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EXAMPLES OF UNIFORMLY HOMEOMORPHIC BANACH SPACES

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ABSTRACT

We give several examples of separable Banach spaces which are non-isomorphic but uniformly homeomorphic. For example, we show that for every $1 < p \neq 2 < \infty$ there are two uniformly homeomorphic subspaces (respectively, quotients) of ℓ_p which are not linearly isomorphic; similarly c_0 has two uniformly homeomorphic subspaces which are not isomorphic. We also give an example of two non-isomorphic separable $\mathcal{L}_\infty$-spaces which are coarsely homeomorphic (i.e. have Lipschitz equivalent nets).

1. Introduction

The first example of two separable uniformly homeomorphic Banach spaces which are not linearly isomorphic was given in 1984 by Ribe [29]. Ribe's basic approach has been used subsequently to create further examples by Aharoni and Lindenstrauss [1] and also Johnson, Lindenstrauss and Schechtman [13]. The aim of this paper is to give further examples by exploiting the same basic technique.

We first develop an abstract approach to the construction of examples; this is not essentially new but gives an overall framework from which examples can be created easily.

Let us now list the examples that we are able to give by this method. We first show that for $1 < p < \infty$ we can find two spaces X and Y which are

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uniformly homeomorphic such that X is an ℓ_p-sum of finite-dimensional spaces and Y contains no subspace isomorphic to ℓ_p. Such an example is significant in that it demonstrates limitations on the preservation of asymptotic structure (see [9] for positive results in this direction). Thus X has a renorming with $\overline{\rho}_X(t) \approx \overline{\delta}_X(t) \approx t^p$ but Y cannot be renormed to have either $\overline{\rho}_Y(t) \approx t^p$ or $\overline{\delta}_Y(t) \approx t^p$. See also [21] for positive results on the uniqueness of uniform structure for certain ℓ_p-sums of finite-dimensional spaces.

Next we give an example of two $\mathcal{L}_\infty$-spaces which are coarsely homeomorphic (i.e. have Lipschitz equivalent nets) but not linearly isomorphic. A similar example of uniformly homeomorphic $\mathcal{L}_1$-spaces can be found in [20].

We also show that each of the spaces ℓ_p for $1 < p \neq 2 < \infty$ or c_0 has two non-isomorphic subspaces which are uniformly homeomorphic; the corresponding result for ℓ_1 is given in [20]. For $1 < p \neq 2 < \infty$, we also find two non-isomorphic quotients of ℓ_p which are uniformly homeomorphic.

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2. Preliminaries from linear and nonlinear Banach space theory

Our notation for Banach spaces is fairly standard (see, e.g., [23], [11] or [2]). All Banach spaces will be real. If X is a Banach space, B_X denotes its closed unit ball and ∂B_X the unit sphere $\{x : \|x\| = 1\}$.

We recall that a separable Banach space X has the **approximation property (AP)** if for every compact subset K of X and every $\epsilon > 0$ there is a bounded finite-rank linear operator $T : X \to X$ with $\|Tx - x\| < \epsilon$ for $x \in K$; X has the **bounded approximation property (BAP)** if there is a bounded sequence of finite-rank operators $T_n : X \to X$ such that $\lim_{n\to\infty} T_n x = x$ for every $x \in X$; X has the **metric approximation property (MAP)** if we can also impose the condition $\|T_n\| \leq 1$. If X^* is separable and, additionally, $\lim_{n\to\infty} T_n^* x^* = x^*$ for every $x^* \in X^*$, we say that X has **shrinking** (BAP) or (MAP); X has a **finite-dimensional decomposition (FDD)** if there is a sequence of finite-rank operators $P_n : X \to X$ such that $P_m P_n = 0$ when

$m \neq n$ and $x = \sum_{n=1}^{\infty} P_n x$ for every $x \in X$. If $\|P_n\| \leq 1$ for all n, the (FDD) is said to be **monotone**. If each P_n has rank one, then X has a **basis.** The (FDD) is called **shrinking** if we also have $x^* = \sum_{n=1}^{\infty} P_n^* x^*$ for every $x^* \in X^*$. If, in addition $x = \sum_{n=1}^{\infty} P_n x$ unconditionally for every $x \in X$, then X has an **unconditional finite-dimensional decomposition (UFDD).** Finally, if $\| \sum_{k=1}^{n} \eta_k P_k \| \leq 1$ for every $n \in \mathbb{N}$ and $\eta_k = \pm 1$ for $1 \leq k \leq n$, then we say that X has a **1-(UFDD)**.

We now discuss asymptotic uniform smoothness and asymptotic uniform convexity. Let X be a separable Banach space. We define the **modulus of asymptotic uniform smoothness** (due to Milman [25]) $\overline{\rho}(t) = \overline{\rho}_X(t)$ by

$$\overline{\rho}(t) = \sup_{x \in \partial B_X} \inf_{E} \sup_{y \in \partial B_E} \{\|x + ty\| - 1\},$$

where E runs through all closed subspaces of X of finite codimension.

The **modulus of asymptotic uniform convexity** is defined by

$$\overline{\delta}(t) = \inf_{x \in \partial B_X} \sup_{E} \inf_{y \in \partial B_E} \{\|x + ty\| - 1\},$$

where E runs through all closed subspaces of X of finite codimension.

We will also use the language of short exact sequences. If $Q : Y \to X$ is a quotient map, then Q induces a short exact sequence

$$\mathcal{S} = 0 \to E \to Y \to X \to 0, \quad \text{where } E = \ker Q.$$

Then $\mathcal{S}$ (or Q) **splits** if there is a projection $P : Y \to E$ or equivalently a linear operator $L : X \to Y$ so that $QL = Id_X$ where $Q : Y \to X$ is the quotient map; $\mathcal{S}$ (or Q) **locally splits** if the dual sequence $0 \to X^* \to Y^* \to E^* \to 0$ splits, or equivalently, if there is a constant $\lambda \geq 1$ so that for every finite-dimensional subspace F of X there is a linear operator $L_F : F \to Y$ with $\|L_F\| \leq \lambda$ and $QL_F = Id_F$. The subspace E is **locally complemented** in Y if there exists a linear operator $L : Y \to E^{**}$ such that $L|_E = Id_E$. Then E is locally complemented in Y if and only if $\mathcal{S}$ locally splits.

We shall frequently deal with ℓ_p-sums of Banach spaces $(X_n)_{n=1}^{\infty}$. We denote by $(\sum_{n=1}^{\infty} X_n)_{\ell_p}$ the space of sequences $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$ and

$$\|(x_n)_{n=1}^{\infty}\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty.$$

Let us now recall in particular the spaces C_p introduced by Johnson [10] and later studied by Johnson and Zippin, [14] and [15]. Let $(G_n)_{n=1}^{\infty}$ be a sequence of finite-dimensional Banach spaces dense in all finite-dimensional Banach spaces for the Banach–Mazur distance. Then we define $C_p = (\sum_{n=1}^{\infty} G_n)_{\ell_p}$; this space is unique up to almost isometry.

A Banach space X is called an $\mathcal{L}_\infty$-space if there is some λ so that for any finite-dimensional space $F \subset X$ there is a finite-dimensional subspace G containing F with $d(G, \ell_\infty^n) \le \lambda$, where $n = \dim G$ and d denotes the Banach–Mazur distance. If X is separable, this is equivalent to the fact that X^* is isomorphic to ℓ_1.

We refer to [3] and [18] for background on nonlinear theory.

Let (M, d) and (M', d') be metric spaces. If $f : M \to M'$ is any mapping we define $\omega_f : [0, \infty) \to [0, \infty]$ by

$$\omega(f; t) = \omega_f(t) := \sup\{d'(f(x), f(y));\ d(x, y) \le t\};$$

f is Lipschitz if $\omega_f(t) \le ct$ for some constant c and **contractive** if $c \le 1$; f is said to be **uniformly continuous** if $\lim_{t\to 0} \omega_f(t) = 0$ and **coarsely continuous** if $\omega_f(t) < \infty$ for every $t > 0$. We also say that f is **coarse Lipschitz** if for some $t_0 > 0$ we have an estimate

$$\omega_f(t) \le ct, \quad t \ge t_0.$$

The notions of coarsely continuous and coarse Lipschitz maps are nontrivial if and only if the metric d' on M' is unbounded. See for example [18].

It will be useful to track the constants for a coarse Lipschitz map. We will say that a map $f : M \to M'$ is of **CL-type** (L, ϵ) if we have an estimate

$$\omega_f(t) \le Lt + \epsilon, \quad t \ge 0.$$

We say that a map $f : M \to M'$ is a **uniform homeomorphism** if f is a bijection and f and f^{-1} are both uniformly continuous. A bijection $f : M \to M'$ is a **coarse homeomorphism** if and only if f and f^{-1} are coarsely continuous. A bijection $f : M \to M'$ is a **coarse Lipschitz homeomorphism** if and only if f and f^{-1} are coarse Lipschitz. In this case we say that f is a **CL-homeomorphism** of type (L, ϵ) if both f and f^{-1} are of CL-type (L, ϵ). Finally, we shall say M and M' are **almost Lipschitz isomorphic** if for some L and every $\epsilon > 0$ there is a CL-homeomorphism $f : M \to M'$ of type (L, ϵ).

If X and Y are Banach spaces, we have that if $f : X \to Y$ is a uniform homeomorphism or a coarse homeomorphism then f is a coarse Lipschitz homeomorphism [18]. In fact X and Y are coarsely homeomorphic if and only if they have Lipschitz equivalent nets.

3. Spaces of homogeneous maps

In this section we develop some ideas first explored in [20].

Let X and Y be Banach spaces. We define $\mathcal{H}(X,Y)$ to be the space of all maps $f : X \to Y$ which are positively homogeneous, i.e.

$$f(\alpha x) = \alpha f(x), \quad x \in X,\ \alpha \geq 0,$$

and bounded, i.e.

$$\|f\| = \sup\{\|f(x)\| :\ \|x\| \leq 1\} < \infty.$$

It is clear that $\mathcal{H}(X,Y)$ is a Banach space with this norm containing the space $\mathcal{L}(X,Y)$ of all bounded linear operators. We define the subspace $\mathcal{HU}(X,Y)$ as the set of f such that the restriction of f to B_X (and hence to any bounded set) is uniformly continuous.

