Representations of Operators between Function Spaces

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1. Introduction. Our objective in this paper is to study conditions on a symmetric quasi-Banach function space X on $[0,1]$ (see Section 2 for the precise definition) so that every (continuous) linear operator $T: X \to L_0\,[0,1]$ can be represented in the form

$$Tf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t) \qquad \text{a.e.} \tag{1.1}$$

for $f \in X$, where the functions $a_n : [0,1] \to \mathbf{R}$ and $\sigma_n : [0,1] \to [0,1]$ are Borel and the series converges absolutely almost everywhere.

In [13] Kwapien obtained a representation of the form (1.1) for operators $T: L_0 \to L_0$, while in [11] we showed that there is such a representation for operators $T: L_p \to L_0$ where $0 < p < 1$ (see also [10] for operators $T: L_p \to L_p$). On the other hand it is clear that no finite-rank operator can be represented in the form of (1.1) so that in general we consider function spaces X with trivial dual.

Operators $T: X \to L_0$ of the form

$$Tf(t) = \int f(s)\, d\mu_t(s) \qquad \text{a.e.} \tag{1.2}$$

for $f \in X$, where $t \to \mu_t$ is a weak*-Borel map, are called pseudo-integral and have been studied by Arveson [1] and Sourour [18], [19]. Clearly if T has a representation (1.1) then it is pseudo-integral. We note (see Theorem 3.2 of this paper or [11]) that $T: X \to L_0$ is pseudo-integral if and only if it is regular, i.e. there is a positive operator $P: X \to L_0$ with $P\,|f| \geq |Tf|$ for $f \in X$.

In this paper we first study conditions under which (1.1) holds only for $f \in L_\infty$. We show that if X has trivial dual then it is necessary and sufficient that T is *controllable* i.e., there exists $h \in L_0$ so that $|Tf| \leq h$ a.e. for $f \in L_\infty$ with $\|f\|_\infty \leq 1$ (Theorem 4.4 below). It follows from this result that a pseudo-integral operator on X can be represented in the form (1.1.).

Next we seek conditions on X so that every operator is controllable, and we show (Theorem 5.1) that if $X \supset L(1,\infty)$ (weak L_1) or more strictly if X contains the closure of the simple functions in $L(1,\infty)$, then this is the case. A converse is also established for a restricted class of spaces X. In particular if X is the Lorentz

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space $L(1,p)$ where $1 < p < \infty$, then there exist operators $T : X \to L_0$ which are not controllable and therefore do not have a representation in the form (1.1) for $f \in L_\infty$.

Finally we turn to the problem of determining when T must have a representation (1.1) valid for all $f \in X$. To answer this we study the more general problem of determining when the space $\mathcal{L}(X,Y)$ of linear operators between two quasi-Banach function spaces is a lattice. In [10] and [11] we used the fact that $\mathcal{L}(L_p,Y)$ is a lattice if Y is a maximal (see Section 2) p-normable function space. Here however we prove a much more general result of this type.

We assume that Y is maximal and L-convex in the sense of [12]. This is a very mild condition on Y; it is in fact rather difficult to construct a non-L-convex function space. In particular if Y is r-concave for some $r < \infty$ (see [14] or Section 2) then Y is L-convex. Under these conditions if $q_X < 1$ where q_X is the upper Boyd index (see [14] or Section 2) of X then $\mathcal{L}(X,Y)$ is lattice. We do not even have to assume that the simple functions are dense in X. As an elementary example $\mathcal{L}(L_p,L_q)$ is a lattice if $0 < q \le p$. More generally if X and Y are Lorentz spaces, $X = L(p,q)$ and $Y = L(r,s)$ then $\mathcal{L}(X,Y)$ is a lattice if $0 < p < 1$.

Based on this, we show that if $q_X < 1$ then every operator $T : X \to L_0$ can be represented in the form (1.1) for $f \in X$. Again, density of the simple functions is not required for this result. A converse is established for a restricted class of spaces. Thus, for example, if $X = \Lambda(1,\infty)$ (the closure of the simple functions in $L(1,\infty)$) then there are operators $T : X \to L_0$ which cannot be represented in the form (1.1) for all $f \in X$ (here $q_X = 1$).

The results proved here have several applications which we will develop in separate papers. For example if X and Y are separable symmetric L-convex function spaces with trivial dual then $\mathcal{L}(X,Y) \neq \{0\}$ implies $X \subset Y$, and this, in turn, can be used to improve some results on symmetric Banach function spaces obtained in [8].

In another direction, we can specialize to the study of endomorphisms of a separable symmetric L-convex function space X. In this case, it turns out that if $X^* = \{0\}$ then every operator $T : X \to X$ is controllable and there exists some p, $0 < p \le 1$ so that T is also an L_p-operator (i.e. T extends continuously to an operator $T : L_p \to L_p$).

2. Function spaces. We recall that a quasi-Banach space X is a complete metrizable vector space whose topology is given by a quasi-norm $x \to \|x\|$ satisfying

$$\|x\| > 0 \qquad x \neq 0$$
$$\|\alpha x\| = |\alpha|\,\|x\| \qquad \alpha \in \mathbf{R},\ x \in X$$
$$\|x_1 + x_2\| \le C(\|x_1\| + \|x_2\|) \qquad x_1, x_2 \in X$$

where C is a constant independent of x_1 and x_2. X is said to be *p-normable* or *topologically p-convex* if it satisfies

$$\|x_1 + \ldots + x_n\| \leq C'(\|x_1\|^p + \ldots + \|x_n\|^p)^{1/p} \qquad x_1, \ldots, x_n \in X$$

for some constant C'.

Let λ denote Lebesgue measure on the Borel subsets $\mathcal{B}$ of [0,1]. We denote, as usual, by $L_0 = L_0\,[0,1]$ the space of (equivalence classes of) Borel measurable functions on [0,1] with the topology of convergence in measure. This topology is given by the metric $d(f,g) = \|f - g\|_0$ where

$$\|f\|_0 = \int_0^1 \min(1,|f(t)|)dt.$$

Throughout this paper we shall use the term *function space* to mean a quasi-Banach space X which is algebraically a subspace of $L_0\,[0,1]$ containing $L_\infty\,[0,1]$ so that

(a) if $f \in X$ and $g \in L_0$ with $|g| \leq |f|$ a.e. then $g \in X$ and $\|g\| \leq \|f\|$
(b) if $0 \leq f_n \uparrow f$ a.e. with $f_n, f \in X$ then $\|f_n\| \uparrow \|f\|$
(c) if $0 \leq f_n \leq 1$ and $f_n \downarrow 0$ a.e. then $\|f_n\| \downarrow 0$.

X will be called *minimal* if L_∞ is dense in X and *maximal* if every norm-bounded increasing sequence is bounded above (cf. [14] page 118). We remark here that X is minimal if and only if it is separable (using arguments developed by Drewnowski [5]). If X is minimal condition (b) above is automatic whenever the quasi-norm is continuous.

For $0 < r < \infty$ we say ([14] page 45) X is (*lattice*) *r-convex* if for some $C < \infty$ we have

$$\left\|\left(\sum_{i=1}^n |f_i|^r\right)^{1/r}\right\| \leq C\left(\sum_{i=1}^n \|f_i\|^r\right)^{1/r}$$

for $f_1, \ldots, f_n \in X$. X is (*lattice*) *q-concave* if for some $C < \infty$

$$\left(\sum_{i=1}^n \|f_i\|^q\right)^{1/q} \leq C\left\|\left(\sum_{i=1}^n |f_i|^q\right)^{1/q}\right\|$$

for $f_i, \ldots, f_n \in X$.

X is to be *L-convex* [12] if for some $0 < \varepsilon < 1$, whenever $f \in X_+$ with $\|f\| = 1$ and $0 \leq f_i \leq f$ $(1 \leq i \leq n)$ with $(1/n)(f_1 + \ldots + f_n) \geq (1 - \varepsilon)f$ then $\max_{1\leq i\leq n} \|f_i\| \geq \varepsilon$. It is shown in [12] that X is L-convex if and only if it is lattice r-convex for some $r > 0$, and that if X is q-concave for some $q < \infty$, then X is L-convex. Unfortunately there exist examples of non-L-convex function spaces (even symmetric function spaces). We shall not construct an explicit counterexample here, but instead we shall observe below that all commonly arising examples are L-convex. Note also that X is lattice r-convex if and only if we can find a Banach function space Y so that $X = \{f \in L_0 : |f|^r \in Y\}$. Here $Y = \{g : |g|^{1/r} \in X\}$ with $\|g\| = \||g|^{1/r}\|^r$ defining the quasi-norm, which is then equivalent to a norm.

