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M-IDEALS OF COMPACT OPERATORS

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1. Introduction

If X is a Banach space and E is a subspace of X then E is called an M-ideal in X if X^* can be decomposed an l_1-sum $X^* = E^\perp \oplus_1 V$ for some closed subspace V of X^*. This notion was introduced by Alfsen and Effros [1].

For any Banach space X we denote by $\mathscr{L}(X)$ the algebra of all bounded operators on X and by $\mathscr{K}(X)$ the ideal of compact operators. The first non-trivial example of an M-ideal obtained (Dixmier [7]) is that $\mathscr{K}(l_2)$ is an M-ideal in $\mathscr{L}(l_2)$. Subsequently, there has been considerable work on studying spaces X for which $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$. It was shown in Lima [20] that $\mathscr{K}(l_p)$ is an M-ideal in $\mathscr{L}(l_p)$ when $1 < p < \infty$ and that similarly $\mathscr{K}(c_0)$ is an M-ideal in $\mathscr{L}(c_0)$. Cho and Johnson [6] (cf. [2], [31]) showed that if a subspace X of l_p for $1 < p < \infty$ has the compact approximation property then $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$. Conversely, it is known that any separable space X for which $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ satisfies the conditions that X^* is separable and has the metric compact approximation property (Harmand-Lima [11]). Further X must be an M-ideal in X^{**} [21] and, if it has the approximation property, it has an unconditional finite-dimensional expansion of the identity [9], [19] from which it can be deduced that X can be $(1+\varepsilon)$—embedded in a space with a shrinking 1-unconditional basis [19].

The aim of this paper is to give a classification of those separable Banach spaces X such that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$. Having achieved this classification, then some outstanding questions can be resolved.

Our main result is Theorem 2.4 which lists six equivalences. The most important conclusion (condition (5)) is that a separable Banach space X has the property that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if X satisfies a structural condition, which we call property (M), and there is a sequence of compact operators (K_n) such that $K_n \to I$ strongly, $K_n^* \to I$ strongly and $\lim_{n\to\infty}\|I - 2K_n\| = 1$. Property (M) is the requirement that if u, v satisfy

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$\|u\| = \|v\|$ and if (x_n) is a weakly null sequence then

$$\limsup_{n\to\infty} \|u + x_n\| = \limsup_{n\to\infty} \|v + x_n\|.$$

Having established this theorem, some conclusions follow. For example, the Cho-Johnson theorem can be extended to show that if Y is a separable Banach space such that $\mathscr{K}(Y)$ is an M-ideal in $\mathscr{L}(Y)$ then for any subspace X with the metric compact approximation property (or just the compact approximation property in the reflexive case) we also have $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$.

Using ideas from the theory of types [18] we can then show that if X has property (M) then it contains a $(1 + \varepsilon)$—copy of some l_p (cf. [27]). Thus, answering a question in [30] we can deduce that the various Tsirelson spaces [5] cannot be renormed so that the compact operators become an M-ideal. On the other hand we show that reflexive Orlicz sequence spaces l_F can be renormed so that $\mathscr{K}(l_F)$ is an M-ideal in $\mathscr{L}(l_F)$. No such renorming can preserve the (isometric) symmetry of the basis unless $l_F = l_p$ for some p by a result of Hennefeld [14]. More generally we show that a Banach space with a shrinking symmetric basis can be renormed so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if it is isomorphic to an Orlicz sequence space (or more strictly the closed linear span of the canonical basis vectors in an Orlicz sequence space).

We also show that a separable order-continuous nonatomic Banach lattice X which is not isomorphic to L_2 can never be renormed so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$. Thus, if $1 < p < \infty$ and $p \neq 2$ then L_p cannot be renormed so that $\mathscr{K}(L_p)$ becomes an M-ideal. The isometric result is well-known [19], [20], [27].

Throughout the paper we will consider only the real case, but the generalization to complex scalars presents no difficulty. We also specialize to separable Banach spaces; this allows us to use sequential arguments. However, most of our results can be extended in some form to the non-separable case.

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2. M-ideals and property (M)

We shall say that a Banach space X has *property* (M) if whenever $u, v \in X$ with $\|u\| = \|v\|$ and (x_n) is weakly null sequence in X then,

$$\limsup_{n\to\infty} \|u + x_n\| = \limsup_{n\to\infty} \|v + x_n\|.$$

We first state a simple lemma.

LEMMA 2.1. *The following conditions are equivalent*:
(1) *X has property (M).*
(2) *If $\|u\| = \|v\|$ and (x_n) is a weakly null sequence such that* $\lim\|u + x_n\|$ *exists then* $\lim\|v + x_n\| = \lim\|u + x_n\|$.
(3) *If $\|u\| \le \|v\|$ and (x_n) is any weakly null sequence then* $\limsup\|u + x_n\| \le \limsup\|v + x_n\|$.
(4) *If $(u_n)(v_n)$ are relatively norm-compact sequences with $\|u_n\| \le \|v_n\|$ for every n, and (x_n) is a weakly null sequence, then*

$$\limsup_{n\to\infty}\|u_n + x_n\| \le \limsup_{n\to\infty}\|v_n + x_n\|.$$

Proof. The equivalence of (1) and (2) is automatic. To deduce (3) from (1) note that

$$\limsup\|u + x_n\| \le \limsup\big(\max(\|\lambda u + x_n\|, \|\lambda u - x_n\|)\big)$$

where $\lambda \ge 1$ is chosen so that $\|\lambda u\| = \|v\|$. The converse is automatic. Let us conclude by deducing (4) from (2). Indeed, if the conclusion is false we may pass to subsequence so that $\limsup\|u_{n_k} + x_{n_k}\| > \limsup\|v_{n_k} + x_{n_k}\|$ and $(u_{n_k}), (v_{n_k})$ are convergent. This quickly leads to a contradiction. ∎

LEMMA 2.2. *Suppose X has property (M) and that $T \in \mathscr{L}(X)$ with $\|T\| \le 1$. If $(u_n), (v_n)$ are relatively norm-compact sequences with $\|u_n\| \le \|v_n\|$ and (x_n) is a weakly null sequence then*

$$\limsup\|u_n + Tx_n\| \le \limsup\|v_n + x_n\|.$$

Proof. First suppose $\|T\| = 1$ and (x_n) is any weakly null sequence. For any $\lambda < 1$ there exists w with $\|w\| = 1$ and $\|Tw\| > \lambda$. Let $w_n = \|v_n\|w$. Hence

$$\begin{aligned}\limsup\|\lambda u_n + Tx_n\| &\le \limsup\|Tw_n + Tx_n\|\\ &\le \limsup\|w_n + x_n\|\\ &\le \limsup\|v_n + x_n\|.\end{aligned}$$

Letting $\lambda \to 1$ we obtain the conclusion for this case.

Now suppose $0 \le \|T\| \le 1$. Let $T = \lambda L$ where $\lambda = \|T\|$ and $\|L\| = 1$. Then

$$\begin{aligned}\limsup\|u_n + Tx_n\| &\le \max\big(\limsup\|u_n + Lx_n\|, \limsup\|u_n - Lx_n\|\big)\\ &\le \max\big(\limsup\|v_n + x_n\|, \limsup\|-v_n + x_n\|\big)\\ &= \limsup\|v_n + x_n\|. \quad ∎\end{aligned}$$

We also introduce condition (M*). We say that X has *property* (M^*) if whenever $u^*, v^* \in X^*$ with $\|u^*\| \le \|v^*\|$ and whenever (x_n^*) is a weak*-null sequence then

$$\limsup \|u^* + x_n^*\| \le \limsup \|v^* + x_n^*\|.$$

As with property (M) there are a number of equivalent definitions.