If $0 < \epsilon < 1$, for $f \in \mathcal{H}(X,Y)$ we define $\|f\|_\epsilon$ to be the least constant L so that

$$\|f(x) - f(x')\| \leq L \max\{\|x - x'\|, \epsilon\|x\|, \epsilon\|x'\|\}, \quad x, x' \in X.$$

It is easy to see that for each $\epsilon > 0$, $\|\cdot\|_\epsilon$ is a norm on $\mathcal{H}(X,Y)$ which is equivalent to the original norm; precisely

$$\|f\| \leq \|f\|_\epsilon \leq 2\epsilon^{-1}\|f\|, \quad f \in \mathcal{H}(X,Y).$$

Observe that $\|f\|_\epsilon$ is decreasing in ϵ and $\sup_{\epsilon>0} \|f\|_\epsilon < \infty$ if and only if f is a Lipschitz map.

We will need the following Lemma, whose proof we omit.

Lemma 3.1: *Let X be a Banach space and suppose $x, z \in X$ with $\|x\| \geq \|z\| > 0$. Then*

$$\left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| \leq 2\frac{\|x - z\|}{\|x\|}$$

and

$$\|x - z\| \leq \|x\| - \|z\| + \|z\| \left\| \frac{x}{\|x\|} - \frac{z}{\|z\|} \right\| \leq 3\|x - z\|.$$

The following result is proved in [20]:

PROPOSITION 3.2: *Suppose $f \in \mathcal{H}(X,Y)$ and $\varphi = f|_{\partial B_X}$. Then:*

(i) *If φ is of CL-type (L,ϵ) where $L \geq 1$, $\epsilon > 0$ and $\|\varphi(x)\| \leq K$ for $x \in \partial B_X$, then we have $\|f\|_\epsilon \leq 2K + 4L$.*

(ii) *If $\|f\|_\epsilon = L$, then φ is of CL-type $(L, L\epsilon)$.*

We will also need the following elementary Lemma about compositions, whose proof we omit:

LEMMA 3.3: *Suppose X,Y,Z are Banach spaces and $f \in \mathcal{H}(X,Y)$ and $g \in \mathcal{H}(Y,Z)$. Then for $0 < \epsilon < 1$ we have*

$$\|gf\|_\epsilon \leq \|g\|_\epsilon \|f\|_\epsilon.$$

LEMMA 3.4: *Suppose X,Y,Z and W are Banach spaces and $f \in \mathcal{H}(X,Y)$, $g \in \mathcal{H}(Z,W)$. Suppose $h_1, h_2 \in \mathcal{H}(Y,Z)$. Let $\omega(t) = \omega(g|_{B_Z}; t)$. Then if $K = \max(\|h_1\|, \|h_2\|)$,*

$$\|gh_1f - gh_2f\| \leq K\|f\|\omega\Big(\frac{\|h_1 - h_2\|}{K}\Big).$$

In particular, if $\epsilon > 0$,

$$\|gh_1f - gh_2f\| \leq \|g\|_\epsilon \|f\|(\|h_1 - h_2\| + K\epsilon).$$

Proof. If $\|x\| = 1$,

$$\|h_1(f(x)) - h_2(f(x))\| \leq \|f\|\|h_1 - h_2\|.$$

Let $K = \max(\|h_1\|, \|h_2\|)$. Then

$$\|gh_1f(x) - gh_2f(x)\| \leq K\|f\|\omega\Big(\frac{\|h_1 - h_2\|}{K}\Big). \qquad \blacksquare$$

The next Lemma is proved in [20] (Lemma 7.4).

LEMMA 3.5: *Let X and Y be Banach spaces and suppose $t \to f_t$ is a map from $[0,\infty)$ into $\mathcal{H}(X,Y)$ with the property that for some constant K we have:*

$$\|f_t\|_{e^{-2t}} \leq K, \quad t \geq 0, \tag{3.1}$$

and

$$\|f_t - f_s\| \leq K(|t-s| + e^{-2t} + e^{-2s}), \quad t,s \geq 0. \tag{3.2}$$

Define $F : X \to Y$

$$F(x) = \begin{cases} f_0(x), & \|x\| \le 1, \\ f_{\log \|x\|}(x), & \|x\| > 1. \end{cases}$$

Then F is coarsely continuous.

If further for every $t \ge 0$, $f_t \in \mathcal{HU}(X,Y)$ and the map $t \mapsto f_t$ is continuous, then F is uniformly continuous.

Now let $\mathcal{G}(X,Y)$ be the subset of $\mathcal{H}(X,Y)$ of all f such that f is a bijection and $f^{-1} \in \mathcal{H}(Y,X)$. Let $\mathcal{GU}(X,Y)$ be the subset of $\mathcal{G}(X,Y)$ of all f such that $f \in \mathcal{HU}(X,Y)$ and $f^{-1} \in \mathcal{HU}(Y,X)$. For $f \in \mathcal{G}(X,Y)$ we will define

$$[[f]] = \max(\|f\|, \|f^{-1}\|), \quad [[f]]_\epsilon = \max(\|f\|_\epsilon, \|f^{-1}\|_\epsilon).$$

We also define a metric Δ on $\mathcal{G}(X,Y)$ by

$$\Delta(f,g) = \max(\|f-g\|, \|f^{-1} - g^{-1}\|).$$

We say that X and Y are **L-close,** respectively **uniformly L-close** for some $L \ge 1$, if for every $\epsilon > 0$ we can find $f \in \mathcal{G}(X,Y)$ (respectively $f \in \mathcal{GU}(X,Y)$) with $[[f]]_\epsilon \le L$; X and Y are (uniformly) close if there exists L such that they are (uniformly) L-close.

To show that this definition is reasonable we need the following Lemma, which follows immediately from Lemma 3.3:

Lemma 3.6: *If X and Y are (respectively, uniformly) close and Y and Z are (respectively, uniformly) close, then X and Z are (respectively, uniformly) close.*

Lemma 3.7: *Let $(X_n)_{n=1}^\infty$, $(Y_n)_{n=1}^\infty$ be two sequences of Banach spaces such that for some L, X_n and Y_n are L-close (respectively, uniformly L-close). Then for $1 \le p < \infty$, $(\sum_{n=1}^\infty X_n)_{\ell_p}$ and $(\sum_{n=1}^\infty Y_n)_{\ell_p}$ are close (respectively, uniformly close); similarly $(\sum_{n=1}^\infty X_n)_{c_0}$ and $(\sum_{n=1}^\infty Y_n)_{c_0}$ are close (respectively, uniformly close). In particular, if X and Y are (uniformly) close then $\ell_p(X)$ and $\ell_p(Y)$ are (uniformly) close.*

Proof. We will treat the uniformly continuous case only and restrict to the more difficult case of $1 \le p < \infty$. The other cases are simpler. For any $\epsilon > 0$ we may pick a sequence (ϵ_n) with $\lim_{n\to\infty} \epsilon_n = 0$ and $\epsilon_n \le \epsilon$ for all n. Pick $f_n \in \mathcal{GU}(X_n, Y_n)$, with $[[f_n]]_{\epsilon_n} \le L$. Define $f : (\sum_{n=1}^\infty X_n)_{\ell_p} \to (\sum_{n=1}^\infty Y_n)_{\ell_p}$

by $f((x_n)_{n=1}^\infty) = (f_n(x_n))_{n=1}^\infty$. Suppose $x = (x_n)_{n=1}^\infty \in (\sum_{n=1}^\infty X_n)_{\ell_p}$ and $x' = (x'_n)_{n=1}^\infty \in (\sum_{n=1}^\infty X_n)_{\ell_p}$. Then

$$\begin{aligned} \|f(x) - f(x')\|^p &\leq L^p \sum_{n=1}^\infty \max(\|x_n - x'_n\|^p, \epsilon^p\|x_n\|^p, \epsilon^p\|x'_n\|^p) \\ &\leq L^p(\|x - x'\|^p + \epsilon^p(\|x\|^p + \|x'\|^p)) \\ &\leq 3L^p \max(\|x - x'\|^p, \epsilon^p\|x\|^p, \epsilon^p\|x'\|^p). \end{aligned}$$

Combining with the estimate for f^{-1} we have $[[f]]_\epsilon \leq 3L$.

To check uniform continuity for f and f^{-1} we note that

$$\omega(f_n|_{B_{X_n}}; t), \omega(f_n^{-1}|_{B_{Y_n}}; t) \leq Lt + \epsilon_n$$

and hence if

$$\omega(t) = \sup_n \max(\omega(f_n|_{B_{X_n}}; t), \omega(f_n^{-1}|_{B_{Y_n}}; t))$$

we have $\lim_{t\to 0} \omega(t) = 0$.

We only need to show that f is uniformly continuous on the surface of the unit ball $(\sum_{n=1}^\infty X_n)_{\ell_p}$. Suppose $\eta > 0$. Pick $\nu > 0$ so that $3L\sqrt{\nu} + \omega(2\sqrt{\nu}) \leq \eta$. Suppose $x, x' \in \partial B_{(\sum_{n=1}^\infty X_n)_{\ell_p}}$ are such that $\|x - x'\| < \nu$. We will show that $\|f(x) - f(x')\| < \eta$.

Let $\mathbb{A} = \{n : \|x_n - x'_n\| > \sqrt{\nu}\max(\|x_n\|, \|x'_n\|)\}$. Then

$$\Big(\sum_{n\in\mathbb{A}} \max(\|x_n\|, \|x'_n\|)^p\Big)^{1/p} \leq \sqrt{\nu}$$

and so

$$\Big(\sum_{n\in\mathbb{A}} \|f_n(x_n)\|^p\Big)^{1/p} \leq L\sqrt{\nu}$$

and

$$\Big(\sum_{n\in\mathbb{A}} \|f_n(x'_n)\|^p\Big)^{1/p} \leq L\sqrt{\nu}.$$

On the other hand, if $n \notin \mathbb{A}$ we have

$$\|f_n(x_n) - f_n(x'_n)\| \leq L\|x_n - x'_n\| + \min(\|x_n\|, \|x'_n\|)\omega\Big(\frac{2\|x_n - x'_n\|}{\max(\|x_n\|, \|x'_n\|)}\Big).$$

Thus

$$\|f_n(x_n) - f_n(x_n')\| \le L\|x_n - x_n'\| + \omega(2\sqrt{\nu})\min(\|x_n\|, \|x_n'\|), \quad n \notin \mathbb{A}.$$

Hence

$$\Big(\sum_{n\notin\mathbb{A}} \|f_n(x_n) - f_n(x_n')\|^p\Big)^{1/p} \le L\nu + \omega(2\sqrt{\nu}).$$

Combining

$$\|f(x) - f(x')\| \le 2L\sqrt{\nu} + L\nu + \omega(2\sqrt{\nu}) < \eta.$$

This and a similar calculation for f^{-1} give the conclusion that $f \in \mathcal{GU}(X,Y)$ and we already know that $[[f]]_\epsilon \le 3L$. ∎

