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UNIQUENESS OF UNCONDITIONAL AND SYMMETRIC STRUCTURES IN FINITE DIMENSIONAL SPACES

BY

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1. Introduction

The main target of the present paper is to study some questions concerning the uniqueness of symmetric and unconditional bases in the framework of the local theory of Banach spaces. Since the spaces under consideration are finite dimensional it is quite obvious that one cannot discuss problems of uniqueness for individual spaces but rather for families of such spaces. As we shall see in the sequel, the case of unconditional bases can be treated from different points of view.

The study of the uniqueness of symmetric bases for finite dimensional spaces was initiated in [8] and continued in [16] and [11] (see also [17]). Results concerning the uniqueness question in the setting of unconditional bases for finite dimensional Banach spaces were obtained in Schütt [16] and in [1].

In order to discuss our results as well as their connection with previously proved ones, we introduce the following definitions.

DEFINITION 1.1. (a) Let $\mathscr{F}$ be a family of finite dimensional Banach spaces each of which has a normalized 1-symmetric basis. We shall say that the members of $\mathscr{F}$ have a unique symmetric basis if there exists a function ψ: $[1,\infty) \to [1,\infty)$ such that, whenever $X \in \mathscr{F}$ has another normalized K-symmetric basis $(y_i)_{i=1}^n$, then $(y_i)_{i=1}^n$ is $\psi(K)$-equivalent to the given 1-symmetric basis.

(b) Let $\mathscr{F}$ be a family of finite dimensional Banach spaces each of which has a normalized 1-unconditional basis. We shall say that the members of $\mathscr{F}$ have an almost (somewhat) unique unconditional basis provided there exists a function φ: $[1,\infty) \times (0,1) \to [1,\infty)$ such that, whenever $X \in \mathscr{F}$ with the given 1-unconditional basis $(x_i)_{i=1}^n$ has also another normalized K-unconditional basis $(y_i)_{i=1}^n$ then, for any (some) $0 < \alpha < 1$, there exists a subset

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$\sigma \subset [n] = \{1, 2, \ldots, n\}$ and a one to one function $\pi: \sigma \to [n]$ so that $|\sigma| \geq \alpha n$ and $(x_i)_{i \in \sigma}$ is $\varphi(K, \alpha)$-equivalent to $(y_{\pi(i)})_{i \in \sigma}$.

Definition 1.1 (a) is similar to that considered in [8]. On the other hand, Definition 1.1 (b) is in the spirit of the "proportional" theory of finite dimensional spaces and quite different from that introduced in [1], where uniqueness, up to a permutation, was considered for the entire basis. The definition of somewhat unique bases already appears in [16] under the name of partial uniqueness.

One of the most natural ways of creating a family of finite dimensional spaces with a 1-symmetric basis is the following. Let X be a rearrangement invariant (r.i.) Banach function space on $[0, 1]$ and, for $n = 1, 2, \ldots,$ let $\mathscr{B}_n$ be the algebra generated by the intervals $[(k-1)/n, k/n)$; $1 \leq k \leq n$ and denote by $X_n = X(\mathscr{B}_n)$ the subspace of X consisting of those functions which are constant on each atom of $\mathscr{B}_n$. The main result of the paper (Theorem 5.6) asserts that, for any r.i. Banach function space X on $[0, 1]$, the members of the corresponding family $\mathscr{E} = \{X_n\}_{n=1}^{\infty}$ have a unique symmetric basis and an almost unique unconditional basis. This theorem can be, of course, applied to many families of Orlicz and Lorentz spaces which are generated as above.

This result is generalized in Section 6 in the following sense: it is shown there that if, for a fixed r.i. function space X on $[0, 1]$, containing some L_q, where $q < \infty$, $0 < \lambda \leq 1$ and n, the corresponding subspace X_n of X contains a normalized K-unconditional basic sequence $(y_i)_{i=1}^m$ with $m \geq \lambda n$ and $[y_i]_{i=1}^m$ contains in turn a sequence $(z_i)_{i=1}^l$ with $l \geq \lambda n$ which is M-equivalent to

$$\left(1_{[(k-1)/n, k/n)} / \|1_{[(k-1)/n, k/n)}\|_X\right)_{k=1}^{l}$$

then there exists a subset $\sigma \subset [l]$ and a one-to-one map $\pi: \sigma \to [m]$ so that $|\sigma| \geq \alpha l$, where $\alpha = \alpha(\lambda, K, M, X) > 0$ and $(z_i)_{i \in \sigma}$ is $L(\lambda, K, M, X)$-equivalent to $(y_{\pi(i)})_{i \in \sigma}$. This result can be improved in the special case when X lies "on one side" of $L_2[0, 1]$; in this case the hypothesis that X contain some L_q can be eliminated and σ can be chosen of the order of $(1 - \varepsilon)l$.

As we have already mentioned above, the question of the uniqueness of symmetric bases for finite dimensional spaces has already been considered in [8] where it was proved that the members of the family $\mathscr{C}_{q,M}$ of all finite dimensional spaces with a 1-symmetric basis which induces a lattice structure with q-concavity constant $\leq M$, for some $q > 2$, have a unique symmetric basis. In [2] Theorem 2.6 it was shown that this result is true also for $q = 2$. We consider in Theorem 5.4 below the family $\mathscr{U}_{q,M}$ of all finite dimensional spaces with a 1-unconditional basis which induces, as above, a lattice structure with q-concavity constant $\leq M$, for some $q < 2$, and show that each member of $\mathscr{U}_{q,M}$ has a somewhat unique unconditional basis. A slightly less general version of this result was proved in [16], Proposition 2.5.

In the symmetric case, the hypothesis of q-concavity for some $q < 2$ was replaced in [14] by the weaker assumption of polynomial euclidean distance. More precisely, it was shown in that paper that, for any value of $r > 0$, the members of the family $\mathscr{F}_r$ of all finite dimensional spaces X with a 1-symmetric basis, whose euclidean distance d_X satisfies $d_X \geq (\dim X)^r$, have a unique symmetric basis. In Theorem 5.5 below we show that the members of $\mathscr{F}_r$ have also an almost unique unconditional basis.

In fact, the results discussed above can be proved in a more general form. For instance, if an n-dimensional Banach space $X \in \mathscr{F}_r$ with a normalized 1-symmetric basis $\{x_i\}_{i=1}^n$ contains an M-complemented subspace of dimension $m \geq \lambda n$ with a normalized K-unconditional basis $\{y_i\}_{i=1}^m$ then, for each $0 < \alpha < 1$, there exists a subset $\eta \subset [m]$ of cardinality $|\eta| \geq \alpha m$ such that $(y_i)_{i \in \eta}$ is $L(r, \lambda, M, K, \alpha)$-equivalent to $(x_i)_{i \in \eta}$ (Theorem 5.5 below). Similar results for complemented subspaces are proved in almost all the cases considered so far.

In addition to the classes of spaces considered above, some interesting results have been proved for families of classical spaces. For example, in [1] Theorem 1.4 it was shown that the members of the family $\mathscr{B} = (l_1^k \oplus l_2^m \oplus l_\infty^n)_{k,m,n=1}^\infty$ have a unique unconditional basis, up to permutation. In the present paper, we study the class $\mathscr{B}_{p,q} = (l_p^m(l_q^n))_{m,n=1}^\infty$, for $1 \leq p, q \leq \infty$, and prove that the members of $\mathscr{B}_{p,q}$ have an almost unique unconditional basis (Theorem 5.7 below). In the case $1 < p < q < 2$, this was proved in [16], Proposition 2.5, although the statement there is slightly less general.

The paper also contains some results on Hilbertian subspaces of a space with a symmetric or unconditional basis, as well as theorems on such spaces which contain "large" Hilbertian subspaces. One such result (Theorem 2.2) asserts that if an n-dimensional Banach space X with a normalized 1-unconditional basis $(x_i)_{i=1}^n$ contains an M-complemented Euclidean subspace of dimension $m \geq \lambda n$ then, for each $0 < \alpha < 1$, there exists a subset σ of $[n]$ of cardinality $|\sigma| \geq \alpha n$ such that $\{x_i\}_{i \in \sigma}$ is well equivalent to the unit vector basis of $l_2^{|\sigma|}$. In particular, if $(x_i)_{i=1}^n$ is already symmetric then, in the above circumstances, X is already well isomorphic to l_2^n (Theorem 2.5 below). If the Euclidean subspace of X of proportional dimension is not well complemented then, of course, X need not be well isomorphic to Hilbert space. However, in this case, we prove in Theorem 2.6 below that X contains, for each $0 < \alpha < 1$, a good copy of a Hilbert space of dimension $\geq \alpha n$. Theorems 2.2 and 2.5 are essentially straightforward consequences of a result of Gordon and Lewis which can be found in [7] and [15]. Our proof of Lemma 2.1 differs very little from the original argument of Gordon and Lewis.

A result of a completely different nature is described in Theorem 2.4: if a direct sum of the form $X_n \oplus l_2^m$ has an unconditional basis then, for each

$\varepsilon > 0$, the dimension of the Euclidean space can be reduced to $\leq (2+\varepsilon)n$ so that $X_n \oplus l_2^{(2+\varepsilon)n}$ still has a good unconditional basis.

For almost every case described above, the proofs of the corresponding uniqueness property are based on the possibility of finding relative large entries in the $n \times n$ matrix which maps one n-dimensional space with a symmetric or unconditional basis onto a space of a similar type. The main difference between our arguments and those used in the original paper of Schütt [16] on this topic is that Schütt considered only vectors whose entries are zeros and ones, where we utilize more general shaped vectors. This situation is formalized in Section 4 where we introduce the so-called property (P).

A family $\mathscr{E}$ of finite dimensional Banach lattices is said to have the property (P) *if, for any $0 < \lambda < 1$ and $M < \infty$, there exists $\nu = \nu(\lambda, M) > 0$ such that, whenever $X \in \mathscr{E}$ and Y are n-dimensional lattices and $A: X \to Y$ and $B: Y \to X$ are linear operators of norm $\leq M$ with* $\operatorname{tr}(BA) \geq \lambda n$, *then* $\max_{1 \leq i,j \leq n} |a_{ij} b_{ji}| \geq \lambda$.

The fact that the above considered classes of finite dimensional spaces have property (P) is proved in Section 4. In many cases, this is achieved by the 2-concavification of the above lattices X and Y and by use of doubly sub-stochastic matrices. The essential part of the arguments is given in Section 3.

2. Large Hilbertian subspaces of spaces with unconditional bases

For any operator T on a Banach space X, we let $\nu(T)$ denote the nuclear norm of T, i.e.,

$$\nu(T) = \inf\left\{ \sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n\| \right\}$$

where the infimum is taken over all sequences $\{x_n\}_{n=1}^{\infty}$ in X and $\{x_n^*\}_{n=1}^{\infty}$ in X^* such that

$$Tx = \sum_{n=1}^{\infty} x_n^*(x) x_n.$$

Furthermore, we shall denote by d_X the Banach-Mazur distance of X to a Euclidean space. We begin with a reformulation of a result of Gordon and Lewis which appears in [7] as Lemma 1.3 and as Proposition 2.4 of [15].

LEMMA 2.1. *Let X be an n-dimensional Banach space and let H be a Hilbert space. Suppose X has a normalized 1-unconditional basis $(u_i)_{i=1}^n$ and that $S: X \to H$, $T: H \to X$ are linear operators.*

For any $0 < \gamma < 1$, *let* $\sigma_\gamma = \{i \in [n]\colon \|Su_i\|\,\|T^*u_i^*\| \geq \gamma n^{-1}\nu(ST)\}$. *Then*

$$d_{[u_i]_{i\in\sigma_\gamma}} \leq \frac{n\|S\|\,\|T\|}{\gamma\nu(ST)} \tag{1}$$

and $(u_i)_{i\in\sigma_\gamma}$ *is* $(n\|S\|\,\|T\|/\gamma\nu(ST))^2$*-equivalent to the canonical basis of* $l_2^{\sigma_\gamma}$. *Moreover,*

$$|\sigma_\gamma| \geq \frac{(1-\gamma)\nu(ST)}{\|S\|\,\|T\|}. \tag{2}$$

Proof. There exists an orthogonal transformation U on H so that $\operatorname{tr}(UST) = \nu(ST)$. Thus

$$\nu(ST) = \operatorname{tr}(TUS) = \sum_{i=1}^{n}(USu_i, T^*u_i^*) \leq \sum_{i=1}^{n}\|Su_i\|\,\|T^*u_i^*\|,$$

where $(u_i^*)_{i=1}^n$ denotes the sequence in X^* biorthogonal to $\{u_i\}_{i=1}^n$.

Now $\|Su_i\|\,\|T^*u_i^*\| \leq \|S\|\,\|T\|$ so that it follows immediately that

$$|\sigma_\gamma| \geq \frac{(1-\gamma)\nu(ST)}{\|S\|\,\|T\|}.$$

For $i \in \sigma_\gamma$, let $s_i = \|Su_i\|$, $t_i = \|Tu_i\|$. Then for any $\{a_i\}_{i\in\sigma_\gamma}$,

$$\begin{aligned}\Big(\sum_{i\in\sigma_\gamma}|a_i|^2 s_i^2\Big)^{1/2} &= \Big(\sum_{i\in\sigma_\gamma}|a_i|^2\|Su_i\|^2\Big)^{1/2}\\ &= \left(\int\Big\|\sum_{i\in\sigma_\gamma}\varepsilon_i a_i Su_i\Big\|^2 d\varepsilon\right)^{1/2}\\ &\leq \|S\|\Big\|\sum_{i\in\sigma_\gamma}a_i u_i\Big\|.\end{aligned}$$

Similarly,

$$\Big(\sum_{i\in\sigma_\gamma}|a_i|^2 t_i^2\Big)^{1/2} \leq \|T\|\Big\|\sum_{i\in\sigma_\gamma}a_i u_i^*\Big\|.$$

Hence

$$\left\| \sum_{i \in \sigma_\gamma} a_i u_i \right\| \le \|T\| \left(\sum_{i \in \sigma_\gamma} |a_i|^2 t_i^{-2} \right)^{1/2}$$
$$\le \frac{n\|T\|}{\gamma \nu(ST)} \left(\sum_{i \in \sigma_\gamma} |a_i|^2 s_i^2 \right)^{1/2}$$
$$\le \frac{n\|S\|\,\|T\|}{\gamma \nu(ST)} \left\| \sum_{i \in \sigma_\gamma} a_i u_i \right\|,$$

which completes the proof. ■

THEOREM 2.2. *Let X be an n-dimensional Banach space with a normalized K-unconditional basis $(u_i)_{i=1}^n$. Let Y be an M-complemented subspace of X with* $\dim Y = m$.

Then, for any $\varepsilon > 0$, there exists a subset $\sigma \subseteq [n]$ so that $|\sigma| \ge (1 - \varepsilon)m$, and

$$d_{[u_i]_{i \in \sigma}} \le \frac{MK^2 d_Y}{\varepsilon\lambda} \quad \text{where } \lambda = \frac{m}{n}.$$

Further $(u_i)_{i \in \sigma}$ is $M^2K^4d_Y^2/\varepsilon^2\lambda^2$-equivalent to the canonical basis of $l_2^{|\sigma|}$.

We also have the further estimate (when λ is close to one) that

$$|\sigma| \ge n - (n - m)(1 + MK)(1 - \varepsilon\lambda)^{-1}.$$

Proof. There exists an n-dimensional Banach space X_1 with a normalized 1-unconditional basis $(v_i)_{i=1}^n$, so that (v_i) is K-equivalent to (u_i). Further there exists a Hilbert space H of dimension m and operators S: $X_1 \to H$, T: $H \to X_1$, with $\|S\|\,\|T\| \le MKd_Y$ and $ST = \mathrm{id}_H$. Thus $\nu(ST) = m$.