PROPOSITION 2.3. *Let X be a separable Banach space with property (M^*). Then X has property (M) and X is an M-ideal in X^{**}.*

Proof. First we show that X has property (M). Suppose $\|u\| = \|v\|$ and that (x_n) is a weakly null sequence such that $\lim \|u + x_n\| > \lim \|v + x_n\|$ (and both limits exist). Then pick $x_n^* \in B_X^*$ so that $\lim x_n^*(u + x_n) = \lim \|u + x_n\|$. By passing to a subsequence we may suppose that x_n^* converges weak* to some x^*. Now pick v^* with $\|v^*\| = \|x^*\|$ and $v^*(v) = \|x^*\| \, \|v\|$. Then

$$\begin{aligned}
\lim_{n\to\infty} \|u + x_n\| &= \lim_{n\to\infty} \langle u + x_n, x_n^* \rangle \\
&= \langle u, x^* \rangle + \lim_{n\to\infty} \langle x_n, x_n^* - x^* \rangle \\
&\le \langle v, v^* \rangle + \lim_{n\to\infty} \langle x_n, x_n^* - x^* \rangle \\
&= \lim_{n\to\infty} \langle v + x_n, v^* + x_n^* - x^* \rangle \\
&\le \limsup_{n\to\infty} \|v^* + x_n^* - x^*\| \, \|v + x_n\| \\
&\le \lim_{n\to\infty} \|v + x_n\|
\end{aligned}$$

since $\limsup \|v^* + x_n^* - x^*\| = \limsup \|x^* + x_n^* - x^*\| = 1$ by property (M^*).

Next we show that X is an M-ideal. Suppose $\phi \in X^\perp \subset X^{***}$ and suppose $\psi \in X^*$ (canonically embedded in X^{***}.) For $\lambda < 1$ we can pick $x^{**} \in B_{X^{**}}$ so that $\phi(x^{**}) > \lambda \|\phi\|$. Pick any x^* in X^* with $\|x^*\| = \|\psi\|$ and $x^{**}(x^*) > \lambda \|\psi\|$. Let y_d^* be a net in X^* converging weak* in X^{***} to ϕ and such that $\|\psi + y_d^*\| \le \|\psi + \phi\|$. We can suppose that $x^{**}(y_d^*) > \lambda \|\phi\|$ for all d. Then y_d^* converges weak* in X^* to zero and so the set $\{y_d^*\}$ contains a sequence v_n^* converging to zero. Now

$$\begin{aligned}
\|\psi + \phi\| &\ge \limsup \|\psi + v_n^*\| \\
&= \limsup \|x^* + v_n^*\| \\
&\ge \limsup \langle x^* + v_n^*, x^{**} \rangle \\
&\ge \lambda(\|\psi\| + \|\phi\|),
\end{aligned}$$

and the result follows. ■

Suppose X is a separable Banach space. We say that (K_n) is a compact approximating sequence for X if each $K_n\colon X \to X$ is a compact operator and $\lim\|K_n x - x\| = 0$ for every $x \in X$. We say (K_n) is shrinking if $\lim\|K_n^* x^* - x^*\| = 0$ for every $x^* \in X^*$. If X has a shrinking compact approximating sequence (K_n) then X^* is separable and further X has a shrinking compact approximating sequence (L_n) with $\|L_n\| \leq 1$, i.e., X has the metric shrinking compact approximation property.

If $\phi \in \mathscr{K}(X)^*$ then by the Hahn-Banach and Riesz Representation Theorems there is a (not necessarily unique) regular Borel measure μ on $B_{X^*} \times B_{X^{**}}$ with the respective weak* topologies such that $\|\phi\| = \|\mu\|$, such that for $S \in \mathscr{K}(X)$,

$$\phi(S) = \int \langle S^* x^*, x^{**} \rangle \, d\mu(x^*, x^{**}).$$

If X has a shrinking compact approximating sequence (K_n) then for each $S \in \mathscr{L}(X)$, we have $\lim\langle K_n^* S^* x^*, x^{**} \rangle = \langle S^* x^*, x^{**} \rangle$ and so the above formula can be used to extend ϕ in a unique norm-preserving way to $\mathscr{L}(X)^*$, and $\phi(S) = \lim \phi(SK_n)$. In this way we can regard $\mathscr{K}(X)^*$ as naturally identified as a subspace of $\mathscr{L}(X)^*$.

If $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ then it has a shrinking compact approximating sequence [11] and $\mathscr{L}(X)^* = \mathscr{K}(X)^* \oplus_1 \mathscr{K}(X)^{\perp}$.

THEOREM 2.4. *Suppose X is a separable Banach space. Then the following conditions are equivalent*:

(1) *$\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$.*

(2) *For any separable subspace $\mathscr{E} \subset \mathscr{L}(X)$ there exists a shrinking compact approximating sequence (K_n) such that*

$$\limsup_{n \to \infty} \| SK_n + T(I - K_n) \| \leq \max(\|S\|, \|T\|)$$

whenever $S, T \in \mathscr{E}$.

(3) *There exists a shrinking compact approximating sequence (K_n) with $\|K_n\| \leq 1$, such that for any $S \in \mathscr{K}(X)$, $T \in \mathscr{L}(X)$ we have*

$$\limsup_{n \to \infty} \| S + T(I - K_n) \| \leq \max(\|S\|, \|T\|).$$

(4) *There exists a shrinking compact approximating sequence (K_n) such that for every compact operator S with $\|S\| \leq 1$, we have*

$$\limsup_{n \to \infty} \|S + I - K_n\| \leq 1.$$

(5) *X has property (M) and a shrinking compact approximating sequence (K_n) such that*

$$\lim \|I - 2K_n\| = 1.$$

(6) *X has property (M^*) and a shrinking compact approximating sequence (K_n) such that*

$$\lim \|I - 2K_n\| = 1.$$

Proof. (1) $\Rightarrow$ (2). This is basically a rewording of a theorem of W. Werner [32] but our proof is somewhat different. We use the fact that since $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ there exists a shrinking compact approximating sequence (K_n). It will suffice to show that if $S, T \in \mathscr{L}(X)$, $n \in \mathbf{N}$, and (L_n) is any shrinking compact approximating sequence then for any $\varepsilon > 0$ there exists $L \in \operatorname{co}\{L_n, L_{n+1}, \dots\}$ with

$$\|SL + T(I - L)\| < \max(\|S\|, \|T\|) + \varepsilon.$$

Once this is achieved the proof can be completed by a simple diagonal argument. In fact if this is not so then by the Hahn-Banach Theorem there exists $\phi \in \mathscr{L}(X)^*$ with $\|\phi\| = 1$ and

$$\phi(SL_k + T(I - L_k)) \geq \max(\|S\|, \|T\|) + \varepsilon,$$

for $k \geq n$. Write $\phi = \psi + \chi$ where $\chi \in \mathscr{K}(X)^*$ and $\psi \in \mathscr{K}(X)^{\perp}$. Then

$$\lim_{k \to \infty} \phi(SL_k + T(I - L_k)) = \chi(S) + \psi(T).$$

However

$$\chi(S) + \psi(T) \leq \max(\|S\|, \|T\|)$$

by the M-ideal property.