A function space X is *symmetric* (or *rearrangement invariant*) if $\|f\|$ depends only on the distribution of f. Precisely if $f \in L_0$ and $d_f(x) = \lambda(|f| > x)$ $(0 \le x < \infty)$ then X is symmetric if whenever $f \in L_0$, $g \in X$ and $d_f \le d_g$ then $f \in X$ and $\|f\| \le \|g\|$.

If X is symmetric we define the dilation operators $D_t : X \to X$ $(0 < t < \infty)$ by

$$\begin{aligned} D_t f(s) &= f(st^{-1}) \qquad 0 \le s \le \min(1,t) \\ &= 0 \qquad \min(1,t) < s \le 1. \end{aligned}$$

Then ([14] page 130) $\|D_{st}\| \le \|D_s\| \cdot \|D_t\|$ and the *Boyd indices* of X are defined by

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|} = \sup_{s>1} \frac{\log s}{\log\|D_s\|}$$

$$q_X = \lim_{s\to 0} \frac{\log s}{\log\|D_s\|} = \inf_{0<s<1} \frac{\log s}{\log\|D_s\|}.$$

If X is p-normable $p \le p_X \le q_X \le \infty$ (cf. [14] page 131).

Finally we discuss some examples. An Orlicz function is a continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and satisfying the Δ_2-condition at ∞ i.e.

$$\phi(2x) \le C\,\phi(x) \qquad x \ge 1.$$

If we suppose (Matuszewska and Orlicz [15]) that for some $p > 0$

$$\inf_{x,y\ge 1} \frac{\phi(xy)}{\phi(x)y^p} > 0$$

then the Orlicz space $L_\phi(0,1)$ is locally bounded and is an example of a minimal (and maximal) symmetric function space. We define

$$\|f\| = \inf\left\{\alpha > 0 : \int \phi(\alpha^{-1}|f|)\, d\lambda \le 1\right\}.$$

The Boyd indices are given by ([14] page 139)

$$p_X = \sup\left\{p : \inf_{x,y\ge 1} \frac{\phi(xy)}{\phi(x)y^p} > 0\right\}$$

$$q_X = \inf\left\{q : \sup_{x,y\ge 1} \frac{\phi(xy)}{\phi(x)y^q} < \infty\right\}.$$

L_ϕ is lattice r-convex for $0 < r < p_X$ and hence is L-convex. It is also q-concave for $q > q_X$.

The Lorentz space $L(p,q)$ ([7]) where $0 < p,q < \infty$ consists of all $f \in L_0$ so that

$$\|f\|_{p,q} = \left\{\frac{q}{p}\int_0^1 t^{q/p-1} f^*(t)^q\, dt\right\}^{1/q} < \infty$$

where $f^*(t)$ is the decreasing rearrangement of $|f|$, i.e.

$$f^*(t) = \inf_{\lambda(E)=t} \sup_{s\in[0,1]\setminus E} |f(s)|.$$

$X = L(p,q)$ is a minimal (and maximal) function space whose Boyd indices are given by $p_X = q_X = q$. $L(p,q)$ is lattice r-convex if $p > r$ and $q \geq r$ or $p = q = r$ and is always L-convex. It is also lattice s-concave if $s > \max(p,q)$.

In the case $q = \infty$, the Lorentz space $X = L(p,\infty)$ (weak L_p) consists of all $f \in L_0$ where $\|f\|_{p,\infty} < \infty$ where

$$\|f\|_{p,\infty} = \sup_{x>0} x\,\lambda\,(|f| > x)^{1/p}.$$

$L(p,\infty)$ is a maximal symmetric function space. We again have $p_X = q_X = p$ and $L(p,\infty)$ is lattice r-convex for $0 < r < p$. $L(p,\infty)$ is not however lattice q-concave for any $q < \infty$.

If we denote by $\Lambda(p,\infty)$ the closure of the simple functions in $L(p,\infty)$ then $\Lambda(p,\infty)$ is a minimal symmetric function space.

3. Controllable and regular operators. Let $\mathcal{M} = \mathcal{M}[0,1]$ be the Banach space of regular Borel measures on [0,1], i.e. $\mathcal{M} = C[0,1]^*$. By an *(abstract) kernel* we shall mean a weak*-Borel map $t \to \mu_t$ ($[0,1] \to \mathcal{M}$) so that $|\mu_t|(B) = 0$ a.e. whenever $B \in \mathcal{B}$ and $\lambda(B) = 0$. Given an abstract kernel $t \to \mu_t$ we can induce a linear map $T : L_\infty[0,1] \to L_0[0,1]$ defined by

$$Tf(t) = \int f\,d\mu_t. \tag{3.1}$$

noting that if f and g are Borel functions with $f = g$ a.e. then $Tf = Tg$ a.e.

Let us say that an operator $T : L_\infty \to L_0$ is *measure-continuous* if whenever $\{f_n\}$ is a bounded sequence in L_∞ and $\|f_n\|_0 \to 0$ then $\|Tf_n\|_0 \to 0$. The following theorem is a refinement of a result of Sourour [18].

Theorem 3.1. *Let $T : L_\infty \to L_0$ be a linear operator. Then in order that there be an abstract kernel $\{\mu_t\}$ representing T in the sense that (3.1) holds for $f \in L_\infty$, it is necessary and sufficient that T is measure-continuous and there exists a non-negative Borel function $h \in L_0$ so that $|Tf| \leq h$* a.e. *whenever $\|f\|_\infty \leq 1$.*

Proof (Sketch). If T is representable by an abstract kernel μ_t then the Bounded Convergence Theorem implies T is measure-continuous. If $h(t) = \|\mu_t\|$ then $|Tf| \leq h$ a.e. wherever $\|f\|_\infty \leq 1$.

Conversely if T is measure-continuous and $|Tf| \leq h$ a.e. for $\|f\|_\infty \leq 1$, then Sourour [18] has shown the existence of a weak*-Borel map $t \to \mu_t$ so that (3.1) holds. A simple direct proof can be given by considering the action of T pointwise on the countable collection of polynomials with rational coefficients.

To complete the proof that $\{\mu_t\}$ is an abstract kernel in our sense we need to show $|\mu_t|(B) = 0$ a.e. whenever $\lambda(B) = 0$. Consider the positive operator $S : L_\infty \to L_0$ defined so that if $f \geq 0$

$$Sf = \sup(|Tg| : |g| \leq f)$$

where the supremum is taken in the complete vector lattice L_0. Clearly $S1 \leq h$ and it can be shown, by considering monotone increasing sequences that S is measure-continuous. Hence there exists a weak*-Borel map $t \to \nu_t$ so that

$$Sf(t) = \int f d\nu_t \qquad \text{a.e.}\ f \in L_\infty.$$

Clearly $\nu_t(B) = 0$ a.e. whenever $\lambda(B) = 0$ and $\nu_t \geq 0$ a.e.

For $f \in C[0,1]$,

$$\int |f|\, d\nu_t \geq \left| \int f d\mu_t \right| \qquad \text{a.e.}$$

and since $C[0,1]$ is separable

$$|\mu_t| \leq \nu_t \qquad \text{a.e.}$$

so that $|\mu_t|(B) = 0$ a.e. whenever $\lambda(B) = 0$. (In fact it is now clear that $\nu_t = |\mu_t|$ a.e.)

We remark that the abstract kernel associated to any operator is unique up to sets of measure zero. This is easily obtained from the separability of $C[0,1]$.

If X is a function space and Y is either a function space or $Y = L_0$ then we shall say that an operator $T : X \to Y$ is *controllable* (by $h \in L_0$) if $|Tf| \leq h$ a.e. whenever $\|f\|_\infty \leq 1$. T is said to be *regular* if there is a positive operator $T : X \to Y$ so that $|Tf| \leq P\,|f|$ for $f \in X$.

The following theorem is proved in [11].

Theorem 3.2. *Let X be a minimal function space and suppose $T : X \to L_0$ is a linear operator. In order that there exists an abstract kernel $\{\mu_t\}$ so that whenever $f \in X$ we have*

$$Tf(t) = \int f d\mu_t \qquad \text{a.e.}$$

(and, in particular $\int |f|d|\mu_t| < \infty$ a.e) it is necessary and sufficient that T be regular.