As in Lemma 2.1, let $\sigma = \{i \in [n]: \|Sv_i\|\,\|T^*v_i^*\| \ge \varepsilon\lambda\}$. Then

$$d_{[v_i]_{i \in \sigma}} \le \frac{MKd_Y}{\varepsilon\lambda}$$

so that

$$d_{[u_i]_{i \in \sigma}} \le \frac{MK^2 d_Y}{\varepsilon\lambda}.$$

Also, $\{u_i\}_{i \in \sigma}$ is $M^2K^4d_Y^2/\varepsilon^2\lambda^2$-equivalent to the unit vector basis of $l_2^{|\sigma|}$. It remains to estimate the size of σ.

Let R_σ: $X_1 \to X_1$ be the map $R_\sigma(\sum_{i=1}^n a_i v_i) = \sum_{i \in \sigma} a_i v_i$. Then, as we noticed in the proof of Lemma 2.1,

$$\nu\big(S(\mathrm{id}_{X_1} - R_\sigma)T\big) \le \sum_{i \in \sigma^c} \|Sv_i\|\,\|T^*v_i^*\| \le \varepsilon\lambda|\sigma^c| \le \varepsilon m.$$

However $SR_\sigma T$ has rank $|\sigma|$ at most. Thus there is an orthogonal projection P on H with rank $P = m - |\sigma|$ and $PSR_\sigma T = 0$. Thus $P = PST = PS(\mathrm{id}_{X_1} - R_\sigma)T$. Hence $\nu(P) \le \varepsilon m$ and so $m - |\sigma| \le \varepsilon m$, i.e., $|\sigma| \ge (1-\varepsilon)m$.

Let us now give an alternative estimate for $|\sigma|$. If $i \in \sigma^c$ then $v_i^*(TSv_i) \le \varepsilon\lambda$ and so

$$\sum_{i=1}^{n} \left| v_i^*\big((\mathrm{id}_{X_1} - TS)v_i\big)\right| \ge (1-\varepsilon\lambda)(n - |\sigma|).$$

Clearly

$$\sum_{i=1}^{n} \left| v_i^*\big((\mathrm{id}_{X_1} - TS)v_i\big)\right| \le \nu\big(\mathrm{id}_{X_1} - TS\big).$$

Now, $\mathrm{id}_{X_1} - TS$ is a projection of rank $n - m$ so that

$$\nu\big(\mathrm{id}_{x_1} - TS\big) \le (n-m)\|\mathrm{id}_{x_1} - TS\| \le (n-m)(1+MK).$$

Thus

$$(1-\varepsilon\lambda)(n-|\sigma|) \le (n-m)(1+MK)$$

so that

$$|\sigma| \ge n - (n-m)\left(\frac{1+MK}{1-\varepsilon\lambda}\right).$$

THEOREM 2.3. *Let X be an n-dimensional space with a normalized K-unconditional basis $(u_i)_{i=1}^n$. Let Y be an m-dimensional Banach space such that $d(X, Y \oplus_2 l_2^{n-m}) = d_1$. Suppose $\sigma \subset [n]$ is such that $d_{[u_i]_{i\in\sigma^c}} = d_2$. Then there is a subset $\tau \subset [n]$ with $\sigma \subset \tau$ and $m \le |\tau| \le m + |\sigma|$ so that $d([u_i]_{i\in\tau}, Y \oplus_2 l_2^{|\tau|-m}) \le 100K^4 d_1^3 d_2^3$.*

Proof. The space X can be decomposed as a direct sum $X = Y_0 \oplus Z_0$ so that $d(Y_0, Y) \le d_1$, $d(Z_0, l_2^{n-m}) \le d_1$ and the natural projections P_{Y_0}, P_{Z_0} satisfy $\|P_{Y_0}\|, \|P_{Z_0}\| \le d_1$. Let

$$Z_1 = Z_0 \cap [u_i]_{i\in\sigma^c}.$$

Then there is a projection P_{Z_1} of $[u_i]_{i\in\sigma^c}$ onto Z_1 with $\|P_{Z_1}\| \le d_2$. Let R be the restriction operator $R(\sum_{i=1}^n a_i u_i) = \sum_{i\in\sigma^c} a_i u_i$. Then $\|R\| \le K$ and $\|P_{Z_1}R\| \le Kd_2$. Hence

$$\|\mathrm{id}_X - P_{Z_1}R\| \le 2Kd_2.$$

Let $V = (\mathrm{id}_X - P_{Z_1}R)(X)$. Then $R|_V$ is a projection on V. In fact $R(V)$ is a subspace of $[u_i]_{i \in \sigma^c}$ and hence is d_2-Hilbertian. Further $(\mathrm{id}_X - R)(V) = [u_i]_{i \in \sigma}$. If we fix ρ to be any subset of σ^c with $|\rho| = \dim R(V)$ then $d(R(V), [u_i]_{i \in \rho}) \leq d_2$. Hence if $\tau = \sigma \cup \rho$, then

$$d(V, [u_i]_{i \in \tau}) \leq 4K^2 d_2.$$

Now consider the projection $(\mathrm{id}_X - P_{Z_1}R)P_{Z_0}$ on V. This projects V onto a subspace Z_2 of Z_0 which is thus d_1-Hilbertian. The complementary projection on V maps V onto

$$Y_1 = (\mathrm{id}_X - P_{Z_1}R)(Y_0).$$

Clearly, if $y \in Y_0$, then $P_{Y_0}(y - P_{Z_1}Ry) = y$ so that we have $\|y\| \leq d_1\|y - P_{Z_1}Ry\|$. Hence Y_0 is $2Kd_1d_2$-isomorphic to Y_1 and so $d(Y_1, Y) \leq 2Kd_1^2d_2$.

If $y \in Y_1$, $z \in Z_2$, then

$$\begin{aligned}(\|y\|^2 + \|z\|^2)^{\frac{1}{2}} &\leq 2\max(\|y\|, \|z\|)\\ &\leq 2\big(\|(\mathrm{id}_X - P_{Z_1})P_{Z_0}\| + 1\big)\|y + z\|\\ &\leq 6Kd_1d_2\|y + z\|\end{aligned}$$

while $\|y + z\| \leq \|y\| + \|z\| \leq 2(\|y\|^2 + \|z\|^2)^{1/2}$.

Thus $d(Y_1 \oplus_2 Z_2, V) \leq 12Kd_1d_2$. Hence

$$d\big(Y \oplus_2 l_2^{|\tau| - m}, [u_i]_{i \in \tau}\big) \leq 100K^4 d_1^3 d_2^3.$$

It remains to estimate $|\tau|$. In fact,

$$\begin{aligned}|\tau| &= \dim V = \dim X - \dim Z_1 = \dim X - \dim\big(Z_0 \cap [u_i]_{i \in \sigma^c}\big)\\ &= \dim X - \dim Z_0 - |\sigma^c| + \dim\big(Z_0 + [u_i]_{i \in \sigma^c}\big)\\ &\leq 2n - (n - m) - |\sigma^c| = m + |\sigma|.\end{aligned}$$

THEOREM 2.4. *Let X be an n-dimensional Banach space and suppose $W = X \oplus_2 l_2^m$ has a K-unconditional basis $(u_i)_{i=1}^{m+n}$. Then, given $\varepsilon > 0$, there is a subset σ of $[m + n]$ with $n \leq |\sigma| \leq (2 + \varepsilon)n$ such that*

$$d_{[u_i]_{i \in \sigma^c}} \leq 2^{17}10^4K^{24}, \tag{1}$$

$$d\big([u_i]_{i \in \sigma}, X \oplus_2 l_2^{|\sigma| - n}\big) \leq 2^{66}10^{22}K^{103}. \tag{2}$$

In particular, $X \oplus_2 l_2^{[n(1+\varepsilon)]}$ has a $2^{66}10^{20}K^{103}$-unconditional basis.

Proof. First we assume that $m \geq 2(1+K)n$ and show that we can reduce the dimension of the required Hilbert space to at most $2(1+K)n$. Applying the last part of Theorem 2.2 with $\varepsilon = \frac{1}{2}$ we find a subset ρ of $[m+n]$ so that

$$d_{[u_i]_{i\in\rho}} \leq 2K^2 m^{-1}(m+n) \leq 4K^2 \quad \text{and} \quad |\rho| \geq (m+n) - 2(1+K)n.$$

Let $\sigma_0 = \rho^c$, so that $|\sigma_0| = m + n - |\rho| \leq 2(1+K)n$. Then, by Theorem 2.3, there exists $\sigma_1, \sigma_0 \subset \sigma_1$, $n \leq |\sigma_1| \leq |\sigma_0| + n$ so that $d([u_i]_{i\in\sigma_1}, X \oplus_2 l_2^{|\sigma_1|-n}) \leq 6400K^{10}$. If $|\sigma_1| \leq (2+\varepsilon)n$, we are done.

If not we estimate $|\sigma_1| \leq |\sigma_0| + n \leq (3+2K)n$. Thus $[u_i]_{i\in\sigma_1}$ has a $6400K^{10}$-complemented subspace Y with $d_Y \leq 6400K^{10}$, and $\dim Y = |\sigma_1| - n = \lambda|\sigma_1|$ where $\lambda \geq \frac{1}{2}$. Apply Theorem 2.2 with ε replaced by $\frac{1}{2}(1+K)^{-1}\varepsilon$. There exist a subset $\tau \subset \sigma_1$ with

$$|\tau| \geq \left(1 - \tfrac{1}{2}(1+K)^{-1}\varepsilon\right)\left(|\sigma_1| - n\right) \quad \text{and} \quad d_{[u_i]_{i\in\tau}} \leq 2^{16}10^4K^{23}.$$

Let $\sigma_2 = \sigma_1 \setminus \tau$ and apply Theorem 2.3. There exists $\sigma_3, \sigma_2 \subset \sigma_3 \subset \sigma_1$ with $n \leq |\sigma_3| \leq |\sigma_2| + n$ so that

$$\begin{aligned} d\left([u_i]_{i\in\sigma_3}, X \oplus_2 l_2^{|\sigma_3|-n}\right) &\leq 100K^4(6400K^{10})^3(2^{16}10^4K^{23})^3 \\ &\leq 2^{66}10^{22}K^{103}. \end{aligned}$$

Next we estimate $|\sigma_3|$. We have

$$\begin{aligned} |\sigma_3| &\leq |\sigma_2| + n = |\sigma_1| - |\tau| + n \\ &\leq |\sigma_1| + n - \left(1 - \frac{\varepsilon}{2(1+K)}\right)\left(|\sigma_1| - n\right) \\ &= 2n + \frac{\varepsilon}{2(1+K)}\left(|\sigma_1| - n\right) \\ &\leq (2+\varepsilon)n. \end{aligned}$$

Finally we note that

$$\begin{aligned} d_{[u_i]_{i\in[m+n]\setminus\sigma_3}} &\leq d_{[u_i]_{i\in\rho\cup\tau}} \\ &\leq 2K\max\left(d_{[u_i]_{i\in\rho}}, d_{[u_i]_{i\in\tau}}\right) \\ &\leq 2^{17}10^4K^{24}. \end{aligned}$$

■

Let us note the following easy corollary of Theorem 2.2.

THEOREM 2.5. *Let X be an n-dimensional Banach space with a normalized symmetric basis $(u_i)_{i=1}^n$. Suppose X has an M-complemented subspace E with* *dimension $m = \lambda n$. Then $d_X \leq 4M\lambda^{-3/2}d_E$ and $(u_i)_{i=1}^n$ is $16M^2\lambda^{-3}d_E^2$-equivalent to the canonical basis of l_2^n.*

Proof. By Theorem 2.2, there is a subset σ of $[n]$ with $|\sigma| \geq \frac{1}{2}m$ so that

$$d_{[u_i]_{i\in\sigma}} \leq 2M\lambda^{-1}d_E.$$

Then

$$d_X \leq \left(1 + \frac{2n}{m}\right)^{1/2} d_{[u_i]_{i\in\sigma}} \leq 4\lambda^{-3/2}Md_E. \qquad \blacksquare$$

THEOREM 2.6. *Let X be an n-dimensional Banach space with a normalized symmetric basis $(u_i)_{i=1}^n$. Suppose E is a subspace of X with* $\dim E = m \geq \lambda n$, *where $0 < \lambda \leq 1$. Then*:

(1) *There is a constant $K = K(\lambda, d_E)$ so that for $a_1, \dots, a_n \in \mathbf{R}$,*

$$\left\|\sum_{i=1}^n a_i u_i\right\| \leq K\left\|\sum_{i=1}^n u_i\right\|\left(\frac{1}{n}\sum_{i=1}^n |a_i|^2\right)^{1/2}.$$

(2) *For any α, $0 < \alpha < 1$, there is a constant $D = D(\alpha, \lambda, d_E)$ so that X contains a subspace E_α with $d_{E_\alpha} \leq D$ and* $\dim E_\alpha \geq \alpha n$.

Proof. Consider the quotient map Q: $X^* \to E^*$. Since $d_{E^*} = d_E$ we have, for any $\{a_i\}_{i=1}^n$,

$$\int\left\|\sum_{i=1}^n \varepsilon_i a_i Q u_i^*\right\|^2 \geq d_E^{-2}\sum_{i=1}^n |a_i|^2 \|Qu_i^*\|^2.$$

Hence

$$\sum_{i=1}^n |a_i|^2 \|Qu_i^*\|^2 \leq d_E^2\left\|\sum_{i=1}^n a_i u_i^*\right\|^2.$$

By averaging over all permutations, if $\gamma = (1/n)\sum_{i=1}^n \|Qu_i^*\|^2$,

$$\gamma\left(\sum_{i=1}^n |a_i|^2\right)^{1/2} \leq d_E\left\|\sum_{i=1}^n a_i u_i^*\right\|.$$

Now if $v_i^* = u_i^*/(\|\sum_{i=1}^n u_i^*\|)$,

$$\sum_{i=1}^n \|Qu_i^*\|^2 \geq d_E^{-2} \int \left\| \sum_{i=1}^n \varepsilon_i Q u_i^* \right\|^2 d\varepsilon$$
$$\geq d_E^{-2} \left\| \sum_{i=1}^n u_i^* \right\|^2 \int \left\| \sum_{i=1}^n \varepsilon_i Q v_i^* \right\|^2 d\varepsilon$$
$$\geq d_E^{-2} \left\| \sum_{i=1}^n u_i \right\|^{-2} n^2 c^{2/\lambda}$$

where $c > 0$ is an absolute constant, by Theorem 1.1 of [2]. Hence

$$\gamma \geq c^{1/\lambda} n^{1/2} d_E^{-1} \left\| \sum_{i=1}^n u_i \right\|^{-1}.$$

Now, by duality, we have, for all $\{a_i\}_{i=1}^n$,

$$\left\| \sum_{i=1}^n a_i u_i \right\| \leq d_E \gamma^{-1} \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}$$
$$\leq c^{-1/\lambda} d_E^2 \left\| \sum_{i=1}^n u_i \right\| \left(\frac{1}{n} \sum_{i=1}^n |a_i|^2 \right)^{1/2}.$$

This proves (1).

For (2), notice that

$$\left\| \sum_{i=1}^n a_i u_i \right\| \geq \left\| \sum_{i=1}^n a_i u_i^* \right\|^{-1} \sum_{i=1}^n |a_i|$$
$$\geq \left\| \sum_{i=1}^n u_i \right\| \left(\frac{1}{n} \sum_{i=1}^n |a_i| \right).$$

Now, by results of [5] or [10], given any α, $0 < \alpha < 1$, there is a constant B_α depending only on α and a subspace E_α of X with $\dim E_\alpha \geq \alpha n$, such that if $x = \sum_{i=1}^n a_i u_i \in E_\alpha$,

$$\left(\frac{1}{n} \sum_{i=1}^n |a_i|^2 \right)^{1/2} \leq B_\alpha \frac{1}{n} \sum_{i=1}^n |a_i|.$$

Then $d_{E_\alpha} \leq B_\alpha c^{-1/\lambda} d_E^2$. ■

COROLLARY 2.7. *For $0 < \lambda < 1$ and $1 \le L < \infty$ there is a constant $K = K(\lambda, L)$ so that if X is an n-dimensional Banach space with a symmetric basis $(u_i)_{i=1}^n$ so that X and X^* both contain L-Hilbertian subspaces of dimension at least λn then $d_X \le K$.*

Proof. In fact, for some $C = C(\lambda, L) < \infty$,

$$\left\| \sum_{i=1}^n a_i u_i \right\| \le C \left\| \sum_{i=1}^n u_i \right\| \left(\frac{1}{n} \sum_{i=1}^n |a_i|^2 \right)^{1/2},$$

$$\left\| \sum_{i=1}^n a_i u_i^* \right\| \le C \left\| \sum_{i=1}^n u_i^* \right\| \left(\frac{1}{n} \sum_{i=1}^n |a_i|^2 \right)^{1/2}$$

from which it follows that

$$C^{-1} \frac{\left(\Sigma |a_i|^2\right)^{1/2}}{n} \le \frac{\|\Sigma a_i u_i\|}{\|\Sigma u_i\|} \le C \frac{\Sigma |a_i|^2)^{1/2}}{n}. \qquad \blacksquare$$

We remark that this corollary can also be obtained by using Theorem 1.2 of [2] and Theorems 2.5 and 2.6 above.