(2) $\Rightarrow$ (4). Since $\mathscr{K}(X)$ is separable this follows immediately from (2).

(4) $\Rightarrow$ (5). Now suppose (x_n) is a weakly null sequence and that $\|u\| \leq \|v\|$. Then there is a rank-one operator S with $\|S\| \leq 1$ and $Sv = u$. Since $\lim_{n \to \infty} \|K_m x_n\| = 0$ we may find an increasing sequence $r_n \to \infty$ such that $\lim \|K_{r_n} x_n\| = 0$. Now

$$\begin{aligned} \limsup_{n \to \infty} \|u + x_n\| &= \limsup_{n \to \infty} \left\| Sv + (I - K_{r_n}) x_n \right\| \\ &= \limsup_{n \to \infty} \left\| S(v + x_n) + (I - K_{r_n})(v + x_n) \right\| \\ &\leq \limsup_{n \to \infty} \left\| S + I - K_{r_n} \right\| \limsup_{n \to \infty} \| v + x_n \| \\ &\leq \limsup_{n \to \infty} \| v + x_n \|. \end{aligned}$$

Thus X has property (M). We now show that X has a shrinking compact approximating sequence (L_n) with such that $\lim\|I - 2L_n\| = 1$. We show that we can pick $L_n \in \operatorname{co}\{K_n, K_{n+1}, \ldots\}$ for every n, so that $\lim\|I - 2L_n\| = 1$. Indeed, if not, then by the Hahn-Banach Theorem there exists $\varepsilon > 0$, $n_0 \in \mathbf{N}$ and $\phi \in \mathscr{L}(X)^*$ with $\|\phi\| = 1$ and $\phi(I - 2K_n) \geq 1 + \varepsilon$ for every $n \geq n_0$. Let $\phi = \psi + \chi$ where $\chi \in \mathscr{K}(X)^*$ and $\psi \in \mathscr{K}(X)^{\perp}$. Then by taking limits we have $\psi(I) - \chi(I) \geq 1 + \varepsilon$. However this implies that there exists m_0 such that if $m \geq m_0$ then $\psi(I) - \chi(K_m) \geq 1 + \frac{1}{2}\varepsilon$ and so if $H \in \operatorname{co}\{K_i \colon i \geq m_0\}$ we have

$$\liminf_{n\to\infty} \phi(-H + I - K_n) \geq 1 + \tfrac{1}{2}\varepsilon.$$

This means in turn that

$$\liminf\| - H + I - K_n\| \geq 1 + \tfrac{1}{2}\varepsilon.$$

Now condition (4) implies that for any compact S $\limsup\|S + I - K_n\| \leq \max(\|S\|, 1)$. Hence we conclude that $\|H\| \geq 1 + \frac{1}{2}\varepsilon$ and another application of the Hahn-Banach theorem produces $\beta \in \mathscr{K}(X)^*$ with $\|\beta\| \leq 1$, such that $\beta(K_m) \geq 1 + \frac{1}{2}\varepsilon$ for $m \geq m_0$. Thus $\beta(I) \geq 1 + \frac{1}{2}\varepsilon$ and we have a contradiction.

(5) $\Rightarrow$ (3). We first prove that for each m,

$$\limsup_{n\to\infty}\|K_m + I - K_n\| \leq \|I - 2K_m\|.$$

In fact if $n \geq m$, $(I - 2K_n)(I - 2K_m) = I - 2K_n - 2K_m + 4K_nK_m$ and hence

$$\|I - K_n - K_m + 2K_nK_m\| \leq \|I - 2K_n\|\,\|I - 2K_m\|$$

so that the assertion follows by taking limits, since $\lim_{n\to\infty}\|K_nK_m - K_m\| = 0$.

Now suppose $S \in \mathscr{K}(X)$, $T \in \mathscr{L}(X)$ with $\|S\|, \|T\| \leq 1$.

We may pick $x_n \in B_X$ so that

$$\limsup_{n\to\infty}\|S + T(I - K_n)\| = \limsup_{n\to\infty}\|Sx_n + T(I - K_n)x_n\|.$$

For any $m \in \mathbf{N}$, (K_mx_n) is contained in a relatively compact set. Thus by Lemma 2.1, and noting as before that $((I - K_n)x_n)$ and $(T(I - K_n)x_n)$ are both weakly null,

$$\limsup_{n\to\infty}\|SK_mx_n + T(I - K_n)x_n\| \leq \limsup_{n\to\infty}\|K_mx_n + T(I - K_n)x_n\|.$$

Now utilizing Lemma 2.2 we conclude that

$$\limsup_{n\to\infty} \| SK_m x_n + T(I-K_n)x_n \| \le \limsup_{n\to\infty} \|(K_m + I - K_n)x_n\|.$$

Thus

$$\limsup_{n\to\infty} \| S + T(I-K_n) \| \le \|S - SK_m\| + \|I - 2K_m\|.$$

Letting $m \to \infty$ yields (3).

(3) $\Rightarrow$ (1). Suppose $\psi \in \mathscr{K}(X)^{\perp}$ and $\chi \in \mathscr{K}(X)^*$. Suppose $S \in \mathscr{K}(X)$ and $T \in \mathscr{L}(X)$. Then

$$\lim_{n\to\infty} (\psi + \chi)(S + T(I-K_n)) = \psi(T) + \chi(S).$$

It follows that

$$\|\psi + \chi\| + \|\psi\| + \|\chi\|$$

and $\mathscr{K}(X)$ is an M-ideal.

(6) $\Rightarrow$ (5). This follows immediately since (M*) implies (M).

(5) $\Rightarrow$ (6). We need only verify that X has property (M*). If not, we may suppose that there are $u^*, v^* \in X^*$ with $\|u^*\| \le \|v^*\|$ and x_n^* converging to zero weak*, so that

$$\lim_{n\to\infty} \|u^* + x_n^*\| > \lim_{n\to\infty} \|v^* + x_n^*\|.$$

Pick $x_n \in B_X$ such that

$$\lim_{n\to\infty} u^*(x_n) + x_n^*(x_n) = \lim_{n\to\infty} \|u^* + x_n^*\|.$$

Now let (K_n) be a shrinking compact approximating sequence with $\lim\|I - 2K_n\| = 1$. We note that by the proof of (5) $\Rightarrow$ (3) we have that, for any $\limsup_n \|K_m + I - K_n\| \le \|I - 2K_m\|$.

Fix any m and $\lambda < 1$. Then there exists $v \in X$ so that $\|v\| = 1$ and $v^*(v) \ge \lambda \|v^*\|$. Let $w_n = \|K_m x_n\| v$. We may also pick a sequence $r_n \to \infty$ such that $\lim \|K_{r_n}^* x_n^*\| = 0$ (since (x_n^*) is weak* null.)

Now

$$\begin{aligned}\limsup_{n\to\infty} \left\| w_n + (I - K_{r_n})x_n \right\| &= \limsup_{n\to\infty} \left\| K_m x_n + (I - K_{r_n})x_n \right\| \\ &\le \|I - 2K_m\|,\end{aligned}$$

since X has property (M).

