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LIPSCHITZ STRUCTURE OF QUASI-BANACH SPACES

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ABSTRACT

We show that the Lipschitz structure of a separable quasi-Banach space does not determine, in general, its linear structure. Using the notion of the Arens-Eells p-space over a metric space for $0 < p \leq 1$ we construct examples of separable quasi-Banach spaces which are Lipschitz isomorphic but not linearly isomorphic.

1. Introduction

Let X and Y be quasi-Banach spaces. A Lipschitz map $f : X \to Y$ is a possibly nonlinear map satisfying an estimate

$$\|f(x_1) - f(x_2)\| \leq C\|x_1 - x_2\|, \quad x_1, x_2 \in X,$$

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for some constant $C > 0$. X and Y are Lipschitz isomorphic if there exists a Lipschitz bijection $f : X \to Y$ such that f^{-1} is also Lipschitz (i.e., f is bi-Lipschitz). Here $\|\cdot\|$ denotes the quasi-norm on X or Y.

It is a well-known open problem whether two separable Lipschitz isomorphic Banach spaces are necessarily linearly isomorphic. Counterexamples are known for non-separable Banach spaces [5, 7, 1]. The aim of this paper is to provide counterexamples for separable quasi-Banach spaces, based on the methods of [7].

Let us remark that rather little is known about the nonlinear structure of quasi-Banach spaces. In general, authors have treated uniform structure rather than Lipschitz structure. For example, Weston [21] has shown that the spaces $L_p(0,1)$ and ℓ_q are not uniformly homeomorphic if $p, q < 1$ and $p \neq q$. See also the recent paper [16]. Let us also note that it is apparently unknown whether for $0 < p < q \leq 1$ the metric spaces (L_p, d_p) and (L_q, d_q) are Lipschitz isomorphic, where $d_p(f,g) = \|f - g\|_p^p$. However, the spaces L_p and L_q are not Lipschitz isomorphic as quasi-Banach spaces in the sense described above [2].

2. Preliminaries

For background on quasi-Banach spaces we refer the reader to [14] or [11]. Let us recall first that a quasi-norm $\|\cdot\|$ on a real vector space X is a map $X \to [0,\infty)$ with the properties:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ if $\alpha \in \mathbb{R}, x \in X$;
(iii) there is a constant $k \geq 1$ so that for any x_1 and $x_2 \in X$ we have

$$\|x_1 + x_2\| \leq k(\|x_1\| + \|x_2\|). \tag{2.1}$$

The least k in equation (2.1) is often referred to as the modulus of concavity of the quasi-norm. A very basic and important result is the Aoki-Rolewicz theorem [4, 17] which can be interpreted as saying that if $0 < p \leq 1$ is given by $k = 2^{1/p-1}$, then there is a constant C such that for any $\{x_k\}_{k=1}^n$ in X we have

$$\Big\|\sum_{j=1}^n x_j\Big\| \leq C\Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p}. \tag{2.2}$$

It is then possible to replace $\|\cdot\|$ by an equivalent quasi-norm $|||\cdot|||$ which is p-subadditive, i.e.,

$$|||x_1 + x_2|||^p \leq |||x_1|||^p + |||x_2|||^p, \quad x_1, x_2 \in X.$$

X is said to be ***p*-normable** if (2.2) holds. X is a ***p*-normed** space if the quasi-norm on X is p-subadditive. We will assume from now on that a quasi-normed space is p-normed for some $0 < p \leq 1$.

A p-subadditive quasi-norm $\|\cdot\|$ induces a metric topology on X. In fact, a metric can be defined by $d(x,y) = \|x-y\|^p$. X is called a **quasi-Banach space** if X is complete for this metric. A quasi-Banach space with an associated p-norm is also called a ***p*-Banach space**.

Let M be an arbitrary set. A **quasimetric** d on M is a symmetric map $d : M \times M \to [0, \infty)$ such that $d(x,y) = 0$ if and only if $x = y$, and for some constant $\kappa \geq 1$, d satisfies

$$d(x,y) \leq \kappa(d(x,z) + d(z,y)), \quad x, y, z \in M.$$

The space (M,d) is then a quasimetric space (see [8, p. 109]).

If (M,d) and $(\mathcal{M},\rho)$ are quasimetric spaces we shall say that a map $f : M \to \mathcal{M}$ is **Lipschitz** if there exists a constant $C > 0$ so that

$$\rho(f(x), f(y)) \leq C d(x,y), \quad x, y \in M.$$

The least such constant C is denoted by $\mathrm{Lip}(f)$. If f is a bijection, and both f and f^{-1} are Lipschitz, then we say that f is **bi-Lipschitz**, and M and $\mathcal{M}$ are called **Lipschitz isomorphic**. A map f from a quasimetric space (M,d) into a quasimetric space $(\mathcal{M},\rho)$ is an **isometry** if

$$\rho(f(x), f(y)) = d(x,y), \quad x, y \in M.$$

Let (M,d) a quasimetric space. We will say that d is a ***p*-metric** for some $0 < p \leq 1$ if d^p is a metric, i.e.,

$$d(x,y)^p \leq d(x,z)^p + d(z,y)^p, \quad x, y, z \in M.$$

We then call (M,d) a p-metric space. An analogue of the Aoki–Rolewicz theorem holds in this context: every quasimetric space can be endowed with an equivalent quasimetric which is p-subadditive for some $0 < p \leq 1$; that is, every quasimetric space is Lipschitz isomorphic to a p-metric space for some choice of $0 < p \leq 1$ ([8, Proposition 14.5]).

We shall say that (M,d) is a **pointed quasimetric space** (or a **pointed p-metric space**, or a **pointed metric space**), if it has a distinguished point that we call the **origin** and denote 0. The assumption of an origin is convenient to normalize Lipschitz functions. We can regard a p-Banach space X as a pointed p-metric space by taking 0 as the origin and the p-metric $d(x,y)=\|x-y\|$.