Proof. As the argument in [11] contains several misprints we briefly indicate the proof. If T is regular there is a positive operator $P : X \to L_0$ with $|Tf| \leq P\,|f|$ for $f \in X$. By Theorem 3.1 we can find abstract kernels $\{\mu_t\}$ and $\{\nu_t\}$ so that

$$Tf(t) = \int f d\mu_t \qquad \text{a.e.}\ f \in L_\infty$$

$$Pf(t) = \int f d\nu_t \qquad \text{a.e.}\ f \in L_\infty$$

and $|\mu_t| \leq \nu_t$ a.e. If $f \in X$ and $f \geq 0$ let $f_n = \min(f,n)$. Since X is minimal

$\|f_n - f\| \to 0$ and so passing to a subsequence (g_n) we can suppose $Tg_n \to Tf$ a.e. Now

$$\int |f|\, d\,|\mu_t| \leq \int |f|\, d\nu_t \qquad \text{a.e.}$$

$$= \lim_{n\to\infty} \int |f_n|\, d\nu_t \qquad \text{a.e.}$$

$$= \lim_{n\to\infty} P|f_n| \qquad \text{a.e.}$$

$$= P|f| < \infty \qquad \text{a.e.}$$

Now by the Dominated Convergence Theorem

$$\lim_{n\to\infty} \int g_n\, d\mu_t = \int f\, d\mu_t \qquad \text{a.e.}$$

so that

$$Tf = \int f\, d\mu_t \qquad \text{a.e.}$$

and the result follows by linearity.

For the converse if T is represented by $\{\mu_t\}$ define $P : X \to L_0$ by

$$Pf(t) = \int f\, d|\mu_t| \qquad f \in X.$$

If $\|f_n\| \leq 2^{-n}$ then $\sum |f_n| \in X$ and so

$$\sum \int |f_n|\, d|\mu_t| < \infty \qquad \text{a.e.}$$

so that $P\,|f_n| \to 0$ a.e. Hence P is continuous and clearly $|Tf| \leq P\,|f|$ a.e. for $f \in X$.

4. Atomic kernels. The following theorem is essentially proved in Theorem 3.2 of [10].

Theorem 4.1. *Let $t \to \mu_t$ be an abstract kernel. Then there are Borel maps $a_n : [0,1] \to \mathbf{R}$, $\sigma_n : [0,1] \to [0,1]$ so that*

(i) $|a_n(t)| \geq |a_{n+1}(t)| \qquad n = 1, 2 \ldots, 0 \leq t \leq 1$

(ii) $\sigma_m(t) \neq \sigma_n(t) \qquad m \neq n$

(iii) $\mu_t = \sum_{n=1}^{\infty} a_n(t)\delta(\sigma_n t) + \nu_t \qquad$ a.e. $0 \leq t \leq 1$

where ν_t is a diffuse (atom-free) measure for $0 \leq t \leq 1$.

We shall say that $t \to \mu_t$ is *atomic* if μ_t is an atomic measure a.e., that is if $\nu_t = 0$ a.e. in the above theorem. If X and Y are function spaces and $T: X \to Y$ is a controllable operator we shall say that T is *atomic* if its kernel is atomic.

Before proceeding we shall need a probabilistic lemma. Suppose $N \in \mathbf{N}$ and suppose $b_i \in \mathbf{R}$ $(1 \leq i \leq N)$ with $b_0 + \ldots + b_N = 0$. Let Π_N denote the group of permutations of $\{1,2,\ldots,N\}$ with its natural probability measure, $P\{\pi\} = (N!)^{-1}$, $\pi \in \Pi_N$. We define random variables ξ_i $(1 \leq i \leq N)$ by

$$\xi_i(\pi) = b_{\pi(i)}.$$

Lemma 4.2. *If $c_1, \ldots, c_N \in \mathbf{R}$, then*

$$\mathscr{E}\left(\left|\sum c_i\xi_i\right|^2\right) \leq \frac{2}{N}\left(\sum_{i=1}^N b_i^2\right)\left(\sum_{i=1}^N c_i^2\right).$$

Proof. We first note $\mathscr{E}(\xi_i^2) = N^{-1}\sum_{i=1}^N b_i^2$ and that $\xi_1 + \ldots + \xi_N = 0$. Hence for $i \neq j$

$$\mathscr{E}(\xi_i\xi_j) = -\frac{1}{N(N-1)}\sum_{i=1}^N b_i^2.$$

Now

$$\begin{aligned}\mathscr{E}\left(\left|\sum c_i\xi_i\right|^2\right) &= \sum_{i=1}^N\sum_{j=1}^N c_ic_j\mathscr{E}(\xi_i\xi_j)\\ &\leq \frac{1}{N}\left(\sum_{i=1}^N b_i^2\right)\left(\sum_{i=1}^N c_i^2 + \sum_{i\neq j}|c_ic_j|\cdot\frac{1}{N-1}\right)\\ &\leq \frac{2}{N}\left(\sum_{i=1}^N b_i^2\right)\left(\sum_{i=1}^N c_j^2\right)\end{aligned}$$

since $|c_ic_j| \leq 1/2\ (c_i^2 + c_j^2)$.

Now denote by $D(n,k)$ the dyadic interval $[(k-1)2^{-n}, k2^{-n})$ for $1 \leq k \leq 2^n - 1$; let $D(n,2^n) = [1 - 2^{-n}, 1]$. Let $\mathscr{D}_n$ denote the algebra generated by the sets $\{D(n,k) : 1 \leq k \leq 2^n\}$ and let $L_0(\mathscr{D}_n)$ be the space of $\mathscr{D}_n$-measurable functions in L_0.

The following lemma quickly leads to the main result of this section.

Lemma 4.3. *Suppose X is a minimal symmetric function space and that $T: X \to L_0$ is a linear operator. Let $\{\mu_t\}$ be an atomic kernel and suppose that for $1 \leq k \leq 2^n$ and $n \in \mathbf{N}$ we define*

$$g_{n,k}(t) = T1_{D(n,k)}(t) - \mu_t(D(n,k))$$

and

$$h_n(t) = \left(\sum_{k=1}^{2^n}|g_{n,k}(t)|^2\right)^{1/2}.$$

Suppose $\liminf_{n\to\infty} \|h_n\|_0 = 0$. *Then for* $f \in L_\infty$

$$Tf\,(t) = \int f\, d\mu_t \qquad \text{a.e.}$$

Proof. It will suffice to show that

$$u(t) = T1(t) - \mu_t[0,1] = 0 \qquad \text{a.e.} \tag{4.1}$$

Then if for $B \in \mathcal{B}$, P_B is the natural projection $f \to f \cdot 1_B$, we can consider the operator $TP_{D(m,\ell)}$ with the atomic kernel $\mu_t'(B) = \mu_t(B \cap D(m,\ell))$. For $n \geq m$ and $1 \leq k \leq 2^n$ if

$$g_{n,k}'(t) = TP_{D(m,\ell)}\, 1_{D(n,k)}(t) - \mu_t'(D(m,\ell))$$

then $g_{n,k}' = g_{n,k}$ if $D(n,k) \subset D(m,\ell)$ and $g_{n,k}' = 0$ otherwise. If (4.1) is established then we will be able to conclude that

$$TP_{D(m,\ell)}\, 1(t) = \mu_t'[0,1] \qquad \text{a.e.}$$

that is

$$T1_{D(m,\ell)}(t) = \mu_t(D(m,\ell)) \qquad \text{a.e.}$$

It follows then quickly that if $f \in L_\infty$

$$Tf\,(t) = \int f\, d\mu_t \qquad \text{a.e.}$$

We therefore turn to establishing (4.1). Fix $\varepsilon > 0$. By Theorem 4.1 we can find Borel maps $a_n : [0,1] \to \mathbf{R}$, $\sigma_n : [0,1] \to [0,1]$ so that (i), (ii) and (iii) hold with $\nu_t = 0$ for all t. Since

$$\sum_{i=1}^{\infty} |a_i| = \|\mu_t\| < \infty \qquad \text{a.e.}$$

there exists $r \in \mathbf{N}$ so that

$$\left\| \sum_{i=r+1}^{\infty} |a_i| \right\|_0 < \frac{1}{4}\,\varepsilon.$$

Now fix $\theta > 0$ so that if $f \in X$ with $\|f\| < \theta$ then $\|Tf\|_0 < (1/4)\,\varepsilon$. Since $X^* = \{0\}$ there exists $m \in \mathbf{N}$ and $f \in L_0(\mathcal{D}_m)$ so that $f \geq 0$, $\lambda(\operatorname{supp} f) < \varepsilon/4r$, $\|f\|_X < \theta$ and

$$\int f(t)\, dt = 1.$$

Now choose $n \geq m$ so that

$$\|\sqrt{2}\,\beta\, h_n\|_0 < \frac{1}{4}\,\varepsilon$$

where

$$\beta^2 = \int_0^1 f(t)^2\,dt.$$

Let $N = 2^n$ and let (Π_N, P) be the group of permutations of $\{1,2,\ldots,N\}$ with its natural probability measure described before Lemma 4.2. Let

$$f - 1 = \sum_{k=1}^{N} b_k\, 1_{D(n,k)}.$$

Clearly $b_1 + \ldots + b_N = 0$, and so we can define random variables ξ_i $(1 \le i \le N)$ as in the discussion before Lemma 4.2.