Now $\lim x_n^*(w_n) = 0$, $\lim v^*(I - K_{r_n})x_n = 0$ and $\|K_{r_n}^* x_n^*\| \to 0$ so that

$$\limsup_{n\to\infty} \left(\lambda \|v^*\| \, \|K_m x_n\| + x_n^*(x_n)\right) \le \|I - 2K_m\| \lim_{n\to\infty} \|v^* + x_n^*\|.$$

This in turn implies, since $\lambda < 1$ is arbitrary and $\|u^*\| \le \|v^*\|$

$$\limsup_{n\to\infty} \left(u^*(K_m x_n) + x_n^*(x_n)\right) \le \|I - 2K_m\| \lim_{n\to\infty} \|v^* + x_n^*\|.$$

Hence

$$\lim_{n\to\infty} \|u^* + x_n^*\| \le \|K_m^* u^* - u^*\| + \|I - 2K_m\| \lim_{n\to\infty} \|v^* + x_n^*\|.$$

Letting $m \to \infty$ produces a contradiction and shows that X has property (M*). ∎

Remark. D. Werner has pointed out to us that the fact that if $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ then X has a shrinking compact approximating sequence (K_n) with $\lim_{n\to\infty}\|I - 2K_n\| = 1$ can be traced to [11] (see Lemma 5.1 and succeeding remarks) by using results of [29]. We would like to stress that the condition $\lim_{n\to\infty}\|I - 2K_n\| = 1$ in (5) and (6) is a very strong unconditionality assumption. As in [4] it can be used to show that there is a sequence of compact operators A_n so that for every $x \in X$ we have $x = \Sigma A_n x$ unconditionally. (In [4] this is done for the case when each K_n is finite-rank, but the modifications are not difficult.) In this setting such results were shown by Li [19]. We refer also to the forthcoming article [8] for further results in the same direction.

The next theorem is now almost immediate. The analogous result for the special case of l_p, $(1 < p < \infty)$ or c_0 is known; see [2], [6], [26], and [31].

THEOREM 2.5. *Let X be a separable Banach space such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Let E be a subspace of a quotient space of X. Then $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$ if and only if E has the metric compact approximation property.*

Proof. First we note that E is an M-ideal (in E^{**}) since X is. It follows that if E has the metric compact approximation property it has a shrinking compact approximating sequence (K_n) (see [10]). We also note that quotient spaces of X inherit property (M*) and subspaces of spaces with property (M) have property (M). Thus E inherits property (M) from X.

It remains to check that if E has the metric approximation property then it has a shrinking compact approximating sequence (H_n) such that $\lim\|I -$

$2H_n\| = 1$. Suppose (L_n) is a shrinking compact approximating sequence for X such that $\lim\|I - 2L_n\| = 1$.

Let us suppose that N is a closed subspace of X and that E is a closed subspace of X/N. Let J_E: $E \to X/N$ be the inclusion map and let Q: $X \to X/N$ be the quotient map. Let $Y = Q^{-1}(E)$ and denote by J_Y: $Y \to X$ the inclusion and Q_Y: $Y \to E$ the restriction of Q so that $QJ_Y = J_E Q_Y$ on Y. Consider the sequence of operators

$$(QL_nJ_Y - J_EK_nQ_Y)_{n=1}^{\infty}$$

in $\mathscr{K}(Y, X/N)$. Then we claim that this sequence is weakly null. In fact (cf. [16]) it suffices to show that for any $y^{**} \in Y^{**}$ and $\xi^* \in (X/N)^*$ we have

$$\lim y^{**}(J_Y^*L_n^*Q^*\xi^* - Q_Y^*K_n^*J_E^*\xi^*) = 0.$$

In fact, $K_n^*J_E^*\xi^*$ converges in norm to $J_E^*\xi^*$ and $L_n^*Q^*\xi^*$ converges in norm to $Q^*\xi^*$ so that this is clear. Now by Mazur's theorem there exists

$$H_n \in \operatorname{co}\{K_n, K_{n+1}, \dots\} \quad \text{and} \quad M_n \in \operatorname{co}\{L_n, L_{n+1}, \dots\}$$

such that $\lim\|QM_nJ_Y - J_EH_nQ_Y\| = 0$. Thus $\lim\|Q(I - 2M_n)J_Y - J_E(I - 2H_n)Q_Y\| = 0$ and so $\lim\|I - 2H_n\| = 1$.

The theorem is now completed by an appeal to Theorem 2.4. ■

Remark. If X is reflexive we need only suppose that E has the compact approximation property (cf. [6]).

Our next result was suggested to us by D. Werner. Actually, as he points out, it is an immediate consequence of condition (3) of Theorem 2.4 and the results of W. Werner [32]. However, we will give a self-contained proof. We denote by $\mathscr{T}(X)$ the subalgebra of $\mathscr{L}(X)$ spanned by $\mathscr{K}(X)$ and the identity.

THEOREM 2.6. *Let X be a separable Banach space. Then $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if $\mathscr{K}(X)$ is an M-ideal in $\mathscr{T}(X)$.*

Proof. One direction is trivial. Suppose for the other that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{T}(X)$ so that $\mathscr{L}(X)^* = V \oplus_1 \mathscr{K}(X)^{\perp}$. Then it is easy to see that every functional $T \to \langle x^{**}, T^*x^*\rangle$ is in V. We show first that there is a shrinking compact approximating sequence (K_n) in X. First we show that I is in the closure of $B_{\mathscr{K}(X)}$ for the strong operator topology. In fact if not there exist by the Hahn-Banach theorem $x_1, \dots, x_n \in X$ and $x_1^*, \dots, x_n^* \in X^*$ such that if $\phi(S) = \sum_{i=1}^n x_i^*(Sx_i)$ then $|\phi(S)| \le \|S\|$ for $S \in \mathscr{K}(X)$ but $\phi(I) > 1$. Let $\phi_0 \in \mathscr{T}(X)^*$ be such that $\|\phi_0\| \le 1$ and $\phi = \phi_0$ on $\mathscr{K}(X)$. Then

$\|\phi_0\| = \|\phi\| + \|\phi_0 - \phi\|$ since $\phi \in V$. This is a contradiction and hence there is a sequence (L_n) in $\mathscr{K}(X)$ with $\|L_n\| \leq 1$ and $L_n x \to x$ for every $x \in X$. Now suppose $x^* \in X^*$; then $L_n^* x^* \to x^*$ weak*. We show that in fact $L_n^* x^* \to x^*$ weakly. Let x^{***} be any weak*-cluster point of $(K_n^* x^*)$ in X^{***}. Then $\|x^{***}\| = \|x^*\|$. We must show $x^{***} = x^*$. Fix any $x^{**} \in X^{**}$ and consider the functional on $\mathscr{T}(X)$ given by $\phi(A) = \langle A^{**}x^{**}, x^{***}\rangle$. Let $\psi(A) = \langle x^*, A^{**}x^{**}\rangle$. If S is compact then $\phi(S) = \psi(S)$ and since $\psi \in V$ we conclude that $\|\phi\| = \|\psi\| + \|\phi - \psi\|$. Hence $\phi = \psi$ and so evaluating at I we have $x^{***}(x^{**}) = x^{**}(x^*)$. We conclude that $x^{***} = x^*$ and hence $L_n^* x^* \to x^*$ weakly. Thus X^* is separable and there is a sequence (H_n) of convex combinations of (L_n) which is a shrinking compact approximating sequence.

