ISRAEL JOURNAL OF MATHEMATICS, Vol. 72, No. 3, 1990

UNIQUENESS OF UNCONDITIONAL BASES IN QUASI-BANACH SPACES WITH APPLICATIONS TO HARDY SPACES

BY

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ABSTRACT

We prove some general results on the uniqueness of unconditional bases in quasi-Banach spaces. We show in particular that certain Lorentz spaces have unique unconditional bases answering a question of Nawrocki and Ortynski. We then give applications of these results to Hardy spaces by showing the spaces $H_p(\mathbf{T}^n)$ are mutually non-isomorphic for differing values of n when $0 < p < 1$.

1. Introduction

The objective of this paper is to give a general result on uniqueness up to permutative equivalence for unconditional bases and then apply this result to show that, when $0 < p < 1$, the Hardy spaces $H_p(\mathbf{T}^m)$ are mutually non-isomorphic for $m \geq 1$.

It is well-known result due to Lindenstrauss, Pełczyński and Zippin ([10],[12]) that precisely three Banach spaces (l_1, l_2 and c_0) have normalized unconditional bases which are unique up to equivalence. For quasi-Banach spaces it was shown in [6] that a wide class of non-locally convex Orlicz sequence spaces including l_p for $0 < p < 1$ share this property. See also [16] and [17]. We significantly extend these results here and, in particular, settle a problem on the uniqueness of unconditional bases in Lorentz sequences spaces raised by Nawrocki and Ortynski [16].

Our techniques enable us to show that in certain quasi-Banach spaces an unconditional basis is close to being unique up to a permutation (cf. [4]). In particular our results apply to the spaces $H_p(\mathbf{T}^m)$ for $p < 1$ which are known to have unconditional bases ([19]). Analysis of these bases shows that the spaces are mutually non-isomorphic; the corresponding result for $p = 1$ is due to Bourgain ([2], [3]) by

†The research of the first two authors was partially supported by NSF-grant DMS 8901636.
Received July 12, 1990

quite different techniques. We might also mention the analogous problem for the Smirnov class has been resolved by Nawrocki [15]. Both the results of Bourgain and Nawrocki depend in some sense on duality arguments, which are not available here since for all m, $H^p(\mathbf{T}^m)^*$ is isomorphic to l_∞.

We recall ([7]) that a quasi-Banach lattice X is said to be *L-convex* if there exists $\epsilon > 0$ so that if $u \geq 0$, $\|u\| = 1$ then for any x_i, $1 \leq i \leq n$ with $0 \leq x_i \leq u$ and such that

$$\frac{1}{n}(x_1 + \cdots + x_n) \geq (1-\epsilon)u$$

we have $\max_{1\leq i\leq n}\|x_i\| \geq \epsilon$. X is said to be *p-convex* where $0 < p < \infty$ if for some C and all $x_1, \ldots, x_n \in X$ we have:

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leq C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}.$$

Here the element $(\sum |x_i|^p)^{1/p}$ of X is defined via the procedure outlined in [11] pp. 40–41. It is shown in [7] that X is L-convex if and only if there exists $p > 0$ so that X is p-convex. A quasi-Banach space Y is called *natural* if it is isomorphic to a subspace of an L-convex quasi-Banach lattice. Every natural quasi-Banach lattice is L-convex ([7]).

Our results apply to natural spaces. In such spaces any unconditional basis induces an L-convex lattice structure; then ([7]) many of the standard techniques of Banach lattice theory can be employed in this more general setting. For most applications it is easy to verify that the spaces of interest are natural either by identifying them as subspaces of L-convex lattices or by showing that some given unconditional basis is already p-convex for some $p > 0$. However it should be pointed out that there are non-natural spaces with unconditional bases ([7]).

Let $\mathbf{T}$ denote the unit circle equipped with its standard normalized Haar measure $(2\pi)^{-1}d\theta$. $\mathbf{T}^m$ denotes the m-fold product with the canonical product measure. The space $H_p(\mathbf{T}^m)$ is defined as the closed linear subspace of $L_p(\mathbf{T}^m)$ generated by the functions $z_1^{n_1} \ldots z_m^{n_m}$ for $n_1, \ldots, n_m \geq 0$. These spaces are, of course, natural being subspaces of L_p.