For each $\pi \in \Pi_N$ define $f_\pi \in L_0(\mathcal{D}_n)$ by

$$f_\pi = 1 + \sum_{k=1}^{N} b_{\pi(k)}\, 1_{D(n,k)}.$$

Then $Tf_\pi = F_\pi + G_\pi + H_\pi$ where

$$F_\pi(t) = \sum_{i=1}^{r} a_i(t) f_\pi(\sigma_i t)$$

$$G_\pi(t) = \sum_{i=r+1}^{\infty} a_i(t) f_\pi(\sigma_i t)$$

$$H_\pi(t) = Tf_\pi(t) - \int f_\pi\, d\mu_t.$$

Now

$$H_\pi(t) = u(t) + \sum_{k=1}^{N} b_{\pi(k)}\, g_{n,k}(t) = u(t) + \sum_{k=1}^{N} \xi_k(\pi)\, g_{n,k}(t).$$

Hence for each t we have

$$\mathscr{E}(|H_\pi(t) - u(t)|^2) \le \frac{2}{N}\left(\sum_{k=1}^{N} b_k^2\right)\left(\sum_{k=1}^{N} g_{n,k}^2(t)\right) = 2\beta^2 h_n^2(t).$$

Thus

$$\begin{aligned}\mathscr{E}(\|H_\pi - u\|_0) &= \int_{\Pi_N}\int_0^1 \min(1, |H_\pi(t) - u(t)|)\, dt\, dP(\pi)\\ &\le \int_0^1 \min(1, \sqrt{2}\,\beta\, h_n(t))\, dt \le \frac{1}{4}\,\varepsilon.\end{aligned}$$

We turn to F_π, where we observe

$$\operatorname{supp} F_\pi \subset \bigcup_{i=1}^{r} \sigma_i^{-1} (\operatorname{supp} f_\pi).$$

Hence

$$\lambda(\operatorname{supp} F_\pi) \le \sum_{i=1}^{r} \lambda(\sigma_i^{-1} \operatorname{supp} f_\pi)$$

and so

$$\mathscr{E}(\|F_\pi\|_0) \le \sum_{i=1}^{r} \mathscr{E}(\lambda(\sigma_i^{-1} \operatorname{supp} f_\pi)) = r\,\lambda(\operatorname{supp} f_\pi) \le \frac{1}{4}\,\varepsilon.$$

Next for G_π,

$$\mathscr{E}(\min(1, G_\pi(t))) \le \mathscr{E}(|g_\pi(t|) \le \left(\sum_{i=r+1}^{\infty} |a_i(t)|\right) \mathscr{E}(|f_\pi(\sigma_i t)|) = \sum_{i=r+1}^{\infty} |a_i(t)|.$$

Thus $\mathscr{E}(\|G_\pi\|_0) < (1/4)\,\varepsilon$.

Finally for all $\pi \in \Pi_N$, $\|f_\pi\| < \theta$ so that $\mathscr{E}(\|Tf_\pi\|) < (1/4)\,\varepsilon$. Combining we conclude

$$u = (u - H_\pi) = F_\pi - G_\pi + Tf_\pi$$

so that

$$\|u\|_0 < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary $u = 0$ as required.

Theorem 4.4. *Let X be a minimal symmetric function space with trivial dual. Let $T: X \to L_0$ be a controllable linear operator. Then there exist Borel maps $a_n : [0,1] \to \mathbf{R}$, $\sigma_n : [0,1] \to [0,1]$ so that*

(i) $|a_n(t)| \ge |a_{n+1}(t)|$ $\qquad n \in \mathbf{N},\ 0 \le t \le 1.$

(ii) $\sigma_m(t) \ne \sigma_n(t)$ $\qquad m \ne n.$

(iii) $\sum_{i=1}^{\infty} |a_i(t) < \infty$ $\qquad$ a.e.

(iv) $Tf(t) = \sum_{i=1}^{\infty} a_i(t) f(\sigma_i(t))$ $\qquad$ a.e. $f \in L_\infty$.

If in addition T is regular, (iv) holds for all $f \in X$.

Proof. By Theorem 3.1, T has an abstract kernel $\{\mu_t\}$ and by Theorem 4.1

$$\mu_t = \sum_{n=1}^{\infty} a_n(t)\, \delta(\sigma_n t) + \nu_t \qquad \text{a.e.}$$

where a_n, σ_n satisfy (i), (ii), (iii) above and ν_t is a.e. diffuse. Now for all n, k

$$T1_{D(n,k)} - \mu_t(D_{n,k}) = \nu_t(D_{n,k})$$

and

$$\left(\sum_{k=1}^{2^n} |\nu_t(D_{n,k})|^2\right)^{1/2} \to 0 \qquad \text{pointwise}$$

since ν_t is diffuse. In fact

$$\left(\sum_{k=1}^{2^n} |\nu_t(D_{n,k})|^2\right)^{1/2} \leq (\max_{k\leq 2^n}|\nu_t(D_{n,k})|)^{1/2}\, \|\nu_t\|^{1/2}.$$

Hence

$$Tf\,(t) = \int f\,d\mu_t \qquad \text{a.e.}\ f \in L_\infty$$

whence (iv) follows. If T is regular (iv) then must hold for all $f \in X$.

Remark. Since $|\mu_t|(B) = 0$ a.e. whenever $\lambda(B) = 0$ we conclude $\lambda(\operatorname{supp} a_n \cap \sigma_n^{-1} B) = 0$ for all $n \in \mathbf{N}$.

5. When is every operator controllable?

Theorem 5.1. *Let $T : \Lambda(1,\infty) \to L_0$ be a linear operator. Then T is controllable.*

Proof. Since T is continuous there is a function $\rho : [0,\infty) \to [0,1]$ with $\lim_{x\to\infty} \rho(x) = 0$ so that if $f \in \Lambda(1,\infty)$ and $\|f\| \leq 1$

$$\lambda(|Tf| > x) < \rho(x).$$

Let $T1_{D(n,k)} = g_{n,k}$ and set

$$h_n = \sum_{k=1}^{2^n} |g_{n,k}|.$$

Then h_n is monotone increasing. Let $h = \sup_n h_n$. We shall show that $h < \infty$ a.e.; clearly we have that if $f \in L_\infty$ then $|Tf| \leq h$ a.e., so we can then conclude T is controllable by h.

Fix $n \in \mathbf{N}$ and $0 < x < \infty$ and let $N = 2^n$. Suppose $c_1, \ldots, c_N \geq 0$ and that $\left\|\sum c_k 1_{D(n,k)}\right\|_{1,\infty} \leq 1$. Let $\varepsilon_1, \ldots, \varepsilon_N$ be independent Bernoulli random variables on some probability space (Ω, P_0) so that

$$P_0(\varepsilon_k = +1) = P_0(\varepsilon_k = -1) = \frac{1}{2}.$$

Then

$$(P_0 \times \lambda)\left\{(\omega,t) : \left|\sum \varepsilon_k(\omega)c_k g_{n,k}(t)\right| > x\right\} < \rho(x).$$

However by the Paley-Zygmund inequality [9] page 24 for each $t \in [0,1]$

$$P_0\left\{\left|\sum \varepsilon_k(\omega)c_k g_{n,k}(t)\right| > \frac{1}{4}\left(\sum c_k^2\, |g_{n,k}(t)|^2\right)^{1/2}\right\} > \frac{1}{4}.$$

Thus

$$\lambda\left\{\left(\sum c_k^2 |g_{n,k}(t)|^2\right)^{1/2} > 4x\right\} < 4\rho(x)$$

and so

$$\lambda\{\max_{k\le N} c_k |g_{n,k}(t)| > x\} < 4\rho\left(\frac{1}{4}x\right). \tag{5.1}$$

Let Π_N be the group of permutations of $\{1,2,\dots N\}$ with the usual measure P on Π_N. If $a_1, \dots, a_N, c_1, \dots, c_N$ are non-negative real numbers we observe

$$\begin{aligned} P[\max_{k\le N} c_k a_{\pi(k)} \le x] &= \sum_{k=1}^{N} P[c_k a_{\pi(k)} \le x|\ c_j a_{\pi(j)} \le x \text{ for } j < k] \\ &\le \sum_{k=1}^{N} P[c_k a_{\pi(k)} \le x] \\ &= \sum_{k=1}^{N} (1 - P(a_{\pi(k)} > c_k^{-1}x)) \\ &\le \exp\left(-\sum_{k=1}^{N} P(a_{\pi(k)} > c_k^{-1}\, x)\right). \end{aligned}$$