3. Doubly substochastic matrices on $\mathbf{R}^n$

Let $e_1, \dots, e_n$ be the canonical basis in $\mathbf{R}^n$ and let $e = e_1 + \cdots + e_n$. Let

$$\|x\|_p = \left(\frac{1}{n} \sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

If $x \in \mathbf{R}^n$ and π is a permutation of $[n]$ (i.e., $\pi \in \Pi_n$) we let $x_\pi = S_\pi x = (x_{\pi(i)})_{i=1}^n$.

Suppose $A = (a_{ij})_{i,j=1}^n$ is a nonnegative $n \times n$-matrix. Let $\delta_A = \max_{1 \le i,j \le n} a_{ij}$.

LEMMA 3.1. *Suppose $x \in \mathbf{R}^n$. Then, for $1 \le p \le 2$,*

$$\left(\int_{\Pi_n} \| Ax_\pi - (x, e) Ae \|_p^p \, d\pi \right)^{1/p} \le 2^{1/p} \delta_A^{(1-1/p)} \|A\|_1^{1/p} \|x\|_p.$$

Proof. Let

$$b_{ij} = a_{ij} - \frac{1}{n} \sum_{k=1}^n a_{ik}.$$

Then $\|B\|_1 \le 2\|A\|_1$, and thus the lemma is obvious for $p = 1$. For $p = 2$ we apply Lemma 4.2 of [9] and deduce that

$$\int_{\Pi_n}\left(\sum_{j=1}^{n} b_{ij}x_{\pi(j)}\right)^2 d\pi \le \frac{2}{n}\left(\sum_{j=1}^{n} b_{ij}^2\right)\left(\sum_{j=1}^{n} x_j^2\right),$$

for all $1 \le i \le n$.

Now, $\sum_{j=1}^{n} b_{ij}^2 \le \sum_{j=1}^{n} a_{ij}^2 \le \delta_A \sum_{j=1}^{n} a_{ij}$. Thus

$$\begin{aligned}\int_{\Pi_n}\|Bx_\pi\|_2^2\, d\pi &\le 2\delta_A\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right)\left(\frac{1}{n}\sum_{j=1}^{n} x_j^2\right)\\ &\le 2\delta_A\|A\|_1\|x\|_2^2.\end{aligned}$$

The general case follows by the Riesz-Thorin interpolation theorem. ∎

Now suppose $\|\cdot\|_E$ is a lattice quasinorm on $\mathbf{R}^n$ which satisfies

$$\|x+y\|_E^{1/2} \le \|x\|_E^{1/2} + \|y\|_E^{1/2}$$

(and hence also $\|x+y\|_E \le (2\|x\|_E + \|y\|_E))$ and

$$\|x\|_{1/2} \le \|x\|_E \le \|x\|_\infty.$$

Let $\|\cdot\|_F$ be a similar lattice quasinorm on $\mathbf{R}^n$ and set $\|A\|_{E\to F} = \max_{\|x\|_E \le 1}\|Ax\|_F$.

Lemma 3.2. *Suppose A is doubly substochastic. Then for $x \ge 0$, with $\|x\|_1 = 1$, and $1 \le p \le 2$ we have*

$$\|Ae\|_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \le \|A\|_{E\to F}^{1/2}\int_{\Pi_n}\|x_\pi\|_E^{1/2}\, d\pi + 2^{1/2p}\delta_A^{(1-1/p)/2}\|x\|_p^{1/2}.$$

Proof. By Lemma 3.1,

$$\begin{aligned}\left(\int_{\Pi_n}\|Ax_\pi - Ae\|_{1/2}^{1/2}\, d\pi\right)^2 &\le \left(\int_{\Pi_n}\|Ax_\pi - Ae\|_p^p\, d\pi\right)^{1/p}\\ &\le 2^{1/p}\delta_A^{1-1/p}\|x\|_p.\end{aligned}$$

Thus, since A is doubly substochastic,

$$\begin{aligned}\|Ae\|_1 &\le \|Ae\|_{1/2}^{1/2}\\ &\le \|A\|_{E\to F}^{1/2}\int_{\Pi_n}\|x_\pi\|_E^{1/2}\,d\pi + 2^{1/2p}\delta_A^{(1-1/p)/2}\|x\|_p^{1/2}.\end{aligned}$$ ∎

LEMMA 3.3. *For any $\beta > 0$, $\theta > 0$, and $M < \infty$, we can find $\delta_0 = \delta_0(\beta, \theta, M) > 0$ so that whenever A is a doubly substochastic $n \times n$-matrix with*

$$\|Ae\|_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \ge \theta$$

and $\|\cdot\|_E$ is a symmetric $\frac{1}{2}$-norm on $\mathbf{R}^n$ such that
(1) $\|x\|_{\frac{1}{2}} \le \|x\|_E \le \|x\|_\infty$,
(2) *there exists $x \in \mathbf{R}^n$, $\|x\|_1 = 1$, and $\|x\|_E \le n^{-\beta}$, and*
(3) $\|A\|_E \le M$
then $\delta_A \ge \delta_0$.

Proof. We may assume that $\theta \le \frac{1}{2}$. If we fix β, θ and M it suffices to provide an estimate for large enough dimensions n. Hence we may assume $n \ge (200\theta^{-7}M)^{2/\beta}$.

If $\|Ae\|_1 \ge \theta$, let σ be the set of $i \in [n]$ so that $\sum_{j=1}^{n} a_{ij} \ge \theta^2$. Then, clearly,

$$|\sigma| \ge n(\theta - \theta^2) \ge \tfrac{1}{2}n\theta,$$

Similarly if τ is the set of $j \in [n]$ for which $\sum_{i=1}^{n} a_{ij} \ge \theta^2$, then $|\tau| \ge \frac{1}{2}n\theta$.

Let m be the integer part of $2\theta^{-1} + 1$. Then we may find permutations $\rho_1, \dots, \rho_m$ and $\rho_1', \dots, \rho_m'$ so that

$$\begin{aligned}(S_{\rho_1} + \cdots + S_{\rho_m})Ae &\ge \theta^2 e,\\ (S_{\rho_1'} + \cdots + S_{\rho_m'})A^t e &\ge \theta^2 e.\end{aligned}$$

where S_ρ is the permutation matrix $(S_\rho)_{ij} = \delta_{i,\rho(i)}$.

Let V be the diagonal matrix such that $V(S_{\rho_1} + \cdots + S_{\rho_m})Ae = e$. Clearly, $\|V\|_E \le \theta^{-2}$ and $Ve \ge m^{-1}e \ge \frac{1}{3}\theta e$. Let

$$W = V\left(\sum_{j=1}^{m} S_{\rho_j} A S_{\rho_j'}^t\right).$$

Then $\|W\|_E \le 9\theta^{-4}M$ (since E is $\frac{1}{2}$-normable) and $\delta_W \le 3\theta^{-3}\delta_A$. Furthermore, $We = e$.

Let $Ux = Wx - (x, e)e$. Then $Ue = 0$ and for any permutation π, $US_\pi Ux = US_\pi Wx$. For any $x \in \mathbf{R}^n$, by Lemma 3.1 we have

$$\int_{\Pi_n} \|US_\pi x\|_2^2 \, d\pi \leq 2\delta_W \|W\|_1 \|x\|_2^2 \leq 6\theta^{-3}\delta_W \|x\|_2^2$$

and so there exists $\pi = \pi(x)$ with

$$(*) \qquad \|US_\pi x\|_2 \leq \left(6\theta^{-3}\delta_W\right)^{\frac{1}{2}} \|x\|_2 .$$

Fix k to be the largest integer such that $(200\theta^{-7}M)^k \leq n^\beta$. Then $k \geq 2$ and $n^\beta \leq (200\theta^{-7}M)^{2k}$. Fix any $x \geq 0$ with $\|x\|_1 = 1$ and $\|x\|_E \leq n^{-\beta}$. Iterating $(*)$ k times we obtain the existence of $\pi_1, \ldots, \pi_k$ with

$$\|US_{\pi_k} US_{\pi_{k-1}} \ldots US_{\pi_1} x\|_2 \leq \left(6\theta^{-3}\delta_W\right)^{k/2} \|x\|_2$$

and hence if $y = US_{\pi_k} WS_{\pi_{k-1}} \ldots WS_{\pi_1} x$

$$\|y\|_E \leq \|y\|_\infty \leq n^{1/2} \|y\|_2 \leq n^{1/2}\left(6\theta^{-3}\delta_W\right)^{k/2} \|x\|_2 \leq n\left(6\theta^{-3}\delta_W\right)^{k/2}.$$

On the other hand $y = WS_{\pi_k} \ldots WS_{\pi_1} x - (S_{\pi_k} W \ldots WS_{\pi_1} x, e)e$. Now

$$\begin{aligned} W^t e &= \left(\sum_{j=1}^m S_{\rho_j} A^t S_{\rho_j}^t \right) V^t e \\ &\geq \frac{1}{3}\theta \sum_{j=1}^m S_{\rho_j} A^t S_{\rho_j}^t e \\ &\geq \frac{1}{3}\theta^3 e. \end{aligned}$$

Hence $S_{\pi_1}^t W^t \ldots W^t S_{\pi_k}^t e \geq (\frac{1}{3}\theta^3)^{k-1} e$ so that

$$\begin{aligned} \left(\frac{1}{3}\theta^3\right)^{k-1} &\leq 2\left(\|y\|_E + \|WS_{\pi_k} \ldots WS_{\pi_1} x\|_E\right) \\ &\leq 2\left(n\left(6\theta^{-3}\delta_W\right)^{k/2} + \|W\|_E^k n^{-\beta}\right) \\ &\leq 4 \max\left(n\left(6\theta^{-3}\delta_W\right)^{k/2}, \|W\|_E^k n^{-\beta}\right). \end{aligned}$$

Thus

$$\frac{1}{3}\theta^3 \leq \max\left(n^{1/k}\left(6\theta^{-3}\delta_W\right)^{1/2}, \|W\|_E n^{-\beta/k}\right).$$

We now recall the choice of k. We have $n^{-\beta/k} \le (200)^{-1}M^{-1}\theta^7$ and $\|W\|_E \le 9\theta^{-4}M$. Thus

$$\|W\|_E n^{-\beta/k} \le \frac{9}{200}\theta^3 \le \frac{1}{3}\theta^3.$$

Hence

$$\left(6\theta^{-3}\delta_W\right)^{1/2} \ge \frac{1}{3}\theta^3 n^{-1/k} \ge \frac{1}{3}\theta^3\left(\frac{\theta^7}{200M}\right)^{2/\beta}$$

which implies an estimate on $\delta_A \ge \frac{1}{3}\theta^3\delta_W$. ■

4. Property (P)

We shall say that a lattice norm $\|\cdot\|_X$ on $\mathbf{R}^n$ satisfies the (p,q)-condition (where $p > q$) if $\|x\|_q \le \|x\|_X \le \|x\|_p$ for $x \in \mathbf{R}^n$. If τ is a subset of $\{1,2,\dots,n\}$ we define R_τ by $(R_\tau x)_i = x_i$ if $i \in \tau$ and $(R_\tau x)_i = 0$ otherwise. The following proposition is a slight extension of Propositions 3.4 and 4.3 of [4] or Proposition 3.3 of [3]

PROPOSITION 4.1. *Let X and Y be n-dimensional Banach lattices satisfying the $(\infty,1)$-condition. Let A: $X \to Y$ be an $n \times n$-matrix. Then, given $\varepsilon > 0$, there exists subsets $\sigma, \tau \subset [n]$ such that $|\sigma|, |\tau| \ge (1-\varepsilon)n$ and $\|R_\tau A R_\sigma\|_2 \le K_G^2\varepsilon^{-1}\|A\|_{X\to Y}$.*

Proof. Let $\|A\| = \|A\|_{X\to Y}$. Consider A: $L_\infty \to L_1$; clearly $\|A\|_{\infty\to 1} \le \|A\|$. By the Grothendieck-Pietsch factorization theorem (cf. [12], pp. 64–70), there exist $(\mu_i)_{i=1}^n$ with $\mu_i \ge 0$, $\Sigma_{i=1}^n \mu_i = 1$ and

$$\|Ax\|_1 \le K_G\|A\|\left(\sum_{i=1}^n \mu_i x_i^2\right)^{1/2}$$

for $x \in X$. Let $\sigma = \{i\colon \mu_i \le 1/\varepsilon n\}$. Then $|\sigma^c| \le \varepsilon n$. If $x \in X$, then

$$\|AR_\sigma x\|_1 \le K_G\|A\|\left(\sum_{i\in\sigma} \mu_i x_i^2\right)^{1/2} \le K_G\|A\|\varepsilon^{-1/2}\|x\|_2.$$

Now consider $R_\sigma A^t$: $L_\infty \to L_2$; we have $\|R_\sigma A^t\|_{\infty\to 2} \le K_G\varepsilon^{-1/2}\|A\|$. Again by the Grothendieck-Pietsch theorem there exists $\tau \subset [n]$, $|\tau| \ge (1-\varepsilon)n$, and

$$\|R_\sigma A^t R_\tau\|_2 \le K_G^2\varepsilon^{-1}\|A\|$$

and the proposition follows. ■

PROPOSITION 4.2. *Let X and Y be n-dimensional Banach lattices satisfying the $(\infty, 1)$-condition. Let A: $X \to Y$ and B: $Y \to X$ be $n \times n$-matrices, with* $\operatorname{tr}(BA) \geq \lambda n$, *for some $\lambda > 0$. Let* $M = \max(\|A\|_{X \to Y}, \|B\|_{Y \to X})$. *Then there are subsets $\sigma, \tau \subset [n]$, with $|\sigma|, |\tau| \geq (1 - \frac{1}{4}\lambda M^{-2})n$ and*

$$(1) \qquad \operatorname{tr}(R_\sigma B R_\tau)(R_\tau A R_\sigma) \geq \frac{1}{2}\lambda n,$$

$$(2) \qquad \|R_\sigma B R_\tau\|_2, \|R_\tau A R_\sigma\|_2 \leq 8K_G^2 \lambda^{-1} M^3.$$

Proof. Let us apply Proposition 4.1 with $\varepsilon = \frac{1}{8}\lambda M^{-2}$. We can find $\tau_1, \sigma_1 \subset [n]$ with $|\tau_1|, |\sigma_1| \geq (1 - \frac{1}{8}\lambda M^{-2})n$ and

$$\|R_{\tau_1} A R_{\sigma_1}\|_2 \leq 8K_G^2 \lambda^{-1} M^3.$$

Similarly we can find τ_2, σ_2 with $|\tau_2|, |\sigma_2| \geq (1 - \frac{1}{8}\lambda M^{-2})n$ and

$$\|R_{\sigma_2} B R_{\tau_2}\|_2 \leq 8K_G^2 \lambda^{-1} M^3.$$

Let $\tau = \tau_1 \cap \tau_2$ and $\sigma = \sigma_1 \cap \sigma_2$. Then $|\sigma|, |\tau| \geq (1 - \frac{1}{4}\lambda M^{-2})$ and obviously, (2) holds. Furthermore

$$\begin{aligned}
\operatorname{tr}(R_\sigma B A R_\sigma) &= \operatorname{tr}(B A R_\sigma) \\
&= \operatorname{tr}(BA) - \operatorname{tr}(B A R_{\sigma^c}) \\
&\geq \lambda n - \|BA\|_X |\sigma^c| \\
&\geq \lambda n - M^2\left(\frac{1}{4}\lambda M^{-2}\right)n \\
&\geq \frac{3}{4}\lambda n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\operatorname{tr}(R_\sigma B R_\tau A R_\sigma) &= \operatorname{tr}(R_\tau A R_\sigma B) \\
&\geq \operatorname{tr}(A R_\sigma B) - \operatorname{tr}(R_{\tau^c} A R_\sigma B) \\
&\geq \frac{3}{4}\lambda n - \frac{1}{4}\lambda n = \frac{1}{2}\lambda n.
\end{aligned}$$

∎

We now consider, under the hypotheses of 4.2, the problem of estimating the size of $\max_{i,j}|a_{ij}b_{ji}|$. To this end we establish the following technical lemma.

