Journal of the Inst. of Math. Jussieu (2003) **2**(3), 401–408 © Cambridge University Press 401
DOI:10.1017/S1474748003000112 Printed in the United Kingdom

TWISTED HILBERT SPACES AND UNCONDITIONAL STRUCTURE

N. J. KALTON

Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA (nigel@math.missouri.edu)

(Received 12 September 2002; accepted 25 September 2002)

Abstract We show that a twisted Hilbert space with an unconditional basis is isomorphic to a Hilbert space.

Keywords: twisted Hilbert spaces; unconditional bases; three-space problem

AMS 2000 *Mathematics subject classification:* Primary 46B10
Secondary 46C15

1. Introduction

In this paper we answer a question raised in [**1**]. We recall that a *twisted Hilbert space* is a Banach space X such that one can construct a short exact sequence $0 \to H_1 \to X \to H_2 \to 0$, where H_1, H_2 are Hilbert spaces (equivalently, X has a subspace E that is isomorphically Hilbertian and such that X/E is also isomorphically Hilbertian). In [**1**] it was asked whether a twisted Hilbert space with unconditional basis is necessarily Hilbertian. We show that this is the case (Theorem 2.3 below).

In fact, the solution requires very little extra work from the results of [**1**]. For the convenience of the reader, we first give a somewhat simpler proof of a result equivalent to [**1**, Theorem 3.9], and then we show that this leads very quickly to the conclusion by using the Rademacher space $\operatorname{Rad} X$ associated to X.

In some ways, our approach is unsatisfactory because it only answers the question of the existence of an unconditional basis for a twisted Hilbert space. We do not know the similar result for local unconditional structure, or even if a twisted Hilbert space which is a Banach lattice is necessarily Hilbertian.

2. The results

Let A be any subset of a Banach space X. We denote by $[A]$ the closed linear span of A. In particular, we denote by $[x_n]_{n=1}^{\infty}$ the closed linear span of the sequence $(x_n)_{n=1}^{\infty}$. We call a sequence $(x_n)_{n=1}^{\infty}$ semi-normalized if $0 < \inf \|x_n\| \leqslant \sup \|x_n\| < \infty$.

We start with a few well-known remarks about complemented unconditional basic sequences.

Let X be a Banach space with an unconditional Schauder decomposition $(E_n)_{n=1}^{\infty}$ and let $R_n : X \to E_n$ be the associated projections. Suppose $x_n \in E_n$ is any sequence of non-zero vectors. (x_n) is then an unconditional basic sequence. If $[x_n]_{n=1}^{\infty}$ is complemented, it is well known that there is a projection P onto $[x_n]_{n=1}^{\infty}$ of the form

$$Px = \sum_{n=1}^{\infty} x_n^*(x) x_n,$$

where $x_n^* \in R_n^*(X^*)$ and $x_n^*(x_n) = 1$. In particular, if $(u_n)_{n=1}^{\infty}$ is an unconditional basis with biorthogonal sequence $(u_n^*)_{n=1}^{\infty}$ and (x_n) is a block basic sequence with $x_n \in [u_{p_{n-1}+1}, \dots, u_{p_n}]$, then we can assume $x_n^* \in [u^*_{p_{n-1}+1}, \dots, u^*_{p_n}]$. These well-known results follow from similar arguments to those of [**3**, Proposition 1.c.8].

If X has an unconditional finite-dimensional Schauder decomposition $(E_n)_{n=1}^{\infty}$ where each $\dim E_n = 2$, then it is shown in [**1**] that X has an unconditional basis (or even local unconditional structure) if and only if we can choose an unconditional basis $(u_n)_{n=1}^{\infty}$ such that $E_n = [u_{2n-1}, u_{2n}]$. These conditions are also easily seen to be equivalent to the existence of non-zero $x_n \in E_n$, so that $[x_n]_{n=1}^{\infty}$ is complemented. On the other hand, if $(u_n)_{n=1}^{\infty}$ is a semi-normalized unconditional basis and $x_n \in [u_{2n-1}, u_{2n}]$ is a semi-normalized sequence and $[x_n]_{n=1}^{\infty}$ is complemented, then $(x_n)_{n=1}^{\infty}$ is equivalent to a subsequence $(u_{j_n})_{n=1}^{\infty}$, where $j_n = 2n-1$ or $j_n = 2n$.