We now verify condition (4) of Theorem 2.5. By a diagonal argument it is only necessary to prove that if (H_n) is a shrinking compact approximating sequence with $\|H_n\| = 1$ and if S is compact with $\|S\| \leq 1$ then for any $n \in \mathbf{N}$ and for any $\varepsilon > 0$ there exists $H \in \mathrm{co}\{H_n, H_{n+1}, \dots\}$ with $\|S + I - H\| < 1 + \varepsilon$. If this is not the case then by Hahn-Banach theorem there exist n_0, $\varepsilon > 0$ and $\phi \in \mathscr{T}(X)^*$ such that $\|\phi\| \leq 1$ and $\phi(S + I - H_n) \geq 1 + \varepsilon$, for $n \geq n_0$. Since (H_n) is weakly Cauchy in $\mathscr{T}(X)$ we can define $\chi(A) = \lim_{n\to\infty} \phi(AH_n)$ for $A \in \mathscr{T}(X)$. Clearly, if $\psi = \chi - \phi$ then $\psi \in \mathscr{K}(X)^\perp$; also χ is clearly in V since $\|\chi\| = \sup\{|\chi(T)|\colon T \in \mathscr{K}(X),\ \|T\| \leq 1\}$. Now taking limits we obtain $\psi(I) + \chi(S) \geq 1 + \varepsilon$ and hence $\|\psi\| + \|\chi\| \geq 1 + \varepsilon$. This contradicts the fact that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{T}(X)$. ∎

3. Spaces with property (M)

We recall that if X is a separable Banach space then a weakly null type τ: $X \to \mathbf{R}$ is a function of the form $\tau(x) = \lim_{n\to\infty}\|x + x_n\|$ where (x_n) is a weakly null sequence. The sequence (x_n) is then said to generate the type τ. We say that τ is trivial if $\tau(0) = 0$ (i.e., $\lim_{n\to\infty}\|x_n\| = 0$) and nontrivial otherwise.

Notice that since X is separable every weakly null sequence contains a subsequence which generates a weakly null type. The following lemma is immediate:

Lemma 3.1. *Suppose X is a separable Banach space. Then X has property (M) if and only if every weakly null type τ is of the form $\tau(x) = f(\|x\|)$ for a suitable function $f\colon \mathbf{R} \to \mathbf{R}$.*

Let us now construct a family of examples. Let $\mathscr{N}$ be the family of all norms N on $\mathbf{R}^2$ satisfying $N(1,0) = 1$ and $N(\alpha, \beta) = N(|\alpha|, |\beta|)$ for all α, β.

If $N_1, \ldots, N_k \in \mathcal{N}$ we define a norm $N_1 * \cdots * N_k$ on $\mathbf{R}^{k+1} = \{(\xi_i)_{i=0}^k\}$ by

$$N_1 * N_2(\xi) = N_2(N_1(\xi_0, \xi_1), \xi_2)$$

and then inductively by the rule

$$N_1 * \cdots * N_k(\xi) = N_k(N_1 * \cdots * N_{k-1}(\xi_0, \ldots, \xi_{k-1}), \xi_k).$$

Next suppose $(N_k)_{k=1}^\infty$ is any sequence of norms in $\mathcal{N}$. We define the sequence space $\tilde{\Lambda}(N_k)$ to be the space of all sequences $(\xi_i)_{i=0}^\infty$ such that

$$\|\zeta\|_{\tilde{\Lambda}(N_k)} = \sup_{1 \le k < \infty} (N_1 * \cdots * N_k)(\xi_0, \ldots, \xi_k) < \infty.$$

$\tilde{\Lambda}(N_k)$ is then a Banach space. We define $\Lambda(N_k)$ to be the closed linear span of the basis vectors $(e_k)_{k=0}^n$ in $\Lambda(N_k)$. It is easy to verify the next result.

PROPOSITION 3.2. (1) *(e_n) is a 1-unconditional basis of $\Lambda(N_k)$.*

(2) *If (e_{m_k}) is a subsequence of (e_n) then (e_{m_k}) is isometrically equivalent to the canonical basis of $\Lambda(N_{m_k})$.*

(3) *$\Lambda(N_k)$ has property (M).*

The proofs of (1) and (2) are trivial and (3) follows from a simple gliding hump argument.

We now turn to identifying the spaces $\Lambda(N_k)$. Let us define $F_k(t) = N_k(1, t) - 1$. For convenience let $F_0(t) = |t|$.

PROPOSITION 3.3. *$\xi \in \tilde{\Lambda}(N_k)$ if and only if some $\alpha > 0$, $\Sigma_{k=0}^\infty F_k(\alpha\xi_k) < \infty$.*

Proof. First suppose $\xi \in \tilde{\Lambda}(N_k)$ and $\|\xi\|_{\tilde{\Lambda}(N_k)} \le 1$. Let $h \ge 0$ be the first index such that $|\xi_h| > 0$. Then for $k \ge h + 1$,

$$N_1 * \cdots * N_k(\xi_0, \ldots, \xi_k) \ge (1 + F_k(\xi_k))N_1 * \cdots * N_{k-1}(\xi_0, \ldots, \xi_{k-1})$$

so that

$$\prod_{k=h+1}^{\infty} (1 + F_k(\xi_k)) < \infty$$

whence $\Sigma_{k=0}^\infty F_k(\xi_k) < \infty$. This quickly establishes one direction.

For the other direction assume $\xi \notin \tilde{\Lambda}(N_k)$. Then there exists h such that

$$N_1 * \cdots * N_h(\xi_0, \ldots, \xi_h) > 1.$$

A similar argument to the above shows that

$$\prod_{k=h+1}^{\infty} (1 + F_k(\xi_k)) = \infty.$$

This completes the proof. ■

We now recall that if $(F_k)_{k=0}^{\infty}$ is a sequence of Orlicz functions the modular space $l_{(F_k)}$ consists of all $(\xi_k)_{k=0}^{\infty}$ such that

$$\|\xi\|_{l_{(F_k)}} = \inf\left\{\rho > 0\colon \sum_{k=0}^{\infty} F_k(\xi_k/\rho) \le 1\right\} < \infty.$$

The closed linear span of (e_n) in $l_{(F_k)}$ is denoted by $h_{(F_k)}$. See Lindenstrauss-Tzafriri [24] for details.

The following is immediate from the Closed Graph Theorem.

PROPOSITION 3.4. *Suppose for* $k \ge 1$, $N_k \in \mathscr{N}$ *and* $F_k(t) = N_k(1, t) - 1$. *Then* $\Lambda(N_k)$ *is canonically isomorphic to* $h_{(F_k)_{k=0}^{\infty}}$ *where* $F_0(t) = |t|$. *The induced norms are equivalent.*

In fact, we obtain slightly more, by noting that Proposition 3.3 holds for every sequence $N_k \in \mathscr{N}$.