Some of the results of this paper form part of the thesis of the second author, currently under preparation at the University of Missouri-Columbia [9].

2. Uniqueness of unconditional bases

Our first result is a simple extension of a result of Maurey [13] (Lindenstrauss-Tzafriri [11], p. 49). Notice, however, that the proof in [11] uses duality and therefore does not extend to the non-locally convex case.

PROPOSITION 2.1. *Let X be an L-convex quasi-Banach lattice with an unconditional basis (x_n). Then there is a constant D (depending on X and (x_n)) such that for all scalars $a_1, \ldots, a_m$,*

$$D^{-1}\left\|\left(\sum_{i=1}^{m}|a_i x_i|^2\right)^{1/2}\right\| \leq \left\|\sum_{i=1}^{m} a_i x_i\right\| \leq D\left\|\left(\sum_{i=1}^{m}|a_i x_i|^2\right)^{1/2}\right\|.$$

PROOF. Let Y be the sequence space of all sequences (a_n) such that $\sum a_n x_n$ converges. Then Y is a quasi-Banach lattice under the quasi-norm $\|a\|_Y = \sup_{|\theta_i|\leq 1}\|\sum \theta_i a_i x_i\|_X$. Then there is an isomorphism $T\colon Y \to X$ such that $Te_i = x_i$ where e_i is the canonical basis of Y. By Theorem 4.2 of [7], Y is L-convex, and so by Theorem 3.3. of [7], there is a constant K so that for all $a_1, \ldots, a_m$,

$$K^{-1}\left\|\left(\sum_{i=1}^{m}|a_i x_i|^2\right)^{1/2}\right\|_X \leq \left\|\sum_{i=1}^{m}|a_i| e_i\right\|_Y \leq K\left\|\left(\sum_{i=1}^{m}|a_i x_i|^2\right)^{1/2}\right\|_X.$$

This quickly gives the result.

PROPOSITION 2.2. *Let X be a p-convex quasi-Banach lattice where $0 < p < 1$. Then there is a constant C depending only on X so that if $(x_{ij})_{i,j=1}^{n}$ is an X-valued matrix then*

$$\left\|\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_{ij}|^2\right)^{1/2}\right\| \leq C\int_0^1\int_0^1\left\|\sum_{i=1}^{n}\sum_{j=1}^{n} r_i(s) r_j(t) x_{ij}\right\| ds\, dt$$

where r_i denote the standard Rademacher functions on $[0,1]$.

PROOF. By Bonami's extension of Khintchine's inequality [1], there is a constant C_0 such that for every matrix (x_{ij})

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_{ij}|^2\right)^{1/2} \leq C_0\left(\int_0^1\int_0^1\left|\sum_{i=1}^{n}\sum_{j=1}^{n} r_i(s) r_j(t) x_{ij}\right|^p ds\, dt\right)^{1/p}.$$

Now by p-convexity, for a suitable constant M,

$$\begin{aligned}&\left\|\left(\int_0^1\int_0^1\left|\sum_{i=1}^{n}\sum_{j=1}^{n} r_i(s) r_j(t) x_{ij}\right|^p ds\, dt\right)^{1/p}\right\| \\ &\quad\leq M\left(\int_0^1\int_0^1\left\|\sum_{i=1}^{n}\sum_{j=1}^{n} r_i(s) r_j(t) x_{ij}\right\|^p ds\, dt\right)^{1/p}\end{aligned}$$

and the result follows.

We now introduce the following definition which will facilitate the statement of our main results. We shall say that an unconditional basis (x_n) in a quasi-Banach space X is *strongly absolute* if, for every $\epsilon > 0$, there is a constant C_ϵ such that for any scalars $a_1, \dots, a_m$ we have

$$\sum_{i=1}^{m} |a_i| \le C_\epsilon \sup_{1 \le i \le m} |a_i| + \epsilon \left\| \sum_{i=1}^{m} a_i x_i \right\|.$$

If (x_n) is a normalized strongly absolute basis of X then it is equivalent to the canonical basis of l_1 in the Banach envelope $\hat{X}$.