We next introduce a normalized version of $f \in \mathcal{G}(X,Y)$. We define

$$\hat{f}(x) = \begin{cases} 0, & x = 0, \\ \|x\|\frac{f(x)}{\|f(x)\|}, & x \neq 0. \end{cases}$$

It is clear that $\hat{f} \in \mathcal{G}(X,Y)$ and is norm-preserving; furthermore, if $f \in \mathcal{GU}(X,Y)$ then $\hat{f} \in \mathcal{GU}(X,Y)$. We give however a simple quantitative estimate:

Lemma 3.8: (i) *If $f \in \mathcal{G}(X,Y)$, then $\hat{f} \in \mathcal{G}(X,Y)$ and*

$$[[\hat{f}]]_\epsilon \le 2[[f]]_\epsilon^2 + 1.$$

(ii) *If $f, g \in \mathcal{G}(X,Y)$, then*

$$\Delta(\hat{f}, \hat{g}) \le 2[[f]]\Delta(f,g).$$

Proof. (i) If $x, x' \in X$ are both nonzero and $\|x\| \ge \|x'\|$, then, using Lemma 3.1,

$$\begin{aligned}\|\hat{f}(x) - \hat{f}(x')\| &\le \|x\| - \|x'\| + \|x\|\Big\|\frac{f(x)}{\|f(x)\|} - \frac{f(x')}{\|f(x')\|}\Big\| \\ &\le \|x - x'\| + 2\|x\|\frac{\|f(x) - f(x')\|}{\|f(x)\|} \\ &\le \|x - x'\| + 2[[f]]_\epsilon\|f(x) - f(x')\| \\ &\le (2[[f]]_\epsilon^2 + 1)\max(\|x - x'\|, \epsilon\|x\|, \epsilon\|x'\|).\end{aligned}$$

A similar calculation can be made with $\hat{f}^{-1} = \widehat{f^{-1}}$.

(ii) If $\|x\| = 1$ we have, again using Lemma 3.1,

$$\|\hat{f}(x) - \hat{g}(x)\| \le 2\frac{\|f(x) - g(x)\|}{\|f(x)\|} \le 2\|f^{-1}\|\|f(x) - g(x)\|.$$

Combining with a similar estimate on the inverses gives the conclusion. ∎

In view of Lemma 3.8 we can rephrase the notion of closeness in terms of almost Lipschitz isomorphisms.

Proposition 3.9: *In order that two Banach spaces X and Y be (uniformly) close, it is necessary and sufficient that ∂B_X and ∂B_Y be (uniformly) almost Lipschitz isomorphic.*

Proof. We treat the uniform case. Suppose X and Y are uniformly close. For some fixed L and all $\epsilon > 0$ there exists $f = f_\epsilon \in \mathcal{GU}(X, Y)$, with $[[f]]_\epsilon \le L$. Then by Lemma 3.8, $[[\hat{f}]]_\epsilon \le 2L^2 + 1$, and by Proposition 3.2, $\hat{f}|_{\partial B_X}$ is a uniform homeomorphism onto ∂B_Y such that $\hat{f}$ and $\hat{f}^{-1}$ both have CL-type $(2L^2 + 1, (2L^2 + 1)\epsilon)$.

Conversely, if $\varphi : \partial B_X \to \partial B_Y$ is a uniform homeomorphism so that φ and φ^{-1} have CL-type (L, ϵ), then the homogeneous extension f of φ belongs to $\mathcal{GU}(X, Y)$, and $[[f]]_\epsilon \le 4L + 2$. ∎

4. Uniformly close and uniformly homeomorphic spaces

We first note a connection between uniform closeness and uniform homeomorphisms essentially due to Nahum [26].

Theorem 4.1: *Let X and Y be coarsely (respectively uniformly) homeomorphic Banach spaces. Then $X \oplus \mathbb{R}$ and $Y \oplus \mathbb{R}$ are close (respectively uniformly close).*

Hence if X and Y are coarsely homeomorphic (respectively uniformly homeomorphic) Banach spaces linearly isomorphic to their hyperplanes, then X and Y are close (respectively uniformly close).

Proof. We prove only the uniform homeomorphism case. Let $\psi : X \to Y$ be a uniform homeomorphism. Then we have estimates

$$\omega(\psi; t), \omega(\psi^{-1}; t) \le Lt + C, \qquad t > 0.$$

In the proof given in [3], p. 211 it is shown that there exists a constant L_1

depending only on L and the spaces X and Y and a bijection

$$\varphi : \partial B_{X\oplus_\infty \mathbb{R}} \to \partial B_{Y\oplus_\infty \mathbb{R}}$$

such that

$$\omega(\varphi;t) \le L_1\omega(\psi;t), \quad \omega(\varphi^{-1};t) \le L_1\omega(\psi^{-1};t).$$

If we replace ψ by $\psi_n : X \to Y$ defined by $\psi_n(x) = n^{-1}\psi(nx)$, we obtain φ_n with

$$\omega(\varphi_n;t) \le L_1 n^{-1}\omega(\psi;nt) \le L_1Lt + L_1Cn^{-1},$$

and

$$\omega(\varphi_n^{-1};t) \le L_1 n^{-1}\omega(\psi^{-1};t) \le L_1Lt + L_1Cn^{-1}.$$

This implies the theorem using Proposition 3.9. ∎

The Ribe construction [29], later developed in [1] and [13] and given in [3] may be regarded as giving a form of converse to Nahum's result.

Proposition 4.2: *Let X and Y be Banach spaces. In order that X and Y be coarsely homeomorphic, it is sufficient that there exist a constant L and maps $f_t \in \mathcal{G}(X,Y)$ for $t \ge 0$ such that we have:*

$$[[f_t]]_{e^{-2t}} \le L, \quad t \ge 0,$$

and

$$\Delta(f_t, f_s) \le L(|t-s| + e^{-2t} + e^{-2s}), \quad t,s \ge 0.$$

If, in addition, each $f_t \in \mathcal{GU}(X,Y)$ for all $t \ge 0$ and the maps $t \mapsto f_t$ and $t \mapsto f_t^{-1}$ are continuous, then X and Y are uniformly homeomorphic.

Proof. It is a consequence of Lemma 3.8, that if we define $g_t = \hat{f}_t$, then g_t obeys the same conditions with the constant L replaced by $2L^2+1$. Hence we may assume that f_t is norm preserving for each t. Then Lemma 3.5 allows one to build a coarsely continuous map $F : X \to Y$. It follows from the same Lemma applied to f_t^{-1} that F^{-1} is also coarsely continuous. It also follows that if each $f_t \in \mathcal{GU}(X,Y)$ and the maps $t \mapsto f_t, t \mapsto f_t^{-1}$ are continuous, then F and F^{-1} are uniformly continuous. ∎

Let us now state our main result on the construction of coarse and uniform homeomorphisms, which is a reworking of the the arguments from [3].

Theorem 4.3: *Let X, Y and Z be three Banach spaces such that X and $X \oplus Y$ are close (respectively, uniformly close) and Y and $Y \oplus Z$ are linearly isomorphic. Then X^2 and $X^2 \oplus Z$ are coarsely (respectively, uniformly) homeomorphic.*

Proof. We treat all direct sums as ℓ_∞-sums for convenience. As usual we will only do the details for the uniform case.

Let us fix a constant L large enough so that for every $\epsilon > 0$ we can find $f \in \mathcal{GU}(X, X \oplus Y)$ with $[[f]]_\epsilon \le L$, and such that there is a linear bijection $T : Y \to Y \oplus Z$ with $\|T\|, \|T^{-1}\| \le L$. We represent T in the form $Ty = (T_Y y, T_Z y)$.

Now suppose $g : X \to X \oplus Y$ is a bijection and $g \in \mathcal{GU}(X, X \oplus Y)$ with $[[g]]_\epsilon \le L$. Let $g(x) = (g_X(x), g_Y(x))$. We define $\phi_g \in \mathcal{G}(X, X \oplus Z)$ by

$$\phi_g(x) = \big(g^{-1}(g_X(x), T_Y(g_Y(x))), T_Z(g_Y(x))\big).$$

We can express ϕ_g as a composition of

$$X \xrightarrow{g} X \oplus Y \xrightarrow{I \oplus T} X \oplus Y \oplus Z \xrightarrow{g^{-1} \oplus I} X \oplus Z.$$

It follows from Lemma 3.3 that $[[\phi_g]]_\epsilon \le L^3$.

Lemma 4.4: *Suppose $g, h : X \to X \oplus Y$ and that $g, h \in \mathcal{GU}(X, X \oplus Y)$, with*

$$\max([[g]]_\epsilon, [[h]]_\epsilon) \le L.$$

Then there is a family of bijections $f_t \in \mathcal{GU}(X^2, X^2 \oplus Z)$ for $0 \le t \le 1$ so that $t \mapsto f_t$ and $t \mapsto f_t^{-1}$ are continuous,

$$[[f_t]]_\epsilon \le 2L^3,$$

$$\Delta(f_t, f_s) \le 10L^3|t - s| + 6L^3\epsilon, \quad 0 \le s, t \le 1,$$

$$f_0(x_1, x_2) = (x_1, \phi_g x_2)$$

and

$$f_1(x_1, x_2) = (x_1, \phi_h x_2).$$

Proof. Let us formulate a principle which we will use repeatedly.

Linking Principle. We will use the following principle several times. Let W be any Banach space. Let $J = J_W : W \oplus W \to W \oplus W$ be the interchange operator defined by $J(w_1, w_2) = (w_2, -w_1)$. We let

$$\Psi(\theta)(w_1, w_2) = (w_1 \cos\theta + w_2 \sin\theta, w_2 \cos\theta - w_1 \sin\theta).$$

Then the maps $\theta \to \Psi(\theta)$ and $\theta \to \Psi(\theta)^{-1}$ from $[0,\pi/2] \to \mathcal{L}(W \oplus W)$ are Lipschitz with constant $\sqrt{2}$. Furthermore, $\|\Psi(\theta)\|, \|\Psi(\theta)^{-1}\| \leq \sqrt{2}$ for all $0 \leq \theta \leq \pi/2$. Clearly a similar family of maps joins the identity to $-J = J^{-1}$.