If we put $c_k = N/k$, then

$$P\left[\max \frac{N}{k} a_{\pi(k)} \le x\right] \le \exp\left(-\sum_{k=1}^{N} P\left(a_{\pi(k)} > \frac{kx}{N}\right)\right).$$

If $0 \le a_k \le x$ for $1 \le k \le N$,

$$\sum_{k=1}^{N} P\left(a_{\pi(k)} > \frac{kx}{N}\right) \ge \frac{N}{x}\mathscr{E}\left(a_{\pi(k)} - \frac{x}{N}\right) \ge \frac{a_1 + \dots + a_N}{x} - 1.$$

Thus

$$P\left[\max \frac{N}{k} a_{\pi(k)} \le x\right] \le \exp\left(1 - \frac{a_1 + \dots + a_N}{x}\right). \tag{5.2}$$

Let $E \subset [0,1]$ be the set of t so that $\max_{k\le N} |g_{n,k}(t)| \le x$. From (5.1) with $c_k = 1$

we deduce that $\lambda(E) \geq 1 - 4\rho\,((1/4)x)$. Let $A \subset \Pi_N \times E$ be the set (π,t) so that

$$\max_{k\leq N} \frac{N}{k}\,|g_{n,\pi(k)}(t)| \leq x.$$

Then from (5.1) with $c_k = N/\pi^{-1}(k)$ we deduce

$$(P \times \lambda)(\Pi_N \times E\backslash A) \leq 4\rho\left(\frac{1}{4}x\right).$$

Hence $(P \times \lambda)\,(A) \geq 1 - 8\rho\,((1/4)\,x)$. However

$$(P \times \lambda)(A) \leq \int_E \min\left(1, \exp\left(1 - \frac{h_n(t)}{x}\right)\right) dt$$

by (5.2). Thus we have

$$\int_0^1 \min\left(1, \exp\left(1 - \frac{h_n(t)}{x}\right)\right) dt \geq 1 - 8\rho\left(\frac{1}{4}x\right).$$

Now let $n \to \infty$;

$$\int_0^1 \min\left(1, \exp\left(1 - \frac{h(t)}{x}\right)\right) dt \geq 1 - 8\rho\left(\frac{1}{4}x\right).$$

Since $\lim_{x\to\infty} \rho(x) = 0$ we conlude that $h < \infty$ a.e. as required.

Corollary 5.2. *If $T : \Lambda(1,\infty) \to L_0$ is a linear operator there exist Borel maps $a_n : [0,1] \to \mathbf{R}$, $\sigma_n : [0,1] \to [0,1]$ so that*

(a) $\sum |a_n(t)| < \infty \qquad 0 \leq t \leq 1$

(b) $\sigma_m(t) \neq \sigma_n(t) \qquad m \neq n,\ 0 \leq t \leq 1$

(c) $Tf(t) = \sum a_n(t) f(\sigma_n t) \qquad$ a.e. $f \in L_\infty$.

We shall now show that under some fairly mild hypotheses on X, the assertion that every operator $T : X \to L_0$ is controllable implies that $X \supset \Lambda(1,\infty)$.

Let ϕ be an Orlicz function satisfying $\phi(st) \leq s^q\phi(t)$ for $s \geq 1$ where $0 < q < 2$. We describe the construction of an operator $T_\phi : L_\phi \to L_0$ first exploited by Bretagnolle and Dacunha-Castelle [3]. Let $M(x) = \phi(x^{-1})$ for $0 < x < \infty$. Then $-dM(x)$ is a Levy measure [3], [6], since

$$\int_0^\infty \min(1,x^2)(-dM(x)) = M(1) + 2\int_0^1 xM(x)dx < \infty.$$

Here

$$\int_0^1 xM(x)dx = \int_1^\infty \frac{\phi(x)}{x^3}\, dx < \infty.$$

Thus ([6] page 76) there is a symmetric random variable η on [0,1] whose distribution is infinitely divisible and such that

$$\log \int_0^1 e^{is\eta(t)}dt = \int_0^\infty (1 - \cos su)dM(u).$$

Using this, we can find a symmetric stochastic process $\{\eta_\tau : 0 \leq \tau \leq 1\}$ with stationary independent increments so that

$$\log \int_0^1 e^{is\eta_\tau(t)}dt = \tau \int_0^\infty (1 - \cos su)dM(u).$$

We then can induce a linear operator $T_\phi : L_\infty \to L_0$ so that $T_\phi(1_{[0,\tau]}) = \eta_\tau$. In fact

$$\log \int_0^1 \exp(is\, T_\phi f(t))dt = \int_0^\infty (1 - \cos su)dM_f(u)$$

where (if $M(\infty) = 0$)

$$M_f(u) = \int_0^1 M\left(\frac{u}{|f(t)|}\right) dt.$$

T_ϕ now extends continuously to a linear operator $T_\phi : L_\phi \to L_0$. Indeed from the hypothesis on ϕ we can deduce that $M_f(tu) \leq t^{-2}M_f(u)$ for $f \in L_\infty$ and hence

$$-\log \int_0^1 \exp(is\, T_\phi f(t))\, dt \leq CM_f(s^{-1})$$

for some constant C. Thus if $\int \phi(|f_n|)\, dt \to 0$ we have $T_\phi f_n \to 0$ in measure.

Theorem 5.2. *T_ϕ is controllable if and only if $\int_1^\infty(\phi(x)/x^2)\, dx < \infty$.*

Proof. T_ϕ is controllable if and only if

$$\sup_n \sum_{k=1}^{\angle} |T_\phi(1_{D(n,k)})| < \infty \qquad \text{a.e.}$$

or if $\tau \to \eta_\tau(t)$ is almost everywhere of bounded variation. This in turn ([2] page 316 Exercise 13) is equivalent to

$$-\int_0^1 xdM(x) < \infty$$

or

$$\int_1^\infty \frac{\phi(x)}{x^2}\,dx < \infty.$$

Now let us say that a symmetric function space X is D-convex if the set of distribution functions $\{d_f : \|f\| \le 1\}$ is convex. Orlicz spaces are clearly D-convex. A Lorentz space $L(p,q)$ is D-convex if $p \le q \le \infty$. To see this simply rewrite the quasi-norm as

$$\|f\|_{p,q} = \left\{ q \int_0^\infty d_f(x)^{q/p} x^{q-1}\,dx \right\}^{1/q}.$$

However if $q < p$, $L(p,q)$ is not D-convex and cannot be equivalently normed to be D-convex.

Theorem 5.4. *Let X be a D-convex maximal symmetric function space. Suppose the upper Boyd index q_X of X satisfies $q_X < 2$. Suppose also every operator $T: X \to L_0$ is controllable. Then $X \supset L(1,\infty)$.*

In order to prove Theorem 5.4 we shall first prove a lemma which will also be useful later.

Lemma 5.5. *Suppose X satisfies the hypotheses of the theorem and that $h \in L_0$ but $h \notin X$. Then if $q_x < r < 2$, there is an Orlicz function ϕ so that $h \notin L_\phi$, $L_\phi \supset X$ and $\phi(sx) \le s^r\phi(x)$ for $s \ge 1$.*

Proof. Let $C_0\,[0,\infty]$ be the Banach space of all real-valued continuous functions on $[0,\infty]$ vanishing at 0. The dual of $C_0\,[0,\infty]$ can be identified with the space $\mathrm{BV}_0[0,\infty]$ of right-continuous functions F of bounded variation on $[0,\infty]$ vanishing at ∞. The duality is given by

$$\langle a,F\rangle = a(\infty)\lim_{x\to\infty} F(x) - \int_0^\infty a(x)\,dF(x)$$

and the norm by

$$\|F\| = \sup_{0=x_0<x_1\ldots<x_n=\infty} \sum_{i=1}^n |F(x_i) - F(x_{i-1})|.$$

For $f \in L_0$ define $d_f(\infty) = 0$. Then we set $C_X = \{d_f : \|f\| \le 1\} \subset \mathrm{BV}_0[0,\infty]$. It is not difficult to show that C_X is weak*-compact, and by hypothesis it is convex. Let

$$Q = \{F \in \mathrm{BV}_0 : F(x) \ge 0, 0 \le x \le \infty\}.$$

Then Q is weak*-closed since $F \in Q$ if and only if $\langle a,F\rangle \ge 0$ whenever a is monotone increasing.

Choose $\varepsilon_n > 0$ so that $\sum \varepsilon_n = 1$. Since $\varepsilon_n d_n \notin C_X - Q$ for any n, we can find $a_n \in C_0[0,\infty]$ so that $\langle a_n, d_h\rangle = 1$ but $\langle a_n, d_f\rangle \le \varepsilon_n$ for $f \in X$ with $\|f\| \le 1$ and $\langle a_n, G\rangle \ge 0$ for $G \in Q$. Hence each a_n is monotone increasing.