THEOREM 2.3. *Let X be a natural quasi-Banach space with a normalized strongly absolute unconditional basis $(x_n)_{n=1}^{\infty}$. Then, if $(u_n)_{n=1}^{\infty}$ is any other normalized unconditional basis of X, there exists a map $\sigma : \mathbf{N} \to \mathbf{N}$ and a partition $S_1, \dots, S_N$ of $\mathbf{N}$ such that σ is injective on each S_k and $(u_n)_{n \in S_k}$ is equivalent to $(x_{\sigma(n)})_{n \in S_k}$ for $k = 1, \dots, N$.*

PROOF. The hypotheses on (x_n) force $\hat{X}$ to be isomorphic to l_1. Then since (u_n) must also be an unconditional basis of $\hat{X}$, it follows from the theorem of Lindenstrauss and Pełczyński [10] that it is equivalent in $\hat{X}$ to the canonical basis of l_1. In particular there is a constant D so that if $a_1, \dots, a_m$ are scalars then

$$\sum_{i=1}^{m} |a_i| \le D \left\| \sum_{i=1}^{m} a_i u_i \right\|.$$

Let (x_n^*) and (u_n^*) be the biorthogonal functions for the bases (x_n) and (u_n). Let $a_k^{(n)} = x_k^*(u_n)$ and let $b_k^{(n)} = u_n^*(x_k)$. Since both $\|x_n^*\|$ and $\|u_n^*\|$ are necessarily bounded there is a constant C_0 such that for every n, k $|a_k^{(n)}| \le C_0$ and $|b_k^{(n)}| \le C_0$.

Now let K be the unconditional basis constant of (x_n) and suppose $\epsilon < (2KC_0)^{-1}$. Then we have:

$$\begin{aligned}
1 = u_n^*(u_n) &= u_n^*\left(\sum_{k=1}^{\infty} a_k^{(n)} x_k \right) \\
&= \sum_{k=1}^{\infty} a_k^{(n)} b_k^{(n)} \le \sum_{k=1}^{\infty} |a_k^{(n)} b_k^{(n)}| \\
&\le C_\epsilon \sup_k |a_k^{(n)} b_k^{(n)}| + \epsilon \left\| \sum_{k=1}^{\infty} a_k^{(n)} b_k^{(n)} x_k \right\| \\
&\le C_\epsilon \sup_k |a_k^{(n)} b_k^{(n)}| + \epsilon K C_0 \left\| \sum_{k=1}^{\infty} a_k^{(n)} x_k \right\| \\
&\le C_\epsilon \sup_k |a_k^{(n)} b_k^{(n)}| + \frac{1}{2}.
\end{aligned}$$

Thus there exists a constant $\gamma > 0$ so that for every n, $\sup_k |a_k^{(n)} b_k^{(n)}| \geq \gamma$.

For each k let A_k be the set of n such that $|a_k^{(n)} b_k^{(n)}| \geq \gamma$. Then, if (u_n) is K'-unconditional,

$$\begin{aligned}
\gamma |A_k| &\leq \sum_{n=1}^{\infty} |a_k^{(n)} b_k^{(n)}| \\
&\leq D \left\| \sum_{n=1}^{\infty} a_k^{(n)} b_k^{(n)} u_n \right\| \\
&\leq DK'C_0 \left\| \sum_{n=1}^{\infty} b_k^{(n)} u_n \right\| \\
&= DK'C_0.
\end{aligned}$$

Thus $|A_k| \leq \gamma^{-1} DK'C_0$ for every k. It follows now that if we define a map $\sigma : \mathbf{N} \to \mathbf{N}$ so that $|a_{\sigma(n)}^{(n)} b_{\sigma(n)}^{(n)}| \geq \gamma$ then we can partition $\mathbf{N}$ into N sets $S_1, \ldots, S_N$, with $N \leq \gamma^{-1} DK'C_0$ and so that σ is injective on each S_k. We also have that for all $n \in \mathbf{N}$, $|a_{\sigma(n)}^{(n)}|$, $|b_{\sigma(n)}^{(n)}| \geq \beta = C_0^{-1}\gamma$.