LEMMA 3.5. *There is a universal constant* C *so that if* $N_1, \dots, N_l$ *is any finite subset of* $\mathscr{N}$ *and* $F_k(t) = N_k(1, t) - 1$ *for* $1 \le k \le l$ *then*:

(1) *If* $N_1 * \cdots * N_l(\xi_0, \dots, \xi_l) \le 1$ *then* $\sum_{k=0}^{l} F_k(\xi_k/C) \le 1$.

(2) *If* $\sum_{k=0}^{l} F_k(C\xi_k) \le 1$ *then* $N_1 * \cdots * N_l(\xi_0, \dots, \xi_1) \le 1$.

If the sequence (N_k) is constant (i.e., $N_k = N$ for all k) then we write $\Lambda(N)$ and this is isomorphic to the Orlicz sequence space h_F where $F(t) = N(1, t) - 1$.

LEMMA 3.6. *Let* X *be a separable Banach space with property* (M). *Suppose* (x_n) *is a weakly null sequence generating a nontrivial type. Let* $N(\alpha, \beta) = \lim\|\alpha x + \beta x_n\|$ *whenever* $\|x\| = 1$. *Then for any* $u \in X$ *with* $\|u\| = 1$ *and any* $\varepsilon > 0$ *there is a subsequence* (y_n) *of* (x_n) *such that*

$$(1+\varepsilon)^{-1}\left\|\xi_0 u + \sum_{i=1}^{\infty} \xi_i y_i\right\| \le \|\xi\|_{\Lambda(N)} \le (1+\varepsilon)\left\|\xi_0 u + \sum_{i=1}^{\infty} \xi_i y_i\right\|$$

for all finitely nonzero sequences $\xi = (\xi_i)_{i=0}^{\infty}$.

The proof of Lemma 3.6 is standard in the theory of types (cf. [18]).

PROPOSITION 3.7. *If $N \in \mathcal{N}$ then there exists p so that $1 \leq p \leq \infty$ and, for every $\varepsilon > 0$, $\Lambda(N)$ contains a subspace E with $d(E, l_p) < 1 + \varepsilon$ (when $p < \infty$) or if $p = \infty$, $d(E, c_0) < 1 + \varepsilon$.*

Proof. If $\Lambda(N)$ contains a copy of either c_0 or l_1 then the result follows from the distortion property for these spaces [15]. Otherwise $\Lambda(N)$ is a reflexive Orlicz space and neither l_1 nor c_0 is finitely representable in $\Lambda(N)$. However by a theorem of Krivine [17] (cf. [28]) we can find a normalized block basic sequence (u_n) such that

$$\lim_{n \to \infty} \|\alpha u_{2n-1} + \beta u_{2n}\| = \left(|\alpha|^p + |\beta|^p\right)^{1/p}$$

for all α, β. But then for any ξ we have $\lim\|\xi + tu_{2n}\| = (\|\xi\|^p + |t|^p)^{1/p}$. It follows by Lemma 3.6, there is a subsequence of (u_{2n}) spanning a space E with $d(E, l_p) < 1 + \varepsilon$. ■

After the initial preparation of this paper, the author learned that a result closely related to the next proposition has been proved in [13]; in fact the main result of [13] implies Proposition 3.8.

PROPOSITION 3.8. *Let X be a separable Banach space with property (M). Then for any infinite-dimensional closed subspace Y of X there exists p, $1 \leq p \leq \infty$ such that for every $\varepsilon > 0$, Y contains a subspace E with $d(E, l_p) < 1 + \varepsilon$ (when $p < \infty$) or if $p = \infty$, $d(E, c_0) < 1 + \varepsilon$.*

Proof. If Y has the Schur property then Y contains a subspace isomorphic to l_1 and hence the assertion holds with $p = 1$. Otherwise there is a weakly null sequence (y_n) in Y with $\|y_n\| = 1$ and we can use Lemma 3.6 and Proposition 3.7 to obtain the result. ■

Remark. Thus we have immediately that Tsirelson type spaces [5] cannot be renormed to have property (M), answering a question of [30].

PROPOSITION 3.9. *Let X be a separable Banach space which contains no copy of l_1 and has property (M). Then there exists $1 < p \leq \infty$ and a weakly null sequence (x_n) so that for every $u \in X$ and every α,*

$$\lim_{n \to \infty} \|u + \alpha x_n\| = \left(\|u\|^p + |\alpha|^p\right)^{1/p}$$

when $p < \infty$ or

$$\lim_{n \to \infty} \|u + \alpha x_n\| = \max(\|u\|, |\alpha|)$$

when $p = \infty$.

Proof. By the preceding Proposition, we can find a subspace Y isomorphic to l_p, for some $p < \infty$ or c_0; for convenience of notation we treat the former case. For every n, Y contains a subspace Z_n with $d(Z_n, l_p) < 1 + \varepsilon_n$ where $\varepsilon_n \to 0$. Thus there is a sequence $(x_{nk})_{k=0}^{\infty}$ in Z_n, generating a weakly null type, which is $(1 + \varepsilon_n)$-equivalent to the canonical basis of l_p. It follows from this and property (M) that for every $u \in X$,

$$(1 + \varepsilon_n)^{-1}(\|u\| + |\alpha|^p)^{1/p} \leq \lim_{k \to \infty} \|u + \alpha x_{nk}\| \leq (1 + \varepsilon_n)(\|u\| + |\alpha|^p)^{1/p}.$$

We then obtain (x_n) by passing to a suitable diagonal sequence $(x_{n,k_n})_{n=1}^{\infty}$. ■

We recall that a separable Banach space X is stable [18] if for any pair of bounded sequences $(u_m), (v_n)$ in X,

$$\lim_{n \to \infty} \lim_{m \to \infty} \|u_m + v_n\| = \lim_{m \to \infty} \lim_{n \to \infty} \|u_m + v_n\|$$

whenever both sides exist.

In [27] a Banach space X is said to have property (M_p) where $1 < p \leq \infty$ if $\mathscr{K}(X \oplus_p X)$ is an M-ideal in $\mathscr{L}(X \oplus_p X)$.

THEOREM 3.10. *Let X be a separable Banach space such that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$. Then the following conditions are equivalent*:

(1) *X has property (M_p) for some $1 < p < \infty$.*
(2) *X is stable.*
(3) *There exists p, $1 < p < \infty$ such that for every weakly null sequence (x_n) and every $u \in X$,*

$$\limsup_{n \to \infty} \|u + x_n\|^p = \|u\|^p + \limsup_{n \to \infty} \|x_n\|^p.$$

Proof. In fact (1) ⇒ (2) is due to Oja-Werner [27]. For (3) ⇒ (1) we suppose (u, v) in $X \oplus_p X$ and that (x_n, y_n) is weakly null sequence so that both (x_n) and (y_n) generate weakly null types. Then

$$\limsup \|(u + x_n, v + y_n)\|^p = \|u\|^p + \|v\|^p + \limsup(\|x_n\|^p + \|y_n\|^p).$$

By passing to suitable subsequences, the fact that $X \oplus_p X$ has property (M) follows trivially. If (K_n) is a shrinking compact approximating sequence for X with $\lim \|I - 2K_n\| = 1$ then $K_n \oplus K_n$ serves the same purpose in $X \oplus_p X$. Thus $\mathscr{K}(X \oplus_p X)$ is an M-ideal in $\mathscr{L}(X \oplus_p X)$.