It is clear that $a_n(x) \leq \varepsilon_n \rho(x)^{-1}$ where $\rho(x)$ is the largest number so that $\rho(x)1_{[0,x]} \in C_X$. Hence $\sum a_n(x)$ converges for finite x to a continuous monotone increasing function $b(x)$ with $b(0) = 0$.

If $F \in C_X$ then

$$\int_0^\infty a_n(x)\, d(-F(x)) \leq \varepsilon_n$$

and hence

$$\int_0^1 b(x)\, d(-F(x)) \leq 1$$

or equivalently

$$\int_0^1 b(|f(t)|)\, dt \leq 1$$

for $f \in X$ with $\|f\| \leq 1$. Similarly

$$\int_0^1 b(|h(t)|)\, dt = \infty.$$

Now since $q_X < r$ we can find u, $q_X < u < r$ and then for some $C < \infty$, $\|D_s\| \leq Cs^{1/u}$ for $0 < s \leq 1$. Now if $\|f\| < C^{-1}$ and $s \geq 1$, $\|D_{s^{-u}}(sf)\| \leq 1$ and hence

$$\int b(s|f(t)|)\, dt \leq s^u.$$

Now let

$$\phi(x) = \int_1^\infty b(sx) s^{-r-1}\, ds.$$

Then ϕ is an Orlicz function satisfying $\phi(sx) \leq s^r \phi(x)$.

Also if $\|f\| \leq C^{-1}$

$$\int_0^1 \phi(|f(t)|)\, dt = \int_0^1 \int_1^\infty b(s|f(t)|)\, s^{-r-1} ds \leq \int_1^\infty s^{u-r-1} ds < \infty$$

so that $X \subset L_\phi$.

Finally

$$\int_0^1 \phi(|h(t)|)\, dt \geq \left(\int_1^\infty s^{-r-1} ds\right) \int_0^1 b(|h(t)|)\, dt = \infty.$$

Proof of Theorem 5.4. If X does not contain $L(1,\infty)$ then the function $h(t) = t^{-1}$, $0 < t \leq 1$ is not in X. Hence there is an Orlicz function ϕ so that $X \subset L_\phi$ but

$$\int_0^1 \phi(t^{-1})\,dt = \int_1^\infty \frac{\phi(t)}{t^2}\,dt = \infty.$$

Now $T_\phi : X \to L_0$ is controllable and we have contradicted Theorem 5.3.

6. When is every operator regular? If X is a function space and $0 < r < \infty$ we define $X_{(r)}$ to be the function space of all $f \in L_0$ so that $|f|^r \in X$. The quasi-norm on $X_{(r)}$ is given by

$$\|f\|_{(r)} = \||f^r|\|^{1/r}.$$

If X is (lattice) r^{-1}-convex then $X_{(r)}$ is a Banach lattice, after an equivalent re-norming.

Let us suppose μ_t is an abstract kernel which is atomic. Thus we may write

$$\mu_t = \sum_{r=1}^{\infty} a_n(t)\, \delta(\sigma_n t) \qquad \text{a.e.} \tag{6.1}$$

where $a_n : [0,1] \to \mathbf{R}$, $\sigma_n : [0,1] \to [0,1]$ are Borel maps and $\sigma_m t \neq \sigma_n t$ whenever $m \neq n$.

Now define a new abstract kernel by

$$\nu_t = \sum_{n=1}^{\infty} |a_n(t)|^2\, \delta(\sigma_n t). \tag{6.2}$$

We say as in Section 3 that $\{\mu_t\}$ represents an operator $T : X \to Y$ if

$$Tf(t) = \int f\, d\nu_t \qquad \text{a.e. } f \in L_\infty.$$

Proposition 6.1. *Let X be a minimal function space and let Y be a maximal L-convex function space. Then there is a constant C so that if μ_t represents an operator $T \in \mathcal{L}(X,Y)$ then ν_t represents an operator $W \in \mathcal{L}(X_{1/2},Y_{1/2})$ with $\|W\| \leq C^2\,\|T\|^2$.*

Proof. For $f \in L_0(\mathcal{D}_n)$ set

$$W_n f = \sum_{k=1}^{\infty} \alpha_k |T 1_{D(n,k)}|^2$$

where $f = \sum \alpha_k 1_{D(n,k)}$. Then in $Y_{1/2}$

$$\|Wf\|_{1/2} \leq \left\| \sum_{k=1}^{2^n} |\alpha_k|\, |T 1_{D(n,k)}|^2 \right\|_{1/2} = \left\| \left(\sum_{k=1}^{2^n} |\alpha_k|\, |T 1_{D(n,k)}|^2 \right)^{1/2} \right\|^2$$

$$\leq C^2 \|T\|^2 \left\| \sum_{k=1}^{2^n} |\alpha_k|^{1/2}\, 1_{D(n,k)} \right\|^2$$

where C depends only on Y (Theorem 3.3 of [12]).

Hence

$$\|W_n f\|_{1/2} \leq C^2 \|T\|^2 \|f\|_{1/2}.$$

Now for $m \geq n$

$$W_m f(t) = \sum_{k=1}^{2^n} \alpha_k \sum_{D(m,\ell)\subset D(n,k)} |T1_{D(m,\ell)}(t)|^2$$
$$= \sum_{k=1}^{2^n} \alpha_k \sum_{D(m,\ell)\subset D(n,k)} \left(\sum_{\alpha_i t\in D(m,\ell)} a_i(t) \right)^2$$

so that a.e. as $m \to \infty$

$$W_m f(t) \to \sum_{k=1}^{2^n} \alpha_k \sum_{\alpha_i t\in D(n,k)} |a_i(t)|^2 = \int f\, d\nu_t.$$

Hence if $Wf = \int f\, d\nu_t$ a.e. then

$$\|Wf\|_{1/2} \leq C^2 \|T\|^2 \|f\|_{1/2}.$$

Here we use the fact that Y and hence $Y_{1/2}$ are maximal so that

$$\|Wf\|_{1/2} \leq \limsup_{m\to\infty} \|W_m f\|_{1/2}.$$

We shall prove a converse for Proposition 6.1. However we first need a technical lemma.

Lemma 6.2. *Let $\sigma_k : [0,1] \to [0,1]$ be Borel maps for $1 \leq k \leq N$ so that for any $s \in [0,1]$ the points $(\sigma_i s : 1 \leq i \leq N)$ are distinct. Let $\mathcal{A}$ be a finite subalgebra of $\mathcal{B}$. Then for any ε, $0 < \varepsilon < 1$, there exist Borel sets $E, B_0, B_1, \ldots, B_m$ so that*

(i) $\lambda(E) \geq 1 - \varepsilon$.
(ii) $\lambda(B_0) \leq \varepsilon$.
(iii) The sets $\{B_0,B_1,\ldots,B_m\}$ partition $[0,1]$ and are each independent of $\mathcal{A}$.
(iv) $\lambda(B_1) = \ldots = \lambda(B_m)$.
(v) The sets $E \cap \sigma_i^{-1}(B_j)$ $(1 \leq i \leq N)$ are pairwise disjoint for $1 \leq j \leq m$.

Proof. According to Theorem 5.4 of [11] we can find $E_0 \in \mathcal{B}$ with $\lambda(E_0) \geq 1 - (1/2)\,\varepsilon$ and a Borel partitioning $(D_1,\ldots,D_u)$ of $[0,1]$ so that for each j, $1 \leq j \leq u$ the sets $E_0 \cap \sigma_i^{-1}(D_j)$, $1 \leq i \leq N$, are pairwise disjoint.

Let $(A_j : 1 \leq j \leq h)$ be the atoms of $\mathcal{A}$. By further partitioning we can define

$$n > 2\,\varepsilon^{-1}(1-\varepsilon)^{-1}\binom{N}{2}^{-1}, \qquad m > (1-\varepsilon)n$$

and Borel sets $\{F_{j\ell} : 1 \leq j \leq h, 1 \leq \ell \leq n\}$ so that:

(a) For each j, the sets $\{F_{j\ell} : 1 \leq \ell \leq n\}$ partition A_j into sets of measure $n^{-1}\lambda(A_j)$.

(b) For each j and each $\ell \leq m$, the sets $E_0 \cap \sigma_i^{-1}(F_{j\ell})$ are pairwise disjoint for $1 \leq i \leq N$.