LEMMA 4.3. *Let X and Y be n-dimensional Banach lattices satisfying the $(\infty, 1)$-condition. Let A: $X \to Y$ and B: $Y \to X$ be $n \times n$-matrices and suppose*

$\operatorname{tr}(BA) \geq \lambda n$ and $M = \max(\|A\|_{X\to Y}, \|B\|_{Y\to X})$. Suppose further that $\nu = \max_{1\le i,j\le n}|a_{ij}b_{ji}|$. Then there is a doubly substochastic $n \times n$-matrix W such that, if E denotes the 2-concavification $X_{(2)}$ of X, we have

$$\|W\|_E \leq 1, \tag{1}$$

$$\delta_W = \max_{1\le i,j\le n} |w_{ij}| \leq 2^{-3}K_G^{-2}\lambda M^{-3}\nu, \tag{2}$$

$$\|We\|_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \geq 2^{-26}K_G^{-12}\lambda^{10}M^{-18}. \tag{3}$$

Proof. First, select σ, τ by Proposition 4.2. We then define a matrix $V = (v_{ij})_{i,j=1}^n$ by

$$v_{ij} = \begin{cases} \min\left(|a_{ij}|^2, |b_{ji}|^2\right), & i \in \tau, j \in \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

For $i \in \tau$,

$$\sum_{i=1}^{n} v_{ij} \leq \sum_{j\in\sigma} |a_{ij}|^2 = n\|R_\sigma A^t R_\tau e_i\|_2^2 \leq 8K_G^2\lambda^{-1}M^3.$$

Similarly for $j \in \sigma$, $\sum_{i=1}^n v_{ij} \leq 8K_G^2\lambda^{-1}M^3$. Thus the matrix $(\frac{1}{8}K_G^{-2}\lambda M^{-3})V$ is doubly substochastic.

Let $E = X_{(2)}$ and $F = Y_{(2)}$. If $x \in E$ with $x \geq 0$ then $x_i = w_i^2$ where $w \in X$ and $\|x\|_E = \|\omega\|_X^2$. Hence,

$$\|Vx\|_F = \left\|\sum_{i=1}^{n} x_i Ve_i\right\|_F \leq \left\|\left(\sum_{i=1}^{n} w_i^2|Ve_i|\right)^{1/2}\right\|_Y^2.$$

Now $Ve_i = \sum_{j=1}^n v_{ji}e_j$ and so

$$\begin{aligned}
\|Vx\|_F &\leq \left\|\sum_{j=1}^{n}\left(\sum_{i=1}^{n} v_{ji}w_i^2\right)^{1/2} e_j\right\|_Y^2 \\
&\leq \left\|\sum_{j=1}^{n}\left(\sum_{i=1}^{n} |a_{ji}|^2 w_i^2\right)^{1/2} e_j\right\|_Y^2 \\
&= \left\|\left(\sum_{i=1}^{n} w_i^2|Ae_i|^2\right)^{1/2}\right\|_Y^2 \\
&\leq K_G^2\|A\|_{X\to Y}^2\|w\|_X^2
\end{aligned}$$

by Theorem 1.f.4 of [13]. Hence $\|V\|_{E\to F} \le K_G^2 M^2$ and similarly $\|V^t\|_{F\to E} \le K_G^2 M^2$.

Now if we let $W = (2^{-6}K_G^{-4}\lambda^2 M^{-6})V^t V$, it follows from the above that $\|W\|_E \le 1$. Next we estimate $\|We\|_1$. Clearly

$$\|We\|_1 = (We, e) = 2^{-6}K_G^{-4}\lambda^2 M^{-6}\|Ve\|_2^2.$$

Let $R_\sigma A R_\tau B R_\sigma = D$. Then for $i \in \sigma$,

$$d_{ii} = \sum_{j\in\tau} a_{ij} b_{ji}$$

so that

$$\begin{aligned}
|d_{ii}| &\le \sum_{j\in\tau} |v_{ij}|^{1/2}\big(|a_{ij}| + |b_{ji}|\big)\\
&\le \Big(\sum_{j\in\tau} v_{ij}\Big)^{1/2}\left(\Big(\sum_{j\in\tau}|a_{ij}|^2\Big)^{1/2} + \Big(\sum_{j\in\tau}|b_{ij}|^2\Big)^{1/2}\right)\\
&= \Big(\sum_{j\in\tau} v_{ij}\Big)^{1/2} n^{1/2}\big(\|R_\sigma A R_\tau e_i\|_2 + \|R_\sigma B^t R_\tau e_i\|_2\big)\\
&\le 2^4 K_G^2 \lambda^{-1} M^3 \Big(\sum_{j\in\tau} v_{ij}\Big)^{1/2}.
\end{aligned}$$

Hence

$$\left(\sum_{j=1}^n v_{ij}\right)^2 \ge 2^{-16}K_G^{-8}\lambda^4 M^{-12}|d_{ii}|^4.$$

However

$$\frac{1}{2}\lambda n \le \sum_{i\in\sigma}|d_{ii}| \le n^{3/4}\Big(\sum_{i\in\sigma}|d_{ii}|^4\Big)^{1/4}.$$

Hence $\sum_{i=1}^n |d_{ii}|^4 \ge 2^{-4}\lambda^4 n$. Thus

$$\|Ve\|_2^2 = \frac{1}{n}\sum_{i=1}^n\left(\sum_{j=1}^n v_{ij}\right)^2 \ge 2^{-20}K_G^{-8}\lambda^8 M^{-12}.$$

Thus

$$\|We\|_1 \ge 2^{-26}K_G^{-12}\lambda^{10}M^{-18}.$$

Finally we estimate δ_W:

$$\begin{aligned} w_{ij} &= 2^{-6}K_G^{-4}\lambda^2 M^{-6}\sum_{k=1}^{n} v_{ki}v_{kj} \\ &\le 2^{-3}K_G^{-2}\lambda M^{-3}\max_{1\le k\le n} v_{ki} \\ &\le 2^{-3}K_G^{-2}\lambda M^{-3}\nu. \end{aligned}$$

■

We now introduce a property of a family of finite-dimensional Banach lattices. Let $\mathscr{E}$ be a collection of finite-dimensional Banach lattices. We say that $\mathscr{E}$ has *property* (P) if given $0<\lambda<1$ and $0<M<\infty$ there exists $\nu=\nu(\lambda,M)>0$ so that whenever $X\in\mathscr{E}$ with $\dim X=n$, Y is any n-dimensional Banach lattice and A, B are $n\times n$-matrices with $\|A\|_{X\to Y}\le M$, $\|B\|_{Y\to X}\le M$ and $\operatorname{tr}(BA)\ge\lambda n$, then $\max_{i,j}|a_{ij}b_{ji}|\ge\nu$. Notice that if $\mathscr{E}$ has property (P) and $\mathscr{E}^*=\{X^*: X\in\mathscr{E}\}$ then $\mathscr{E}\cup\mathscr{E}^*$ also has property (P).

We make use of a further remark. If X and Y are n-dimensional Banach lattices, then by Lozanovskii's theorem [14] there exist invertible positive diagonal matrices D_X, D_Y so that the lattice norms $\|\cdot\|_{\tilde X}$ on X and $\|\cdot\|_{\tilde Y}$ on Y defined by $\|x\|_{\tilde X}=\|D_X x\|_X$ and $\|y\|_{\tilde Y}=\|D_Y y\|_Y$ verify the $(\infty,1)$-condition. Then if A: $X\to Y$ and B: $Y\to X$ are bounded linear maps, set

$$\tilde A = D_Y^{-1}AD_X, \qquad \tilde B = D_X^{-1}BD_Y.$$

It follows that $\|\tilde A\|_{\tilde X\to\tilde Y}=\|A\|_{X\to Y}$ and $\|\tilde B\|_{\tilde Y\to\tilde X}=\|B\|_{Y\to X}$ and

$$\operatorname{tr}(\tilde B\tilde A)=\operatorname{tr}\big(D_X^{-1}BAD_X^{-1}\big)=\operatorname{tr}(BA).$$

Furthermore for fixed i, j, $\tilde a_{ij}\tilde b_{ji}=a_{ij}b_{ji}$. Hence, in all our arguments, we will be free to renorm both X and Y by a diagonal transformation to satisfy the $(\infty,1)$-condition.

Further, if X is p-convex, we can renorm X to satisfy the (∞,p)-condition; if X is q-concave, we can renorm, using duality, so that X satisfies the $(q,1)$-condition.

PROPOSITION 4.4. *Suppose $\beta>0$. Let $\mathscr{E}$ be the collection of finite-dimensional* 1*-symmetric spaces X for which $d_X\ge n^\beta$, where $n=$* dim X. *Then $\mathscr{E}$ has property* (P).

Proof. In view of the above remarks, it suffices to consider the case when X is a 1-symmetric space satisfying the $(\infty,1)$-condition and such that for some $0\ne x\in X$,

$$\|x\|_X\le n^{-\beta}\|x\|_2.$$

Let Y be any n-dimensional Banach lattice satisfying the $(\infty, 1)$-condition. Suppose A, B are $n \times n$-matrices with $\operatorname{tr}(BA) \geq \lambda n$, and $\|A\|_{X \to Y} \leq M$ and $\|B\|_{Y \to X} \leq M$.

By Lemma 4.3, if $E = X_{(2)}$, there is a substochastic matrix W with $\|W\|_E \leq 1$, and

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \geq 2^{-26} K_G^{-12} \lambda^{10} M^{-18},$$

$$\max_{1 \leq i, j \leq n} w_{ij} \leq 2^{-3} K_G^{-2} \lambda M^{-3} \max_{1 \leq i, j \leq n} |a_{ij} b_{ji}|.$$

Further, in E there is a vector u with $\|u\|_E \leq n^{-2\beta}$ and $\|u\|_1 = 1$. It follows immediately from Lemma 3.3 that $\max_{1 \leq i, j \leq n} |a_{ij} b_{ji}| \geq \nu$ where $\nu = \nu(\lambda, M) > 0$. ∎

Now let $\mathscr{E}$ be a collection of finite-dimensional Banach lattices satisfying the $(\infty, 1)$-condition. We shall say that $\mathscr{E}$ has *property* (Q) if there exists p, $2 < p < \infty$, such that, given any $\varepsilon > 0$, there exist $K(\varepsilon), N(\varepsilon) > 0$ so that if $X \in \mathscr{E}$, with $\dim X = n \geq N(\varepsilon)$ then there exists $x \in X$ with $\|x\|_2 = 1$,

$$\int_{\Pi_n} \|x_\pi\|_X \, d\pi \leq \varepsilon$$

and $\|x\|_p \leq K(\varepsilon)$.

PROPOSITION 4.5. *If $\mathscr{E}$ has property* (Q) *then $\mathscr{E}$ has property* (P).

Proof. We may assume $p \leq 4$. If $\lambda > 0$ and $M > 1$ are given, select

$$\varepsilon = 2^{-27} K_G^{-12} \lambda^{10} M^{-18}.$$

Suppose $X \in \mathscr{E}$ with $\dim X = n \geq N(\varepsilon)$. Let Y be any n-dimensional Banach lattice satisfying the $(\infty, 1)$-condition. Suppose A, B are $n \times n$-matrices with $\|A\|_{X \to Y}$, $\|B\|_{Y \to X} \leq M$ and $\operatorname{tr}(BA) \geq \lambda n$.

Let $E = X_{(2)}$. By Lemma 4.3, there is a doubly substochastic matrix W with $\|W\|_E \leq 1$,

$$\|We\|_1 \geq 2^{-26} K_G^{-12} \lambda^{10} M^{-18}$$

and

$$\delta_W \leq 2^{-3} K_G^{-2} \lambda M^{-3} \max_{1 \leq i, j \leq n} |a_{ij} b_{ij}|.$$

There exists $x \in E$ with $x \geq 0$, $\|x\|_1 = 1$ and such that

$$\int_{\Pi_n} \|x_\pi\|_E^{1/2} \, d\pi = \int_{\Pi_n} \|x_\pi^2\|_X \, d\pi \leq \varepsilon$$

and

$$\|x\|_{p/2}^{1/2} = \|x^2\|_p \leq K(\varepsilon).$$

Hence, by Lemma 3.2,

$$2^{-26} K_G^{-12} \lambda^{10} M^{-18} \leq \varepsilon + 2^{1/2p} \delta_W^{1/2-1/2p} K(\varepsilon).$$

By choice of ε this implies $\delta_W \geq f(\lambda, M) > 0$ and hence $\max_{1 \leq i,j \leq n} |a_{ij} b_{ji}| \geq \nu(\lambda, M) > 0$. Of course if $n \leq N(\varepsilon)$ we obtain a trivial bound on $\max_{1 \leq i,j \leq n} |a_{ij} b_{ji}|$ so that the proposition follows. ■

PROPOSITION 4.6. *Suppose* $1 \leq q < 2$.

(a) *Let* $\mathscr{E}$ *be the collection of finite-dimensional Banach lattices satisfying the condition* $(q, 1)$. *Then* $\mathscr{E}$ *has property* (P).

(b) *Suppose* $\mathscr{E}$ *is the collection of finite-dimensional Banach lattices, with* q*-concavity constant one (or, by duality, with* p*-convexity constant one where* $1/p + 1/q = 1$). *Then* $\mathscr{E}$ *has property* (P).

Proof. (a). We verify (Q). For $\varepsilon > 0$, let $N(\varepsilon) = 2\varepsilon^{-2q/(2-q)}$. For $n \geq N(\varepsilon)$, pick k so that $\frac{1}{2}\varepsilon^{2q/(2-q)} < k/n \leq \varepsilon^{2q/(2-q)}$. If $X \in \mathscr{E}$ with $\dim X = n$, let $x = n^{1/2} k^{-1/2} 1_{[k]}$. Then

$$\int_{\Pi_n} \|x_\pi\|_X \, d\pi \leq \|x\|_q \leq \left(\frac{k}{n}\right)^{1/q-1/2} \leq \varepsilon,$$

by choice of k. However,

$$\|x\|_4 = \left(\frac{k}{n}\right)^{-1/4} \leq 2^{1/4} \varepsilon^{-q/(4-2q)} = K(\varepsilon).$$

(b). This follows from (a) immediately by the remarks preceding Proposition 4.4. ■

PROPOSITION 4.7. *Let* $X[0,1]$ *be an r.i. Banach function space and let* $X_n = X(\mathscr{B}_n)$ *where* $\mathscr{B}_n$ *is the algebra generated by the sets* $[(k-1)/n, k/n]$ *for* $1 \leq k \leq n$. *Then if* $X \neq L_2$, *the collection* $\mathscr{E} = (X_n)_{n=1}^\infty$ *has property* (P).

