$

In particular,  $T$  is not SS. On the other hand, the operator  $T$  is  $\ell_p$ -singular because so is the inclusion  $J : L_p \hookrightarrow L_r$ .

Let us now prove that  $T$  does not preserve an isomorphic copy of  $\ell_2$ . To see this, it suffices to show that  $RL$  preserves no isomorphic copy of  $\ell_2$ . Indeed, if this were not the case, then let  $(g_n)$  be a sequence equivalent to the unit vector basis of  $\ell_2$  in  $L_r$ , so that  $(RL(g_n))$  is also equivalent to it. Since  $g_n \rightarrow 0$  weakly, without loss of generality, we can suppose that  $(g_n)$  is a block basis of the Haar system. In fact, we can extract a subsequence (still denoted by  $(g_n)$ ) such that

$$g_n = \sum_{k=p_{n-1}+1}^{p_n} \psi_k^n,$$

where each  $\psi_k^n \in [h_{q_{j_k-1}+1}, \dots, h_{q_{j_k}}]$  for a certain increasing sequence  $(j_k)$  in  $\mathbb{N}$  (notice that the sequence  $(q_n)$  has already been fixed in the definition of the operator  $R$ ).

Now, the sequence  $(\psi_k^n)_{n=1, k=p_{n-1}+1, \dots, p_n}^\infty$  forms an unconditional basic sequence since it is a sequence of blocks of the Haar basis, which is unconditional in  $L_r(0, 1)$ . Therefore, for every finitely non-zero sequence of scalars  $(a_n)$ , we have

$$\begin{aligned} \left\| \left( \sum_{n=1}^\infty \sum_{k=p_{n-1}+1}^{p_n} |a_n \psi_k^n|^2 \right)^{1/2} \right\|_{L_r} &\sim \left\| \sum_{n=1}^\infty \sum_{k=p_{n-1}+1}^{p_n} a_n \psi_k^n \right\|_{L_r} \\ &= \left\| \sum_{n=1}^\infty a_n g_n \right\|_{L_r} \\ &\sim \left( \sum_{n=1}^\infty a_n^2 \right)^{1/2} \end{aligned}$$

(see [20, Theorem 1.d.6]). Let us consider

$$f_n = \left( \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \right)^{1/q}.$$

Since  $q > 2$ , we have

$$\begin{aligned} \left\| \left( \sum_{n=1}^\infty a_n^2 f_n^2 \right)^{1/2} \right\|_{L_r} &= \left\| \left( \sum_{n=1}^\infty a_n^2 \left( \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \right)^{2/q} \right)^{1/2} \right\|_{L_r} \\ &\leq \left\| \left( \sum_{n=1}^\infty a_n^2 \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^2 \right)^{1/2} \right\|_{L_r} \sim \left( \sum_{n=1}^\infty a_n^2 \right)^{1/2}. \end{aligned}$$

Now, by Lemma 3.5, we obtain

$$\inf_{|A|=m} \left\| \left( \sum_{n \in A} |f_n|^q \right)^{1/q} \right\|_{L_r} = o(\sqrt{m}).$$

However, by hypothesis, there exists some constant  $C > 0$  such that

$$\begin{aligned} C\sqrt{m} &\leq \inf_{|A|=m} \left\| \sum_{n \in A} RLg_n \right\|_{L_r(\ell_q)} \\ &= \inf_{|A|=m} \left\| \left( \sum_{n \in A} \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \right)^{1/q} \right\|_{L_r} \\ &= \inf_{|A|=m} \left\| \left( \sum_{n \in A} |f_n|^q \right)^{1/q} \right\|_{L_r} = o(\sqrt{m}). \end{aligned}$$

This is a contradiction; thus, the operator  $RL$  is  $\ell_2$ -singular, and so is  $T = RLJ$ .  $\square$

**REMARK 3.6.** Note that, if the sequence  $(q_n)$ , appearing in the definition of the operator  $T : L_p \rightarrow L_r(\ell_q)$  given above, increases fast enough, then it can be seen that the operator  $T$  is not  $\ell_{s_n}$ -singular, where  $(s_n)$  is a countable dense set in the interval  $(p, 2)$ . From this fact, using an approximation argument for  $s$ -stable random variables, we can conclude that  $T$  is not  $\ell_s$ -singular for any  $s \in (p, 2)$ .

**REMARK 3.7.** The hypothesis of order continuity of the range Banach lattice  $Y$  in Theorem B cannot be removed.

Indeed, consider the operator  $T : L_p \rightarrow L_r(\ell_q)$  constructed in Theorem C and the canonical isomorphic embedding  $j : L_r(\ell_q) \rightarrow \ell_\infty$ . Now, the composition  $jT : L_p \rightarrow \ell_\infty$  is a regular operator (cf. [21, Theorem 1.5.11]) that is DSS and  $\ell_2$ -singular, but it is not SS (because  $T$  is not SS and  $j$  is an isomorphic embedding).

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