Let Π_m be the set of permutations of $\{1,2,\ldots,m\}$. Let P be the natural probability measure on Π_m^h. For each $\pi = \{\pi_j : 1 \leq j \leq h\} \in \Pi_m^h$, we define for $1 \leq \ell \leq m$,

$$B_{\ell,\pi} = \bigcup_{j=1}^{h} F_{j,\pi_j(\ell)}.$$

For $\ell = 0$, we set

$$B_{0,\pi} = [0,1]\setminus\bigcup_{\ell=1}^{m} B_{\ell,\pi}.$$

The sets $\{B_{\ell,\pi} : 0 \leq \ell \leq m\}$ satisfy (ii), (iii) and (iv). It remains to select E and π so that (i) and (v) hold.

If $s \in E_0$, $1 \leq \ell \leq m$ and $1 \leq i_1 < i_2 \leq N$,

$$P\{\sigma_{i_1} s \in B_{\ell,\pi}, \sigma_{i_2} s \in B_{\ell,\pi}\} \leq m^{-2}$$

$$P\{s \in \sigma_{i_1}^{-1}(B_{\ell,\pi}) \cap \sigma_{i_2}^{-1}(B_{\ell,\pi}), \text{ some } \ell \leq m, 1 \leq i_1 < i_2 \leq N\} \leq m^{-1}\binom{N}{2} \leq \varepsilon/2.$$

Let G_π be the set of $s \in E_0$ belonging to

$$\sigma_{i_1}^{-1}(B_{\ell,\pi}) \cap \sigma_{i_2}^{-1}(B_{\ell,\pi})$$

for some $1 \leq i_1 < i_2 \leq N$ and $1 \leq \ell \leq m$. Then by the above and Fubini's theorem

$$\mathscr{E}(\lambda(G_\pi)) \leq \frac{1}{2}\varepsilon.$$

Now fix $\pi \in \Pi_m^h$ so that $\lambda(G_\pi) \leq (1/2)\varepsilon$. Then we set $E = E_0\setminus G_\pi$ and $B_j = B_{j,\pi}$ for $0 \leq j \leq m$.

Proposition 6.3. *Let X be a minimal symmetric function space for which $q_X < 1$; let Y be a maximal L-convex function space. Then there is a constant M so that if ν_t represents a positive operator $W \in \mathscr{L}(X_{1/2}, y_{1/2})$ then $|\mu_t|$ represents an operator $|T| \in \mathscr{L}(X,Y)$ with $\||T|\| \leq M\,\|W\|^{1/2}$.*

Proof. We first reduce the proposition to the case $Y = L_p$ where $0 < p < 1$. Indeed suppose this case has been established. For a general L-convex Y, then for some $r > 0$, $Y_{(r)}$ is a maximal Banach lattice. Thus, there is a subset A of $(L_0)_+$ so that $f \in Y_{(r)}$ if and only if

$$|||f||| = \sup_{w\in A} \int |f|w(t)dt < \infty$$

and $|||\cdot|||$ is an equivalent norm on $Y_{(r)}$. Hence Y is equivalently quasi-normed by

$$\|f\|^* = \sup_{w\in A}\left\{\int_0^1 |f(t)|^p w(t)dt\right\}^{1/p}$$

and $f \in Y$ if and only if $\|f\|^* < \infty$.

Now for each $w \in A$ the operator $f \to w^{2/p}$. Wf maps $X_{1/2}$ into $L_{p/2}$ with norm bounded by $C\|W\|$ where C depends on the renorming by $\| \,\|^*$. Hence the operator $f \to w^{1/p}\,Tf$ maps X into L_p with the norm bounded by $A^{1/2}\,\|W\|^{1/2}$ where A depends on C and p.

Thus for $f \in L_\infty$

$$\|Tf\|^* \le \sup_{w\in A} \|w^{1/p}Tf\|_p \le A^{1/2}\|W\|^{1/2}\|f\|$$

and the result will follow.

Now we consider the case $Y = L_p$. We may also suppose that on X, $\|D_t\| \le t^{1/s}$ for $0 < t < 1$ where $q_X < s < 1$. Otherwise renorm X by

$$\|f\| = \sup_{0<t<1} t^{-1/s}\|D_t f\|.$$

For each $k \in \mathbf{N}$ let $U_k : X \to Y$ be defined by

$$U_k f(t) = |a_k(t)|f(\sigma_k t).$$

It is easily seen that $\|U_k\| \le \|W\|^{1/2}$ so that $U_k \in \mathscr{L}(X,Y)$. For any $N \in \mathbf{N}$ let $S = U_1 + \ldots + U_N$. The operator S is of *finite type* in the sense of [11] page 330 since

$$Sf(t) = \sum_{k=1}^{N} |a_k(t)|f(\sigma_k t).$$

Now suppose $f \ge 0$ is a simple function, measurable with respect to some finite algebra $\mathscr{A}$. For any $\varepsilon > 0$ we can find Borel sets $E, B_0, \ldots, B_m$ as in Lemma 6.2. Let $\eta_1, \ldots, \eta_m$ be independent Bernoulli random variables on some probability space (Ω,P) so that

$$P(\eta_i = +1) = P(\eta_i = -1) = 1/2.$$

Let

$$g_i = P_E S(f 1_{B_i}).$$

Then

$$\mathscr{E}\left(\left\|\sum \eta_i g_i\right\|_p^p\right) = \int \mathscr{E}\left|\sum \eta_i g_i\right|^p dt \le \int \left(\sum \left|g_i\right|^2\right)^{p/2} dt$$

by Khintchine's inequality.

Now

$$g_i(t) = |a_j(t)|f(\sigma_j t), \qquad t \in E \cap \sigma_j^{-1}(B_i)$$

and so

$$|g_i|^2 \leq W(f^2 1_{B_i}).$$

Thus

$$\sum_{i=1}^{m} |g_i|^2 \leq W(f^2)$$

and hence

$$\left\| \sum_{i=1}^{m} |g_i|^2 \right\|_{p/2} \leq \|W\| \, \|f\|^2.$$

Hence

$$\min_{\delta_i = \pm 1} \left\| \sum_{i=1}^{m} \delta_i g_i \right\|_p \leq \|W\|^{1/2} \, \|f\|.$$

Thus there is a subset, H, of $\{1,2,\ldots m\}$ with $|H| \leq (1/2)m$ so that

$$\left\| \sum_{i\in H} g_i - \sum_{i \notin H} g_i \right\|_p \leq \|W\|^{1/2} \, \|f\|.$$

Hence

$$\left\| \sum_{i=1}^{m} g_i \right\|_p^p \leq \left\| 2\sum_{i\in H} g_i \right\|_p^p + \|W\|^{p/2} \|f\|^p$$

$$\leq 2^p \|S\|^p \left\| \sum_{i\in H} f 1_{B_i} \right\|^p + \|W\|^{p/2} \, \|f\|^p.$$

Now $\|\Sigma_{i\in H} f 1_{B_i}\|^p \leq \|D_{1/2}\|^p \, \|f\|^p \leq 2^{-p/s} \, \|f\|^p$.

Hence

$$\|P_E S(f 1_{[0,1]\setminus B_0})\|^p \leq (2^{p-p/s} \|S\|^p + \|W\|^{p/2}) \, \|f\|^p.$$

Now $\|f 1_{B_0}\| \leq \varepsilon^{1/s} \, \|f\|$ and so

$$\|P_E S f\|^p \leq (2^{p-p/s} \, \|S\|^p + \|W\|^{p/2} + \varepsilon^{p/s}) \, \|f\|^p.$$

As $\varepsilon > 0$ is arbitrary and L_p is maximal,

$$\|Sf\|^p \leq (2^{p-p/s} \|S\|^p + \|W\|^{p/2}) \, \|f\|^p.$$

Now f is arbitrary, subject to being simple. Hence

$$\|S\|^p \leq 2^{p-p/s} \, \|S\|^p + \|W\|^{p/2}$$

so that

$$\|S\| \leq M \, \|W\|^{1/2}$$

where $M^p = (1 - 2^{p-p/s})^{-1}$.

Returning to the definition of S we now let $N \to \infty$ and obtain that $|T|$ is bounded linear operator from X into Y with $\| |T| \| \leq M \|W\|^{1/2}$.

In order to apply these results we shall prove a preparatory theorem of some independent interest. We note that it is false for $X = L(1,\infty)$ since then $X^* \neq \{0\}$ see [4]).

Theorem 6.4. *Let X be a symmetric function space for which $q_X < 1$. Let X_0 be the closure of the simple functions in X. Let $0 < p < \infty$ and let $T : X \to L_p$ be a linear operator so that $Tf = 0$ for $f \in X_0$. Then $T = 0$.*

Proof. We may suppose that $\|D_t\| \leq t^{1/r}$ for $0 < t \leq 1$ where $q_X < r < 1$. We also suppose that in X, $\|f + g\| \leq C(\|f\| + \|g\|)$.