Now fix $1 \leq j \leq N$ and suppose α_n is a finitely non-zero sequence of scalars. Then we observe that the unconditional basis (u_n) induces an L-convex lattice structure on X (since X is natural), and so by Proposition 2.1, for a suitable constant C_1 we have

$$\begin{aligned}
\left\| \sum_{n \in S_j} \alpha_n x_{\sigma(n)} \right\| &\leq K\beta^{-1} \left\| \sum_{n \in S_j} \alpha_n a_{\sigma(n)}^{(n)} x_{\sigma(n)} \right\| \\
&\leq C_1 \left\| \left(\sum_{n \in S_j} |\alpha_n|^2 |a_{\sigma(n)}^{(n)}|^2 |x_{\sigma(n)}|^2 \right)^{1/2} \right\| \\
&\leq C_1 \left\| \left(\sum_{n \in S_j} |\alpha_n|^2 \sum_{l \in S_j} |a_{\sigma(l)}^{(n)}|^2 |x_{\sigma(l)}|^2 \right)^{1/2} \right\|.
\end{aligned}$$

Now we can apply Proposition 2.2 to deduce that, for a suitable constant C_2

$$\begin{aligned}
\left\| \sum_{n \in S_j} \alpha_n x_{\sigma(n)} \right\| &\leq C_2 \int_0^1 \int_0^1 \left\| \sum_{l \in S_j} \sum_{n \in S_j} \alpha_n a_{\sigma(l)}^{(n)} r_l(s) r_n(t) x_{\sigma(l)} \right\| ds\, dt \\
&\leq KC_2 \int_0^1 \left\| \sum_{l \in S_j} \sum_{n \in S_j} \alpha_n a_{\sigma(l)}^{(n)} r_n(t) x_{\sigma(l)} \right\| dt \\
&\leq K^2 C_2 \int_0^1 \left\| \sum_{n \in S_j} \alpha_n r_n(t) u_n \right\| dt \\
&\leq K^2 K' C_2 \left\| \sum_{n \in S_j} \alpha_n u_n \right\|.
\end{aligned}$$

We may now apply similar reasoning, interchanging the roles of the two bases, to deduce that there is a constant C_3 so that for all such (α_n)

$$\left\| \sum_{n \in S_j} \alpha_n u_n \right\| \leq C_3 \left\| \sum_{n \in S_j} \alpha_n x_{\sigma(n)} \right\|.$$

This completes the proof.

Corollary 2.4. *Under the hypotheses of the Theorem, there is also a map $\tau : \mathbf{N} \to \mathbf{N}$ and a partition $R_1, \ldots, R_M$ of $\mathbf{N}$ so that τ is injective on each R_k and $(x_n)_{n \in R_k}$ is equivalent to $(u_{\tau(n)})_{n \in R_k}$ for each $1 \leq k \leq M$.*

Proof. It suffices to observe that Theorem 2.3 also implies that (u_n) is strongly absolute.

Corollary 2.5. *If X is a natural quasi-Banach space with a symmetric strongly absolute basis then all normalized unconditional bases of X are equivalent.*

This is immediate.

There are several immediate applications of Corollary 2.5. In all the applications the fact that X is natural follows quickly from verifying that the given unconditional basis is p-convex for a suitable $p > 0$. The simplest example is the space l_p for $0 < p < 1$. The uniqueness of the unconditional basis for these spaces was first established in [6]; an alternative proof was given in [17], but this appears to be incorrect. In [6] the uniqueness of the unconditional basis of an Orlicz sequence space l_F is considered and Corollary 2.5 above can be used to prove Theorem 7.6 of [6], although it does not imply the stronger Theorem 7.5. Lorentz sequence spaces were similarly investigated by Nawrocki and Ortynski [16] and Corollary 2.5 allows us to answer a question raised in [16]. We recall that if $w = (w_n) \in l_\infty \backslash l_1$ is a monotone decreasing nonnegative sequence, then the Lorentz sequence space $d(w,p)$ is defined to be the space of all sequences $\xi = (\xi_n)$ such that

$$\| \xi \|_{w,p} = \sup_{\pi \in \Pi} \left(\sum_{n=1}^{\infty} | \xi_{\pi(n)} |^p w_n \right)^{1/p} < \infty,$$

where Π is the group of all permutations of $\mathbf{N}$.