We observe first that f_0 can be obtained as the composition of the maps

$$F_1 : X^2 \to (X \oplus Y)^2,\ F_2 : (X \oplus Y)^2 \to (X \oplus Y)^2 \oplus Z,$$

$$F_3 : (X \oplus Y)^2 \oplus Z \to X^2 \oplus Z,$$

where

$$F_1(x_1,x_2) = (h(x_1), g(x_2)) = (h_X(x_1), h_Y(x_1), g_X(x_2), g_Y(x_2)),$$

$$F_2(x_1,y_1,x_2,y_2) = (x_1,y_1,x_2,Ty_2) = (x_1,y_1,x_2,T_Y y_2, T_Z y_2),$$

and

$$F_3(x_1,y_1,x_2,y_2,z) = (h^{-1}(x_1,y_1), g^{-1}(x_2,y_2), z).$$

Thus $f_0 = F_3F_2F_1$. Now $f_1 = J_4F_3J_3F_2J_2F_1J_1$ where

$$J_1(x_1,x_2) = (x_2,-x_1),$$

$$J_2(x_1,y_1,x_2,y_2) = (x_1,-y_2,x_2,y_1),$$

$$J_3(x_1,y_1,x_2,y_2,z) = (x_1,y_2,x_2,-y_1,z)$$

and $J_4 = -J_1 \oplus I_Z$.

Note that $[[F_i]]_\epsilon \leq L$ $(i = 1,2,3)$, while J_1, J_2, J_3, J_4 are isometries.

Let us define $f_{1/4} = F_3F_2F_1J_1$, $f_{1/2} = F_3F_2J_2F_1J_1$ and $f_{3/4} = F_3J_3F_2J_2F_1J_1$. Now use the Linking Principle on each interval $[(j-1)/4, j/4]$ for $j = 1,2,3,4$ we can construct f_t for $0 \leq t \leq 1$. On the interval $[(j-1)/4, j/4]$, f_t takes the form $f_t = G_j\Phi(t)H_j$ where $\Phi(t)$ are linear maps and verify the estimates

$$\max(\|\Phi(t)-\Phi(s)\|, \|\Phi(t)^{-1}-\Phi(s)^{-1}\|) \leq 2\sqrt{2}\pi|t-s| \leq 10|t-s|$$

and

$$\max(\|\Phi(t)\|, \|\Phi(t)^{-1}\|) \leq \sqrt{2}.$$

Further, we have

$$[[G_j]]_\epsilon [[H_j]]_\epsilon \leq L^3.$$

This implies that

$$[[f_t]] \leq 2L^3$$

and also, using Lemma 3.4,

$$\|f_t - f_s\| \leq \|H_j\|(10\|G_j\|_\epsilon |t-s| + \sqrt{2}\epsilon), \quad (j-1)/4 \leq s,t \leq j/4$$

and

$$\|f_t^{-1} - f_s^{-1}\| \le \|G_j^{-1}\|(10\|H_j^{-1}\|_\epsilon |t-s| + \sqrt{2}\epsilon), \quad (j-1)/4 \le s,t \le j/4.$$

Combining these give

$$\Delta(f_t, f_s) \le 10L^3|t-s| + \sqrt{2}L^3\epsilon, \quad (j-1)/4 \le s,t \le j/4.$$

This implies that

$$\Delta(f_t, f_s) \le 10L^3|t-s| + 4\sqrt{2}L^3\epsilon \le 10L^3|t-s| + 6L^3\epsilon, \quad 0 \le s,t \le 1.$$

We also note that Lemma 3.4 implies the continuity of the map $t \mapsto f_t$. ∎

The proof of the Theorem is now completed easily. We may make a sequence $(g_n)_{n=0}^\infty \in \mathcal{GU}(X, X \oplus Y)$ with $[[g_n]]_{e^{-2n-2}} \le L$. Then using Lemma 4.4 we can make a family $f_t \in \mathcal{GU}(X^2, X^2 \oplus Z)$ with $f_n = I \oplus \phi_{g_n}$,

$$[[f_t]]_{e^{-2t}} \le 2L^3,$$

$$\Delta(f_t, f_s) \le 10L^3|t-s| + 12L^3(e^{-2t} + e^{-2s}), \quad 0 \le s,t < \infty$$

and such that $t \mapsto f_t$ and $t \mapsto f_t^{-1}$ are continuous. We can now complete the proof by Proposition 4.2. ∎

THEOREM 4.5: *If X and Y are close (respectively uniformly close) and we also have $X \approx X^2$ and $Y \approx Y^2$, then X is coarsely (respectively, uniformly) homeomorphic to Y.*

Proof. (The uniform case.) We have that X is uniformly close to Y and hence $X \oplus Y$ is uniformly close to $X \oplus X$ and thus to X. If we take $Z = Y$ in Theorem 4.3 we obtain that X^2 is uniformly homeomorphic to $X^2 \oplus Y$, i.e. X is uniformly homeomorphic to $X \oplus Y$. The proof is completed by reversing the roles of X and Y. ∎

Part (ii) of the following result ought to have an easier direct proof, but we do not see it except in the case of c_0-products!

THEOREM 4.6: (i) *Let X and Y be close (respectively, uniformly close). Then for any $1 \le p < \infty$, $\ell_p(X)$ and $\ell_p(Y)$ are coarsely (respectively, uniformly) homeomorphic. Similarly, $c_0(X)$ and $c_0(Y)$ are coarsely (respectively, uniformly) homeomorphic.*

(ii) *Let X and Y be coarsely (respectively, uniformly) homeomorphic. Then for any $1 \leq p < \infty$, $\ell_p(X)$ and $\ell_p(Y)$ are coarsely (respectively, uniformly) homeomorphic. Similarly, $c_0(X)$ and $c_0(Y)$ are coarsely (respectively, uniformly) homeomorphic.*

Proof. We treat the uniform case for $1 \leq p < \infty$ only.

(i) By Lemma 3.7, $\ell_p(X)$ and $\ell_p(Y)$ are uniformly close. The result then follows by Theorem 4.5.

(ii) By Theorem 4.1, $X \oplus \mathbb{R}$ and $Y \oplus \mathbb{R}$ are uniformly close. Hence by (i) $\ell_p(X \oplus \mathbb{R})$ and $\ell_p(Y \oplus \mathbb{R})$ are uniformly homeomorphic. But these spaces are isomorphic to $\ell_p(X)$ and $\ell_p(Y)$ respectively. ∎

We also have the following:

Theorem 4.7: *Suppose X and Y are close (respectively, uniformly close) Banach spaces and X is linearly isomorphic to $\ell_p(X)$ where $1 \leq p < \infty$ or to $c_0(X)$. Then Y^2 is coarsely (respectively, uniformly) homeomorphic to X.*

Proof. We treat the uniform case. In this case, by Theorem 4.6, X is uniformly homeomorphic to $\ell_p(Y)$ and hence to $X \oplus Y^2$.

On the other hand, Y is uniformly close to X and hence to X^2 and thus to $X \oplus Y$. Since X^2 is isomorphic to X^3, Theorem 4.3 gives that Y^2 and $Y^2 \oplus X$ are uniformly homeomorphic. ∎

5. Asymptotic smoothness and convexity

In this section we construct an example to show that the moduli of asymptotic smoothness and convexity are not preserved under uniform homeomorphism. In fact, for $1 < p < \infty$, we give examples of ℓ_p-sums of finite-dimensional spaces which are uniformly homeomorphic to spaces not containing an isomorphic copy of ℓ_p. To achieve this we first discuss the results of Odell and Schlumprecht [27] on uniform homeomorphisms between the unit balls of Banach spaces with unconditional bases. Most of our discussion is, at least implicitly, known and can be deduced from work in [27], [17], [22] and [7].

We work with sequence spaces, i.e. linear subspaces of the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences. We will use the term Banach sequence space to denote a Banach space, $(X, \|\cdot\|_X)$, of sequences with a norm so that the canonical basis $(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis of X. For $x = (x_i)_{i=1}^{\infty} \in X$, we denote

$|x| = (|x_i|)_{i=1}^\infty \in X$. Then, X is said to be p**-convex** with constant M for $1 < p < \infty$ if

$$\left\| \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \right\|_X \leq M \left(\sum_{j=1}^n \|x_j\|_X^p \right)^{1/p}, \quad x_1, \dots, x_n \in X$$

and **strictly** p**-convex** if $M = 1$; X is q**-concave** with constant M for $1<q<\infty$ if

$$\left(\sum_{j=1}^n \|x_j\|_X^q \right)^{1/q} \leq M \left\| \left(\sum_{j=1}^n |x_j|^q \right)^{1/q} \right\|_X$$

and **strictly** q**-concave** if $M = 1$. If X is strictly p-convex and strictly q-concave where $1 < p \leq q < \infty$, then X is uniformly convex and uniformly smooth ([24] p. 80). In this case X^* can be identified as a Banach sequence space which is strictly q'-convex and strictly p'-concave where p', q' are conjugate to p, q. It should also be mentioned that if X is only p-convex and q-concave for $1 \leq p \leq q \leq \infty$, then it can be equivalently renormed to be a strictly p-convex and strictly q-concave Banach sequence space (see [24] p. 54).

If $\mathbb{A}$ is an infinite subset of $\mathbb{N}$, we denote by $X(\mathbb{A})$ the sequence space obtained by considering the basis $(e_j)_{j\in\mathbb{A}}$.

If X_0 and X_1 are two Banach sequence spaces, then, for $0 < \theta < 1$, the Calderon space $X_\theta = X_0^{1-\theta} X_1^\theta$ is described as the space of all x so that $|x| = |x_0|^{1-\theta}|x_1|^\theta$ with $x_0 \in X_0,\ x_1 \in X_1$ under the norm

$$\|x\|_{X_\theta} = \inf\{\max(\|x_0\|_{X_0}, \|x_1\|_{X_1}) :\ |x| = |x_0|^{1-\theta}|x_1|^\theta\}.$$

If both X_0 and X_1 are uniformly convex, then the infimum is attained uniquely.

Now let $(G_n)_{n=1}^\infty$ be a sequence of Banach spaces. If X is a Banach sequence space, we define $X(G_n)_{n=1}^\infty$ or $(\sum_{n=1}^\infty G_n)_X$ to be the space of sequences $x = (x_n)_{n=1}^\infty$ such that $x_n \in G_n$ and $(\|x_n\|)_{n=1}^\infty \in X$ with the norm

$$\|x\| = \left\| (\|x_n\|)_{n=1}^\infty \right\|_X.$$

Lemma 5.1: *Suppose $1 < p < \infty$ and $\epsilon > 0$. Then there exist $1 < r < p < s < \infty$ so that if X is a Banach sequence space which is strictly r-convex and strictly s-concave, and if $(G_n)_{n=1}^\infty$ is any sequence of Banach spaces, then there is a bijection $f \in \mathcal{GU}((\sum_{n=1}^\infty G_n)_{\ell_p}, (\sum_{n=1}^\infty G_n)_X)$ such that $[[f]]_\epsilon \leq 2$.*

Proof. We apply Corollary I.5 p. 452 of [3] to deduce that we can choose r, s so that for every such X there is a pair of uniformly convex Banach sequence spaces X_0, X_1 such that $X_0^{1/2}X_1^{1/2} = \ell_p$ and $X_0^{1-\theta}X_1^{1-\theta} = X$, where $0 < \theta < 1$ is such that $|\cot \pi\theta| < \epsilon/8$.