3. Lipschitz maps between quasi-Banach spaces

A classical theorem of Mazur and Ulam from 1932 [15] establishes that a surjective isometry between two (real!) Banach spaces that takes 0 to 0 is linear, i.e., the linear structure of a Banach space is completely determined by its structure as a metric space. The generalization of this result to quasi-Banach spaces was obtained by Rolewicz in 1968 (note that we are assuming that every quasi-norm is a p-norm for some p):

Theorem 3.1 (Rolewicz, [18], [19, p. 397]): *If $U : X \to Y$ is a bijective isometry between the (real) quasi-Banach spaces X and Y with $U(0)=0$ then U is linear.*

In [7] it is shown that if X is a separable Banach space and Y is any Banach space such that X embeds isometrically into Y, then X will also embed linearly and isometrically. In the quasi-Banach case the corresponding result fails:

Theorem 3.2: *Suppose $0<p<1$. There exists a separable p-normed quasi-Banach space X and a p-normed quasi-Banach space Y such that:*

(i) *X embeds isometrically into Y.*

(ii) *If $T : X\to Y$ is a bounded linear operator then $T=0$.*

In order to prove Theorem 3.2 first we prove:

Proposition 3.3: *Suppose $0<p<1$. Let (M,d) be a pointed p-metric space. Let $Y=\ell_\infty(M;L_p(0,\infty))$ be the p-Banach space of bounded maps from M into the real space $L_p(0,\infty)$, with the associated p-norm $\|f\|_Y = \sup\{\|f(x)\| : x\in M\}$. Then M embeds isometrically into Y.*

Proof. We will define a map $f : M\to Y$ with $f(0)=0$ which is an isometric embedding. For $x\in M$ put

$$f(x)(y)=\chi_{(0,d(x,y)^p)}-\chi_{(0,d(0,y)^p)},\quad y\in M.$$

Then $f(0) = 0$. Then for any x_1, x_2 in M,

$$\|f(x_1)(y) - f(x_2)(y)\|_{L_p} = |d(x_1, y)^p - d(x_2, y)^p|^{1/p} \le d(x_1, x_2), \quad y \in M,$$

while

$$\|f(x_1)(x_2) - f(x_2)(x_2)\|_{L_p} = d(x_1, x_2).$$

Thus $\|f(x_1) - f(x_2)\|_Y = d(x_1, x_2)$. ∎

Remark: Proposition 3.3 asserts that every metric space can be embedded isometrically into a p-Banach space X with the metric $\|x - y\|^p$.

Proof of Theorem 3.2. Consider the complex space $L_p(\mathbb{T}; \mathbb{C})$ and let $H_p(\mathbb{T})$ be the usual Hardy subspace. Let X be the quotient space $L_p(\mathbb{T})/H_p(\mathbb{T})$ regarded as a real quasi-Banach space.

Using Proposition 3.3 with $M = X$, we can embed X isometrically in the space $Y = \ell_\infty(X; L_p(0, \infty))$, hence (i) follows.

To see (ii), if there exists a nonzero bounded linear operator $T : X \to Y$, then there exists a nonzero bounded linear operator $S : X \to L_p(0, \infty)$, e.g., letting $Sx = Tx(y)$ for some $y \in X$. We can then induce a bounded complex-linear map $\tilde{S} : X \to L_p((0, \infty); \mathbb{C})$ by $\tilde{S}(x) = Sx - iS(ix)$. But then $\tilde{S} = S = 0$ (the fact that there is no nonzero bounded complex-linear map $S : L_p/H_p \to L_p$ follows as a consequence of the F. & M. Riesz theorem, [9]). ∎

Remark: A quasi-Banach space X is called **natural** if it is linearly isomorphic to a closed subspace of a quasi-Banach lattice which is p-convex for some $p > 0$ (see [10]). Proposition 3.3 shows that every p-normed space $(0 < p < 1)$, and, in particular, the real quasi-Banach space $X = L_p/H_p$, embeds isometrically into a natural space, namely, $\ell_\infty(M; L_p(0, \infty))$. But X cannot be linearly embedded into any natural space since it fails to be natural. Thus we could replace X by any nonnatural space.

4. Arens-Eells p-spaces

If (M, d) is a pointed quasimetric space, let $\mathbb{R}^M$ be the space of all functions (not necessarily continuous) $f : M \to \mathbb{R}$ so that $f(0) = 0$. We then define $\mathcal{P}(M)$ to be linear span in the linear dual $(\mathbb{R}^M)^\#$ of the evaluations $\delta(x)$, where x runs through M, defined by

$$\langle \delta(x), f \rangle = f(x), \quad f \in \mathbb{R}^M.$$

Note that $\delta(0)=0$.

Definition: If $0<p\le 1$ and d is a p-metric on M, we define the **Arens-Eells p-space over M**, denoted by $\text{Æ}_p(M)$, as follows. If $\mu=\sum_{j=1}^N a_j\delta(x_j)\in\mathcal{P}(M)$ put

$$\|\mu\|_{\text{Æ}_p(M)}=\sup\Big\|\sum_{j=1}^N a_jf(x_j)\Big\|_Y, \tag{4.3}$$

the supremum being taken over all p-normed spaces Y and all maps $f:M\to Y$ with $f(0)=0$, and satisfying the inequality

$$\|f(x)-f(y)\|_Y\le d(x,y),\quad x,y\in M.$$

Then $\|\cdot\|_{\text{Æ}_p(M)}$ is a p-seminorm which induces a p-norm on $\mathcal{P}(M)/Z$ where $Z=\{\mu\in\mathcal{P}(M):\|\mu\|_{\text{Æ}_p(M)}=0\}$. Then $\text{Æ}_p(M)$ is the completion of $\mathcal{P}(M)/Z$ under this p-norm.

Remarks: We do not know if $\|\cdot\|_{\text{Æ}_p(M)}$ is actually a p-norm on $\mathcal{P}(M)$ except in the case when $p=1$ (see below); equivalently, we do not know if Z intersects $\mathcal{P}(M)$. Of course, if M is a subset of a p-Banach space, then $\|\cdot\|_{\text{Æ}_p(M)}$ is trivially a p-norm. It will be convenient for us to regard $\mathcal{P}(M)$ as a subset of $\text{Æ}_p(M)$ by identifying each $\mu\in\mathcal{P}(M)$ with the corresponding equivalence class in $\text{Æ}_p(M)$.