Theorem 2.6. *If $0 < p < 1$ and $\lim_{n \to \infty} (1/n)(w_1 + \cdots + w_n)^{1/p} = \infty$ then all normalized unconditional bases of $d(w,p)$ are equivalent.*

Proof. It suffices to note that Theorem 1 and Lemma 4 of [16] together imply that the canonical basis of $d(w,p)$ is strongly absolute.

Our final application of this section is to spaces of the form $l_p(l_q)$ where $0 < p, q < 1$. Such a space has a canonical unconditional basis e_{nk} such that

$$\left\| \sum_{n,k} \alpha_{nk} e_{nk} \right\|^p = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |\alpha_{nk}|^q \right)^{p/q}.$$

THEOREM 2.7. *If $0 < p, q < 1$ then any normalized unconditional basis is permutatively equivalent to the canonical basis of $l_p(l_q)$.*

PROOF. First observe that Theorem 2.3 implies that any normalized unconditional basis (u_n) is (permutatively) equivalent to a subset of the canonical basis. It is also easy to see that it will suffice to show that (u_n) contains a subset equivalent to the canonical basis. To do this we use Corollary 2.4 to partition $\mathbf{N} \times \mathbf{N}$ into finitely many sets $S_1, \ldots, S_N$ so that $(e_{nk})_{(n,k) \in S_j}$ is equivalent to a subset of (u_n) for each j. By standard Ramsey arguments there is an infinite subset of A of $\mathbf{N}$ and a fixed j so that $(a,b) \in S_j$ as long as $a \neq b$ and $a, b \in A$. Write $A = B_0 \cup B_1$ where B_0, B_1 are infinite and disjoint. Then $(e_{nk})_{n \in B_0, k \in B_1}$ is equivalent to a subset of (u_n) and this will complete the proof.

The argument above can be easily extended to a large class of "matrix" spaces. For the Banach space analogues of this theorem see [4].

3. Applications to Hardy spaces

We now use the ideas of Section 2 to show the non-isomorphism of certain Hardy spaces. Let X be a quasi-Banach space with an unconditional basis (x_n). We denote by $l_p(x_n)$ the unconditional basis of $l_p(X)$ obtained by repeating (x_n) in each co-ordinate.

PROPOSITION 3.1. *Let X be a natural quasi-Banach space with a strongly absolute normalized unconditional basis (x_n) and suppose $0 < p < 1$. Then if (y_n) is any other normalized unconditional basis of X, the unconditional bases $l_p(x_n)$ and $l_p(y_n)$ of $l_p(X)$ are permutatively equivalent.*

PROOF. Clearly Theorem 2.3 implies that (y_n) is equivalent to a subset of $l_p(x_n)$, and hence that $l_p(y_n)$ is equivalent to a subset of $l_p(x_n)$. Conversely Corollary 2.4 allows us to show that $l_p(x_n)$ is equivalent to a subset of $l_p(y_n)$. It now follows from a version of the Pełczyński decomposition argument that $l_p(x_n)$ is permutatively equivalent to $l_p(y_n)$. Thus

$$l_p(x_n) \sim l_p(l_p(x_n)) \sim l_p(l_p(y_n) \oplus (f_n)) \sim l_p(y_n) \oplus l_p(x_n),$$

etc., where we use $\sim$ for permutative equivalence and (f_n) is some suitable subsequence of $l_p(x_n)$.