In our first proposition, we give another sufficient condition for such a sequence to span a complemented subspace. It is equivalent to [**1**, Theorem 3.9], but the proof is more direct.

Proposition 2.1. *Let X be a Banach space with a normalized unconditional basis $(u_n)_{n=1}^{\infty}$ with biorthogonal functionals $(u_n^*)_{n=1}^{\infty}$. Suppose that $(x_n)_{n=1}^{\infty}$ is a normalized sequences of the form*

$$x_n = a_{2n-1}u_{2n-1} + a_{2n}u_{2n}.$$

Let $E = [x_n]_{n=1}^{\infty}$ and $Q : X \to X/E$ denote the quotient map. Let $(y_n)_{n=1}^{\infty}$ be a normalized unconditional basis of X/E with $y_n \in [Qu_{2n-1}, Qu_{2n}]$. Assume $(y_n)_{n=1}^{\infty}$ is equivalent to $(x_n)_{n=1}^{\infty}$, i.e. there exists C such that, for any $\xi_1, \dots, \xi_n$, we have

$$C^{-1}\left\|\sum_{k=1}^{n} \xi_k y_k\right\| \leqslant \left\|\sum_{k=1}^{n} \xi_k x_k\right\| \leqslant C\left\|\sum_{k=1}^{n} \xi_k y_k\right\|.$$

Then $[x_n]$ is equivalent to a sequence $[u_{j_n}]_{n=1}^{\infty}$ where $j_n = 2n-1$ or $j_n = 2n$ and $[x_n]$ is complemented in X.

Proof. In this proof we use c_{00} to denote the space of finitely non-zero sequences and $c_{00}(\mathbb{A})$ to denote the subset of such sequences supported on a subset $\mathbb{A}$ of $\mathbb{N}$.

We may suppose by renorming that the basis (u_n) is 1-unconditional. We also suppose, for notational convenience, that (u_n) is reordered so that $|a_{2n-1}| \leqslant |a_{2n}|$ for all n. We will show that (x_n) is equivalent to $(u_{2n})_{n=1}^{\infty}$. Once this is done, the projection P given by

$$Px = \sum_{n=1}^{\infty} a_{2n}^{-1} u_{2n}^*(x) x_n$$

is easily seen to be bounded.

Let $\delta = 10^{-2}C^{-2}$. We first split $\mathbb{N}$ into two complementary sets by setting

$$\mathbb{A} = \{n : |a_{2n-1}| \leqslant \delta\} \quad \text{and} \quad \mathbb{B} = \mathbb{N} \setminus \mathbb{A}.$$

Note first that $|a_{2n}| \geqslant \frac{1}{2}$ by the triangle law and so, for any $(\xi_n)_{n=1}^{\infty} \in c_{00}$, we have

$$\Big\| \sum_{n=1}^{\infty} \xi_n u_{2n} \Big\| \leqslant 2 \Big\| \sum_{n=1}^{\infty} \xi_n x_n \Big\|. \tag{2.1}$$

Let us show that, for any $(\xi_n)_{n=1}^{\infty} \in c_{00}(\mathbb{A})$, we have

$$\Big\| \sum_{n\in\mathbb{A}} \xi_n x_n \Big\| \leqslant 10C \Big\| \sum_{n\in\mathbb{A}} \xi_n u_{2n} \Big\|. \tag{2.2}$$