If $p=1$ (so that d is a metric), then it follows from the Hahn-Banach theorem that $\text{Æ}_1(M)$ is the space denoted by $\mathcal{F}(M)$ in [12] (or [7]); however, the terminology here dates back to Weaver [20], who denotes this space by $\text{Æ}(M)$. In this case $\text{Æ}_1(M)^*$ can be identified naturally with $\text{Lip}_0(M)$, the space of all Lipschitz functions $f:M\to\mathbb{R}$ with $f(0)=0$ and the norm

$$\|f\|_{\text{Lip}_0(M)}=\sup\left\{\frac{|f(x)-f(y)|}{d(x,y)}:\ x\ne y\right\}.$$

Note that if $p<r\le 1$ and M is a pointed r-metric space, then it is also a pointed p-metric space. From definition we then have

$$\|\mu\|_{\text{Æ}_r(M)}\le\|\mu\|_{\text{Æ}_p(M)},\quad \mu\in\mathcal{P}(M),$$

and from this it follows that there is a natural map $J_{p,r}:\text{Æ}_p(M)\to\text{Æ}_r(M)$ with $\|J_{p,r}\|\le 1$, which is induced by the identity map on $\mathcal{P}(M)$. We do not know if this map is always injective. We will need these remarks in the case $r=1$, when we consider $\text{Æ}_p(X)$ for X a Banach space.

LEMMA 4.1: *Suppose* $0 < p \leq 1$. *If* (M, d) *is a pointed* p*-metric space, then the map* $\delta : (M, d) \to (\text{Æ}_p(M), \|\cdot\|_{\text{Æ}_p(M)})$ *is an isometric embedding and* $\text{Æ}_p(M)$ *is the closed linear span of* $\delta(M)$.

Proof. This follows directly from Proposition 3.3 since we can take f to be an isometric embedding. ∎

Remark: Let us highlight one difference between the cases $p < 1$ and $p = 1$. If $p = 1$ and M is a pointed metric space, then if M_0 is a subset of M containing 0 we have

$$\|\mu\|_{\text{Æ}_1(M_0)} = \|\mu\|_{\text{Æ}_1(M)}, \quad \mu \in \mathcal{P}(M_0).$$

This follows from the fact that every $f_0 \in \text{Lip}_0(M_0)$ can be extended to some $f \in \text{Lip}_0(M)$ with the same Lipschitz constant. It follows that $\text{Æ}_1(M_0)$ can be naturally identified with a subspace of $\text{Æ}_1(M)$. In the case $p < 1$, it is unclear whether one has a corresponding result.

LEMMA 4.2: *Suppose* (M, d) *is a* p*-metric space. Suppose* $\mu \in \text{Æ}_p(M)$. *Then for any* $\epsilon > 0$ *we can write* μ *in the form*

$$\mu = \sum_{n=1}^{\infty} \alpha_n(\delta(x_n) - \delta(y_n)), \quad x_n, y_n \in M,$$

where

$$\sum_{n=1}^{\infty} |\alpha_n|^p d(x_n, y_n)^p < (1+\epsilon)^p \|\mu\|^p_{\text{Æ}_p(M)}.$$

Proof. Let K be the absolutely p-convex hull of $\{d(x,y)^{-1}(\delta(x) - \delta(y)) : x, y \in M,\ x \neq y\}$ in $\mathcal{P}(M)$. Then we may define a p-seminorm on $\mathcal{P}(M)$ via the Minkowski functional of K, i.e.,

$$\|\mu\|_K = \inf\{\lambda > 0 : \mu \in \lambda K\}.$$

Clearly,

$$\|\mu\|_{\text{Æ}_p(M)} \leq \|\mu\|_K, \quad \mu \in \mathcal{P}(M).$$

However,

$$\|\delta(x) - \delta(y)\|_K \leq d(x,y), \quad x, y \in M.$$

This means $\delta : M \to (\mathcal{P}(M), \|\cdot\|_K)$ is a permissible map in the definition of $\|\cdot\|_{\text{Æ}_p(M)}$ and so $\|\mu\|_{\text{Æ}_p(M)} = \|\mu\|_K$ for $\mu \in \mathcal{P}(M)$. The lemma follows immediately from this. ∎

Lemma 4.3: *Suppose Y is a p-Banach space, M is a pointed p-metric space, and $f : M \to Y$ is a map satisfying $f(0) = 0$ and*

$$\|f(x) - f(y)\|_Y \leq Cd(x, y), \quad x, y \in M.$$

Then f induces a bounded linear operator $T_f : Æ_p(M) \to Y$ with $\|T_f\| \leq C$ and

$$T_f(\delta(x)) = f(x), \quad x \in M.$$

Conversely, if $T : Æ_p(M) \to Y$ is a bounded linear operator, then $T = T_f$ where $f(x) = T(\delta(x))$ for $x \in M$.

In particular, if X is a p-Banach space, the identity map $Id_X : X \to X$ induces a linear operator $\beta_X : Æ_p(X) \to X$ which is a quotient map. Thus X is (isometrically) a quotient of $Æ_p(X)$.

Proof. This is purely formal and can be proved in the same way as the corresponding result in [7, Lemma 2.5]. ∎

Remarks: If X is a p-Banach space, then for $\mu = \sum_{j=1}^{n} a_j \delta(x_j) \in \mathcal{P}(X)$ we have

$$\beta_X(\mu) = \sum_{j=1}^{n} a_j x_j.$$

If X is a Banach space we should distinguish between $\beta_X = \beta_X^{(p)} : Æ_p(X) \to X$ and $\beta_X = \beta_X^{(1)} : Æ_1(X) \to X$. These are related by $\beta_X^{(p)} = \beta_X^{(1)} \circ J_{p,1}$, where $J_{p,1} : Æ_p(X) \to Æ_1(X)$ is the canonical map, described above.

Definition: Suppose $0 < p \leq 1$. Let us say that a p-Banach space X has the ***p*-Lipschitz lifting property** if whenever

$$0 \longrightarrow E \longrightarrow Y \stackrel{q}{\longrightarrow} X \to 0$$

is a short exact sequence of p-Banach spaces such that there exists a Lipschitz map $f : X \to Y$ with $q \circ f = Id_X$, then the sequence splits linearly, i.e., there is a bounded linear operator $S : X \to Y$ with $qS = Id_X$. For Banach spaces this concept appears implicitly in [7]; similar ideas are also studied in [6].