THEOREM 3.2. *Suppose X is a p-normed natural quasi-Banach space, where $0 < p < 1$, and has a strongly absolute normalized unconditional basis (x_n). Suppose (u_n) is any other normalized unconditional basis and define b_N and b'_N to be the greatest constants such that*

$$\left\| \sum_{n\in A} \alpha_n x_n \right\| \geq b_N \left(\sum_{n\in A} |\alpha_n|^p \right)^{1/p}$$

and

$$\left\| \sum_{n\in A} \alpha_n u_n \right\| \geq b'_N \left(\sum_{n\in A} |\alpha_n|^p \right)^{1/p}$$

whenever $|A| \leq N$. Then there is a constant C so that $C^{-1}b_N \leq b'_N \leq Cb_N$ for all N.

PROOF. Clearly the quantities b_N and b'_N are unchanged if the bases $l_p(x_n)$ and $l_p(y_n)$ are used in place of (x_n) and (y_n). The result then follows from Proposition 3.1.

Thus the asymptotic behavior of the sequence $b_N = b_N(X)$ is an isomorphic invariant of X.

We now turn to considering the space $H_p(\mathbf{T})$. This space has an unconditional basis, described in [19], which we denote by $(\psi_{m,k})$, $1 \leq k \leq 2^m$, $1 \leq m < \infty$. Denote by $E_{m,k}$, $1 \leq k \leq 2^m$, $1 \leq m < \infty$ the dyadic interval in [0,1], $E_{m,k} = [(k-1)2^{-m}, k2^{-m})$ and let $\chi_{m,k}$ be the corresponding characteristic function. Then, there is a constant C so that for any finitely nonzero family of complex numbers $(\alpha_{m,k})$

$$C^{-1}\left(\int_0^1 \left(\sum_{m,k} |\alpha_{m,k}|^2 \chi_{m,k}(t) \right)^{p/2} dt \right)^{1/p} \leq \left\| \sum_{m,k} \alpha_{m,k}\psi_{m,k} \right\|_p$$
$$\leq C\left(\int_0^1 \left(\sum_{m,k} |\alpha_{m,k}|^2 \chi_{m,k}(t) \right)^{p/2} dt \right)^{1/p}.$$

Let us denote by (ϕ_n) an enumeration of the corresponding normalized unconditional basis $\|\psi_{m,k}\|^{-1}\psi_{m,k}$.

We remark that the q-Banach envelope of H_p is isomorphic to l_q for $p < q \leq 1$ (see [5],[8],[19]). Thus, ([6]), the unconditional basis ϕ_n is equivalent in the q-Banach envelope to the canonical l_q-basis. In particular in H_p we have the lower estimate:

$$\left\| \sum_{n=1}^{\infty} \alpha_n \phi_n \right\|_p \geq \gamma_q \left(\sum_{n=1}^{\infty} |\alpha_n|^q \right)^{1/q}$$

for a suitable constant $\gamma_q > 0$. This implies that (ϕ_n) is strongly absolute. We thus turn to estimating the invariants b_N for this basis.

PROPOSITION 3.3. *For the space H_p, where $0 < p < 1$, we have $b_N(H_p) \sim (\log N)^{1/2-1/p}$ (i.e. for a suitable constant C we have $C^{-1}(\log N)^{1/2-1/p} \le b_N \le C(\log N)^{1/2-1/p}$ for $N \ge 2$).*

PROOF. We first prove that there is a constant $\delta > 0$ so that for any finite subset A of $\mathbf{N}$, we have:

$$\left\| \sum_{n\in A} \phi_n \right\| \ge \delta |A|^{1/p}.$$

To see this we can consider a finite subset B of $\{(m,k) : 1 \le k \le 2^m,\ 1 \le m < \infty\}$ of cardinality N and estimate:

$$\left\| \sum_{(m,k)\in B} 2^{m/p}\psi_{m,k} \right\|_p \ge C^{-1}\left(\int_0^1 \left(\sum_{(m,k)\in B} 2^{2m/p}\chi_{m,k}(t)\right)^{p/2} dt\right)^{1/p}.$$

Now, for $0 \le t < 1$, let $M(t) = \max\{m : t \in E_{m,k},\ (m,k) \in B\}$. We let $M(t) = -\infty$ if this set is empty. Then we have

$$\begin{aligned}\int_0^1 \left(\sum_{(m,k)\in B} 2^{2m/p}\chi_{m,k}(t)\right)^{(p/2)} dt &\ge \int_0^1 2^{M(t)}\, dt \\ &\ge \frac{1}{2}\int_0^1 \sum_{(m,k)\in B} 2^m \chi_{m,k}(t)\, dt \\ &\ge \frac{1}{2} N,\end{aligned}$$

and our first claim follows easily.