(2) ⇒ (3). Since X contains no copy of l_1 and is stable it is necessarily reflexive. Now by Proposition 3.9, there exists p, $1 < p < \infty$, such that there

is a normalized weakly null sequence (y_n) such that for every $u \in X$ with $\|u\| = 1$,

$$\lim_{n\to\infty} \|\alpha u + \beta y_n\| = \left(|\alpha|^p + |\beta|^p\right)^{1/p}.$$

Now let (x_n) be any normalized weakly null sequence generating a type, i.e., such that $\lim\|u + x_n\|$ exists for all n. Then we can define $N(\alpha, \beta) = \lim_{n\to\infty}\|\alpha u + \beta x_n\|$ where u satisfies $\|u\| = 1$. Then by stability, if $\alpha, \beta \in \mathbf{R}$,

$$\begin{aligned} N(\alpha,\beta) &= \lim_{n\to\infty}\lim_{m\to\infty} \|\alpha y_n + \beta x_m\| \\ &= \lim_{m\to\infty}\lim_{n\to\infty} \|\alpha y_n + \beta x_m\| \\ &= \left(|\alpha|^p + |\beta|^p\right)^{1/p}. \end{aligned}$$

From this (3) follows easily. ■

4. Renormings and M-ideals

This section is devoted to results on renorming a space X so that $\mathscr{K}(X)$ in an M-ideal in $\mathscr{L}(X)$ and the related question of renorming X so that it has property (M).

PROPOSITION 4.1. *A modular sequence space $X = h_{(G_k)}$ can be equivalently normed to have property (M). If X^* is separable then X can be equivalently normed so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$.*

Proof. We may assume that $G_k(1) = 1$ for all k. Note that the right derivative of each G_k at $1/2$ is at most 2. It suffices to replace $(G_k)_{k=1}^\infty$ by an equivalent sequence $(F_k)_{k=1}^\infty$ of the form $F_k(t) = N_k(1,t) - 1$ for suitable $N_k \in \mathscr{N}$. We will choose each F_k to be convex and satisfy the condition that $t^{-1}(F_k(t) + 1)$ is monotone decreasing for $t > 0$. This is achieved by putting $F_k(t) = G_k(t)$ for $0 \le t \le 1/2$ and $F_k(t) = G_k(1/2) + 2(t - 1/2)$ for $t \ge 1/2$. Now let

$$N_k(\alpha,\beta) = |\alpha|\big(1 + F_k\big(|\beta|\,|\alpha|^{-1}\big)\big) \quad \text{for } \alpha \neq 0$$

and

$$N_k(0,\beta) = 2|\beta|.$$

Then it may be verified that N_k is an absolute norm on $\mathbf{C}^2$.

In fact, suppose $s_1, s_2 > 0$ and $t_1, t_2 \geq 0$. Then

$$\begin{aligned} N_k(s_1 + s_2, t_1 + t_2) &= (s_1 + s_2) + (s_1 + s_2)F_k\left(\frac{t_1 + t_2}{s_1 + s_2}\right) \\ &\leq s_1 + s_2 + s_1 F_k\left(\frac{t_1}{s_1}\right) + s_2 F_k\left(\frac{t_2}{s_2}\right) \\ &= N_k(s_1, t_1) + N_k(s_2, t_2) \end{aligned}$$

by the convexity of F_k. This and a similar calculation when one or both s_j vanishes ensure that N_k is a norm. To check the required absolute property since $N(s,t)$ is monotone increasing in t we need only check that N is monotone in s for t fixed. If $0 < s_1 \leq s_2$ and $t > 0$,

$$\frac{s_1}{t}\left(1 + F_k\left(\frac{t}{s_1}\right)\right) \leq \frac{s_2}{t}\left(1 + F_k\left(\frac{t}{s_2}\right)\right)$$

whence it follows that $N_k(s_1, t) \leq N_k(s_2, t)$; again the other case when one or more of s_1, t vanish is treated as a limiting case.

If X^* is separable, since $\Lambda(N_k)$ has a shrinking 1-unconditional basis and has property (M), Theorem 2.4 gives the result. ■

Remark. Notice that Proposition 3.8 gives another proof of the classical result of Lindenstrauss and Tzafriri [22], [23] that every Orlicz sequence space h_F contains a copy of some l_p. The proof above depends on the Krivine theorem whereas the original proof uses a fixed point argument. We also note that it is now easy to give examples of unstable reflexive spaces X for which $\mathscr{K}(X)$ is an M-ideal answering a question in [27] and [30]. In fact any reflexive Orlicz sequence space $l_F \neq l_p$ can be renormed to have this property but by Theorem 3.10 no such renorming could be stable (since a subsequence of the canonical basis would then have to be equivalent to some l_p). Such a space must also fail condition (M_p) for every p, thus answering another question in [27].

Remark. D. Werner points out that an Orlicz space $X = l_F$ with separable dual can be equivalently normed so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ and the canonical basis is subsymmetric (by Proposition 3.2). Thus Hennefeld's result [14] that $X = l_p$, $1 < p < \infty$, and c_0 are the only spaces with symmetric bases so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ cannot be extended to subsymmetric bases.

THEOREM 4.2. *Let X be a Banach space with a subsymmetric basis (u_n). In order that X can be equivalently normed so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$*

it is necessary and sufficient that X is isomorphic to an Orlicz sequence space l_F and X^* is separable.

Proof. We have already observed one direction in Proposition 4.1. For the other if X can be so renormed then X^* is separable and (u_n) is weakly null sequence. By Lemma 3.6, (u_n) has a subsequence equivalent to the basis of an Orlicz sequence space l_F. ∎

THEOREM 4.3. *Let X be a separable order-continuous nonatomic Banach lattice. If X has an equivalent norm with property (M) then X is lattice-isomorphic to L_2.*

Proof. We can using standard representations represent X as a Köthe function space on $[0,1]$ with $L_\infty \subset X \subset L_1$ (see [25], page 25, combined with the discussion of measure spaces on page 114). The lattice norm on X we denote by $\| \ \|_L$. Let us assume that X has an equivalent norm, $\| \ \|_M$ with property (M). We can assume that for some A we have $\|x\|_L \le \|x\|_M \le A\|x\|_L$.

Now if $x \in X$ with $\|x\| = 1$ then $r_n x$ converges to zero weakly where r_n denotes the sequence of Rademacher functions on $[0,1]$. We may pass to a subsequence n_k so that for every $u, x \in X$, $\lim \|u + xr_{n_k}\|_M$ exists. For convenience we set $R_k x = xr_{n_k}$. We then define, taking some fixed u with $\|u\|_M + 1$,

$$N_x(\alpha, \beta) = \lim_{n\to\infty} \|\alpha u + \beta R_n x\|_M$$

and

$$F_x(t) = N_x(1,t) - 1.$$

Of course these definitions are independent of the choice of u. Now let $\mathcal{N}_X$ be the collection of all N_x for which $\|x\|_L = 1$.