If (2.2) fails, there is a $\xi \in c_{00}(\mathbb{A})$ such that $m = |\operatorname{supp}\xi|$ is minimized and (2.2) fails. Note that $m \geqslant 2$. Let $\mathbb{D}$ be a subset of $\operatorname{supp}\xi$ with $|\mathbb{D}| = m-1$. For any $\eta \in c_{00}(\mathbb{D})$, we have

$$\Big\| \sum_{n\in\mathbb{D}} \eta_n y_n \Big\| \leqslant C \Big\| \sum_{n\in\mathbb{D}} \eta_n x_n \Big\| \leqslant 10C^2 \Big\| \sum_{n\in\mathbb{D}} \eta_n u_{2n} \Big\|.$$

Let $(y_n^*)_{n=1}^{\infty}$ be the sequence biorthogonal to (y_n) in $(X/E)^* = E^{\perp} \subset X^*$. Then we have $y_n^* = c_n(a_{2n}u_{2n-1}^* - a_{2n-1}u_{2n}^*)$, where $\frac{1}{2} \leqslant |c_n| \leqslant 2$ by use of the triangle law, since $\|y_n^*\| = \|u_{2n}^*\| = \|u_{2n-1}^*\| = 1$. It follows by duality that, for any $\eta \in c_{00}(\mathbb{D})$, we have

$$\Big\| \sum_{n\in\mathbb{D}} \eta_n u_{2n}^* \Big\| \leqslant 10C^2 \Big\| \sum_{n\in\mathbb{D}} \eta_n y_n^* \Big\|.$$

Hence

$$\Big\| \sum_{n\in\mathbb{D}} \eta_n c_n a_{2n-1} u_{2n}^* \Big\| \leqslant 20C^2\delta \Big\| \sum_{n\in\mathbb{D}} \eta_n y_n^* \Big\|.$$

Since $20C^2\delta \leqslant \frac{1}{2}$, we conclude that

$$\Big\| \sum_{n\in\mathbb{D}} \eta_n c_n a_{2n} u_{2n-1}^* \Big\| \geqslant \frac{1}{2} \Big\| \sum_{n\in\mathbb{D}} \eta_n y_n^* \Big\|.$$

Since $|c_n a_{2n}| \geqslant \frac{1}{4}$, this implies

$$\Big\| \sum_{n\in\mathbb{D}} \eta_n y_n^* \Big\| \leqslant 8 \Big\| \sum_{n\in\mathbb{D}} \eta_n u_{2n-1}^* \Big\|.$$

Now duality gives the inequality

$$\Big\|\sum_{n\in\mathbb{D}}\eta_n u_{2n-1}\Big\| \leqslant 8\Big\|\sum_{n\in\mathbb{D}}\eta_n y_n\Big\|.$$

Hence

$$\Big\|\sum_{n\in\mathbb{D}}\eta_n a_{2n-1}u_{2n-1}\Big\| \leqslant 8\delta\Big\|\sum_{n\in\mathbb{D}}\eta_n y_n\Big\|,$$

and this implies, since $8\delta < \frac{1}{2}$, that

$$\Big\|\sum_{n\in\mathbb{D}}\eta_n a_{2n}u_{2n}\Big\| \geqslant \frac{1}{2}\Big\|\sum_{n\in\mathbb{D}}\eta_n y_n\Big\|.$$

In particular, since $|a_{2n}| \geqslant \frac{1}{2}$,

$$\Big\|\sum_{n\in\mathbb{D}}\xi_n y_n\Big\| \leqslant 4\Big\|\sum_{n\in\mathbb{D}}\xi_n u_{2n}\Big\|.$$

Applying this inequality for every subset $\mathbb{D}$ with cardinality $m-1$ of $\operatorname{supp}\xi$ and averaging gives

$$\Big\|\sum_{n=1}^{\infty}\xi_n y_n\Big\| \leqslant \frac{4m}{m-1}\Big\|\sum_{n=1}^{\infty}\xi_n u_{2n}\Big\| \leqslant 8\Big\|\sum_{n=1}^{\infty}\xi_n u_{2n}\Big\|.$$

Hence

$$\Big\|\sum_{n=1}^{\infty}\xi_n x_n\Big\| \leqslant 8C\Big\|\sum_{n=1}^{\infty}\xi_n u_{2n}\Big\|,$$

and this contradicts our choice of ξ.