Lemma 4.4: *Suppose $0 < p \leq 1$ and let (M, d) be a pointed p-metric space. Then the p-Banach space $Æ_p(M)$ has the p-Lipschitz lifting property.*

Proof. (Compare with [7, Lemma 2.10].) Suppose

$$0 \longrightarrow E \longrightarrow Y \xrightarrow{q} \text{Æ}_p(M) \to 0$$

is a short exact sequence and $f : \text{Æ}_p(M) \to Y$ is a Lipschitz map such that $q \circ f = Id_{\text{Æ}_p(M)}$. We can assume $f(0) = 0$ by translation. Then $f \circ \delta : M \to Y$ satisfies $f \circ \delta(0) = 0$ and

$$\|f \circ \delta(x) - f \circ \delta(y)\| \leq Cd(x, y), \quad x, y \in M.$$

Let $T : \text{Æ}_p(M) \to Y$ be the associated linear operator such that

$$T(\delta(x)) = f(\delta(x)), \quad x \in M.$$

Then $q \circ T = Id_{\text{Æ}_p(M)}$. ∎

Proposition 4.5: *Let X be a p-Banach space for some $0 < p \leq 1$. Then the following statements are equivalent:*

(i) *X has the p-Lipschitz lifting property;*

(ii) *the short exact sequence of p-Banach spaces*

$$0 \longrightarrow \ker \beta_X \longrightarrow \text{Æ}_p(X) \xrightarrow{\beta_X} X \longrightarrow 0$$

splits (linearly);

(iii) *X is linearly isomorphic to a complemented subspace of $\text{Æ}_p(M)$ for some pointed p-metric space M.*

Proof. (i) ⇒ (ii): By Lemma 4.1, the map $\delta : (X, \|\cdot\|_X) \to (\text{Æ}_p(X), \|\cdot\|_{\text{Æ}_p(X)})$ is an isometry, and $\beta_X \circ \delta = Id_X$.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (i): $\text{Æ}_p(M)$ has the p-Lipschitz lifting property (Lemma 4.4) so any complemented subspace of $\text{Æ}_p(M)$ has the p-Lipschitz lifting property as well. Since X is linearly isomorphic to a complemented subspace of $\text{Æ}_p(M)$ we are done. ∎

In [7], it was shown that in the case $p = 1$ every separable Banach space has the 1-Lipschitz lifting property, while for nonseparable spaces the property is rather rare. However, for $0 < p < 1$, the situation is somewhat different; in the next section we will show that a separable Banach space which fails the Schur property also fails the p-Lipschitz lifting property for $0 < p < 1$. In fact, it seems that, if $0 < p < 1$, the set of separable p-Banach spaces with the p-Lipschitz lifting property is probably quite small. Of course ℓ_p has this property (as it is projective for the category of p-Banach spaces; see [14]). If $M = [0,1]$ with the p-metric $d(x,y) = |x-y|^{1/p}$, then $\text{Æ}_p(M) = L_p(0,1)$. If M is the unit circle with arc length to the power $1/p$ as the p-metric we obtain that $\text{Æ}_p(M)$ is isomorphic to the quotient of L_p by a one-dimensional space (which is not isomorphic to $L_p(0,1)$, [13]). If one takes, say, $M = [0,1]$ with $d(x,y) = |x-y|^a$ where $0 < a < 1/p$ one obtains some other non-classical spaces.

To conclude the section, let us also observe that Theorem 3.1 of [7] holds in somewhat more generality with a very similar proof (which we omit):

Theorem 4.6: *Let*

$$0 \longrightarrow X \longrightarrow Y \stackrel{q}{\longrightarrow} Z \longrightarrow 0$$

be a short exact sequence of quasi-Banach spaces such that Z is separable, X is a Banach space and there exists a Lipschitz map $f : Z \to Y$ such that $q \circ f = Id_Z$. Then the sequence splits, i.e., there exists a bounded linear operator $T : Z \to Y$ with $q \circ T = Id_X$.

5. The Example

Throughout this section M will be a complete pointed metric space. If $\mu \in \mathcal{P}(M)$ we define the **support** of μ to be the smallest subset F of M which contains 0 and such that $\mu \in [\delta(x)]_{x\in F}$ (the linear span of $\{\delta(x) : x \in F\}$).

If $g : M \to [0,\infty)$ is any Lipschitz function with Lipschitz constant at most one, we can induce a pseudo-metric

$$d_g(x,y) = \min(d(x,y), g(x) + g(y)), \quad x,y \in M.$$

(We recall that a pseudo-metric satisfies all the conditions of a metric except that we allow $d(x,y) = 0$ when $x \neq y$.)

We recall that a **type** on a metric space M is a function of the form

$$\tau(x) = \lim_{n\in\mathcal{U}} d(x,a_n),$$

where $(a_n)_{n=1}^\infty$ is a metrically bounded sequence in M (i.e., $\sup_n d(a_n,0) < \infty$) and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. We refer to [3] for a discussion of types in the context of Banach space theory. We note the following properties of types:

$$d(x,y) \le \tau(x) + \tau(y), \quad x,y \in M,$$

and

$$|\tau(x) - \tau(y)| \le d(x,y), \quad x,y \in M. \tag{5.4}$$

Let $\sigma_\tau = \inf_{x\in M} \tau(x)$. We say that τ is **principal** if $\sigma_\tau = 0$. Clearly, since M is complete, τ is principal if and only if there exists $a \in M$ so that

$$\tau(x) = d(x,a), \quad x \in M.$$

Lemma 5.1: *Suppose $(\mu_n)_{n=1}^\infty$ is a sequence in $\mathcal{P}(M)$. Let F_n be the support of μ_n. Suppose that:*

(i) $\sup_n \|\mu_n\|_{Æ_1(M)} < \infty$,
(ii) $\|\mu_m - \mu_n\|_{Æ_1(M)} \geq 1$ *whenever* $m \neq n$, *and*
(iii) $\sup_n |F_n| < \infty$.