Proof. Note that $X_n^* = X^*(\mathscr{B}_n)$. We show that either $\mathscr{E}$ or $\mathscr{E}^*$ has property (Q). In fact we may assume by replacing $\mathscr{E}$ by $\mathscr{E}^*$ if necessary that $\inf_{\|f\|_2=1}\|f\|_X = 0$. Then for given $\varepsilon > 0$ there exists a simple function $f_\varepsilon \in X$ such that $\|f_\varepsilon\|_X < \varepsilon$ but $\|f\|_2 > 1$. Let $K(\varepsilon) = \|f\|_4$, and notice that for large enough n the conditional expectation f_n of f_ε on $\mathscr{B}_n$ satisfies $\|f_n\|_X < \varepsilon$, $\|f_n\|_2 > 1$, and $\|f_n\|_4 \le K(\varepsilon)$. This means that $\mathscr{E}$ has property (Q) and hence also (P). ∎

Our final example concerns the matrix spaces $l_p^m(l_q^n)$. We need first a preparatory lemma (which is due to Gluskin [6]).

LEMMA 4.8. *For each $p > 0$, there is a constant C_p so that if $A \subset [n]$, and if $\xi_i(\pi) = 1$ if $\pi(i) \in A$ and $\xi_i(\pi) = 0$ otherwise then*

$$\left(\int_{\Pi_n} |\xi_1(\pi) + \cdots + \xi_r(\pi)|^p \, d\pi\right)^{1/p} \le r\frac{|A|}{n} + C_p r^{1/2}\left(\frac{|A|}{n}\right)^{\beta}.$$

where $\beta = \min(1/2p, 1/2)$.

Remark. $\xi_1 + \cdots + \xi_r$ has a hypergeometric distribution.

Proof. Let $\eta_1, \ldots, \eta_{2r}$ be independent $\{0,1\}$-valued random variables defined on some probability space (Ω, P) with $P(\eta_i = 1) = |A|/n$, $P(\eta_i = 0) = 1 - |A|/n$. Then for $p \ge 2$,

$$\begin{aligned}
\left\|\sum_{i=1}^r \left(\eta_i - \frac{|A|}{n}\right)\right\|_p &\le \left\|\sum_{i=1}^r (\eta_i - \eta_{i+r})\right\|_p \\
&= \int \left\|\sum_{i=1}^r \varepsilon_i(\eta_i - \eta_{i+r})\right\|_p d\varepsilon \\
&\le C_0 \left\|\left(\sum_{i=1}^r (\eta_i - \eta_{i+r})^2\right)^{1/2}\right\|_p \\
&\le C_0 \left(\sum_{i=1}^r \|\eta_i - \eta_{i+r}\|_p^2\right)^{1/2} \\
&\le C r^{1/2}\left(\frac{|A|}{n}\right)^{1/p}
\end{aligned}$$

where C_0 and C depend only on p. Thus

$$\left\| \sum_{i=1}^{r} \eta_i \right\|_p \leq \frac{|A|}{n} + Cr^{1/2}\left(\frac{|A|}{n}\right)^{1/p}.$$

To prove the lemma, first suppose $p \geq 2$ is an integer. Clearly for any $1 < i_1, \ldots, i_k \leq r$,

$$\begin{aligned} \int_{\Pi_n} \xi_{i_1} \ldots \xi_{i_k} \, d\pi &= \binom{|A|}{k} \Big/ \binom{n}{k} \\ &\leq \left(\frac{|A|}{n}\right)^k \\ &= \int_{\Omega} \eta_{i_1} \ldots \eta_{i_k} \, dP. \end{aligned}$$

Hence, since ξ_i and η_i are $\{0, 1\}$-valued it follows that:

$$\|\xi_1 + \cdots + \xi_r\|_p \leq \|\eta_1 + \cdots + \eta_r\|_p.$$

The lemma thus follows easily for $p \geq 2$ an integer. For the general case pick q to be an integer so that $p < q \leq 1/\beta$ and use the estimate $\|\xi_1 + \cdots + \xi_r\|_p \leq \|\xi_1 + \cdots + \xi_r\|_q$. ∎

PROPOSITION 4.9. (a) *Suppose* $2 \notin \{p, q\}$. *Then the collection* $\{l_p^m(l_q^n)\}_{m,n=1}^{\infty}$ *has property* (P).

(b) *If* $q \neq 2$ *and* $n_k \to \infty$ *then any sequence of spaces* $\{l_2^{m_k}(l_q^{n_k})\}_{k=1}^{\infty}$ *has property* (P).

Proof. (a). By duality we may assume that $q < 2$. It then suffices to show that any sequence of spaces in the collection has a subsequence with property (P). Clearly if $\sup_k n_k < \infty$ then the spaces $l_p^{m_k}(l_q^{n_k})$ are uniformly isomorphic to $l_p^{m_k n_k}$ and so has (P). Therefore suppose $n_k \to \infty$. The spaces $l_p^{m_k}(l_q^{n_k})$ can be renormed to obey the $(\infty, 1)$-condition by

$$\|x\|_0 = m_k^{-1/p} n_k^{-1/q} \|x\|.$$

Let $N = N_k = n_k m_k$. Now let A be any subset of $[m_k] \times [n_k]$ of cardinality $|A| = \theta N$. Consider Π_N acting on $[m_k] \times [n_k]$ in the obvious way. If we let

$x = \theta^{-\frac{1}{2}} 1_A$ then $\|x\|_2 = 1$, $\|x\|_4 = \theta^{-1/4}$ and

$$\int_{\Pi_N} \|x_\pi\|_0 \, d\pi \le \left(\int_{\Pi_N} \|x_\pi\|_0^p \right)^{1/p}$$
$$= \theta^{-1/2} n_k^{-1/q} \left(\int_{\Pi_N} (\xi_1 + \cdots + \xi_{n_k})^{p/q} \, d\pi \right)^{1/p}$$

where $\xi_1, \dots, \xi_{n_k}$ are given as in Lemma 4.8. Thus

$$\int_{\Pi_N} \|x_\pi\|_0 \, d\pi \le \theta^{-1/2} n_k^{-1/q} \left(n_k \theta + C n_k^{1/2} \theta^{\beta} \right)^{1/q}.$$

where $C = C(p)$ and $\beta = \min(1/2, q/2p)$.

Now, given $\varepsilon > 0$, it is clear that for large enough k we may choose $|A|$ so that

$$\int_{\Pi_N} \|x_\pi\|_0 \, d\pi < \varepsilon$$

and $\|x\|_4 < 2\varepsilon^{q/(4-2q)}$. Thus the collection $l_p^{m_k}(l_q^{n_k})$ has property (P).

(b) Same proof as (a). ■

5. Applications of property (P)

LEMMA 5.1. *Let $\mathscr{E}$ be a collection of finite-dimensional Banach lattices with property* (P). *Then, given any $1 \le M < \infty$, $0 < \lambda < 1$ and $\gamma \ge 1$, there exist $\alpha = \alpha(M, \lambda, \gamma) > 0$ and $\delta = \delta(M, \lambda, \gamma) > 0$ so that if $X \in \mathscr{E}$ with* $\dim X = n$ *and Y is any m-dimensional Banach lattice, where $m \le \gamma n$, and A: $X \to Y$, B: $Y \to X$ are linear maps with $\|A\|_{X \to Y}$, $\|B\|_{Y \to X} \le M$, and* $\operatorname{tr}(BA) \ge \lambda n$, *then there exists a subset $\sigma \subset [n]$ with $|\sigma| \ge \alpha n$, and a one-one map π: $\sigma \to [m]$ so that $|a_{\pi(i), i} b_{i, \pi(i)}| \ge \delta$, for $i \in \sigma$.*

Proof. The lattice Y is spanned by a set $(y_i)_{i=1}^m$ of normalized atoms; for $\tau \subset [m]$ we shall denote by R_τ the natural projection from Y onto $[y_i]_{i \in \tau}$.

First we note that if $m \ge n$,

$$\operatorname{Ave}_\tau \operatorname{tr}(R_\tau AB) = \frac{n}{m} \operatorname{tr}(AB) \ge \frac{\lambda}{\gamma} n$$

where the average is computed over all subsets τ with $|\tau| = n$. We thus can fix τ so that

$$\operatorname{tr}(R_\tau AB) \geq \lambda\gamma^{-1}n,$$

and consider the maps $R_\tau A\colon X \to Y_\tau(=R_\tau Y)$, $B\colon Y_\tau \to X$, where $\dim Y_\tau = n$. If $m \leq n$ we may simply expand Y to have dimension n, and let $\tau = [m]$. Let $\delta = \nu(\frac{1}{2}\lambda\gamma^{-1}, M)$ be given by (P) and let σ be a maximal subset of $[n]$ so that there is a one-one map $\pi\colon \sigma \to \tau$ with $|a_{\pi(i),i}b_{i,\pi(i)}| \geq \delta$.

Consider the maps $R_\rho AR_{\rho^c}\colon X \to Y_\tau$ and $B\colon Y_\tau \to X$, where $\rho = \tau \setminus \pi(\sigma)$. Clearly,

$$|a_{ij}b_{ji}| < \delta \quad \text{if } j \in \sigma^c,\ i \in \rho.$$

Thus

$$\operatorname{tr}\left(R_\rho AR_{\sigma^c}B\right) \leq \frac{1}{2}\lambda\gamma^{-1}n.$$

However,

$$\begin{aligned}\lambda\gamma^{-1}n \leq \operatorname{tr}(R_\tau AB) &= \operatorname{tr}\left(R_\rho AB\right) - \operatorname{tr}\left(R_{\pi(\sigma)}AB\right)\\ &= \operatorname{tr}\left(R_\rho AR_{\sigma^c}B\right) + \operatorname{tr}\left(R_\rho AR_\sigma B\right) - \operatorname{tr}\left(R_{\pi(\sigma)}AB\right)\\ &< \frac{1}{2}\lambda\gamma^{-1}n + 2|\sigma|M^2,\end{aligned}$$

as in Proposition 4.2. Thus $2|\sigma|M^2 \geq \frac{1}{2}\lambda\gamma^{-1}n$ which yields that $|\sigma| \geq (4\gamma M^2)^{-1}n$ so that we may take $\alpha = (4\gamma M^2)^{-1}$ and $\delta = \nu(\frac{1}{2}\lambda\gamma^{-1}, M)$. ∎

PROPOSITION 5.2. *Let $\mathscr{E}$ be a collection of finite-dimensional Banach lattices having property* (P). *Then*

(1) *Given $1 \leq M < \infty$, $1 \leq K < \infty$ and $0 < \lambda \leq 1$, there exists $\beta = \beta(\lambda, M, K) > 0$ and $L = L(\lambda, M, K) < \infty$ so that if $X \in \mathscr{E}$ with $\dim X = n$ is spanned by normalized atoms $(e_i)_{i=1}^n$ and if $(u_i)_{i=1}^m$ is an M-complemented normalized K-unconditional basic sequence in X with $m \geq \lambda n$, then there is a subset $\sigma \subset [n]$ and a one-one map $\pi\colon \sigma \to [m]$ with $|\sigma| \geq \beta n$ so that $(e_i)_{i\in\sigma}$ is L-equivalent to $(u_{\pi(i)})_{i\in\sigma}$.*

(2) *Given $1 \leq M < \infty$, $1 \leq K < \infty$ and $1 < \gamma < \infty$, there exists $\beta = \beta(\gamma, M, K) > 0$ and $L = L(\gamma, M, K) < \infty$ so that if $X \in \mathscr{E}$ with $\dim X = n$ is K-isomorphic to an M-complemented subspace of a space Y with $\dim Y = m \leq \gamma n$, having a normalized 1-unconditional basis $(u_i)_{i=1}^m$ then there is a subset $\sigma \subset [n]$ with $|\sigma| \geq \beta n$ and a one-one map $\pi\colon \sigma \to [m]$ for which $(e_i)_{i\in\sigma}$ is L-equivalent to $(u_i)_{i\in\pi(\sigma)}$.*

Proof. These follow immediately from Lemma 5.1. We prove only (1). First note that $(u_i)_{i=1}^m$ is K-equivalent to a normalized 1-unconditional basis

$(v_i)_{i=1}^m$ of some Banach lattice Y. Hence there exists A: $X \to Y$ with $\|A\|_{X\to Y} \le MK$ and B: $Y \to X$ with $\|B\|_{Y\to X} \le 1$ such that $BA = P$, the projection of X onto $[u_i]_{i=1}^m$. It follows that $\mathrm{tr}(BA) = m \ge \lambda n$, and by Lemma 5.1, we can find $\sigma \subset [n]$ with $|\sigma| \ge \alpha(MK, \lambda, 1)$ and a map π: $\sigma \to [m]$ with $|a_{\pi(i), i} b_{i, \pi(i)}| \ge \delta$, where $\delta = \delta(MK, \lambda, 1)$.

Since the bases are normalized, $|a_{ij}| \le MK$ and $|b_{ij}| \le 1$. Hence $\delta \le |a_{\pi(i), i}| \le MK$ and $\delta/MK \le |b_{i, \pi(i)}| \le 1$, for all $i \in \sigma$.

Now, if

$$\tilde{A}\left(\sum_{i=1}^n \xi_i e_i\right) = \sum_{i\in\sigma} \xi_i a_{\pi(i), i} v_{\pi(i)}$$

then $\|\tilde{A}\|_{X\to Y} \le \|A\|_{X\to Y}$ by a standard "diagonal" argument [12, p. 20]. Hence

$$\left\|\sum_{i\in\sigma} \xi_i v_{\pi(i)}\right\|_Y \le \frac{1}{\delta}\left\|\sum_{i\in\sigma} \xi_i e_i\right\|_X,$$

for all $\{\xi_i\}_{i\in\sigma}$. In exactly the same manner, one can show that also

$$\left\|\sum_{i\in\sigma} \xi_i e_i\right\|_X \le \frac{MK}{\delta}\left\|\sum_{i\in\sigma} \xi_i v_{\pi(i)}\right\|_Y,$$

again for every $\{\xi_i\}_{i\in\sigma}$. It follows that $(e_i)_{i\in\sigma}$ in X is MK/δ^2-equivalent to $(v_{\pi(i)})_{i\in\sigma}$ in Y and thus is also MK^2/δ^2-equivalent to $(u_{\pi(i)})_{i\in\sigma}$. ■

In the case when $\mathscr{E}$ is a family of spaces with a symmetric basis, the set σ, whose existence is asserted by Proposition 5.2, can be chosen to have almost maximal cardinality.