If $n \in \mathbb{B}$, we note that $|c_n a_{2n-1}| > \frac{1}{2}\delta$, and so

$$\Big\|\sum_{n\in\mathbb{B}}\eta_n u^*_{2n}\Big\| \leqslant \frac{2}{\delta}\Big\|\sum_{n\in\mathbb{B}}\eta_n y^*_n\Big\|, \qquad \eta \in c_{00}(\mathbb{B}).$$

Hence, by duality,

$$\Big\|\sum_{n\in\mathbb{B}}\xi_n y_n\Big\| \leqslant \frac{2}{\delta}\Big\|\sum_{n\in\mathbb{B}}\xi_n u_{2n}\Big\|, \qquad \xi \in c_{00}(\mathbb{B}).$$

Hence

$$\Big\|\sum_{n\in\mathbb{B}}\xi_n x_n\Big\| \leqslant \frac{2C}{\delta}\Big\|\sum_{n\in\mathbb{B}}\xi_n u_{2n}\Big\|. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) gives the proof. □

We use the term *sequence space* to denote a Banach space of sequences such that the canonical basis vectors form a normalized 1-unconditional basis.

Proposition 2.2. *Let S be a fixed sequence space. Let X be a Banach space with an unconditional Schauder decomposition $(E_n)_{n=1}^{\infty}$.*

For every n, let F_n be a closed non-trivial subspace of E_n. Let $Y = [F_n]_{n=1}^{\infty}$, so that (F_n) is an unconditional Schauder decomposition of Y and E_n/F_n is an unconditional Schauder decomposition of X/Y. Suppose that there is a constant C such that, if $y_n \in F_n$ and $z_n \in E_n/F_n$ are finitely non-zero sequences, then

$$C^{-1}\|(\|y_k\|)_{k=1}^{\infty}\|_S \leqslant \left\|\sum_{k=1}^{\infty} y_k\right\|_Y \leqslant C\|(\|y_k\|)_{k=1}^{\infty}\|_S$$

and

$$C^{-1}\|(\|z_k\|)_{k=1}^{\infty}\|_S \leqslant \left\|\sum_{k=1}^{\infty} z_k\right\|_{X/Y} \leqslant C\|(\|z_k\|)_{k=1}^{\infty}\|_S.$$

Now suppose $(u_n)_{n=1}^{\infty}$ is a normalized sequence with $u_n \in E_n$. Then, if $[u_n]_{n=1}^{\infty}$ is complemented, we have that $(u_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of S.

Proof. Let (e_n) be the canonical basis of S. Let $R_n : X \to E_n$ be the projections associated to the Schauder decomposition. If $[u_n]_{n=1}^{\infty}$ is complemented, we can find a projection P of the form

$$Px = \sum_{n=1}^{\infty} u_n^*(x)u_n,$$

where $u_n^* \in R_n^*(X^*)$. If we let $\mathbb{A} = \{n : u_n \in F_n\}$, it is clear that $(u_n)_{n\in\mathbb{A}}$ is equivalent to $(e_n)_{n\in\mathbb{A}}$. Now let $\mathbb{B} = \{n : u_n^* \in F_n^{\perp}\}$. Then, for any $y \in Y$ and $\xi \in c_{00}(\mathbb{B})$, we have

$$\left\|\sum_{n\in\mathbb{B}} \xi_n u_n\right\| \leqslant \|P\|\left\|\sum_{n\in\mathbb{B}} \xi_n u_n + y\right\|,$$

and so, denoting by Q the quotient map onto X/Y,

$$\left\|\sum_{n\in\mathbb{B}} \xi u_n\right\| \leqslant \|P\|\left\|\sum_{n\in\mathbb{B}} \xi_n Q u_n\right\|.$$