Assume $\mathcal{T}$ is a finite set of types on M and let $g(x) = \min_{\tau\in\mathcal{T}} \tau(x)$. Then for any $\epsilon > 0$ and $n_0 \in \mathbb{N}$ there exist $f \in Lip_0(M)$ and $n \in \mathbb{N}$ with $n > n_0$ such that

$$|f(x) - f(y)| \leq d_g(x,y), \quad x,y \in M,$$

and

$$\langle \mu_n, f\rangle > \tfrac{1}{2}(1-\epsilon).$$

Proof. We pass to a subsequence to ensure that each F_n has the same cardinality, say N, and write $F_n = \{a_{nk} : 1 \leq k \leq N\}$. Fix a nonprincipal ultrafilter $\mathcal{U}$ and let V be the subset of $\{1,\ldots,n\}$ such that $\lim_{n\in\mathcal{U}} a_{nj} = b_j$ exists in M for $j \in V$. By solving an appropriate linear programming problem, for each n we may write

$$\mu_n = \sum_{1\leq j<k\leq N} \alpha_{njk}(\delta(a_{nj}) - \delta(a_{nk}))$$

with $\alpha_{njk} \geq 0$ in such a way that

$$h_n = \sum_{1\leq j<k\leq N} \alpha_{njk} d_g(a_{nj}, a_{nk})$$

is minimized. The space (F_n, d_g) is a pseudo-metric space and it is possible to define $\|\cdot\|_{Æ_1(F_n,d_g)}$ in the same way as in the case of a metric space. Then (cf. Lemma 4.2) h_n is simply the norm of μ_n in the space $Æ_1(F_n,d_g)$. Hence $h_n \leq \|\mu_n\|_{Æ_1(F_n,d)} = \|\mu_n\|_{Æ_1(M)}$, using the remark following Lemma 4.1. Thus $\sup_n h_n < \infty$.

By the properties of an ultrafilter, we may select a set $\mathbb{A}$ in the ultrafilter $\mathcal{U}$ so that for every $1 \leq j,k \leq N$ with $j \neq k$ we can find $\sigma = \sigma(j,k), \tau = \tau(j,k)$ both in $\mathcal{T}$ so that either:

$$d_g(a_{nj}, a_{nk}) = d(a_{nj}, a_{nk}), \quad \forall\, n \in \mathbb{A}, \tag{5.5}$$

or

$$d_g(a_{nj}, a_{nk}) = \sigma(a_{nj}) + \tau(a_{nk}), \quad \forall\, n \in \mathbb{A}. \tag{5.6}$$

We let

$$\theta_{jk} = \lim_{m\in\mathcal{U}} \alpha_{mjk} d_g(a_{mj}, a_{mk}).$$

We show now that as long as $j \neq k$, and either $\lim_{m\in\mathcal{U}} \alpha_{mjk} < \infty$ or (5.5) holds, then we have

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|\alpha_{mjk}(\delta(a_{mj}) - \delta(a_{mk})) - \alpha_{njk}(\delta(a_{nj}) - \delta(a_{nk}))\|_{\text{Æ}_1(M)} \leq 2\theta_{jk}. \tag{5.7}$$

First we consider the case when (5.5) holds. Then for $m \neq n$ both in $\mathbb{A}$,

$$\begin{aligned}\|\alpha_{mjk}(\delta(a_{mj}) - \delta(a_{mk})) - \alpha_{njk}(\delta(a_{nj}) - \delta(a_{nk}))\|_{\text{Æ}_1(M)}& \\ \leq \alpha_{mjk} d_g(a_{mj}, a_{mk}) + \alpha_{njk} d_g(a_{nj}, a_{nk})&.\end{aligned}$$

Thus (5.7) follows.

If $\lim_{m\in\mathcal{U}} \alpha_{mjk} < \infty$ and (j,k) does not satisfy (5.5), then for suitable types $\sigma, \tau \in \mathcal{T}$ we have $d_g(a_{mj}, a_{mk}) = \sigma(a_{mj}) + \tau(a_{mk})$ for all $m \in \mathbb{A}$. Suppose first that $\lim_{m\in\mathcal{U}} d_g(a_{mj}, a_{mk}) < \infty$; then if $m \neq n \in \mathbb{A}$,

$$\begin{aligned}&\|\alpha_{mjk}(\delta(a_{mj}) - \delta(a_{mk})) - \alpha_{njk}(\delta(a_{nj}) - \delta(a_{nk}))\|_{\text{Æ}_1(M)} \\ &\leq |\alpha_{mjk} - \alpha_{njk}| d(a_{mj}, a_{mk}) + \alpha_{njk}(d(a_{mj}, a_{nj}) + d(a_{mk}, a_{nk})) \\ &\leq |\alpha_{mjk} - \alpha_{njk}| d(a_{mj}, a_{mk}) + \alpha_{njk}(\sigma(a_{mj}) + \sigma(a_{nj}) + \tau(a_{mk}) + \tau(a_{nk})) \\ &= |\alpha_{mjk} - \alpha_{njk}| d(a_{mj}, a_{mk}) + \alpha_{njk}(d_g(a_{mj}, a_{mk}) + d_g(a_{nj}, a_{nk})) \\ &\leq 2|\alpha_{mjk} - \alpha_{njk}| d(a_{mj}, a_{mk}) + \alpha_{mjk} d_g(a_{mj}, a_{mk}) + \alpha_{njk} d_g(a_{nj}, a_{nk}).\end{aligned}$$

Since $\lim_{m\in\mathcal{U}} \alpha_{mjk} < \infty$ this also implies (5.7).

On the other hand, if $\lim_{m\in\mathcal{U}} d_g(a_{mj}, a_{mk}) = \infty$ and (5.6) holds, we have $\lim_{m\in\mathcal{U}} \alpha_{mjk} = 0$. Note that

$$d_g(a_{nj}, a_{nk}) \geq d(a_{nj}, 0) + d(a_{nk}, 0) - \sigma(0) - \tau(0) \geq d(a_{nj}, a_{nk}) - \sigma(0) - \tau(0),$$

so

$$\lim_{n\in\mathcal{U}} \alpha_{njk}(d(a_{nj}, a_{nk}) - d_g(a_{nj}, a_{nk})) = 0$$

and again (5.7) follows. This completes the proof of (5.7) in all the claimed cases.