We now define f. If $x \in (\sum_{n=1}^{\infty} G_n)_{\ell_p}$, write $x = (x_n)_{n=1}^{\infty} = (\|x_n\|u_n)_{n=1}^{\infty}$ where $\|u_n\| = 1$ and $(\|x_n\|)_{n=1}^{\infty} \in \ell_p$. Let $\|x_n\| = \sqrt{a_n b_n}$ be the unique optimal factorization with $a = (a_n)_{n=1}^{\infty} \in X_0,\ b = (b_n)_{n=1}^{\infty} \in X_1$ and $\|a\|_{X_0} = \|b\|_{X_1} = \|x\|_{X(G_n)}$. Then we set

$$f(x) = (a_n^{1-\theta} b_n^{\theta} u_n)_{n=1}^{\infty}.$$

Then f is norm-preserving.

In the scalar case when $G_n = \mathbb{R}$, let us denote $f = f_0$. Then f_0 is a bijection which is a uniform homeomorphism between B_{ℓ_p} and B_X. Indeed it is precisely the uniform homeomorphism given by the complex interpolation method (Theorem 9.12 of [3], p. 204, or see [7]); this is exactly the same as the original uniform homeomorphism given by Odell and Schlumprecht [27] based on different arguments. It follows easily that f is also a norm-preserving bijection in the vector case for general spaces $(G_n)_{n=1}^{\infty}$.

We next prove that f is uniformly continuous on $(\sum_{n=1}^{\infty} G_n)_{\ell_p}$. Let ω_0 be the modulus of continuity of $f_0|_{B_{\ell_p}}$ in the scalar case. Now suppose $x, y \in (\sum_{n=1}^{\infty} G_n)_{\ell_p}$ with $\|x\|, \|y\| \le 1$ and let $t = \|x - y\|$. We let $\mathbb{A}$ be the set of all $n \in \mathbb{N}$ for which $\|x_n - y_n\| \ge \sqrt{t}\max(\|x_n\|, \|y_n\|)$. Let $\mathbb{B} = \mathbb{N} \setminus \mathbb{A}$. We denote by $\chi_{\mathbb{A}}x$ the sequence $(\chi_{\mathbb{A}}(n)x_n)$ coinciding with x on $\mathbb{A}$ and zero on $\mathbb{B}$, etc.

We clearly have

$$\|\chi_{\mathbb{A}}x\| \le t^{-1/2}\|x - y\| \le \sqrt{t}$$

and similarly $\|\chi_{\mathbb{A}}y\| \le \sqrt{t}$. Hence

$$\|f(x) - f(\chi_{\mathbb{B}}x)\|, \|f(y) - f(\chi_{\mathbb{B}}y)\| \le \omega_0(\sqrt{t}).$$

Let us write $\xi_n = \chi_{\mathbb{B}}(n)\|x_n\|$ and $\eta_n = \chi_{\mathbb{B}}(n)\|y_n\|$. Let u_n, v_n be normalized vectors such that $x_n = \xi_n u_n$ and $y_n = \eta_n v_n$ for $n \in \mathbb{N}$. Then by Lemma 3.1

$$\|u_n - v_n\| \le \max(\xi_n, \eta_n)^{-1}\|x_n - y_n\| \le \sqrt{t}, \quad n \in \mathbb{B}$$

and hence

$$\|f_0(\xi)_n u_n - f_0(\eta)_n v_n\| \le |f_0(\xi)_n - f_0(\eta)_n| + \min(f_0(\xi)_n, f_0(\eta)_n)\sqrt{t}, \quad n \in \mathbb{B}.$$

This implies that

$$\|f(\chi_{\mathbb{B}}x) - f(\chi_{\mathbb{B}}y)\| \le \omega_0(t) + \sqrt{t}.$$

Combining we have

$$\|f(x)-f(y)\| \leq \omega_0(t) + 2\omega_0(\sqrt{t}) + \sqrt{t}$$

and f is uniformly continuous. The argument for f^{-1} is similar.

It remains to show that $[[f]]_\epsilon \leq 2$. To do this we first consider $\|f(x)-f(y)\|$ when $\|x\|, \|y\| \in B_{(\sum G_n)_{\ell_p}}$, $\|x-y\| = t$ and x, y have finite supports (this last restriction is not really necessary but avoids technical discussions). Let us suppose we have found $a, c \in X_0,\ b, d \in X_1$ with $a, b, c, d \geq 0$ and such that $\|a\|_{X_0} = \|b\|_{X_1} = \|x\|$ and $\|c\|_{X_0} = \|d\|_{X_1} = \|y\|$ and $\|x_n\| = \sqrt{a_n b_n},\ \|y_n\| = \sqrt{c_n d_n}$. Then

$$f(x) - f(y) = a_n^{1-\theta} b_n^{\theta} u_n - c_n^{1-\theta} d_n^{\theta} v_n.$$

Let us choose $x^* \in X^*(G_n^*)$ with $\|x^*\| = 1$ and $x^*(f(x)-f(y)) = \|f(x)-f(y)\|$. Then we can write

$$x_n^* = \alpha_n^{1-\theta} \beta_n^{\theta} w_n^*,$$

where $\alpha \in X_0^*,\ \beta \in X_1^*,\ \alpha, \beta \geq 0$ are such that $\|\alpha\|_{X_0^*} = \|\beta\|_{X_1^*} = 1$.

Finally, consider the analytic function

$$F(z) = \sum_{n=1}^{\infty} \alpha_n^{1-z} \beta_n^z (a_n^{1-z} b_n^z w_n^*(u_n) - c_n^{1-z} d_n^z w_n^*(v_n)), \quad 0 < \Re z < 1.$$

This function is bounded on the strip and extends continuously to the boundary. On the boundary we can estimate

$$|F(it)| \leq \|\alpha\|_{X_0^*}(\|a\|_{X_0} + \|c\|_{X_0}) \leq 2$$

and

$$|F(1+it)| \leq \|\beta\|_{X_1^*}(\|b\|_{X_1} + \|d\|_{X_1}) \leq 2.$$

Furthermore

$$|F(1/2)| \leq \|x-y\| = t \leq 2.$$

Hence the function $|F(z) - F(1/2)|$ is bounded by 4 on the boundary. We can write

$$F(z) - F(1/2) = \cot(\pi z) G(z),$$

where G is analytic on the strip and $|G|$ is also bounded by 4 on the boundary. Thus $|G(z)| \leq 4$ throughout the strip and in particular $|G(\theta)| \leq 4$. Hence

$$|F(\theta)| \leq t + 4|\cot \pi\theta| \leq t + \epsilon/2 \leq 2\max(t, \epsilon).$$

The argument for f^{-1} is very similar, reversing the roles of θ and $1/2$. In this case one should use

$$\varphi(z) = \frac{\sin \pi(z-\theta)}{\cos \pi(z+\theta)}$$

in place of $\cot(\pi z)$. Note that $|\varphi(1/2)| = |\cot(\pi\theta)|$. ∎

Let T denote Tsirelson space (actually the dual of the original space of Tsirelson). We refer to [6] for the background on Tsirelson space. For $1 < p < \infty$ let T_p denote the p-convexification of T and $\tilde{T}_p = T_{p'}^*$. We also denote $T_p([N+1,\infty))$ the closed linear span of $\{e_i,\ i > N\}$ in T_p. The following fact is proved in [13] (see also [3] Proposition 10.33). Note that T_p is strictly p-convex and $\tilde{T}_p$ is strictly p-concave.

Proposition 5.2: *There is a constant M so that if $q > p$ there exists an integer $N = N(q)$ so that $T_p([N+1,\infty))$ has q-concavity constant at most M.*

We remark that this is proved in [13] for a space $\mathcal{T}_p$ which can be shown using results from [6] to coincide with T_p in an equivalent norm, as explained in both [13] and [3].

By duality we have immediately:

Proposition 5.3: *There is a constant M so that if $q < p$ there exists an integer $N = N(q)$ so that $\tilde{T}_p([N+1,\infty))$ has q-convexity constant at most M.*

We are now ready for our example.

Theorem 5.4: *Let $(G_n)_{n=1}^\infty$ be a sequence of finite-dimensional normed spaces dense for the Banach–Mazur distance in all finite-dimensional normed spaces. Then for $1 < p < \infty$, $((\sum_{n=1}^\infty G_n)_{T_p})^2$, $((\sum_{n=1}^\infty G_n)_{\tilde{T}_p})^2$ and $C_p = (\sum_{n=1}^\infty G_n)_{\ell_p}$ are all uniformly homeomorphic.*

Remark: Of course if the $(G_n)_{n=1}^\infty$ are chosen carefully (e.g. replaced by $(G_1, G_1, G_2, G_2, \ldots)$) there is no need for the squares in this statement.

Proof. We only prove the statement for T_p as the proof for $\tilde{T}_p$ is similar. Suppose $\epsilon > 0$. Then we may find an integer N such that $T_p([N+1,\infty))$ is M-isomorphic to a sequence space X which is strictly p-convex and strictly q-concave with q close enough to p so that there is a map

$$f \in \mathcal{GU}\Big(\Big(\sum_{n=N+1}^\infty G_n\Big)_X, \Big(\sum_{n=N+1}^\infty G_n\Big)_{\ell_p}\Big)$$

with $[[f]]_\epsilon \leq 2$. It follows that as a member of

$$\mathcal{GU}\Big(\Big(\sum_{n=N+1}^{\infty} G_n\Big)_{T_p}, \Big(\sum_{n=N+1}^{\infty} G_n\Big)_{\ell_p}\Big)$$

we have $[[f]]_\epsilon \leq 2M$.

Next we extend it to a map

$$g : \Big(\sum_{n=1}^{N} G_n\Big)_{T_p} \oplus_\infty \Big(\sum_{n=N+1}^{\infty} G_n\Big)_{T_p} \to \Big(\sum_{n=1}^{N} G_n\Big)_{T_p} \oplus_\infty \Big(\sum_{n=N+1}^{\infty} G_n\Big)_{\ell_p}$$

given by $g(x_1, x_2) = (x_1, f(x_2))$. Then $g \in \mathcal{GU}$ and $[[g]]_\epsilon \leq 2M$. Hence considering g as a map: $(\sum_{n=1}^{\infty} G_n)_{T_p} \to (\sum_{n=1}^{N} G_n)_{T_p} \oplus_p (\sum_{n=N+1}^{\infty} G_n)_{\ell_p})$, we easily get a crude estimate $[[g]]_\epsilon \leq 8M$. But the latter space is certainly 2-isomorphic to C_p, so we have $[[g]]_\epsilon \leq 16M$ when g is considered as a member of $\mathcal{GU}((\sum_{n=1}^{\infty} G_n)_{T_p}, C_p)$.