Thus $(u_n)_{n\in\mathbb{B}}$ is equivalent to $(Qu_n)_{n\in\mathbb{B}}$ and hence to $(e_n)_{n\in\mathbb{B}}$. We can thus reduce the problem to the case when $u_n \notin F_n$ and $u_n^* \notin F_n^{\perp}$. We may make a further reduction by replacing E_n by $[F_n, u_n]$ and so we may assume that $\dim E_n/F_n = 1$.

Let $H_n = \ker u_n^* \cap F_n$, so that H_n has codimension one in F_n. Pick $x_n \in F_n$ with $\|x_n\| < 2$ and $d(x_n, H_n) = 1$.

Let T_n be a projection of F_n onto H_n with $\|T_n\| \leqslant 2$. Then we can define a bounded projection $T : Y \to [H_n]_{n=1}^{\infty}$ by

$$T\left(\sum_{n=1}^{\infty} y_n\right) = \sum_{n=1}^{\infty} T_n y_n \quad \text{if } y_n \in F_n.$$

In fact,

$$\left\|\sum_{n=1}^{\infty} T_n y_n\right\| \leqslant 2C^2 \left\|\sum_{n=1}^{\infty} y_n\right\|.$$

Let $X_0 = X/[H_n]_{n=1}^{\infty}$ and Q_0 be the quotient map. Pick $x_n \in \ker T_n$ with $\|Q_0 x_n\| = 1$. It follows that $(Q_0 x_n)_{n=1}^{\infty}$ is an unconditional basic sequence equivalent to $(x_n)_{n=1}^{\infty}$ and hence to $(e_n)_{n=1}^{\infty}$.

From our construction, the projection P factors to a bounded projection on X_0. Furthermore, X_0 has a two-dimensional (UFDD) $(G_n)_{n=1}^{\infty}$ given by $G_n = Q_0(E_n)$. Hence $(Q_0 u_n)_{n=1}^{\infty}$ is a complemented unconditional basic sequence with $Q_0 u_n \in G_n$; furthermore, $(Q_0 u_n)_{n=1}^{\infty}$ is equivalent to $(u_n)_{n=1}^{\infty}$. It follows that we can form an unconditional basis $(Q_0 u_1, v_1, Q_0 u_2, v_2, \dots)$ of X_0 with $v_n \in G_n$.

We will now verify the conditions of Proposition 2.1. We have seen that $(Q_0 x_n)_{n=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$. Now $X_0/[Q_0 x_n]_{n=1}^{\infty}$ is naturally isomorphic to X/Y. Denote by $Q_1 : X_0 \to X_0/[Q_0 x_n]_{n=1}^{\infty}$ the quotient map. If $y_n \in Q_1(G_n)$ with $\|y_n\| = 1$, then $(y_n)_{n=1}^{\infty}$ is also equivalent to $(e_n)_{n=1}^{\infty}$. By Proposition 2.1, we see that $(Q_0 x_n)_{n=1}^{\infty}$ spans a complemented subspace of X. It thus follows that we can pick $z_n \in G_n$ such that $\{Q_0 x_1, z_1, Q_0 x_2, z_2, \dots\}$ is an unconditional basis of X; furthermore, $(z_n)_{n=1}^{\infty}$ is equivalent to $(y_n)_{n=1}^{\infty}$ and hence to $(e_n)_{n=1}^{\infty}$.

It now follows that since $Q_0 u_n \in [Q_0 x_n, z_n] = G_n$, then $(Q_0 u_n)_{n=1}^{\infty}$ is also equivalent to $(e_n)_{n=1}^{\infty}$. ∎

We are now in position to prove the main result.