Let us define P to be the set of all pairs (j,k) with $j \neq k$ so that we have both that $\lim_{m\in\mathcal{U}} \alpha_{mjk} = \infty$ and (5.5) fails. In this case we claim that $j,k \in V$ and

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|\alpha_{mjk}(\nu_{mj} - \nu_{mk}) - \alpha_{njk}(\nu_{nj} - \nu_{nk})\|_{\text{Æ}_1(M)} \leq 2\theta_{jk}, \tag{5.8}$$

where $\nu_{nj} = \delta(a_{nj}) - \delta(b_j)$.

We can find $\sigma, \tau \in \mathcal{T}$ such that $d_g(a_{nj}, a_{nk}) = \sigma(a_{nj}) + \tau(a_{nk})$ for $n \in \mathbb{A}$. Since $\lim_{m\in\mathcal{U}} \alpha_{mjk} = \infty$ we must have $\lim_{m\in\mathcal{U}} d_g(a_{mj}, a_{mk}) = 0$ and so $\lim_{m\in\mathcal{U}} \sigma(a_{mj}) = \lim_{m\in\mathcal{U}} \tau(a_{mk}) = 0$. Thus both types σ and τ are principal and the sequences $(a_{nj}), (a_{nk})$ are both convergent, i.e., $j,k \in V$. In fact, we must have $\sigma(x) = d(x, b_j)$ while $\tau(x) = d(x, b_k)$. Hence, for $m \neq n$ both in $\mathbb{A}$,

$$\begin{aligned}
&\|\alpha_{mjk}(\nu_{mj} - \nu_{mk}) - \alpha_{njk}(\nu_{nj} - \nu_{nk})\|_{\text{Æ}_1(M)} \\
&\qquad\qquad \leq \alpha_{mjk}(\sigma(a_{mj}) + \tau(a_{mk})) + \alpha_{njk}(\sigma(a_{nj}) + \tau(a_{nk})) \\
&\qquad\qquad = \alpha_{mjk} d_g(a_{mj}, a_{mk}) + \alpha_{njk} d_g(a_{nj}, a_{nk}).
\end{aligned}$$

Thus (5.8) holds.

Now let

$$\lambda_n = \sum_{(j,k)\in P} \alpha_{njk}(\delta(b_j) - \delta(b_k)).$$

Combining (5.7) and (5.8) we get

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|\mu_m - \mu_n - \lambda_m + \lambda_n\|_{\text{Æ}_1(M)} \leq 2\sum_{j\neq k} \theta_{jk} = 2 \lim_{n\in\mathcal{U}} h_n. \tag{5.9}$$

Together with $\sup_n \|\mu_n\|_{\text{Æ}_1(M)} < \infty$ this implies that

$$\lim_{m\in\mathcal{U}} \|\lambda_m\|_{\text{Æ}_1(M)} < \infty$$

and, since $(\lambda_m)_{m=1}^{\infty}$ is contained in a finite-dimensional subspace,

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|\lambda_m - \lambda_n\|_{\text{Æ}_1(M)} = 0.$$

We therefore deduce from (5.9) that

$$\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \|\mu_m - \mu_n\|_{\text{Æ}_1(M)} \leq 2 \lim_{n\in\mathcal{U}} h_n. \tag{5.10}$$

Hence,

$$1/2 \leq \lim_{n\in\mathcal{U}} h_n.$$

Thus we may pick $n > n_0$ so that $h_n > \frac{1}{2}(1-\epsilon)$. Now by a simple application of the Hahn-Banach theorem (from the definition of h_n) there is a linear functional

φ on $[\delta(x)]_{x\in F_n}$ so that $\varphi(\delta(a_{nj})-\delta(a_{nk})) \le d_g(a_{nj}, a_{nk})$ for all j, k and $\varphi(\mu_n) = h_n > \frac{1}{2}(1-\epsilon)$. If we put $f_0(x) = \varphi(\delta(x))$ for $x \in F_n$, then $|f_0(x) - f_0(y)| \le d_g(x, y)$ and, since d_g is a pseudo-metric, f_0 has an extension f to M satisfying the same inequality. ∎

THEOREM 5.2: *Let $(\mu_n)_{n=1}^{\infty}$ be as in Lemma 5.1. Then for any $\epsilon > 0$ there exist an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $f \in Lip_0(M)$ with $\|f\|_{Lip_0(M)} \le 1$ such that*

$$\langle \mu_n, f\rangle > 1/4 - \epsilon, \quad n \in \mathbb{M}.$$

Proof. Since F_n, the support of μ_n, is finite for all n, it suffices to consider the case when M is separable. As we did before in Lemma 5.1, by passing to a subsequence we may suppose that $F_n = \{a_{n1}, \ldots, a_{nN}\}$ for some fixed N. We may pass to a further subsequence and assume that for each $1 \le k \le N$ either $\lim_{n\to\infty} d(x, a_{nk}) = \infty$ for all x or $\lim_{n\to\infty} d(x, a_{nk})$ exists and is finite for all x. In the latter case we can define a type by

$$\tau_k(x) = \lim_{n\to\infty} d(x, a_{nk}).$$

This yields a finite set of types $\mathcal{T}$ with $|\mathcal{T}| \le N$. Notice that, since 0 is assumed to be in every F_n, $\mathcal{T}$ is always nonempty because the type given by $d(x, 0)$ belongs to $\mathcal{T}$.

We will build by induction two increasing sequences of natural numbers, $\mathbb{M} = \{m_1, m_2, \ldots\}$ and $\mathbb{P} = \{p_1, p_2, \ldots\}$, and a sequence of Lipschitz functions $(f_n)_{n=1}^{\infty} \subset \mathrm{Lip}_0(M)$ so that

(i) $\|f_n\|_{\mathrm{Lip}_0(M)} \le 1$ for all n;
(ii) $f_1(x) = 0$ for all $x \in \bigcup_{l=p_1+1}^{\infty} F_l$;
(iii) $f_n(x) = 0$ for all $x \in \big(\bigcup_{j=1}^{n-1} F_{m_j}\big) \cup \big(\bigcup_{l=p_n+1}^{\infty} F_l\big)$ and $n \ge 2$;
(iv) $\langle \mu_{m_n}, f_n\rangle > \frac{1}{2}(1-\epsilon)$ for each $n \ge 1$; and
(v) $m_n > p_{n-1}$ if $n \ge 2$, and $m_n \le p_n$ if $n \ge 1$.