PROPOSITION 5.3. *Let $\mathscr{E}$ be a collection of finite-dimensional symmetric spaces having property* (P). *Then*:

(1) *Given $1 \le M < \infty$, $1 \le K < \infty$, $0 < \lambda < 1$ and $0 < \varepsilon < 1$ there exists $L' = L'(\lambda, M, K, \varepsilon)$ so that if $X \in \mathscr{E}$ with* $\dim x = n$ *has a symmetric basis $(e_i)_{i=1}^n$ and $(u_i)_{i=1}^m$ is an M-complemented normalized K-unconditional basic sequence in X with $m \ge \lambda n$ then there is a subset $\sigma \subset [m]$ so that $(u_i)_{i\in\sigma}$ is L'-equivalent to $(e_i)_{i=1}^{|\sigma|}$ and $|\sigma| \ge (1-\varepsilon)m$.*

(2) *Given $1 \le M < \infty$, $1 \le K < \infty$, $1 < \gamma < \infty$ and $0 < \varepsilon < 1$ there exists $L' = L'(\gamma, M, K, \varepsilon)$ so that if $X \in \mathscr{E}$ with* $\dim X = n$ *has a symmetric basis $(e_i)_{i=1}^n$ and is K-isomorphic to an M-complemented subspace of a space Y with* $\dim Y = m \le \gamma n$ *and a* 1*-unconditional normalized basis $(u_i)_{i=1}^m$ then there is a subset $\sigma \subset [m]$ with $|\sigma| \ge (1-\varepsilon)n$ so that $(u_i)_{i\in\sigma}$ is L'-equivalent to $(e_i)_{i=1}^{|\sigma|}$.*

Proof. Again we prove only (1). Let $\beta(\lambda, M, K)$ and $L(\lambda, M, K)$ be determined as in Proposition 5.2. Thus if $\sigma \subset [m]$ and $|\sigma_1| \geq \varepsilon m$ then σ_1 contains a subset σ_2 with

$$|\sigma_2| \geq \beta(\varepsilon\lambda, MK, K)n$$

so that $(u_i)_{i\in\sigma_2}$ is $L(\varepsilon\lambda, MK, K)$-equivalent to $(e_i)_{i=1}^{|\sigma_2|}$. If we iterate this it is clear that we can find disjoint subsets $\tau_1, \tau_2, \dots, \tau_l$ where $l \leq \lambda\beta(\varepsilon, MK, K)^{-1}$ so that each $|\tau_k| \geq \alpha n$, $\Sigma_{j=1}^{l}|\tau_j| \geq m(1-\varepsilon)$ and $(u_i)_{i\in\tau_j}$ is L-equivalent to $(e_i)_{i=1}^{|\tau_j|}$. By the symmetry of $(e_i)_{i=1}^{n}$ this implies if $\sigma = \tau_1 \cup \cdots \cup \tau_l$ then $(u_i)_{i\in\sigma}$ is $C(l)L$-equivalent to $(e_i)_{i=1}^{|\sigma|}$ and $|\sigma| \geq m(1-\varepsilon)$. ■

THEOREM 5.4. *Suppose $p > 2$. Then for $0 < \lambda < 1$, $1 \leq K < \infty$, and $1 \leq M < \infty$ there exist $\alpha = \alpha(\lambda, M, K, p) > 0$ and $L = L(\lambda, M, K, p) < \infty$ so that if X and Y are finite-dimensional Banach spaces with* $\dim X = n$, $\dim Y = m \geq \lambda n$ *and X has a 1-unconditional normalized basis $(x_i)_{i=1}^{n}$, Y has a 1-unconditional normalized basis $(y_i)_{i=1}^{m}$ and Y is K-isomorphic to an M-complemented subspace of X and either $(x_i)_{i=1}^{n}$ or $(y_i)_{i=1}^{m}$ is p-convex (with p-convexity constant one) or q-concave (with q-concavity constant one), where $p^{-1} + q^{-1} = 1$, then there is a subset $\sigma \subset [n]$ and a one-one map $\pi\colon \sigma \to [m]$ so that $|\sigma| \geq \alpha n$ and $(x_i)_{i\in\sigma}$ is L-equivalent to $(y_i)_{i\in\pi(\sigma)}$.*

Proof. By duality only the case of q-concavity need be considered. The result then follows immediately from Propositions 4.6 and 5.2. ■

THEOREM 5.5. *Suppose $r > 0$. Then for $0 < \lambda < 1$, $1 \leq K < \infty$, $1 \leq M < \infty$ and $0 < \varepsilon < 1$ there exists $L = L(\lambda, M, K, \varepsilon, r) < \infty$ so that whenever X and Y are finite-dimensional Banach spaces with* $\dim X = n$, $\dim Y = m \geq \lambda n$, *$X$ has a 1-unconditional normalized basis $(x_i)_{i=1}^{n}$, Y has a 1-unconditional normalized basis $(y_i)_{i=1}^{m}$, Y is K-isomorphic to an M-complemented subspace of X and either $(x_i)_{i=1}^{n}$ is symmetric and $d_X \geq n^r$ or $(y_i)_{i=1}^{m}$ is symmetric and $d_Y \geq m^r$, then there is asubset $\sigma \subset [n]$ with $|\sigma| \geq m(1-\varepsilon)$ and a one-one map $\pi\colon \sigma \to [m]$ so that $(x_i)_{i\in\sigma}$ is L-equivalent to $(y_i)_{i\in\pi(\sigma)}$.*

Proof. Use Propositions 4.4 and 5.3. ■

Remark. As we have pointed out in the introduction, Theorems 5.4 and 5.5 are generalizations of results of Schütt [16].

In order to state the next theorem we introduce some notation. If X is a rearrangement-invariant Banach function space on $[0,1]$ we denote by X_n the n-dimensional subspace $X(\mathscr{B}_n)$ where $\mathscr{B}_n$ is the algebra generated by the sets $[j-1/n, j/n)$ for $1 \leq j \leq n$. We then let $e_j = 1_{[j-1/n, j/n)}$ and $x_j = e_j/\|e_j\|$. Thus $(x_j)_{j=1}^{n}$ is the canonical normalized symmetric basis of X_n.

THEOREM 5.6. *Let $X = X[0,1]$ be a rearrangement-invariant Banach function space on* $[0,1]$. *Then, given $0 < \varepsilon < 1$, $0 < \lambda \le 1$, $1 \le K < \infty$ and $1 \le M < \infty$, there is a constant $L = L(\varepsilon, \lambda, K, M, X)$ so that if $(y_i)_{i=1}^m$ is any M-complemented K-unconditional normalized basic sequence in X_n with $m \ge \lambda n$, then there is a subset $\sigma \subset [n]$ with $|\sigma| \ge m(1-\varepsilon)$ so that $(y_i)_{i\in\sigma}$ is L-equivalent to $(x_i)_{i=1}^{|\sigma|}$.*

Proof. If $X = L_2$ this is obvious. For $X \neq L_2$, it is a direct consequence of Propositions 4.7 and 5.3. ∎

THEOREM 5.7. *Suppose $1 \le p \le \infty$ and $1 \le q \le \infty$. Then given $\varepsilon > 0$ and $1 \le K < \infty$ there exists $L = L(\varepsilon, K, p, q) < \infty$ so that if $(y_i)_{i=1}^{mn}$ is a K-unconditional normalized basis of $l_p^m(l_q^n)$ and $(x_i)_{i=1}^{mn}$ is the canonical basis then there is a subset $\sigma \subset [mn]$ and a one-one map $\pi\colon \sigma \to [mn]$ so that $(y_i)_{i=1}^{mn}$ is L-equivalent to $(y_{\pi(i)})_{i\in\sigma}$ and $|\sigma| \ge (1-\varepsilon)mn$.*

Remark. The case $1 < p < q < 2$ is due to Schütt [16]. For the sake of completeness, we consider all cases in the proof below.

Proof. Case (1). $2 \notin \{p, q\}$. In this case the collection $\mathscr{E} = l_p^m(l_q^n)$ for $m, n \in \mathbf{N}$ has property (P) by Proposition 4.9. Thus Proposition 5.2 and the iteration argument of 5.3 show that we can find disjoint subsets $\sigma_1, \ldots, \sigma_l$ of $[mn]$, where $l = l(\varepsilon, K, p, q)$ so that

$$\sum_{i=1}^{l} |\sigma_j| \ge \left(1 - \frac{1}{2}\varepsilon\right) mn$$

and $(y_i)_{i\in\sigma_j}$ is L_1-equivalent to a subset of the canonical basis $(x_i)_{i=1}^{mn}$, for each $1 \le j \le l$ where $L_1 = L_1(\varepsilon, K, p, q)$. Thus $(y_i)_{i\in\sigma_j}$ is L_1-equivalent to the canonical basis of

$$\left(l_q^{n_{j_1}} \oplus l_q^{n_{j_2}} \oplus \cdots \oplus l_q^{n_{j_m}}\right)_p$$

where $n_{j_1} + \cdots + n_{j_m} = |\sigma_j|$. Hence if $\sigma = \sigma_1 \cup \cdots \cup \sigma_l$, $(y_i)_{i\in\sigma}$ is L_2-equivalent to the canonical basis of $(l_q^{n_1} \oplus \cdots \oplus l_q^{n_m})_p$ where $n_k \le ln$, $n_1 + \cdots + n_m = |\sigma| \le mn$, and $L_2 = L_2(\varepsilon, K, p, q)$. If we eliminate all basis members corresponding to an $n_k \le \frac{1}{4}\varepsilon n$ we are still left with a set σ' with $|\sigma'| \ge (1-\varepsilon)mn$ and so we can assume $n_k \ge \frac{1}{4}\varepsilon n$ or $n_k = 0$ for $1 \le k \le m$. Thus, by breaking up each non-zero n_k, the canonical basis of $(l_q^{n_1} \oplus \cdots \oplus l_q^{n_m})_p$ is $L_3(\varepsilon, K, p, q)$-equivalent to that of $(l_q^{h_1} \oplus \cdots \oplus l_q^{h_N})_p$ where $\frac{1}{4}\varepsilon n \le h_j \le \frac{1}{2}\varepsilon n$ for $1 \le j \le N$ and hence by recombining is $L_4(\varepsilon, K, p, q)$-equivalent to that of $(l_q^{k_1} \oplus \cdots \oplus l_q^{k_M})_p$ where $(1 - \frac{1}{2}\varepsilon)n \le k_j \le n$ for $1 \le j \le M-1$, and $k_M \le n$. Thus

$$(M-1)(1-\varepsilon/2) \le (1-\varepsilon)m$$

so that $M-1<m$ and $M \leq m$. It follows that the canonical basis of $(l_q^{k_1} \oplus \cdots \oplus l_q^{k_M})_p$ is 1-equivalent to a subset of the basis of $l_p^m(l_q^n)$. Hence $(y_i)_{i \in \sigma'}$ is $L(\varepsilon, K, p, q)$-equivalent to a subset of the canonical basis as required.

Case (2). $q \neq 2$, $p=2$. Here it is enough to consider a sequence $l_2^{m_k}(l_q^{n_k})$ where $m_k n_k \to \infty$. As in Proposition 4.9(b) if $n_k \to \infty$ the sequence has property (P) and so the preceding argument for Case (1) applies. If not, we pass to a subsequence where $4n_k$ is bounded and then the spaces are uniformly isomorphic to $l_2^{m_k n_k}$ so that the result is trivial.

Case (3). $p=q=2$. Trivial.

Case (4). $q=2$, $p \neq 2$. In this case we must proceed somewhat differently. We can assume, by duality, that $1<p<2$. We will need to prove the following lemma.

LEMMA 5.8. *Given $1<p<2$, $0<\lambda \leq 1$ and $0<M<\infty$, there exist $\alpha = \alpha(\lambda, M, p)<\infty$ and $\theta=\theta(\lambda, M, p)$, $0<\theta \leq 1$, such that if $X=l_p^m(l_2^n)$ and its canonical basis is denoted by $(x_i)_{i=1}^N$, $N=mn$, Y is an N-dimensional Banach lattice spanned by atoms $(y_i)_{i=1}^N$ and $A\colon X \to Y$, $B\colon Y \to X$ are linear operators satisfying $\|A\|_{X \to Y}, \|B\|_{Y \to X} \leq M$ and $\operatorname{tr}(BA) \geq \lambda N$ then there are subsets σ and η of $[N]$ with $|\sigma|=n$, $[x_i]_{i \in \sigma}$ isometric to l_2^n, $|\eta| \leq \alpha n$ and signs $(\varepsilon_i)_{i \in \eta}$ so that if*

$$Q_{\varepsilon, \eta}\left(\sum_{i=1}^{N} a_i y_i\right)=\sum_{i \in \eta} \varepsilon_i a_i y_i$$

then $\nu(R_\sigma B Q_{\varepsilon, \eta} A R_\sigma) \geq \theta n$.

Let us first assume Lemma 5.8 and complete the proof of Theorem 5.7. It suffices to consider an isomorphism $A\colon X \to Y$ where Y has a 1-unconditional basis $(y_i)_{i=1}^N$ and let $B=A^{-1}$ where $\|A\|_{X \to Y}, \|B\|_{Y \to X} \leq K$. We will show that for any $0<\gamma<1$ there exists $\beta=\beta(\gamma, K, p)>0$ and a constant $L=L(\gamma, K, p)<\infty$ so that if $\tau \subset [N]$ with $|\tau| \geq \gamma N$ then τ contains a subset η with $|\eta| \geq \beta N$ so that $(y_i)_{i \in \eta}$ is L-equivalent to a subset of $(x_i)_{i=1}^N$. The argument is then completed easily as in Case (1).

Let $\alpha=\alpha(\frac{1}{2}\gamma, K, p)$ and $\theta=\theta(\frac{1}{2}\gamma, K, p)$ be given by Lemma 5.8. Let $\sigma_1, \ldots, \sigma_m$ be the partition of $[N]$ into sets of cardinality n so that each $[x_i]_{i \in \sigma_j}$ is isometric to l_2^n. We may determine inductively a maximal collection of distinct $\sigma_{h_1}, \ldots, \sigma_{h_k}$ and corresponding disjoint subsets $\eta_1, \ldots, \eta_k$ of τ so that $|\eta_j| \leq \alpha n$ for $j=1,2,\ldots,k$, and there are signs $\varepsilon_i = \pm 1 (i \in \eta_1 \cup \ldots \cup \eta_k)$ so that

$$\nu\left(R_{\sigma_{h_j}} B Q_{\varepsilon, \eta_j} R_\tau A R_{\sigma_{h_j}}\right) \geq \theta n$$

where

$$Q_{\varepsilon,\eta_j}\left(\sum_{i=1}^{N} a_i y_i\right) = \sum_{i\in\eta_j} \varepsilon_i a_i y_i.$$

When this is complete set $\rho_X = [N] \setminus (\sigma_{h_1} \cup \cdots \cup \sigma_{h_k})$ and $\rho_Y = \tau \setminus (\eta_1 \cup \cdots \cup \eta_k)$. It follows that if $\sigma_j \subset \rho_X$ and $\eta \subset \rho_Y$ with $|\eta| \le \alpha n$ then for every choice of signs $\varepsilon_i = \pm 1$ $(i \in \eta)$,

$$\nu\left(R_{\sigma_j} B Q_{\varepsilon,\eta} R_\tau A R_{\sigma_j}\right) < \theta n$$

and hence for every j and every $\eta \subset [N]$ with $|\eta| \le \alpha n$ and every choice of signs $\varepsilon_i = \pm 1$ $(i \in \eta)$,

$$\nu\left(R_{\sigma_j} R_{\rho_X} B R_{\rho_Y} Q_{\varepsilon,\eta} R_{\rho_Y} R_\tau A R_{\rho_X} R_{\sigma_j}\right) < \theta n.$$

Thus, by Lemma 5.8,

$$\operatorname{tr}\left(R_{\rho_X} B R_{\rho_Y} A R_{\rho_X}\right) < \frac{1}{2}\gamma N.$$

Now $\operatorname{tr}(BR_{\rho_Y}A) = \operatorname{tr} R_{\rho_Y} = |\rho_Y|$. Hence

$$\begin{aligned}\operatorname{tr}\left(BR_{\rho_Y} A R_{\rho_X}\right) &\ge \operatorname{tr}\left(BR_{\rho_Y}A\right) - \left(N - |\rho_X|\right)K^2 \\ &= |\rho_Y| + K^2|\rho_X| - K^2 N.\end{aligned}$$

Now $|\rho_Y| \ge |\tau| - k\alpha n$ and $|\rho_X| \ge N - kn$. Thus

$$\begin{aligned}\operatorname{tr}\left(BR_{\rho_Y} A R_{\rho_X}\right) &\ge |\tau| - kK^2 n - k\alpha n \\ &\ge \gamma N - k(K^2 + \alpha)n.\end{aligned}$$

Hence $k(K^2+\alpha)n \ge \frac{1}{2}\gamma N$ so that $k \ge \frac{1}{2}\gamma(K^2+\alpha)^{-1}m$.
Now define $A_1\colon X \to Y$ and $B_1\colon Y \to X$ by

$$A_1 = \sum_{j=1}^{k} Q_{\varepsilon,\eta_j} A R_{\sigma_{h_j}}, \qquad B_1 = \sum_{j=1}^{k} R_{\sigma_{h_j}} B R_{\eta_j}.$$

Then $\|A_1\|, \|B_1\| \le K$ by a diagonal argument [12, p. 20].