Theorem 2.3. *Let Y be a complemented subspace of a twisted Hilbert space X. If Y has an unconditional basis, then Y is isomorphic to a Hilbert space.*

Proof. We start with the short exact sequence

$$0 \to H_1 \xrightarrow{J} X \xrightarrow{Q} H_2 \to 0,$$

where H_1 and H_2 are Hilbert spaces. This clearly induces a short exact sequence

$$0 \to L_2([0,1], H_1) \to L_2([0,1], X) \to L_2([0,1], H_2) \to 0.$$

Let $\epsilon_n(t)$ denote the standard Rademacher functions on $[0,1]$. For any Banach space E, denote by $\operatorname{Rad} E$ the subspace of $L_2([0,1], E)$ of functions of the form

$$f = \sum_{n=1}^{\infty} \epsilon_n \otimes e_n$$

(with convergence in norm). It is a theorem of Pisier [**4**] that if E has non-trivial Rademacher type, then the canonical projection of $L_2(E)$ onto $\operatorname{Rad} E$, given by

$$Pf = \sum_{n=1}^{\infty} \epsilon_n \otimes \left(\int_0^1 f(t)\epsilon_n(t)\,\mathrm{d}t\right),$$

is bounded.

Now Y has non-trivial type (see [**2**]) and the canonical projection P restricts to the canonical projection from $L_2(H_1)$ to $\operatorname{Rad} H_1$ and similarly factors to the canonical projection from $L_2(H_2)$ to $\operatorname{Rad} H_2$. We thus obtain a short exact sequence

$$0 \to \operatorname{Rad} H_1 \xrightarrow{\tilde{J}} \operatorname{Rad} X \xrightarrow{\tilde{Q}} \operatorname{Rad} H_2 \to 0.$$

Note that

$$\tilde{J}\left(\sum_{n=1}^{\infty} \epsilon_n \otimes h_n\right) = \sum_{n=1}^{\infty} \epsilon_n \otimes Jh_n$$

and

$$\tilde{Q}\left(\sum_{n=1}^{\infty} \epsilon_n \otimes x_n\right) = \sum_{n=1}^{\infty} \epsilon_n \otimes Qx_n.$$

Now assume that $(u_n)_{n=1}^{\infty}$ is a normalized unconditional basic sequence and that $R : X \to Y = [u_n]_{n=1}^{\infty}$ is a bounded projection. Then R takes the form

$$Rx = \sum_{n=1}^{\infty} u_n^*(x)u_n,$$

where $(u_n^*)_{n=1}^{\infty}$ is a (complemented) unconditional basic sequence in X^*. Note that if $\mathbb{A} = \{n : Qu_n = 0\}$, then $u_n \in J(H_1)$ for $n \in \mathbb{A}$; thus $(u_n)_{n\in\mathbb{A}}$ is equivalent to the canonical basis of ℓ_2. Similarly, if $\mathbb{B} = \{n : J^*u_n^* = 0\}$, then $(u_n^*)_{n\in\mathbb{B}}$ is equivalent to the canonical basis of ℓ_2 and the same is then true for $(u_n)_{n\in\mathbb{B}}$. It follows that we need only consider the case when $Qu_n \neq 0$ and $J^*u_n^* \neq 0$ for every n.

Let us define $\tilde{R} : \operatorname{Rad} X \to \operatorname{Rad} X$ by

$$\tilde{R}\left(\sum_{n=1}^{\infty} \epsilon_n \otimes x_n\right) = \sum_{n=1}^{\infty} u_n^*(x_n)\epsilon_n \otimes u_n.$$

Then $\tilde{R}$ is a bounded projection onto the subspace $[\epsilon_n \otimes u_n]_{n=1}^{\infty}$. The proof that $\tilde{R}$ is bounded is standard. There is a constant C so that we have

$$\left\|\sum_{k=1}^{\infty} \epsilon_k(s)u_k^*(x)u_k\right\| \leqslant C\|x\|, \quad 0 \leqslant s \leqslant 1, \quad x \in X.$$