Suppose $m_1, \ldots, m_{n-1}, p_1, \ldots, p_{n-1}, f_1, \ldots, f_{n-1}$ have been constructed (if $n = 1$ this set is vacuous). We then put

$$g_1(x) = \min_{\tau\in\mathcal{T}} \tau(x),$$

and for $n \ge 2$,

$$g_n(x) = \min\left(g_1(x), d\left(x, \bigcup_{j=1}^{n-1} F_{m_j}\right)\right).$$

Applying Lemma 5.1 we can find m_n with $m_n > p_{n-1}$ if $n \geq 2$, and a function $\tilde{f}_n \in \mathrm{Lip}_0(M)$ so that

$$|\tilde{f}_n(x) - \tilde{f}_n(y)| \leq d_{g_n}(x,y), \quad x, y \in M$$

and

$$\langle \mu_{m_n}, \tilde{f}_n \rangle > \frac{1}{2}(1-\epsilon).$$

Let $\hat{f}_n = \theta \tilde{f}_n$ where $0 < \theta < 1$ is chosen so that

$$\langle \mu_{m_n}, \hat{f}_n \rangle > \frac{1}{2}(1-\epsilon).$$

For any $p > m_n$, define

$$h_p(x) = d\bigg(x, \Big(\bigcup_{j=1}^{n-1} F_{m_j}\Big) \cup \Big(\bigcup_{l=p+1}^{\infty} F_l\Big)\bigg).$$

We have that $h_p(x) \leq g_n(x)$ and $h_p(x)$ is increasing for $p > m_n$. Suppose that $\lim_{p\to\infty} h_p(x) = \xi < g_n(x)$. Then, clearly, there is a sequence $(y_p) \subset \bigcup_{l=p+1}^{\infty} F_l$ such that $d(x, y_p) \leq \xi$. But this implies the existence of a sequence $(a_{n_p,k_p})_{p>m_n}$ so that $d(x, a_{n_p,k_p}) \leq \xi$ and $n_p > p,\ 1 \leq k_p \leq N$. Thus there exists $\tau \in \mathcal{T}$ with $\tau(x) \leq \xi$ which gives a contradiction. Thus

$$g_n(x) = \lim_{p\to\infty} h_p(x), \quad x \in M$$

and hence

$$d_{g_n}(x,y) = \lim_{p\to\infty} d_{h_p}(x,y), \quad x, y \in M.$$

It follows, since F_{m_n} is finite, that for some $p_n > m_n$ we have

$$|\hat{f}_n(x) - \hat{f}_n(y)| \leq \theta d_{g_n}(x,y) \leq d_{h_{p_n}}(x,y), \quad x, y \in F_{m_n},$$

and so we can extend $\hat{f}_n|_{F_{m_n}}$ to a function $f_n \in \mathrm{Lip}_0(M)$ such that

$$|f_n(x) - f_n(y)| \leq d_{h_{p_n}}(x,y), \quad x, y \in M.$$

In particular, f_n vanishes on $\bigcup_{j=1}^{n-1} F_{m_j}$ and on $\bigcup_{l=p_n+1}^{\infty} F_l$. This completes the inductive construction.

Now let $f_n^+(x) = \max(f_n(x), 0)$ and $f_n^-(x) = \max(-f_n(x), 0)$. Then for each n we can find $\varphi_n = f_n^+$ or f_n^- so that

$$|\langle \mu_{m_n}, \varphi_n \rangle| > \frac{1}{4}(1-\epsilon), \quad n = 1, 2, \ldots$$

Let

$$\varphi(x) = \sup_n \varphi_n(x), \quad x \in M,$$

so that $\|\varphi\|_{\mathrm{Lip}_0(M)} \leq 1$ and $\varphi|_{F_{m_n}} = \varphi_n$. Thus

$$|\langle \mu_{m_n}, \varphi\rangle| > \frac{1}{4}(1-\epsilon), \quad n = 1, 2, \ldots .$$

Taking a further subsequence and $f = \pm\varphi$ gives the conclusion. ∎

Theorem 5.3: *Let X be an infinite-dimensional separable Banach space which is not a Schur space. Then X fails the p-Lipschitz lifting property for any $0 < p < 1$.*

Proof. Let us assume that X has the p-Lipschitz lifting property for some $0 < p < 1$. Then there exists a bounded linear operator $S : X \to Æ_p(X)$ with $\beta_X S = Id_X$. Since X is not a Schur space, there is a bounded sequence $(x_n)_{n=1}^\infty$ in X such that $\|x_m - x_n\|_X \geq 1$ for $m \neq n$ and $\lim_{n\to\infty} x_n = 0$ weakly. Thus $(Sx_n)_{n=1}^\infty$ is bounded in $Æ_p(X)$. By Lemma 4.2 we can write

$$Sx_n = \sum_{j=1}^\infty \alpha_{nj}(\delta(y_{nj}) - \delta(z_{nj})),$$

where for some constant K,

$$\sum_{j=1}^\infty |\alpha_{nj}|^p \|y_{nj} - z_{nj}\|_X^p \leq K^p, \quad n = 1, 2, \ldots .$$

Further, suppose that for each n the sequence $(|\alpha_{nj}|\|y_{nj} - z_{nj}\|_X)_{j=1}^\infty$ is decreasing. Thus for any $N \in \mathbb{N}$

$$|\alpha_{nN}|\|y_{nN} - z_{nN}\|_X \leq KN^{-1/p}, \quad n = 1, 2, \ldots$$

and so

$$\begin{aligned}\sum_{j=N+1}^\infty |\alpha_{nj}|\|y_{nj} - z_{nj}\|_X &\leq (KN^{-1/p})^{1-p} \sum_{j=N+1}^\infty |\alpha_{nj}|^p \|y_{nj} - z_{nj}\|_X^p \\ &\leq KN^{1-1/p}, \quad n = 1, 2, \ldots .\end{aligned}$$

Fix N so that $KN^{1-1/p} < 1/10$ and let

$$\mu_n = \sum_{j=1}^N \alpha_{nj}(\delta(y_{nj}) - \delta(z_{nj})), \quad n \in \mathbb{N}.$$