If we fix j, A_1 maps $[x_i]_{i\in\sigma_j}$ into $[y_i]_{i\in\eta_j}$ and B_1 maps $[y_i]_{i\in\eta_j}$ into $[x_i]_{i\in\sigma_j}$ with $\nu(B_1A_1)\geq\theta n\geq\theta\alpha^{-1}|\eta_j|$. Let

$$\tilde{\eta}_j=\left\{i\in\eta_j:\ \|B_1y_i\|\,\|A_1^*y_i^*\|\geq\frac{\theta}{2\alpha}\right\}.$$

Then by Lemma 2.1, $|\tilde{\eta}_j|\geq\frac{1}{2}K^{-2}\theta n$ and for any scalars $(a_i)_{i\in\tilde{\eta}_j}$,

$$\frac{1}{C_1}\left(\sum_{i\in\tilde{\eta}_j}|a_i|^2\right)^{1/2}\leq\left\|\sum_{i\in\tilde{\eta}_j}a_iy_i\right\|\leq C_1\left(\sum_{i\in\tilde{\eta}_j}|a_i|^2\right)^{1/2}$$

where $C_1=2K^2\alpha\theta^{-1}$.

By a further reduction we may suppose $|\tilde{\eta}_j|\leq n$. Let $\tilde{\eta}=\tilde{\eta}_1\cup\cdots\cup\tilde{\eta}_k$. Then

$$|\tilde{\eta}|\geq\frac{1}{2}kK^{-2}\theta n\geq c_2N\quad\text{where } c_2=\frac{1}{4}K^{-2}(K^2+\alpha)^{-1}\theta.$$

Since Y has type p constant K^2 at most

$$\begin{aligned}\left\|\sum_{i\in\eta}a_iy_i\right\|&\leq K^2\left(\sum_{j=1}^{k}\left\|\sum_{i\in\tilde{\eta}_j}a_iy_i\right\|^p\right)^{1/p}\\&\leq C_3\left(\sum_{j=1}^{k}\left(\sum_{i\in\tilde{\eta}_j}|a_i|^2\right)^{p/2}\right)^{1/2}\end{aligned}$$

where $C_3=C_3(\theta,K,\alpha)=C_3(\gamma,K,p)$.

Conversely,

$$\begin{aligned}\int\left\|\sum_{i\in\tilde{\eta}}\varepsilon_ia_iB_1y_i\right\|^p d\varepsilon&=\sum_{j=1}^{k}\int\left\|\sum_{i\in\tilde{\eta}_j}\varepsilon_ia_iB_1y_i\right\|^p d\varepsilon\\&\geq c_4\sum_{j=1}^{k}\left(\int\left\|\sum_{i\in\tilde{\eta}_j}\varepsilon_ia_iB_1y_i\right\|^2 d\varepsilon\right)^{p/2}\end{aligned}$$

where $c_4>0$ depends only on p. Thus

$$\left\|\sum_{i\in\tilde{\eta}}a_iy_i\right\|\geq c_5\left(\sum_{j=1}^{k}\left(\sum_{i\in\tilde{\eta}_j}|a_i|^2\|B_1y_i\|^2\right)^{p/2}\right)^{1/p}.$$

However $\|B_1 y_i\| \geq \frac{1}{2}K^{-1}\theta\alpha^{-1}$. Thus

$$\left\|\sum_{i\in\tilde{\eta}} a_i y_i\right\| \geq c_6\left(\sum_{j=1}^{k}\left(\sum_{i\in\tilde{\eta}_j}|a_i|^2\right)^{p/2}\right)^{1/p}$$

where $c_6 = c_6(\gamma, K, p) > 0$. Hence $(y_i)_{i\in\tilde{\eta}}$ is $C_7(\gamma, K, p)$-equivalent to a subset of $(x_i)_{i=1}^N$ (and recall that $|\tilde{\eta}| \geq c_2 N$). This will complete the proof. ∎

It remains to establish the lemma.

Proof of Lemma 5.8. Here it is necessary to use the methods of Sections 3 and 4. First we renormalize X, Y to be lattices satisfying the $(\infty, 1)$-condition. Thus the norm on X is defined by

$$\|x\|_X = \frac{1}{m^{1/p}n^{1/2}}\left(\sum_{j=1}^{m}\left(\sum_{k\in\sigma_j}|\xi_k|^2\right)^{p/2}\right)^{1/p},$$

where $\{\sigma_j\}_{j=1}^m$ is a suitable partition of $[N]$, and $x = (\xi_i)_{i=1}^N$. We regard A and B as $N\times N$-matrices. Select subsets ρ and τ as in Proposition 4.2 so that $|\rho|, |\tau| \geq (1 - \frac{1}{4}\lambda M^{-2})N$, and

$$\operatorname{tr}(R_\rho B R_\tau)(R_\tau A R_\rho) \geq \frac{1}{2}\lambda N,$$

$$\|R_\rho B R_\tau\|_2, \|R_\tau A R_\rho\|_2 \leq 8K_G^2\lambda^{-1}M^3.$$

Now as in Lemma 4.3 define the matrix $V = (v_{ij})_{i,j=1}^N$ by $v_{ij} = \min(|a_{ij}|^2, |b_{ji}|^2)$ for $i\in\tau$, $j\in\rho$ and $v_{ij} = 0$ otherwise. Then if $W = (2^{-6}K_G^{-4}\lambda^2 M^{-6})(V^t V)$, W is doubly substochastic,

$$\|We\|_1 \geq 2^{-26}K_G^{-12}\lambda^{10}M^{-18},$$

and $\|W\|_E \leq 1$, where E is the 2-concavification of X.

Consider the $m\times m$-matrix $T = (t_{ij})_{i,j=1}^m$ defined by

$$t_{ij} = \frac{1}{n}\sum_{k\in\sigma_i}\sum_{l\in\sigma_j} w_{kl},$$

where $W = (w_{kl})_{k,l=1}^N$.

If $x = \sum_{j=1}^m a_j 1_{\sigma_j} \in X$ with $a_j \geq 0$, for all $1 \leq j \leq m$, then the vector $Wx = z$ satisfies

$$\|x\|_E \geq \|z\|_E = m^{-2/p} n^{-1} \left(\sum_{i=1}^m \left(\sum_{k \in \sigma_i} |z_k| \right)^{p/2} \right)^{2/p}$$

where

$$z_k = \sum_{j=1}^m a_j \sum_{l \in \sigma_j} w_{k,l}.$$

Hence

$$\begin{aligned} \|x\|_E &\geq m^{-2/p} n^{-1} \left(\sum_{i=1}^m \left(n \sum_{j=1}^m t_{ij} a_j \right)^{p/2} \right)^{2/p} \\ &= m^{-2/p} \left(\sum_{i=1}^m \left(\sum_{j=1}^m t_{ij} a_j \right)^{p/2} \right)^{2/p} \\ &= \|Ta\|_{p/2}. \end{aligned}$$

Thus $\|T\|_{p/2} \leq 1$. Since $\|Te\|_1 = \|We\|_1 \geq 2^{-26} K_G^{-12} \lambda^{10} M^{-18}$ we can apply Lemma 3.3 for T as an operator on $L_{p/2}^m$ (notice that any atom normalized in L_1^m has norm in $L_{p/2}^m$ equal to $m^{1-2/p}$) to deduce that $\max_{1 \leq i,j \leq m} t_{ij} \geq \delta$, for a suitable $\delta = \delta(\lambda, M, p) > 0$. Hence there exist $\delta_1 = \delta_1(\lambda, M, p) > 0$ and $i, j \in [m]$ with

$$\sum_{k \in \sigma_i} \sum_{l \in \sigma_j} (V^t V)_{kl} \geq n\delta_1$$

or

$$\sum_{k \in \sigma_i} \sum_{l \in \sigma_j} \sum_{r=1}^N v_{kr} v_{lr} \geq n\delta_1.$$

This yields

$$\sum_{r=1}^N \left(\sum_{k \in \sigma_i} v_{kr} \right) \left(\sum_{l \in \sigma_j} v_{lr} \right) \geq n\delta_1$$

and hence with $\sigma = \sigma_i$ or $\sigma = \sigma_j$, we have

$$\sum_{r=1}^{N} \left(\sum_{k \in \sigma} v_{kr} \right)^2 \geq n\delta_1.$$

Notice that

$$\sum_{r=1}^{N} \left(\sum_{k \in \sigma} v_{kr} \right) \leq 8K_G^2 \lambda^{-1} M^3 n$$

and put

$$\eta = \left\{ r \colon \sum_{k \in \sigma} v_{kr} \geq \frac{\lambda \delta_1}{16 K_G^2 M^3} \right\}.$$

Then $|\eta| \leq 2^7 K_G^4 M^6 \lambda^{-2} \delta^{-1} n = \alpha n$ where $\alpha = \alpha(\lambda, M, p)$ and

$$\sum_{r \in \eta^c} \left(\sum_{k \in \sigma} v_{kr} \right)^2 \leq \frac{1}{2} n\delta_1.$$

Thus

$$\sum_{r \in \eta} \left(\sum_{k \in \sigma} v_{kr} \right)^2 \geq \frac{1}{2} n\delta_1.$$

For arbitrary signs $\varepsilon_i = \pm 1$, $i \in \eta$, consider the map $R_\sigma B Q_{\varepsilon,\eta} A R_\sigma$: $[x_i]_{i \in \sigma} \to [x_i]_{i \in \sigma}$, where Q is defined in the statement. This operator has Hilbert-Schmidt norm equal to

$$\left(\sum_{k,l \in \sigma} \left(\sum_{r \in \eta} \varepsilon_r b_{kr} a_{rl} \right)^2 \right)^{1/2}.$$

Hence

$$\begin{aligned} \int \|R_\sigma B Q_{\varepsilon,\eta} A R_\sigma\|_{HS}^2 \, d\varepsilon &= \sum_{k \in \sigma} \sum_{l \in \sigma} \sum_{r \in \eta} |b_{kr}|^2 |a_{rl}|^2 \\ &\geq \sum_{k \in \sigma} \sum_{l \in \sigma} \sum_{r \in \eta} v_{rl} v_{rk} \\ &= \sum_{r \in \eta} \left(\sum_{k \in \sigma} v_{rk} \right)^2 \\ &\geq \frac{1}{2} n\delta_1. \end{aligned}$$

Thus there is a choice of signs $\varepsilon_i = \pm 1$, $i \in \eta$, so that $\|R_\sigma B Q_{\varepsilon,\eta} A R_\sigma\|_{HS} \geq (\frac{1}{2} n \delta_1)^{1/2}$. But then, since for any operator S on a Hilbert space $\|S\|_{HS} \leq \nu(S)^{1/2} \|S\|^{1/2}$ we have

$$\nu(R_\sigma B Q_{\varepsilon,\eta} A R_\sigma) \geq \frac{n\delta_1}{2M^2} = n\theta$$

where $\theta = \theta(\lambda, M, p) > 0$. ■

6. Rearrangement-invariant Banach function spaces

In this section we consider some special results concerning families of spaces of the form $X_n = X(\mathscr{B}_n)$ where $X = X[0,1]$ is a rearrangement-invariant Banach function space on $[0,1]$. We assume without loss of generality that X satisfies the $(\infty, 1)$-condition and hence so does each X_n if we take for the canonical basis the vectors (atoms) $e_k = 1_{[(k-1)/n, k/n]}$ for $1 \leq k \leq n$. For notational convenience we also need to identify the normalized symmetric basis $x_k = e_k / \|e_k\|$ of X_n.

LEMMA 6.1. *Let X be a rearrangement-invariant Banach function space on* $[0,1]$ *such that $X \neq L_2$. Then, given $\eta > 0$, there exists $\alpha = \alpha(\eta, X) > 0$ so that if A is an $n \times n$-matrix with $\|A\|_{X_n} \leq 1$ and if $\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \geq \eta n$ then there exists $\pi \in \Pi_n$ so that $\sum_{i=1}^n |a_{\pi(i),i}| \geq \alpha n$.*

Proof. By a simple duality argument it suffices to consider the case when there exists $f \in X \setminus L_2$ so that the collection $\mathscr{E} = (X_n)_{n=1}^\infty$ has property (Q). Thus there exists $p > 2$ and functions $K(\varepsilon), N(\varepsilon)$ so that if $n \geq N(\varepsilon)$ there exists $x \in X_n$ with $\|x\|_2 = 1$, $\|x\|_X \leq \varepsilon$ and $\|x\|_p \leq K(\varepsilon)$.

We fix $\varepsilon = \frac{1}{3}\eta$ and then choose $\beta > 0$ so that $2^{1/p}\beta^{1/2-1/p}K(\varepsilon) < \varepsilon$.

If A is an $n \times n$ -matrix where $n \geq N(\varepsilon)$ which satisfies the hypotheses of the lemma, we may pick a maximal subset $\sigma \subset [n]$ so that there is a one-one map $\pi: \sigma \to [n]$ with $|a_{\pi(i),i}| \geq \beta$. Let $\tau = \pi(\sigma)$ and let E_n denote the 2-concavification of X_n. As in Lemma 4.3 the matrix V given by $v_{ij} = |a_{ij}|^2$ for $i \in \tau^c$ and $j \in \sigma^c$ and $v_{ij} = 0$ otherwise, satisfies $\|V\|_E \leq 1$, $\|V\|_1 \leq 1$ and $\|V\|_\infty \leq 1$. Now pick $x \in X_n$ with $\|x\|_X \leq \varepsilon$, $\|x\|_2 = 1$ and $\|x\|_p \leq K(\varepsilon)$. If we put $u = |x|^2$ then $\|u\|_E \leq \varepsilon^2$, $\|u\|_1 = 1$, and $\|u\|_{p/2} \leq K(\varepsilon)^2$.

By Lemma 3.2, with $e = \sum_{j=1}^n e_j$ we have

$$\|Ve\|_1 \leq \|u\|_E^{1/2} + 2^{1/p}\beta^{1/2-1/p}K(\varepsilon),$$

since $\max_{1 \le i,j \le n} v_{ij} < \beta$. Thus $\|Ve\|_1 < \frac{2}{3}\eta$. Now

$$\begin{aligned}\eta n &\le \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \\ &\le \sum_{i \in \sigma} \sum_{j=1}^{n} |a_{ij}|^2 + \sum_{i=1}^{n} \sum_{j \in \tau} |a_{ij}|^2 + \|Ve\|_1 \\ &\le |\sigma| + |\tau| + 2\varepsilon n.\end{aligned}$$

Hence as $|\sigma| = |\tau|$ we have $|\sigma| \ge \frac{1}{6}\eta n$. Then if we extend π to an element of Π_n we have

$$\sum_{i=1}^{n} |a_{\pi(i),i}| \ge \frac{1}{6}\beta\eta n.$$

On the other hand if $n \le N(\varepsilon)$,

$$\begin{aligned}\max_{\pi \in \Pi_n} \sum_{i=1}^{n} |a_{\pi(i),i}| &\ge \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \\ &\ge \frac{1}{n} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \\ &\ge \frac{1}{n} \eta^{1/2} n^{1/2} \\ &\ge \eta^{1/2} N(\varepsilon)^{-3/2} n.\end{aligned}$$

and this completes the proof of the lemma. ■

THEOREM 6.2. *Let X be a rearrangement-invariant Banach function space on $[0,1]$ containing $L_q[0,1]$ where $q < \infty$. Then, given $0 < \lambda \le 1$, $1 \le K < \infty$ and $1 \le M < \infty$, there exist $\alpha = \alpha(\lambda, K, M, X) > 0$ and $L = L(\lambda, K, M, X) < \infty$ so that if $m \ge l \ge \lambda n$, and $(y_i)_{i=1}^m$ is a K-unconditional normalized basic sequence in X_n so that $[y_i]_{i=1}^m$ contains a subspace M-isomorphic to $[x_i]_{i=1}^l$, where, as usual, $(x_i)_{i=1}^n$ denotes the canonical normalized symmetric basis of X_n, then there is a subset $\sigma \subset [m]$ with $|\sigma| \ge \alpha n$ so that $(y_i)_{i \in \sigma}$ is L-equivalent to $(x_i)_{i=1}^{|\sigma|}$.*