This means that C_p and $(\sum_{n=1}^{\infty} G_n)_{T_p}$ are uniformly close; since C_p is isomorphic to its ℓ_p-sum, $\ell_p(C_p)$, we can apply Theorem 4.7 to obtain the conclusion. ∎

Remarks: Notice that the space C_p has the property that its asymptotic moduli of convexity and smoothness satisfy the conditions $\overline{\delta}_{C_p}(t) \approx \overline{\rho}_{C_p}(t) \approx t^p$. On the other hand, the space $X = ((\sum_{n=1}^{\infty} G_n)_{\tilde{T}_p})^2$ cannot have any equivalent norm for which $\overline{\rho}_X(t) \leq Ct^p$. This follows from the fact that it certainly has an equivalent norm so that $\overline{\delta}_X(t) \approx t^p$ and a result of [12] Proposition 2.11 can be used to imply that the existence of a similar norm for the asymptotic smoothness modulus would imply that X embeds into C_p and hence is hereditarily ℓ_p. In contrast, results in [9] show that if X is uniformly homeomorphic to a space Y where $\overline{\rho}_Y(t) \leq Ct^p$ and $r < p$, then there is a renorming of X such that $\overline{\rho}_X(t) \leq Ct^r$.

The space $X = ((\sum_{n=1}^{\infty} G_n)_{T_p})^2$ yields a similar example for the modulus of asymptotic convexity. Here results in [21] give some positive information for this modulus. Finally, note that C_p is uniformly homeomorphic to

$$X = \Big(\Big(\sum_{n=1}^{\infty} G_n\Big)_{T_p}\Big)^2 \oplus \Big(\Big(\sum_{n=1}^{\infty} G_n\Big)_{\tilde{T}_p}\Big)^2$$

which cannot be renormed to have either $\overline{\rho}_X(t) \leq Ct^p$ or $\overline{\delta}_X(t) \geq ct^p$ where $0 < c, C < \infty$ are constants.

We also remark that one can easily make super-reflexive examples by varying the collection of spaces G_n. For example, fixing p and then $r < p < s$ we could take G_n to be a dense sequence in the set of finite-dimensional spaces with a 1-unconditional basis which is r-convex and s-concave with both constants one.

6. Preparatory results on spaces with a (UFDD)

The following result is Lemma 4.2 of [20].

Lemma 6.1: *Let M be a metric space and let Y be a Banach space. Suppose $\psi : M \to Y$ is a uniformly continuous map with range contained in a compact set K. Let F be a compact convex set such that $\sup_{y\in K} d(y, F) < \epsilon$, for some positive $\epsilon > 0$. Then there is a uniformly continuous map $\psi' : M \to F$ with finite-dimensional relatively compact range such that $\|\psi(x) - \psi'(x)\| < \epsilon$ for $x \in M$ and $\omega_{\psi'}(t) < \omega_\psi(t) + 2\epsilon$ for $t > 0$.*

Lemma 6.2: *Let Y be a Banach space with a shrinking monotone (FDD) and suppose X is a closed subspace of Y. Let $(S_n)_{n=1}^\infty$ denote the partial sum operators of the (FDD) of Y. Then given $m \in \mathbb{N}$, a finite-dimensional subspace F of X and $\epsilon > 0$, there exists $T \in \text{co}\,\{S_k\}_{k=m}^\infty$ and a uniformly continuous function $\psi : B_X \to (1+\epsilon)B_X$ with finite-dimensional range and such that*

$$\|\psi(x) - Tx\| < \epsilon, \quad x \in B_X,$$

$$S_m\psi(x) = S_m x, \quad x \in B_X,$$

$$\psi(x) = x, \quad x \in B_F$$

and

$$\omega_\psi(t) \le t + \epsilon, \quad t \ge 0.$$

Proof. Let $P_F : X \to F$ be any bounded projection. Similarly, let $G = S_m^*(Y^*)|_X$ and let $P_G : X \to X$ be a bounded projection so that $P_G^*(X^*) = G$. Let $Q : Y \to Y/X$ denote the quotient map. Suppose $\eta > 0$ is chosen so that

$$20\|P_F\|\|P_G\|\eta < \epsilon.$$

Then for $x^{**} \in X^{**} = X^{\perp\perp} \subset Y^{**}$ and $u^* \in X^\perp \subset Y^*$ we have

$$\lim_{n\to\infty} x^{**}(S_n^* u^*) = x^{**}(u^*) = 0$$

and this implies that $\lim_{n\to\infty} QS_n = 0$ weakly in the space $\mathcal{K}(X, Y/X)$ (see [16] Corollary 3). Hence we can find an operator $T \in \text{co}\,\{S_k : k \geq m\}$ such that $\|QT\| < \eta$ and $\|Tx - x\| \leq \eta\|x\|$ for $x \in F$.

Then there is a finite-dimensional subspace H of X so that

$$\sup_{x\in\|P_F\|B_X} d(Tx, \|P_F\|B_H) \leq 2\eta\|P_F\|.$$

Applying Lemma 6.1, $\psi_0 : \|P_F\|B_X \to \|P_F\|B_H$ is a uniformly continuous map such that

$$\omega_{\psi_0}(t) \leq t + 4\eta\|P_F\|, \quad t \geq 0$$

and

$$\|\psi_0(x) - Tx\| \leq 2\eta\|P_F\|, \quad x \in \|P_F\|B_X. \tag{6.3}$$

In particular

$$\|\psi_0(x) - x\| \leq 3\eta\|P_F\|, \quad x \in \|P_F\|B_F.$$

Let

$$\psi_1(x) = P_F x - \psi_0(P_F x) + \psi_0(x), \quad x \in B_X,$$

and then

$$\psi(x) = \psi_1(x) - P_G\psi_1(x) + P_G x \quad x \in B_X.$$

Hence, ψ has finite-dimensional range.

Therefore

$$\begin{aligned}\|\psi_1(x) - \psi_0(x)\| &\leq \|P_F x - \psi_0(P_F x)\| \\ &\leq 2\eta\|P_F\| + \|P_F x - TP_F x\| \\ &\leq 3\|P_F\|\eta.\end{aligned}$$

Thus by (6.3),

$$\|\psi_1(x) - Tx\| \leq 5\|P_F\|\eta, \quad x \in B_X. \tag{6.4}$$

Next, if $y^* \in B_{Y^*}$, let $z^* \in S_m^*(Y^*)$ be such that $z^*|_X = P_G^*(y^*|_X)$ and $\|z^*\| \leq \|P_G^*\|$. The existence of such a z^* follows from the Hahn–Banach Theorem and the fact that $\|S_m\| = 1$. Then for $x \in B_X$, $z^*(x - Tx) = 0$ and so

$$|y^*(\psi(x) - \psi_1(x))| = |z^*(x - \psi_1(x))| = |z^*(Tx - \psi_1(x))| \leq 5\|P_F\|\|P_G\|\eta.$$

Thus

$$\|\psi(x) - Tx\| \leq 10\|P_F\|\|P_G\|\eta < \epsilon/2, \quad x \in B_X. \tag{6.5}$$

Clearly ψ is uniformly continuous and $\omega_\psi(t) \le t + \epsilon$.

If $x \in F$ then $\psi(x) = \psi_1(x) = x$. Finally, if $y^* \in Y^*$ we have $S_m^* y^*|_X \in G$ and so $y^*(S_m P_G v) = y^*(S_m v)$ for $v \in X$. Hence

$$\begin{aligned} y^*(S_m\psi(x)) &= y^*(S_m\psi_1(x)) - y^*(S_m P_G \psi_1(x)) + y^*(S_m P_G x) \\ &= y^*(S_m x), \quad x \in B_X, \end{aligned}$$

and thus

$$S_m\psi(x) = S_m x, \quad x \in B_X. \qquad \blacksquare$$

Before we state the next result, let us say that a quotient map $Q : Y \to X$ has a **locally uniformly continuous section** if there exists a uniformly continuous map $\varphi : B_X \to Y$ such that $Q \circ \varphi = Id_{B_X}$. Note that φ may be assumed to be homogeneous on B_X and then can be extended into a homogeneous section defined on X and uniformly continuous on every bounded subset of X.

Theorem 6.3: *Let X be a closed subspace of a Banach space Y with a shrinking (UFDD). Then there are closed subspaces V_1 and V_2 of X each with a (UFDD) and a quotient map $Q : V_1 \oplus V_2 \to X$ with a locally uniformly continuous section.*

If, further, X^* has the approximation property, then X is isomorphic to a complemented subspace of $V_1 \oplus V_2$.

Proof. We will assume that Y has a 1-(UFDD). Let S_n be the partial sum operators associated to the 1-UFDD. Let $S_0 = 0$.

Fix any decreasing sequence $(\epsilon_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty \epsilon_n < 1/8$. Let $(H_n)_{n=1}^\infty$ be an increasing sequence of finite-dimensional subspaces of X whose union is dense.

Let $m_0 = 0$. We will define an increasing sequence of integers $(m_k)_{k=0}^\infty$, a sequence of operators $(T_k)_{k=1}^\infty$ with $T_k \in \operatorname{co}\{S_{m_{k-1}+1}, \ldots, S_{m_k}\}$, and a sequence $(f_k)_{k=0}^\infty$ of uniformly continuous maps $f_k : B_X \to (1+\epsilon_k)B_X$ with finite-dimensional range such that $f_0(x) = 0$ and

$$S_{m_{k-1}}(x - f_k(x)) = 0, \quad x \in B_X, \tag{6.6}$$

$$\|f_k(x) - T_k(x)\| < \epsilon_k, \quad x \in B_X, \tag{6.7}$$

$$f_k(x) = x, \quad x \in B_{H_k}, \tag{6.8}$$

$$\|S_{m_k}x - x\| \le \epsilon_k\|x\|, \quad x \in [f_j(u);\ j \le k,\ u \in B_X], \tag{6.9}$$

where $[y,\ y \in I]$ denotes the closed linear span in X of $\{y,\ y \in I\}$. This notation will be used throughout this section.