Now suppose α_n is a sequence with at most $N = 2^r$ nonzero terms. Let β_n be decreasing rearrangement of $|\alpha_n|$. Then since H_p has cotype 2 we have the estimate that for suitable constants c_0, $c > 0$, a suitable injection σ of $\{1,2,\ldots,N\}$ into $\mathbf{N}$, and some η_n with $|\eta_n| = 1$,

$$\begin{aligned}\left\| \sum_{n=1}^{\infty} \alpha_n \phi_n \right\|_p &= \left\| \sum_{n=1}^{N} \eta_n \beta_n \phi_{\sigma(n)} \right\|_p \\ &\ge c_0 \left(\sum_{k=1}^{r} \left\| \sum_{n=2^{k-1}}^{2^k-1} \eta_n \beta_n \phi_{\sigma(n)} \right\|_p^2 \right)^{1/2} \\ &\ge c\left(\sum_{k=0}^{r} 2^{2k/p} \beta_{2^k}^2 \right)^{1/2}.\end{aligned}$$

However, by Holder's inequality, we have:

$$\begin{aligned}\sum_{n=1}^{\infty}|\alpha_n|^p &= \sum_{k=1}^{N}\beta_k^p \\ &\le \sum_{k=0}^{r}2^k\beta_{2^k}^p \\ &\le r^{1-p/2}\left(\sum_{k=0}^{r}2^{2k/p}\beta_{2^k}^2\right)^{p/2}.\end{aligned}$$

Thus we deduce that for a suitable $c'>0$,

$$\left\|\sum_{n=1}^{\infty}\alpha\phi_n\right\|_p \ge c'(\log N)^{1/2-1/p}\left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{1/p}$$

so that $b_N \ge c'(\log N)^{1/2-1/p}$.

To complete the proof we observe that

$$\left\|\sum_{m=1}^{r}\sum_{k=1}^{2^m}\psi_{m,k}\right\|_p \le Kr^{1/2}$$

for a suitable constant K. However,

$$\left(\sum_{m=1}^{r}\sum_{k=1}^{2^m}\|\psi_{m,k}\|_p^p\right)^{1/p} \ge cr^{1/p}$$

for a suitable $c>0$. This implies an upper estimate $b_N \le C(\log N)^{1/2-1/p}$ for some $C<\infty$.

Proposition 3.4. *Suppose (f_n), (g_n) are normalized unconditional basic sequences in $L_p[0,1]$. Then the double sequence $(f_m\otimes g_n)_{m,n}$ is an unconditional basic sequence in $L_p[0,1]^2$, and we have for every $N\in\mathbf{N}$,*

$$b_{N^2}(f_n)b_{N^2}(g_n) \le b_{N^2}(f_m\otimes g_n) \le b_N(f_n)b_N(g_n).$$

(Here $f_m\otimes g_n(s,t)=f_m(s)g_n(t)$.)

Proof. The fact that $f_m\otimes g_n$ is an unconditional basic sequence is essentially proved in [18]. We sketch the argument. Suppose $\alpha_{m,n}$ is finitely nonzero. Then

$$\begin{aligned}\left\|\sum_{m,n}\alpha_{m,n}f_m\otimes g_n\right\|^p &\sim \int_0^1\int_0^1\left\|\sum_{m,n}r_m(s)r_n(t)\alpha_{m,n}f_m\otimes g_n\right\|^p ds\,dt \\ &\sim \iiiint\left|\sum_{m,n}\alpha_{m,n}r_m(s)r_n(t)f_m(u)g_n(v)\right|^p ds\,dt\,du\,dv \\ &\sim \iint\left(\sum_{m,n}|\alpha_{m,n}|^2|f_m(u)|^2|g_n(v)|^2\right)^{p/2} du\,dv\end{aligned}$$

using again Bonami's extension of the Khintchine inequality [1]. This quickly shows unconditionality of $(f_m \otimes g_n)$.