Suppose $f \in X$ is any countably simple function of the form $f = \sum_{k=1}^{\infty} a_k 1_{B_k}$ where $(B_1,\ldots,B_k,\ldots)$ are disjoint Borel sets. Let $(\eta_m : m \in \mathbf{N})$ be a sequence of independent Bernoulli random variables with

$$P(\eta_m = +1) = P(\eta_m = -1) = \frac{1}{2},$$

which are mutually independent and independent of the σ-algebra generated by $(B_n : n \in \mathbf{N})$.

Now it is possible to choose strictly increasing sequences of natural numbers $m_s = (m(s,k))_{k=1}^{\infty}$ for $0 \leq s \leq 1$ so that if $s \neq t$ then the set of k such that $m(s,k) = m(t,k)$ is finite (see [20]).

For $s \in [0,1]$ let

$$g_s = \sum_{k=1}^{\infty} a_k \, \eta_{m(s,k)} \, 1_{B_k}.$$

Then $(Tg_s : 0 \leq s \leq 1)$ is bounded in L_p. If for every $s \in [0,1]$, $Tg_s \neq 0$ then since L_p is separable there is a sequence s_n so that for some $h \in L_p$, with $h \neq 0$, $\|Tg_{s_n} - h\| \leq 2^{-n}$.

Let

$$\phi_n = \frac{1}{n}(g_{s_1} + \ldots + g_{s_n}).$$

For each $n \in \mathbf{N}$ there is a finite set $F \subset \mathbf{N}$ so that for $k \notin F$ the integers $\{m(s_i,k) : 1 \leq i \leq n\}$ are distinct. Noting that

$$\lambda\left(\left| \frac{1}{n}(\eta_1 + \ldots + \eta_n) \right| \geq n^{-1/3} \right) \leq n^{-1/3}$$

we have

$$\|\phi_n 1_{[0,1] \setminus \bigcup_{k \in F} B_k}\| \leq C(n^{-1/3r}\|f\| + n^{-1/3}\|f\|).$$

Hence $\|T\phi_n\| \to 0$ which implies $h = 0$ contrary to assumption. Now for some

$s \in [0,1]$, $Tg_s = 0$. However $\|f + g_s\| \le 2^{1-1/r}\|f\|$ or $\|f - g_s\| \le 2^{1-1/r}\|f\|$, so that

$$\|Tf\| \le 2^{1-1/r}\|T\|\,\|f\|.$$

Applying to all countably simple $f \in X$ we obtain

$$\|T\| \le 2^{1-1/r}\|T\|$$

so that $T = 0$.

Theorem 6.5. *Let X be a symmetric function space for which* $q_x < 1$. *Then*
(i) If $T : X \to L_0$ *is a linear operator than T is regular and there exist Borel maps* $a_n : [0,1] \to \mathbf{R}$, $\sigma_n : [0,1] \to [0,1]$ *such that*

$$(a)\quad \sum |a_n(t)| < \infty \qquad \text{a.e.}$$

$$(b)\quad \sigma_m t \neq \sigma_n t \qquad m \neq n$$

$$(c)\quad Tf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t) \qquad \text{a.e. } f \in X$$

where the series converges absolutely a.e.
(ii) If Y is any maximal L-convex function space then $\mathcal{L}(X,Y)$ *is a lattice.*

Remark. We do *not* assume that X is minimal.

Proof. By Nikishin's theorem [16], [17], if $T : X \to L_0$ is a linear operator there is a strictly positive $w \in L_0$ so that the map $f \to w.\ Tf$ maps X into L_p for $p < 1$. Hence to prove both (i) and (ii) we shall assume that $T \in \mathcal{L}(X,Y)$ where Y is any maximal L-convex function space.

Let $X_0 \subset X$ be the closure of the simple functions. Then $X_0 \supset L(1,\infty)$. Indeed since $q_X < 1$, there exist r, $q_X < r < 1$ and $C < \infty$ so that

$$\|1_{[2^{-n},2^{-(n-1)}]}\| \le C2^{-n/r}.$$

Hence

$$\sum 2^{n}1_{[2^{-n},2^{-(n-1)}]} \in X_0.$$

Thus $T|X_0$ is controllable and hence there exist Borel maps a_n, σ_n satisfying (i), (a) and (b) so that

$$Tf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t) \qquad \text{a.e. } f \in L_\infty.$$

Now by combining Propositions 6.1 and 6.3, we can define a positive operator $P_0 : X_0 \to Y$ by

$$P_0 f(t) = \sum_{n=1}^{\infty} |a_n(t)| f(\sigma_n t) \qquad \text{a.e. } f \in L_\infty.$$

Now if $f \in X_+$ there exist $f_n \in L_\infty$ with $0 \leq f_n \uparrow f$ pointwise. As $P_0 f_n$ is monotone in Y, then $\sup_n P_0 f_n \in Y$. It follows quickly that $\sum_n |a_n(t)| f(\sigma_n t)$ converges a.e., and that we can define operators $P \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(X,Y)$ by

$$Pf(t) = \sum_{n=1}^{\infty} |a_n(t)| f(\sigma_n t) \qquad \text{a.e. } f \in X$$

$$Sf(t) = \sum_{n=1}^{\infty} a_n(t) f(\sigma_n t) \qquad \text{a.e. } f \in X.$$

Now $(T - S)(X_0) = 0$ so that $T = S$. Clearly $P \geq T$ and $P \geq -T$. If $Q \in \mathcal{L}(X,Y)$ and $Q \geq T$ and $Q \geq -T$ then clearly $Qf \geq Pf$ for $f \in (X_0)_+$. If $f \in X_+$ then choosing $f_n \in X_0$ with $0 \leq f_n \uparrow f$, we have

$$Pf = \sup_n Pf_n \leq \sup_n Qf_n \leq Qf.$$

Hence $P = |T|$ and $\mathcal{L}(X,Y)$ is a lattice.

Again the converse to Theorem 6.5 is "almost" true. First we shall need a lemma. If X is a symmetric function space then $X([0,1]^n)$ for $n \geq 1$ may be defined in the natural way and are isometrically lattice isomorphic to X. We shall say that $f \in L_0$ is an *independent multiplier* of X if the map $U_f : X[0,1]^2$ is bounded where $U_f g(s,t) = f(s)g(t)$.

Lemma 6.6. *If $f(s) = s^{-1}$ $(0 < s \leq 1)$ is an independent multiplier of X then $q_X < 1$.*

Proof. If $g \in X$ and $\|U_f\| \leq C$ then $\|g(s)t^{-1}u^{-1}\| \leq C^2 \|g\|$ where $g(s)t^{-1}u^{-1} \in X([0,1]^3)$. Now for $x \geq 1$,

$$(\lambda \times \lambda)\{t^{-1}u^{-1} \geq x\} = x^{-1}\left(\log \frac{1}{x} + 1\right).$$

Hence for $0 < t \leq 1$, and $x \geq 1$,

$$\|D_t\| \leq C^2 x^{-1}$$

where $t = x^{-1}(\log(1/x) + 1)$. Thus $\inf_{0<t<1} t^{-1} \|D_t\| = 0$ so that $q_X < 1$.

Theorem 6.7. *Let X be a D-convex symmetric function space for which $q_X < 2$. Then in order that every $T \in \mathcal{L}(X,L_0)$ is regular it is necessary and sufficient that $q_X < 1$.*

Proof. If $q_X \geq 1$ there exists $f \in X$ so that $g(s,t) = f(s)t^{-1} \notin X([0,1]^2)$. Hence there is an Orlicz function ϕ satisfying $\phi(sx) \leq s^r \phi(x)$ (where $r < 2$) for $s \geq 1$ and such that $g \notin L_\phi([0,1]^2)$ but $X \subset L_\phi$.

Now the operator $T_\phi : X \to L_0$ is regular and induces an operator $T'_\phi : X([0,1]^2) \to L_0$ by identifying the measure spaces $[0,1]$ and $[0,1]^2$. Define

$S : L_\infty [0,1] \to L$ by $Sh = T'_\phi (U_f h)$. Then S can be identified with an operator T_ψ where

$$\psi(x) = \int_0^1 \phi(x|f(t)|)\, dt.$$

But S must be controllable since if $|h| \leq 1$ a.e. then $|U_f h| \leq |f(s)|$ for all s,t and T'_ϕ is regular. Hence

$$\int_0^1 \psi(x^{-1})\, dx < \infty$$

or

$$\int_0^1 \int_0^1 \phi(|g(s,t)|)\, ds dt < \infty$$

contrary to assumption.

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