Proof. We may assume $X \ne L_2[0,1]$, and that X satisfies the $(q,1)$-condition. It suffices then to consider the following situation. Let Y be an m-dimensional Banach lattice spanned by normalized atoms $(u_i)_{i=1}^m$ K-equivalent to $(y_i)_{i=1}^m$ and suppose A: $X_n \to Y$ and B: $Y \to X_n$ are linear maps satisfying $\|A\|_{X_n \to Y} \le MK$, $\|B\|_{Y \to X_n} \le 1$ and:

$$\|x\|_X \le \|Ax\|_Y, \quad x \in [e_i]_{i=1}^l, \tag{1}$$

$$\|y\|_Y \le K\|By\|_X, \quad y \in Y. \tag{2}$$

Now, by Proposition 4.1, there exist subsets $\sigma, \tau \subset [n]$ so that $|\sigma^c|, |\tau^c| \le \frac{1}{4}\lambda n$, and $\|R_\tau BAR_\sigma\|_2 \le 4K_G^2\lambda^{-1}MK$. Let $\sigma_0 = \sigma \cap [l]$ so that $|\sigma_0| \ge \frac{3}{4}\lambda n$ and consider the vectors $\{BAe_i\}_{i\in\sigma_0}$. Notice that for $\{t_i\}_{i\in\sigma_0}$,

$$\left\|\sum_{i\in\sigma_0} t_i BAe_i\right\|_X \le MK \max_{i\in\sigma_0}|t_i|$$

and

$$\begin{aligned}\left\|\sum_{i\in\sigma_0} t_i BAe_i\right\|_X &\ge K^{-1}\frac{1}{n}\sum_{i\in\sigma_0}|t_i| \\ &\ge \frac{3}{4}\lambda K^{-1}|\sigma_0|^{-1}\sum_{i\in\sigma_0}|t_i|.\end{aligned}$$

Now R_τ is a quotient map from X_n onto a subspace $[e_i]_{i\in\tau}$ of dimension $|\tau|$. Since

$$|\tau| + |\sigma_0| - n \ge \frac{1}{2}\lambda n,$$

we may apply Theorem 1.1 of [2] to deduce that

$$\int\left\|\sum_{i\in\sigma_0}\varepsilon_i R_\tau BAe_i\right\|_X d\varepsilon \ge c_0(\lambda, K, M) > 0.$$

Thus

$$\int\left\|\sum_{i\in\sigma_0}\varepsilon_i R_\tau BAe_i\right\|_q d\varepsilon \ge c_0(\lambda, K, M).$$

Hence

$$\left\|\left(\sum_{i\in\sigma_0}|R_\tau BAe_i|^2\right)^{1/2}\right\|_q \ge c_1(\lambda, K, M, q) > 0.$$

Let $D = R_\tau BAR_\sigma$ and let $(d_{ij})_{1\le i,j\le n}$ be its matrix with respect to the canonical basis $(e_i)_{i=1}^n$ of X_n. Then

$$\left\| \sum_{j\in\tau} \left(\sum_{i\in\sigma_0} |d_{ij}|^2 \right)^{1/2} e_j \right\|_q \ge c_1.$$

However, for fixed j, $\sum_{i\in\sigma_0} |d_{ij}|^2 \le 2^4 K_G^4 \lambda^{-2} M^2 K^2$ so that we also obtain, by a simple interpolation argument,

$$\left\| \sum_{j\in\tau} \left(\sum_{i\in\sigma_0} |d_{ij}|^2 \right)^{1/2} e_j \right\|_2 \ge c_2(\lambda, K, M, q) > 0$$

or

$$\sum_{j\in\tau} \sum_{i\in\sigma_0} |d_{ij}|^2 \ge c_2^2 n.$$

Now, by applying Lemma 6.1 to D we obtain that for some $\pi \in \Pi_n$ and some signs $\varepsilon_i = \pm 1$, $1 \le i \le n$ we have $\operatorname{tr}(S_{\varepsilon,\pi} R_\tau BAR_\sigma) \ge \beta n$ where $\beta = \beta(\lambda, K, M, X) > 0$ and

$$S_{\varepsilon,\pi}\left(\sum_{i=1}^n \xi_i e_i \right) = \sum_{i=1}^n \varepsilon_i \xi_i e_{\pi(i)}.$$

Since $\mathscr{E} = \{X_n\}_{n=1}^\infty$ has property (P) we can appeal to Lemma 5.1 to deduce the existence of a subset $\sigma_1 \subset [n]$, with $|\sigma_1| \ge \alpha n$, where $\alpha = \alpha(\lambda, K, M, X) > 0$ and one-one maps $\pi_1\colon \sigma_1 \to [m]$ and $\pi_2\colon \sigma_1 \to [n]$ so that $|a_{\pi_1(i),i} b_{\pi_2(i),\pi_1(i)}| \ge \delta(\lambda, K, M, X) > 0$. It then follows easily that $(y_i)_{i\in\pi_1(\sigma_1)}$ is $L(\lambda, K, M, X)$-equivalent to $(x_i)_{i=1}^{|\pi_1(\sigma_1)|}$ completing the proof of the theorem. ∎

If we make the assumption that X lies on one side of L_2 then we can achieve a rather stronger statement and eliminate the hypothesis that X contains some L_q.

THEOREM 6.3. *Let X be a rearrangement-invariant Banach function space on $[0,1]$ so that either $L_2 \subset X$ or $X \subset L_2$. Then, given $0 < \lambda \le 1$, $1 \le K < \infty$, $1 \le M < \infty$ and $0 < \varepsilon < 1$ there exists $L = L(\varepsilon, \lambda, K, M, X)$ so that if $m \ge l$*

$\geq \lambda n$ and $(y_i)_{i=1}^m$ is a normalized K-unconditional basic sequence in X_n such that $[y_i]_{i=1}^m$ contains a subspace M-isomorphic to $[x_i]_{i=1}^l$ then there is a subset $\sigma \subset [m]$ for which $|\sigma| \geq (1-\varepsilon)l$ and $(y_i)_{i\in\sigma}$ is L-equivalent to $(x_i)_{i=1}^{|\sigma|}$.

Proof. As in Theorem 6.2, we can assume that $X \neq L_2$. We may further assume that X satisfies either the $(\infty, 2)$-condition or the $(2,1)$-condition. It will suffice, again as before, to consider the situation when Y is an m-dimensional Banach lattice spanned by atoms $(u_i)_{i=1}^m$ K-equivalent to $(y_i)_{i=1}^m$ and $A\colon X_n \to Y$, $B\colon Y \to X_n$ are linear maps such that $\|A\|_{X_n \to Y} \leq MK$, $\|B\|_{Y\to X_n} \leq 1$ and

$$\|x\|_X \leq \|Ax\|_Y, \quad x \in [e_i]_{i=1}^l, \tag{1}$$

$$Ax = 0, \quad x \in [e_i]_{i=l+1}^n, \tag{2}$$

$$\|y\|_Y \leq K\|By\|_X, \quad y \in Y. \tag{3}$$

To obtain the conclusion it will suffice to prove that there exist $\alpha = \alpha(\varepsilon, \lambda, K, M, X) > 0$ and $L_1 = L_1(\varepsilon, \lambda, K, M, X) < \infty$ so that if $\sigma \subset [m]$ with $|\sigma| \geq m - (1-\varepsilon)l$ then there exists $\sigma_0 \subset \sigma$ with $|\sigma_0| \geq \alpha l$ so that $(u_i)_{i\in\sigma_0}$ is L_1-equivalent to $(x_i)_{i=1}^{|\sigma_0|}$. The result will then follow by an obvious induction process.

Case 1. Assume first that X satisfies the $(2,1)$-condition. Then by the results of [5] and [10] there exists $c_0 = c_0(\varepsilon, \lambda) > 0$ so that X_n contains a subspace Z with $\dim Z \geq (1 - \frac{1}{3}\varepsilon\lambda)n$ and

$$\|z\|_2 \geq \|z\|_X \geq \|z\|_1 \geq c_0\|z\|_2, \quad z \in Z.$$

Suppose $\sigma \subset [m]$ with $|\sigma| \geq m - (1-\varepsilon)l$. Then, by applying one half of the argument of Proposition 4.1, we may find a subset $\tau \subset [n]$ with $|\tau| \geq (1 - \frac{1}{3}\varepsilon\lambda)n$ so that

$$\|R_\tau B R_\sigma A\|_2 \leq C_0(\varepsilon, \lambda, K, M).$$

Let $Z_1 = Z \cap A^{-1}([u_i]_{i\in\sigma}) \cap [e_i]_{i=1}^l$. Then

$$\dim\left(A^{-1}[u_i]_{i\in\sigma} \cap [e_i]_{i=1}^l\right) \geq l - (m - |\sigma|) \geq \varepsilon l.$$

Thus

$$\dim Z_1 \geq \left(1 - \frac{1}{3}\varepsilon\lambda\right)n + \varepsilon l - n \geq \frac{2}{3}\varepsilon l.$$

Let $h = \dim Z_1$, and let $(f_i)_{i=1}^h$ be an orthonormal basis of $(Z_1, \|\ \|_2)$. Then for any $\{t_i\}_{i=1}^h$,

$$\left\| \sum_{i=1}^{h} t_i B R_\sigma A f_i \right\|_X \le MK \left(\sum_{i=1}^{h} t_i^2 \right)^{1/2} \le MKh^{1/2} \max_{1 \le i \le h} |t_i|,$$

$$\left\| \sum_{i=1}^{h} t_i B R_\sigma A f_i \right\|_X \ge K^{-1} \left\| \sum_{i=1}^{h} t_i f_i \right\|_X \ge c_0 K^{-1} \left(\sum_{i=1}^{h} t_i^2 \right)^{1/2} \ge c_0 K^{-1} h^{1/2} \left(\frac{1}{h} \sum_{i=1}^{h} |t_i| \right).$$

Thus we can apply Theorem 1.1 of [2] (since $|\tau| + h + n \ge \frac{1}{3}\varepsilon l \ge \frac{1}{3}\varepsilon\lambda n$) to deduce that

$$\int \left\| \sum_{i=1}^{h} \varepsilon_i R_\tau B R_\sigma A f_i \right\|_X d\varepsilon \ge c_1 h^{1/2}$$

where $c_1 = c_1(\varepsilon, \lambda, K, M) > 0$. Thus

$$\sum_{i=1}^{h} \| R_\tau B R_\sigma A f_i \|_2^2 = \int \left\| \sum_{i=1}^{h} \varepsilon_i R_\tau B R_\sigma A f_i \right\|_2^2 d\varepsilon \ge c_1^2 h.$$

We conclude that if $D = R_\tau B R_\sigma A$: $L_2^n \to L_2^n$ then D has Hilbert-Schmidt norm at least $c_1\sqrt{h}$. Thus if D has matrix $(d_{ij})_{1 \le i, j \le n}$ with respect to the basis $(e_i)_{i=1}^n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |d_{ij}|^2 \ge c_1^2 h \ge \frac{2}{3} c_1^2 \varepsilon \lambda n.$$

Now we can use Lemma 6.1 to deduce the existence of an operator $S_{\varepsilon,\pi}$ which is an isometry on X_n so that $\operatorname{tr}(S_{\varepsilon,\pi} D) \ge c_2 n$ where $c_2 = c_2(\varepsilon, \lambda, K, M, X) > 0$. As in the proof of Theorem 6.2 this implies by Lemma 5.1 that σ has a subset σ_0 with $|\sigma_0|/l \ge \alpha(\varepsilon, \lambda, K, M) > 0$ so that $[u_i]_{i \in \sigma_0}$ is $L_1(\varepsilon, \lambda, K, M, X)$-equivalent to $(x_i)_{i=1}^{|\sigma_0|}$.

Case 2. Assume X satisfies the $(\infty, 2)$-condition, so that X^* satisfies the $(2,1)$-condition. As before there is a subspace Z of X_n^* with $\dim Z \geq (1 - \frac{1}{3}\varepsilon\lambda)n$ and

$$\|z\|_2 \geq \|z\|_{X^*} \geq \|z\|_1 \geq c_0\|z\|_2 \qquad z \in Z$$

where $c_0 = c_0(\varepsilon, \lambda, K, M) > 0$. As before, suppose $\sigma \subset [m]$ with $|\sigma| \geq m - (1-\varepsilon)l$. We may now find a subset $\tau \subset [n]$ with $|\tau| \geq (1 - \frac{1}{3}\varepsilon\lambda)n$ so that $\|R_\tau A^t R_\sigma B^t\|_2 \leq C_0(\varepsilon, \lambda, K, M)$ where B^t and A^t are the adjoints (transposes) of B and A.

Let $Z_1 = Z \cap (B^t)^{-1}[u_i]_{i\in\sigma}$. Then if $h = \dim Z_1$ we have

$$\begin{aligned} h &\geq |\sigma| + \dim B^t + \left(1 - \frac{1}{3}\varepsilon\lambda\right)n - n \\ &\geq |\sigma| + n - \operatorname{rank} B - \frac{1}{3}\varepsilon\lambda n \\ &\geq n - \left(1 - \frac{2}{3}\varepsilon\right)l. \end{aligned}$$

Let $(f_i)_{i=1}^h$ be an orthonormal basis of $(Z_1, \|\ \|_2)$. Let $S = R_\tau A^t B^t$; then

$$\operatorname{rank} S \geq l - \frac{1}{3}\varepsilon\lambda n \geq \left(1 - \frac{1}{3}\varepsilon\right)l.$$

Suppose $x^* \in X_n^*$ and $\|Sx^*\|_{X^*} = 1$. Then $\|x^*|_{BAR_\tau(X_n)}\| \geq K^{-1}$. It follows that

$$\operatorname{dist}(x^*, \ker(S)) \geq K^{-1}$$

so that there is a linear operator $T\colon X_n^*/\ker S \to X_n^*$ with

$$\|T\| \leq MK \quad \text{and} \quad \|Tg\| \geq K^{-1}\|g\| \quad \text{for } g \in X_n^*/\ker S,$$

and such that $S = TQ$ where Q is the quotient map.

For any $\{t_i\}_{i=1}^h$ we have

$$c_0 h^{1/2}\left(\frac{1}{h}\sum_{i=1}^h |t_i|\right) \leq \left\|\sum_{i=1}^h t_i f_i\right\|_{X^*} \leq h^{1/2} \max_{1\leq i\leq h} |t_i|.$$

Notice that $\dim(X_n^*/\ker S) \geq (1 - \frac{1}{3}\varepsilon)l$ and so $\dim(X_n^*/\ker S) + h - n \geq \frac{1}{3}\varepsilon l$. Hence by Theorem 1.1 of [2],

$$\int\left\|\sum_{i=1}^h \varepsilon_i Q f_i\right\| d\varepsilon \geq c_1 h^{1/2}$$

where $c_1 = c_1(\varepsilon, \lambda, K, M) > 0$. Thus

$$\int \left\| \sum_{i=1}^{h} \varepsilon_i S f_i \right\|_{X^*} d\varepsilon \geq K^{-1} c_1 h^{1/2}$$

and so

$$\int \left\| \sum_{i=1}^{h} \varepsilon_i R_\tau A^t R_\sigma B^t f_i \right\|_2^2 d\varepsilon \geq K^2 c_1^2 h.$$

Hence if $D = R_\tau A^t R_\sigma B^t$ we deduce that there exists an $S_{\varepsilon,\pi}$ with

$$\operatorname{tr}(S_{\varepsilon,\pi} D) \geq c_2 n \quad \text{where } c_2 = c_2(\varepsilon, \lambda, K, M) > 0$$

and the argument is completed as in Case 1. ■

If X is 2-convex and has some concavity, we may weaken the hypotheses further. The following theorem is a mild extension of a result proved in [2] (Theorem 2.3). We omit the proof which employs techniques from [2] and this paper.

THEOREM 6.4. *Suppose $q > 2$ and that X is a 2-convex, q-concave rearrangement-invariant Banach function space on* $[0,1]$. *Then, given $0 < \lambda < 1$, $0 < \varepsilon < 1$, and $1 \leq K < \infty$ there exists $L = L(\varepsilon, \lambda, K, X)$ so that if $m \geq \lambda n$ and $(y_i)_{i=1}^m$ is a K-unconditional normalized basic sequence in X_n then there is a subset $\sigma \subset [m]$ with $|\sigma| \geq (1-\varepsilon)m$ so that $(y_i)_{i\in\sigma}$ is L-equivalent to $(x_i)_{i=1}^{|\sigma|}$.*

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