Now suppose α_n and β_n are two sequences with at most N nonzero entries. Then

$$\left\| \sum_{m,n} \alpha_m \beta_n f_m \otimes g_n \right\|_p = \left\| \sum_m \alpha_m f_m \right\|_p \left\| \sum_n \beta_n g_n \right\|_p$$

from which we deduce that

$$b_{N^2}(f_m \otimes g_n) \leq b_N(f_n) b_N(g_n).$$

Conversely suppose $\alpha_{m,n}$ has at most N^2 nonzero terms. Then

$$\int_0^1 \int_0^1 \left| \sum_{m,n} \alpha_{m,n} f_m(s) g_n(t) \right|^p ds\, dt \geq b_{N^2}^p(g_n) \int_0^1 \sum_n \left| \sum_m \alpha_{m,n} f_m(s) \right|^p ds$$

$$\geq b_{N^2}^p(f_n) b_{N^2}^p(g_n) \sum_{m,n} |\alpha_{m,n}|^p$$

and the proposition follows.

Theorem 3.5. *For each $m \in \mathbf{N}$ the space $H_p(\mathbf{T}^m)$ has a strongly absolute unconditional basis for which $b_N = b_N(H_p(\mathbf{T}^m)) \sim (\log N)^{m(1/2-1/p)}$.*

Proof. It follows by induction from Proposition 3.4 that if (ϕ_n) is the basis of H_p considered above, then $(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m})$ for $i_1, \ldots, i_m \in \mathbf{N}$ is an unconditional basis of $H_p(\mathbf{T}^m)$. Further, for this basis we clearly have $b_N \sim (\log N)^{m(1/2-1/p)}$. But this also implies the basis is strongly absolute. In fact if (g_n) denotes this basis, α_n is finitely nonzero, and β_n is the decreasing rearrangement of $|\alpha_n|$ we quickly deduce that $|\beta_N| \leq C b_N^{-1} N^{-1/p} \| \sum \alpha_n g_n \|_p$ for all N whence we get an estimate

$$\left\| \sum \alpha_n g_n \right\|_p \geq c_q \left(\sum |\alpha_n|^q \right)^{1/q}$$

for some $c_q > 0$ whenever $p < q \leq 1$.

Theorem 3.6. *The spaces $H_p(\mathbf{T}^m)$ are mutually non-isomorphic when $p < 1$.*

Proof. This is now immediate.

For $p = 1$, the result analogous to Theorem 3.6 is due to Bourgain [2] and [3]. For $p > 1$, it is false.

COROLLARY 3.7. *For $0 < p < 1$ and $m < n$, the space $H_p(\mathbf{T}^n)$ is not isomorphic to a complemented subspace of $H_p(\mathbf{T}^m)$.*

PROOF. In fact $H_p(\mathbf{T}^m)$ is isomorphic to a complemented subspace of $H_p(\mathbf{T}^n)$. This is well-known, and can be observed as a consequence of tensoring of unconditional bases. Since each space is isomorphic to its own square, it follows from standard Pełczyński decomposition arguments that if $H_p(\mathbf{T}^n)$ is complemented in $H_p(\mathbf{T}^m)$ then the two spaces are isomorphic.

REMARK 1. In fact $H_p(\mathbf{T}^m)$ is isomorphic to a subspace of $H_p(\mathbf{T})$. More generally any subspace of L_p with an unconditional basis can be embedded in H_p. This fact, for $p = 1$, is due to Maurey [14], but easily extends to $p < 1$.

REMARK 2. The argument for Proposition 3.3 works for the Haar basis in $L_p[0,1]$ for $1 < p < 2$ and this and a duality argument show that if $1 < p < \infty$ then the tensored Haar bases of $L_p([0,1]^n)$ are not permutatively equivalent for differing choices of n.

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