Thus

$$\left\|\sum_{k=1}^{\infty} \epsilon_k(s)u_k^*\left(\sum_{j=1}^{\infty} \epsilon_j(s)x_j\right)u_k\right\| \leqslant C\left\|\sum_{j=1}^{\infty} \epsilon_j(s)x_j\right\|$$

whenever $(x_j)_{j=1}^{\infty}$ is a finitely non-zero sequence in X. Hence

$$\begin{aligned}\left(\int_0^1\left\|\sum_{k=1}^{\infty}\epsilon_k(t)u_k^*(x_k)u_k\right\|^2 \mathrm{d}t\right)^{1/2} &= \left(\int_0^1\int_0^1\left\|\sum_{k=1}^{\infty}\epsilon_k(t)u_k^*(x_k)u_k\right\|^2 \mathrm{d}t\,\mathrm{d}s\right)^{1/2}\\ &\leqslant \left(\int_0^1\int_0^1\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\epsilon_j(s)\epsilon_k(s)\epsilon_k(t)u_j^*(x_k)u_j\right\|^2 \mathrm{d}s\,\mathrm{d}t\right)^{1/2}\\ &\leqslant C\left(\int_0^1\int_0^1\left\|\sum_{k=1}^{\infty}\epsilon_k(s)\epsilon_k(t)x_k\right\|^2 \mathrm{d}s\,\mathrm{d}t\right)^{1/2}\\ &= C\left\|\sum_{k=1}^{n}\epsilon_k\otimes x_k\right\|.\end{aligned}$$

The space $\mathrm{Rad}\,X$ has an unconditional Schauder decomposition $E_n = (\epsilon_n \otimes X)_{n=1}^{\infty}$. If we let $F_n = \epsilon_n \otimes JH_1$, then it is trivial to see that the assumptions of Proposition 2.2 are satisfied with $S = \ell_2$. Hence $(\epsilon_n \otimes u_n)_{n=1}^{\infty}$ is equivalent to the canonical basis of ℓ_2; the same is clearly then true for $(u_n)_{n=1}^{\infty}$. □

Let us note that this argument can be phrased in purely finite-dimensional terms. Thus we have a result for finite-dimensional Banach spaces. For X a finite-dimensional Banach space of dimension n, let d_X be its Euclidean distance, i.e. the Banach–Mazur distance $d(X, \ell_2^n)$.

Theorem 2.4. *Given any constant C, there is a constant K such that, if X is finite-dimensional Banach space with a C-unconditional basis and a subspace E with $d_E, d_{X/E} \leqslant C$, then $d_X \leqslant K$.*

As pointed out by the referee, to extend our results requires a refinement of this theorem where X is merely assumed to be well complemented in a space with unconditional basis. More precisely, we have the following conjecture.

Conjecture 2.5. *Given any constant C, there is a constant K such that, if X is a finite-dimensional Banach space with a C-unconditional basis, Y is a C-complemented subspace of X and E is a subspace of Y for which $d_E, d_{Y/E} \leqslant C$, then $d_Y \leqslant K$.*

Acknowledgements. The author is grateful to the referee for careful reading of the manuscript and several important corrections. The author was supported by NSF grant DMS-9870027.

References

1. P. G. Casazza and N. J. Kalton, Unconditional bases and unconditional finite-dimensional decompositions in Banach spaces, *Israel J. Math.* **95** (1996), 349–373.
2. P. Enflo, J. Lindenstrauss and G. Pisier, On the 'three space problem', *Math. Scand.* **36** (1975), 199–210.
3. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, vol. 1, *Sequence spaces* (Springer, 1977).
4. G. Pisier, Holomorphic semigroups and the geometry of Banach spaces, *Ann. Math.* **115** (1982), 375–392.