The induction can be carried out using Lemma 6.2. Notice that by construction $\lim_{k\to\infty} f_k(x) = x$ for $x \in B_X$. We let $F_k = [f_k(x) - f_{k-1}(x);\ x \in B_X]$.

We now show that the sequences $(F_{2k})_{k=1}^{\infty}$ and $(F_{2k-1})_{k=1}^{\infty}$ each define 2-UFDD's. Let us treat the even case. Note that

$$S_{m_{2k-2}}(f_{2k}(x) - f_{2k-1}(x)) = 0, \quad x \in B_X.$$

Thus if $x \in F_{2k}$ we have

$$\|x - S_{m_{2k}}x\| \le \epsilon_{2k}\|x\|$$

and $S_{m_{2k-2}}x = 0$.

Now suppose $x_j \in F_{2j}$ for $1 \le j \le r$ are such that $\|x\| = 1$, where $x = x_1 + \cdots + x_r$. Let $x' = \theta_1 x_1 + \cdots + \theta_r x_r$ be chosen to maximize $\|x'\| = d$ say. Then $\|x_j\| \le d$ and $\|x_1 + \cdots + x_j\| \le d$ for $1 \le j \le r$.

Now

$$\|x - S_{m_{2r}}x\| \le \epsilon_{2r} \le d/8$$

and

$$\|x' - S_{m_{2r}}x'\| \le \epsilon_{2r}\|x'\| \le d/8.$$

Then

$$\begin{aligned}
\|x'\| &\le \|S_{m_{2r}}x'\| + d/8 \\
&= \Big\| \sum_{j=1}^{r} (S_{m_{2j}} - S_{m_{2j-2}})x' \Big\| + d/8 \\
&\le \Big\| \sum_{j=1}^{r} \theta_j (S_{m_{2j}} - S_{m_{2j-2}})x_j \Big\| + \sum_{j=1}^{r} \epsilon_{2j}\|x_j\| + d/8 \\
&\le \Big\| \sum_{j=1}^{r} \theta_j (S_{m_{2j}} - S_{m_{2j-2}})x \Big\| + \sum_{j=1}^{r} \epsilon_{2j}\|x_j\| + \sum_{j=1}^{r-1} \epsilon_{2j}\|x_1 + \cdots + x_j\| + d/8 \\
&\le 1 + d/2.
\end{aligned}$$

Hence $d \le 1 + d/2$ so that $d \le 2$, and so $(F_{2j})_{j=1}^{\infty}$ is a 2-UFDD of its closed linear span V_2. Similarly, $(F_{2j-1})_{j=1}^{\infty}$ is a 2-UFDD of its closed linear span V_1.

We let $Z = V_1 \oplus V_2$. Let $Q : Z \to X$ be defined by $Q(v_1, v_2) = v_1 + v_2$.

Note that

$$\|(f_k x - f_{k-1}x) - (T_k x - T_{k-1}x)\| < \epsilon_k + \epsilon_{k-1}, \quad x \in B_X.$$

Hence $\sum_{k=1}^{\infty}(f_k(x)-f_{k-1}(x))$ converges unconditionally for every $x\in B_X$. Let us define $g_1 : X\to V_1$ and $g_2 : X\to V_2$ such that

$$g_1(x)=\sum_{k=1}^{\infty}(f_{2k-1}(x)-f_{2k-2}(x)),\quad x\in B_X,$$

and

$$g_2(x)=\sum_{k=1}^{\infty}(f_{2k}(x)-f_{2k-1}(x)),\quad x\in B_X.$$

Let $g : B_X\to V_1\oplus V_2$ be defined by $g(x)=(g_1(x),g_2(x))$.

We observe that $Qg(x)=x$ for $x\in B_X$ and this implies that Q is surjective and hence a quotient map. We next show that g is uniformly continuous on B_X. Indeed the series

$$\sum_{k=1}^{\infty}(f_{2k-1}(x)-f_{2k-2}(x)-T_{2k-1}x+T_{2k-2}x)$$

converges uniformly on B_X and hence defines a uniformly continuous function (with values in Y). Since $x\to\sum_{k=1}^{\infty}(T_{2k-1}x-T_{2k-2}x)$ defines a bounded linear operator of norm one, this implies g_1 is uniformly continuous and so is g_2 by a similar argument. Hence g is uniformly continuous on B_X.

Let us now describe the modifications when X^* has the approximation property. In this case X^* is separable and has the (MAP) (see e.g. [23] p. 39 and [5]). It follows that for any finite-dimensional subspace F of X^* and any $\epsilon>0$ there is a finite rank operator $R : X\to X$ with $R^*x^*=x^*$ for $x^*\in F$ and $\|R^*\|<1+\epsilon$.

We consider the partial sum operators S_n as operators from X to Y. We may find a sequence of finite-rank operators $R_n : X\to X\subset Y$ so that $\|R_n\|<2$ and $R_n^*S_n^*=S_n^*$. Then for $y^*\in Y^*$ we have

$$\lim_{n\to\infty}\|S_n^*y^*-y^*|_X\|=0\quad\text{and}\quad\lim_{n\to\infty}\|R_n^*y^*-y^*|_X\|=0.$$

Hence $\lim_{n\to\infty}\|S_n^*y^*-R_n^*y^*\|=0$. Now, again apply [16] Corollary 3, S_n-R_n converges weakly to zero in $\mathcal{K}(X,Y)$. In particular, for any m and any $\epsilon>0$ we can find $T\in\text{co}\,\{S_m,S_{m+1},\ldots\}$ and $V\in\text{co}\,\{R_m,R_{m+1},\ldots\}$ with $\|V-T\|<\epsilon$.

Let $m_0=0$. We will define an increasing sequence of integers $(m_k)_{k=0}^{\infty}$, a sequence of operators $(T_k)_{k=1}^{\infty}$ with $T_k\in\text{co}\,\{S_{m_{k-1}+1},\ldots,S_{m_k}\}$, and a sequence of operators $(U_k)_{k=0}^{\infty}$ with $U_0=0$ and then $U_k\in co\{R_{m_{k-1}+1},\ldots,R_{m_k}\}$ for

$k \geq 1$, such that in place of (6.6), (6.7), (6.9) and (6.8) we have

$$S_{m_{k-1}}(x - U_k(x)) = 0, \quad x \in B_X, \tag{6.10}$$

$$\|U_k - T_k\| < \epsilon_k, \quad k \geq 1, \tag{6.11}$$

$$\|U_k x - x\| < \epsilon_k, \quad x \in B_{H_k}, \tag{6.12}$$

$$\|S_{m_k} x - x\| \leq \epsilon_k \|x\|, \quad x \in [U_j(X);\ j \leq k]. \tag{6.13}$$

Now the calculations are the same, with the nonlinear map f_k replaced by U_k (the slight change in (6.12) over (6.8) plays no role). We let $F_k = (U_k - U_{k-1})(X)$ and define V_1 and V_2 as before. This time however the map

$$Lx = \left(\sum_{k=1}^{\infty} (U_{2k-1} - U_{2k-2})x, \sum_{k=1}^{\infty} (U_{2k} - U_{2k-1})x \right)$$

defines a bounded linear section of the quotient map A. ∎

7. Applications

LEMMA 7.1: *Suppose $Q : X \to Y$ is a quotient map and that $E = \ker Q$. For $0 < \mu < 1$ define X_μ to be the space X with the equivalent norm*

$$\|x\|_{X_\mu} = \max\{\|Qx\|, \mu\|x\|\}.$$

Then the map $Q : X_\mu \to Y$ is also a quotient map. If there is a locally uniformly continuous section $f \in \mathcal{HU}(Y, X)$, then there exists $g \in \mathcal{GU}(X_\mu, Y \oplus_\infty E)$ with $[[g]]_\mu \leq 2\|f\| + 1$.

Proof. Let $f \in \mathcal{H}(Y, X)$ be a positively homogeneous bounded section for Q, i.e. $Qf = Id_Y$. Let $\|f\| = M$. Then we define $g : X_\mu \to Y \oplus_\infty E$ by $g(x) = (Qx, \mu(x - fQx))$. Then $g \in \mathcal{H}(X_\mu, Y \oplus_\infty E)$. Furthermore, g is a bijection and $g^{-1}(y, e) = f(y) + \mu^{-1}e$.

Note that if $\|x\|_{X_\mu}, \|x'\|_{X_\mu} \leq 1$ we have

$$\|g(x) - g(x')\| \leq \max(\|Q(x - x')\|, \mu(\|x - x'\| + 2M)),$$

so that

$$\|g\|_\mu \leq 2M + 1.$$

On the other hand, if $y, y' \in B_Y$ and $e, e' \in B_E$ we have

$$\begin{aligned}\|g^{-1}(y,e) - g^{-1}(y',e')\|_{X_\mu} &= \max(\|y-y'\|, \mu\|f(y) - f(y') + \mu^{-1}(e-e')\|) \\ &\le \max(\|y-y'\|, 2\mu M + \|e-e'\|),\end{aligned}$$

so that

$$\|g^{-1}\|_\mu \le 2M + 1.$$

We conclude that $g \in \mathcal{G}(X_\mu, Y \oplus_\infty E)$ and $[[g]]_\mu \le 2M+1$. This proves both assertions in the Lemma. ∎

THEOREM 7.2: *Suppose $\mathcal{E} = 0 \to E \to X \to Y \to 0$ is a short exact sequence of Banach spaces. Then there is a sequence of Banach spaces $(X_n)_{n=1}^\infty$ such that if $Z_p = (\sum_{n=1}^\infty X_n)_{\ell_p}$ for $1 \le p < \infty$ and $Z_\infty = (\sum_{n=1}^\infty X_n)_{c_0}$, then:*

1. *If $1 \le p < \infty$, Z_p is isomorphic to a subspace of $\ell_p(X \oplus Y)$ and a quotient space of $\ell_p(X \oplus E)$; Z_∞ is isomorphic to a subspace of $c_0(X \oplus Y)$ and a quotient space of $c_0(X \oplus E)$.*
2. *If $1 \le p \le \infty$, Z_p is coarsely homeomorphic to both $Z_p \oplus E$ and $Z_p \oplus Y$.*
3. *If the quotient map $Q : X \to Y$ admits a locally uniformly continuous section, then, for $1 \le p \le \infty$, Z_p is uniformly homeomorphic to both $Z_p \oplus E$ and $Z_p \oplus Y$.*

Furthermore, if $\mathcal{E}$ locally splits, then for some constant C we have $d(X_n^*, X^*) \le C$ for every n and so Z_∞^* is isomorphic to $\ell_1(X^*)$.