Suppose $x_1, \dots, x_m$ satisfy $\|x_i\|_L = 1$ for all i. Suppose $\xi_1, \dots, \xi_m \in \mathbf{R}$. Then

$$\lim_{n_1\to\infty} \cdots \lim_{n_m\to\infty} \left\| \sum_{i=1}^m \xi_i R_{n_i} x_i \right\|_M = N_{x_1} * \cdots * N_{x_m}(0, \xi_1, \dots, \xi_m).$$

It follows that if $x_1, \dots, x_m$ are, in addition, disjointly supported,

$$\left\| \sum_{i=1}^m \xi_i x_i \right\|_L \le N_{x_1} * \cdots * N_{x_m}(0, \xi_1, \dots, \xi_m) \le A \left\| \sum_{i=1}^m \xi_i x_i \right\|_L.$$

We now turn to estimating N for $N \in \mathcal{N}_X$. By a well-known application of Khintchine's inequality ([25], pp. 50–51) if $\|x\|_L = 1$ and $\xi_i \in \mathbf{R}$ for $1 \le i \le m$ then

$$\left(\sum_{i=1}^{m} |\xi_i|^2\right)^{1/2} \le \sqrt{2}\ \mathrm{Ave}\left\|\sum_{i=1}^{m} \varepsilon_i \xi_i R_{n_i} x\right\|_L$$

where the average is taken over all choices of sign $\varepsilon_i = \pm 1$.

It follows that

$$\left(\sum_{i=1}^{m} |\xi_i|^2\right)^{1/2} \le \sqrt{2} N_x * \cdots * N_x(0, \xi_1, \dots, \xi_m),$$

and this implies that

$$\|\xi\|_2 \le \sqrt{2}\left\|\left(0, (\xi_i)_{i=1}^{\infty}\right)\right\|_{\Lambda(N_x)}$$

for all finitely non-zero sequences. Now suppose $F_x(\xi) = N_x(1, \xi) - 1 \le 2^{-n}$. Then if C is the universal constant of Lemma 3.5, $\xi^2 \le 2C^2 2^{-n}$. Since we have an estimate $F_x(\xi) \ge A^{-1}|\xi| \dot{-} 1$ for large ξ we conclude the existence of a constant $c > 0$ independent of x so that if $\|x\|_L = 1$,

$$F_x(t) \ge (1 + c^2 t^2)^{1/2} - 1$$

or

$$N_x(\alpha, \beta) \ge \left(|\alpha|^2 + c^2|\beta|^2\right)^{1/2}.$$

It follows that for every $x_1, x_2, \dots, x_m$ with $\|x_i\|_L = 1$, we have

$$N_{x_1} * \cdots * N_{x_m}(0, \xi_1, \dots, \xi_m) \ge c\left(\sum_{i=1}^{m} |\xi_i|^2\right)^{1/2}$$

Thus if $y_1, y_2, \dots, y_m$ are disjoint we get an estimate

$$\left\|\sum_{i=1}^{m} y_i\right\|_L \ge cA^{-1}\left(\sum_{i=1}^{m} \|y_i\|_L^2\right)^{1/2},$$

i.e., X satisfies a lower 2-estimate. Thus X is in fact q-concave for every $q > 2$ and by Theorem 1.d.6 of [25] there is a constant B so that if

$y_1, \ldots, y_m \in X$ then

$$\operatorname{Ave}\left\| \sum_{i=1}^{m} \varepsilon_i y_i \right\|_L \leq B \left\| \left(\sum_{i=1}^{m} |y_i|^2 \right)^{1/2} \right\|_L .$$

We thus deduce that if $\|x\|_L = 1$ then for any $n_1, \ldots, n_m$,

$$\operatorname{Ave}\left\| \sum_{i=1}^{m} \varepsilon_i R_{n_i} x_i \right\|_L \leq B \left(\sum_{i=1}^{m} |\xi_i|^2 \right)^{1/2} .$$

Taking limits we deduce

$$N_x * \cdots * N_x(0, \xi_1, \ldots, \xi_m) \leq AB \left(\sum_{i=1}^{m} |\xi_i|^2 \right)^{1/2} .$$

Exactly the same reasoning as above yields the existence of a constant D independent of x so that

$$N_x(\alpha, \beta) \leq \left(|\alpha|^2 + D^2 |\beta|^2 \right)^{1/2},$$

and hence that X satisfies an upper 2-estimate, i.e, for disjoint $y_1, \ldots, y_m \in X$ we have

$$\left\| \sum_{i=1}^{m} y_i \right\|_L \leq AD \left(\sum_{i=1}^{m} \|y_i\|^2 \right)^{1/2} .$$

Finally a nonatomic Banach lattice with both an upper and lower 2-estimate is lattice-isomorphic to L_2 (see [25] p. 22). ∎

The following corollary extends a theorem of Lindenstrauss and Tzafriri [23], [25] who showed the same result for rearrangement invariant spaces.

COROLLARY 4.4. *Let X be a nonatomic order-continuous Banach lattice which is isomorphic to a subspace of an Orlicz sequence space h_F. Then X is lattice-isomorphic to L_2.*

Proof. h_F can be renormed to have property (M) (Proposition 4.1). ∎

COROLLARY 4.5. *Let X be a separable nonatomic order-continuous Banach lattice which has a renorming so that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$. Then X is lattice-isomorphic to L_2.*

Remark. In particular if $p \neq 2$ and $1 < p < \infty$ then the spaces L_p cannot be renormed so that $\mathscr{K}(L_p)$ is an M-ideal in $\mathscr{L}(L_p)$. See [19], [20], and [27] for the isometric theorem and [12] for the case of H_p (which is isomorphic to L_p). (Although we have treated only the real case, there is no basic change in our argument for the complex version.)

We now give another result in the same spirit.

THEOREM 4.6. *The space $Y = l_p(l_r)$ for $1 < p$, $r < \infty$ has a renorming so that $\mathscr{K}(Y)$ is an M-ideal in $\mathscr{L}(Y)$ if and only if $p = r$.*

Proof. Suppose Y has a renorming $\| \ \|_M$ for which $\mathscr{K}(Y)$ is an M-ideal in $\mathscr{L}(Y)$. Let X_k be the space of sequences (x_n) with $x_n \in l_r$ and $x_n = 0$ when $n \neq k$. Then X_k is isomorphic to l_r. Thus we can pick a sequence in each X_k say $(u_{kn})_{n=1}^{\infty}$ such that for every $y \in Y$,

$$\lim_{n \to \infty} \|y + \alpha u_{kn}\|_M = \left(\|y\|^r + |\alpha|^r\right)^{1/r}.$$

Thus for every $\alpha_1, \dots, \alpha_k$,

$$\lim_{n_1 \to \infty} \dots \lim_{n_i \to \infty} \left\| \sum_{i=1}^{k} \alpha_i u_{in_i} \right\|_M = \left(\sum_{i=1}^{k} |\alpha_i|^r \right)^{1/r}.$$

However the left-hand side is equivalent to $(\Sigma |\alpha_i|^p)^{1/p}$ and we get a contradiction unless $p = r$. ∎

Remark. This theorem can, of course, be extended to a more general setting but we shall not pursue this point. Notice that this gives an alternative proof that if $p \neq 2$ then L_p cannot be renormed so that $\mathscr{K}(L_p)$ is an M-ideal, since $l_p(l_2)$ is isomorphic to a subspace of L_p.

In conclusion we note that we do not know any example of a separable Banach space X with property (M) such that X^* is separble and has the approximation property where $\mathscr{K}(X)$ fails to be an M-ideal in $\mathscr{L}(X)$. It would be of interest to have such an example with X reflexive.

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