Then

$$\|\beta_X \mu_n - x_n\|_X \leq KN^{1-1/p} < 1/10, \quad n \in \mathbb{N}.$$

Thus

$$\|\beta_X\mu_n - \beta_X\mu_m\|_X \geq 4/5, \quad m \neq n.$$

This implies that

$$\|\mu_n - \mu_m\|_{\AE_1(X)} \geq 4/5, \quad m \neq n,$$

and so, after scaling, by Theorem 5.2 there exists $f \in \mathrm{Lip}_0(X) = \AE_1(X)^*$ with $\|f\|_{\mathrm{Lip}_0(X)} \leq 1$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that

$$|\langle \mu_n, f\rangle| \geq 1/6, \quad n \in \mathbb{M}. \tag{5.11}$$

However, the canonical map $J_{p,1} : \AE_p(X) \hookrightarrow \AE_1(X)$ is norm-decreasing and therefore $(J_{p,1}Sx_n)_{n=1}^{\infty}$ is weakly null in $\AE_1(X)$,

$$\lim_{n\to\infty} \langle J_{p,1}Sx_n, f\rangle = 0,$$

and also

$$|\langle J_{p,1}Sx_n - \mu_n, f\rangle| \leq \sum_{j=N+1}^{\infty} |\alpha_{nj}|\|y_{nj} - z_{nj}\| < 1/10,$$

a contradiction with (5.11). ∎

Remark: Although this is not made explicit in the argument, we are essentially using the fact that $\AE_1(X)$ can be identified with the Banach envelope of $\AE_p(X)$ and the quotient $\beta_X : \AE_p(X) \to X$ factors through $\AE_1(X)$.

THEOREM 5.4: *Let X be an infinite-dimensional separable Banach space which is not a Schur space. Then for $0 < p < 1$, $\AE_p(X)$ is Lipschitz isomorphic but not linearly isomorphic to the space $\ker\beta_X \oplus_p X$ equipped with the p-norm $\|\cdot\| = (\|\cdot\|^p_{\AE_p(X)} + \|\cdot\|^p_X)^{1/p}$.*

Proof. The map $\mu \to (\mu - \delta\beta_X(\mu), \beta_X(\mu))$ is a Lipschitz isomorphism between the p-Banach spaces $\AE_p(X)$ and $\ker\beta_X \oplus_p X$.

On the other hand, X fails the p-Lipschitz lifting property by Theorem 5.3 and so it cannot be isomorphic to a complemented subspace of $\AE_p(X)$; this would contradict (iii) of Proposition 4.5 with $M = (X, \|\cdot\|_X)$. We conclude that $\AE_p(X)$ cannot be linearly isomorphic to $\ker\beta_X \oplus_p X$. ∎

Remarks: If the reader prefers, this theorem may be rephrased in the following terms: the separable complete metric linear spaces $\AE_p(X)$ and $\ker\beta_X \oplus_p X$ are Lipschitz isomorphic with the metrics induced by their respective p-norms but not linearly isomorphic.

It is very likely that the hypothesis that X is not a Schur space in the theorem can be eliminated. To establish this, one needs to prove that no infinite-dimensional Banach space can have the p-Lipschitz lifting property for $0 < p < 1$. We conjecture that this is true. Indeed, it is very likely that if $p < q \leq 1$, no infinite dimensional q-Banach space has the p-Lipschitz lifting property.

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References

[1] I. Aharoni and J. Lindenstrauss, *Uniform equivalence between Banach spaces*, American Mathematical Society. Bulletin. New Series **84** (1978), 281–283.

[2] F. Albiac, *Nonlinear structure of some classical quasi-Banach spaces and F-spaces*, Journal of Mathematical Analysis and Applications **340** (2008), 1312–1325.

[3] D. Alspach and E. Odell, *L_p spaces*, in *Handbook of the Geometry of Banach Spaces, Vol. I* (Amsterdam), North-Holland, 2001, pp. 123–159.

[4] T. Aoki, *Locally bounded linear topological spaces*, Proceedings of the Imperial Academy, Tokyo **18** (1942), 588–594.

[5] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[6] F. Cabello Sánchez and J. M. F. Castillo, *Banach space techniques underpinning a theory for nearly additive mappings*, Dissertationes Mathematicae (Rozprawy Matematyczne) **404** (2002), 1–73.

[7] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Mathematica **159** (2003), 121–141.

[8] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer-Verlag, New York, 2001.

[9] N. J. Kalton, *The endomorphisms of L_p $(0 \leq p \leq 1)$*, Indiana University Mathematics Journal **27** (1978), 353–381.

[10] N. J. Kalton, *Convexity conditions on non-locally convex lattices*, Glasgow Mathematical Journal **25** (1984), 141–152.

[11] N. J. Kalton, *Quasi-Banach spaces*, in *Handbook of the Geometry of Banach Spaces*, Vol. 2 (Amsterdam), North-Holland, 2003, pp. 1099–1130.

[12] N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collectanea Mathematica **55** (2004), 171–217.

[13] N. J. Kalton and N. T. Peck, *Quotients of $L_p(0, 1)$ for $0 \leq p < 1$*, Studia Mathematica **64** (1979), 65–75.

[14] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Mathematical Society Lecture Note Series, vol. 89, Cambridge University Press, Cambridge, 1984.

[15] S. Mazur and S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, Comptes Rendus Mathématique. Académie des Sciences. Paris **194** (1932), 946–948.

[16] E. Prassidis and A. Weston, *Uniform Banach groups and structures*, Comptes Rendus Mathématiques de l'Académie des Sciences. La Société Royale du Canada **26** (2004), 25–32.

[17] S. Rolewicz, *On a certain class of linear metric spaces*, Bulletin de l'Academie Polonaise des Sciences. Cl. III. **5** (1957), 471–473, XL (English, with Russian summary).

[18] S. Rolewicz, *A generalization of the Mazur-Ulam theorem*, Studia Mathematica **31** (1968), 501–505.

[19] S. Rolewicz, *Metric linear spaces*, 2 ed., Mathematics and its Applications (East European Series), vol. 20, D. Reidel Publishing Co., Dordrecht, 1985.

[20] N. Weaver, *Lipschitz Algebras*, World Scientific Publishing Co. Inc., River Edge, NJ, 1999.

[21] A. Weston, *Some non-uniformly homeomorphic spaces*, Israel Journal of Mathematics **83** (1993), 375–380.