Proof. To avoid unnecessary duplication, we will treat the case $p < \infty$ only; the details are the same for the case $p = \infty$. Partition the natural numbers into infinitely many infinite sets $\mathbb{A}_k$ and set $\mu_n = 2^{-k}$ if $n \in \mathbb{A}_k$. We define X_n to be the space X with the equivalent norm

$$\|x\|_{X_n} = \max\{\|Qx\|, \mu_n\|x\|\}.$$

Now X_n is isometric to the subspace of $X \oplus_\infty Y$ of all $(\mu_n x, Qx)$ for $x \in X$. We will define Z_p as the subspace of $\ell_p(X \oplus_\infty Y)$ of all sequences of the form $(\mu_n x_n, Qx_n)_{n=1}^\infty$. Thus Z_p is isomorphic to $\ell_p(X_{\mu_n})$ where X_{μ_n} is defined in Lemma 7.1.

Now define $T : \ell_p(X \oplus_\infty E) \to Z_p$ by

$$T((x_n, e_n))_{n=1}^\infty = (\mu_n x_n + e_n, Qx_n)_{n=1}^\infty.$$

Then T is bounded. If $(\mu_n x_n, Qx_n)_{n=1}^\infty \in Z_p$, then we may find $(x_n')_{n=1}^\infty \subset X$ so that $Qx_n' = Qx_n$ for all n and

$$\|x_n'\| \leq 2\|Qx_n\|, \quad n \in \mathbb{N}.$$

Let $e_n = \mu_n x_n - \mu_n x_n'$. Then

$$\begin{aligned}\|(e_n)_{n=1}^\infty\|_{\ell_p(E)} &\leq \|(\mu_n x_n)_{n=1}^\infty\|_{\ell_p(X)} + 2\|(Qx_n)_{n=1}^\infty\|_{\ell_p(Y)} \\ &\leq 3\|(\mu_n x_n, Qx_n)_{n=1}^\infty\|_{\ell_p(X\oplus_\infty Y)}.\end{aligned}$$

This establishes (1).

Now fix any n. Then by Lemma 7.1 there exists $g \in \mathcal{G}(X_n, E \oplus_\infty Y)$ with $[[g]]_{\mu_n} \leq 5$. This trivially induces $\tilde{g} \in \mathcal{G}(Z_p \oplus_\infty X_n, Z_p \oplus_\infty Y \oplus_\infty E)$ with $[[\tilde{g}]]_{\mu_n} \leq 5$. However, the first space is 2-isomorphic to Z_p. It follows that Z_p is close to $Z_p \oplus E \oplus Y$. Now by Theorem 4.6 we have that Z_p is coarsely homeomorphic to $Z_p \oplus \ell_p(Y) \oplus \ell_p(E)$ and from this (2) follows immediately.

If the short exact sequence $\mathcal{E}$ admits a locally uniformly continuous section, then again using Lemma 7.1 we can use the same reasoning to obtain that Z_p is uniformly homeomorphic to $Z_p \oplus E$ and to $Z_p \oplus Y$ and therefore prove (3).

For the final remark observe first that

$$\|x\|_{X_n} \leq \|x\|_X \leq \mu_n^{-1}\|x\|_{X_n}, \quad x \in X.$$

Hence

$$\mu_n\|x^*\|_{X_n^*} \leq \|x^*\|_{X^*} \leq \|x^*\|_{X_n^*}, \quad x^* \in X^*.$$

Assume there is a bounded projection $P : X^* \to E^\perp$. Note that $\|x^*\|_{X_n^*} = \|x^*\|_{X^*}$ if $x^* \in E^\perp$. Thus $\|P\|_{X_n^* \to X_n^*} \leq \|P\|_{X^* \to X^*} = K$, say.

If $x^* = (I-P)x^*$, let β denote the norm of $x^*|_E$ (for the X-norm). We may find $u^* \in X^*$ with $u^*|_E = x^*|_E$ and $\|u^*\|_{X^*} = \beta$. Thus $u^* - x^* \in E^\perp$ and so $(I-P)u^* = (I-P)x^*$. Hence

$$\beta \leq \|x^*\|_{X^*} \leq (K+1)\beta.$$

However, if we work with respect to the X_n-norm, the restriction of x^* to E has norm β/μ_n and the same calculation gives

$$\beta/\mu_n \leq \|x^*\|_{X_n^*} \leq (K+1)\beta/\mu_n.$$

This shows that $d((I-P)X^*, (I-P)X_n^*) \leq (K+1)^2$ and hence provides a uniform bound on $d(X_n^*, X^*)$, where d still denotes the Banach–Mazur distance. ∎

THEOREM 7.3: *There are two separable $\mathcal{L}_\infty$-spaces which are coarsely homeomorphic but not linearly isomorphic.*

Proof. We start by observing that $\mathcal{C}[0,1]$ has a subspace E which is both a $\mathcal{L}_\infty$-space, and a Schur space ([4]). Then since E is a $\mathcal{L}_\infty$-space, the short exact sequence $0 \to E \to \mathcal{C}[0,1] \to \mathcal{C}[0,1]/E \to 0$ locally splits. We now use Theorem 7.2 in the case $p = \infty$. The space Z_∞ of Theorem 7.2 is then a c_0-product of spaces X_n isomorphic to $\mathcal{C}[0,1]$ and such that X_n^* are uniformly linearly isomorphic to $\mathcal{C}[0,1]^*$; thus Z_∞ is a $\mathcal{L}_\infty$-space. Also, Z_∞ and $Z_\infty \oplus E$ are coarsely homeomorphic. On the other hand, by a well-known result of Pełczyński [28] and the fact that E is a Schur space, every bounded operator $T : \mathcal{C}[0,1] \to E$ is compact. Hence every bounded operator $T : Z_\infty \to E$ is also compact and so Z_∞ and $Z_\infty \oplus E$ are not linearly isomorphic. ∎

REMARKS: The author does not know if Z_∞ and $Z_\infty \oplus E$ are uniformly homeomorphic. To apply Theorem 7.2 one would need to know the existence of a locally uniformly continuous section for the quotient map $\mathcal{C}[0,1] \to \mathcal{C}[0,1]/E$. The argument of Theorem 8.3 of [20] can be used to show that if B_{Z_∞} is an AUR, then such a section must exist; B_{Z_∞} is an AUR if and only if there is a uniform retraction from $B_{Z_\infty^{**}}$ onto B_{Z_∞}. It remains an open problem whether for every separable Banach space X there is a uniform retraction of $B_{X^{**}}$ onto B_X or even a Lipschitz retraction of X^{**} onto X; see [3], [18], [20] and [19].

THEOREM 7.4: *Let Y be a Banach space with a shrinking (UFDD) and suppose X is an infinite-dimensional closed subspace of Y. Then for $1 < p < \infty$ there is a closed subspace W_p of $\ell_p(X)$ with a (UFDD) so that W_p is uniformly homeomorphic to $W_p \oplus X$. Similarly, there is a closed subspace W_∞ of $c_0(X)$ with a (UFDD) so that W_∞ is uniformly homeomorphic to $W_\infty \oplus X$.*

Proof. We consider the case $1 < p < \infty$; only notational changes are necessary for the other case. We first note that by Theorem 6.3 we can find a short exact sequence $0 \to E \to V \to X \to 0$ which admits a locally uniformly continuous section and such that V has a (UFDD) and linearly embeds into X^2. Hence by Theorem 7.2 we can find a space $Z_p = (\sum_{n=1}^\infty V_n)_{\ell_p}$ where each V_n is a subspace of X^2 and is linearly isomorphic to V and such that Z_p and $Z_p \oplus X$ are uniformly homeomorphic. Since the (UFDD) of V is necessarily shrinking, we obtain that Z_p^* has the approximation property and so by Theorem 6.3 again we can find a subspace W_p of $\ell_p(X)$ with a (UFDD) and so that Z_p is complemented in W_p. Clearly W_p has the required properties. ∎

The following Corollary is now obvious and improves Theorem 4.11 of [20].

COROLLARY 7.5: *Let X be a separable Banach space which embeds in a space with a shrinking (UFDD). Then there is a Banach space W with a shrinking (UFDD) so that $W \oplus X$ is uniformly homeomorphic to W. In particular, X is a uniform retract of a Banach space with a shrinking (UFDD). If X is reflexive (respectively superreflexive), then W may be chosen to be reflexive (respectively super-reflexive).*

We conclude with our promised examples of uniformly homeomorphic subspaces and quotients of ℓ_p when $1 < p < \infty$. Let us observe that ℓ_p has unique uniform structure for $1 < p < \infty$ [13]. It is shown in [9] and [21] that if X and Y are uniformly homeomorphic and X is a subspace (respectively, quotient) of ℓ_p, then Y is also a subspace (respectively, quotient) of ℓ_p. We also note that in [20] we gave examples of subspaces of ℓ_1 which are uniformly homeomorphic but not linearly isomorphic; these examples are of a rather different character, since they both are $\mathcal{L}_1$-spaces and therefore each has a basis.

THEOREM 7.6: *For $1 < p < \infty$ with $p \neq 2$ there exist two closed subspaces Y and Z of ℓ_p which are uniformly homeomorphic but not linearly isomorphic. Similarly, there exist two closed subspaces Y and Z of c_0 which are uniformly homeomorphic but not linearly isomorphic.*

Proof. Take X to be a subspace of ℓ_p $(p \neq 2)$ failing the approximation property [8], [30]. Then take $Y = W_p$ and $Z = W_p \oplus X$ in Theorem 7.4; since Y has a (UFDD) and Z fails (AP) these cannot be linearly isomorphic. ∎

THEOREM 7.7: *For $1 < p < \infty$ with $p \neq 2$ there exist two quotients Y and Z of ℓ_p which are uniformly homeomorphic but not linearly isomorphic.*

Proof. Let X be a quotient space of ℓ_p failing (AP). Then by Theorem 6.3 we can find a subspace H of ℓ_q where $1/p + 1/q = 1$ with a (UFDD) so that X^* is a quotient of H. Dualizing we have a short exact sequence $0 \to X \to Y \to E \to 0$ where $Y = H^*$ is a quotient of ℓ_p. Since Y is super-reflexive, there is a locally uniformly continuous section of the quotient map $Y \to E$ (see [3] p. 28). Hence we may find a sequence of spaces Y_n each of which is 2-isomorphic to a quotient of $X \oplus_p Y$ and isomorphic to Y, so that if $Z = (\sum_{n=1}^{\infty} Y_n)_{\ell_p}$ then Z is uniformly homeomorphic to $Z \oplus X$. Now Z is a quotient of ℓ_p, since both X and Y are quotients of ℓ_p and has the approximation property which $Z \oplus X$ fails. ∎

Remark: We do not know if this theorem holds for quotients of c_0.

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