FACTORIZATION THEOREMS
FOR QUASI-NORMED SPACES

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ABSTRACT. We extend Pisier's abstract version of Grothendieck's theorem to general non-locally convex quasi-Banach spaces. We also prove a related result on factoring operators through a Banach space and apply our results to the study of vector-valued inequalities for Sidon sets. We also develop the local theory of (non-locally convex) spaces with duals of weak cotype 2.

1. Introduction.

In [16] (see also [18]) Pisier showed that if X and Y are Banach spaces so that X^* and Y have cotype 2 then any approximable operator $u : X \to Y$ factors through a Hilbert space. This result (referred to as the abstract version of Grothendieck's theorem in [18]) implies the usual Grothendieck theorem by taking the special case $X = C(\Omega)$ and $Y = L_1$ as explained in [18].

Our main result is that the abstract form of Grothendieck's theorem is valid for quasi-Banach spaces. To make this precise let us say that an operator $u : X \to Y$ between two quasi-Banach spaces is *strongly approximable* if it is in the smallest subspace $\mathcal{A}(X,Y)$ of the space $\mathcal{L}(X,Y)$ of all bounded operators which contains the finite-rank operators and is closed for pointwise convergence of bounded nets. We define the dual X^* of a quasi-Banach space as the space under all bounded linear functionals; this is always a Banach space. Then suppose X, Y are quasi-Banach spaces so

that X^* and Y have cotype 2; we prove that if $u : X \to Y$ is strongly approximable then u factors through a Hilbert space.

Some approximability assumption is necessary even for Banach spaces (cf. [17]). However, in our situation, such an assumption is transparently required because X^* could have cotype 2 for the trivial reason that $X^* = \{0\}$; then the only strongly approximable operator on X is identically zero. We remark that there are many known examples of nonlocally convex spaces X with cotype 2 (e.g. $L_p, L_p/H_p$ and the Schatten ideals S_p when $p < 1$ ([20],[23]). Examples of nonlocally convex spaces whose dual have cotype 2 are less visible in nature, but in [7] there is an example of such a space X with an unconditional basis so that $X^* \sim \ell_1$.

We also give a similar result for factorization through a Banach space; in this case we require that X^* embeds into an L_1–space and that Y has nontrivial cotype. These results are then applied to the study of Sidon sets. We say a quasi-Banach space X is Sidon-regular if for every compact abelian group G and every Sidon subset $E = \{\gamma_n\}_{n=1}^{\infty}$ of the dual group Γ and for every $0 < p \le \infty$ we have $\| \sum_{k=1}^{n} x_k \gamma_k \|_{L_p(G,X)} \sim \| \sum_{k=1}^{n} \epsilon_k x_k \|_{L_p(X)}$ where (ϵ_k) are the Rademacher functions on $[0,1]$. It is a well-known result of Pisier [15] that Banach spaces are Sidon-regular but in [9] it is shown that not every quasi-Banach space is Sidon-regular. We show as a consequence of the above factorization theorems that any space with nontrivial cotype is Sidon-regular; this includes such spaces as the Schatten ideals S_p and the quotient spaces L_p/H_p when $0 < p < 1$.

Our final section is motivated by the fact that the main factorization theorem suggests that quasi-Banach spaces whose duals have cotype 2 have special properties. On an intuitive level there is no reason to suspect that properties of the dual space will influence the original space very strongly in the absence of local convexity. However we show that quasi-Banach spaces whose duals have weak cotype 2 can be characterized internally by conditions dual to the standard characterizations of weak cotype 2 spaces. We show that, for example, that X has a dual of weak cotype 2 if and only if its finite-dimensional quotients have uniformly bounded outer-volume ratios.

We refer to [11] for the essential background on quasi-Banach spaces. We will need the fact that every quasi-normed space can be equivalently normed with an r-norm where $0 < r \le 1$ (the Aoki-Rolewicz theorem) i.e. a quasi-norm satisfying $\|x + y\|^r \le \|x\|^r + \|y\|^r$.

Let us recall that the Banach envelope $\hat{X}$ of a quasi-Banach space X

is defined to be the closure of $j(X)$ where $j : X \to X^{**}$ is the canonical map (which is not necessarily injective). If j is injective (i.e. X has a separating dual) we regard $\hat{X}$ as the completion of X with respect to the norm induced by the convex hull of the closed unit ball B_X. If X is locally convex then $d(X, \hat{X}) = \|I_X\|_{\hat{X} \to X}$ is equal to the minimal Banach-Mazur distance between X and a Banach space.

2. The main factorization theorems.

Let us suppose that X and Y are r-Banach spaces where $0 < r \leq 1$. Suppose $u : X \to Y$ is a bounded linear operator. We will define $\gamma_2(u)$ to be the infimum of $\|v\|\|w\|$ over all factorizations $u = vw$ where $w : X \to H$ and $v : H \to Y$ for some Hilbert space H. We define $\delta(u)$ to be the infimum of $\|v\|\|w\|$ where $u = vw$ and $w : X \to B$ and $v : B \to Y$ for some Banach space B. In the special case when $u = I_X$ is the identity operator then $\gamma_2(I_X) = d_X$ is the Euclidean distance of X and $\delta(I_X) = \delta_X = d(X, \hat{X})$ (cf. [5], [14]) is the distance of X to its Banach envelope.

Let $D_N = \{-1, +1\}^N$ be equipped with normalized counting measure λ and define the Rademacher functions $\epsilon_i(t) = t_i$ on D_N for $1 \leq i \leq N$. We define $T_2^{(N)}(u)$ to be the least constant such that

$$\left(\int \| \sum_{i=1}^N \epsilon_i u(x_i) \|^2 d\lambda \right)^{1/2} \leq T_2^{(N)}(u) \left(\sum_{i=1}^N \|x_i\|^2 \right)^{1/2}$$

for $x_1, \ldots, x_N \in X$. We define $C_2^{(N)}(u)$ to be the least constant so that

$$\left(\sum_{i=1}^N \|u(x_i)\|^2 \right)^{1/2} \leq C_2^{(N)}(u) \left(\int \| \sum_{i=1}^N \epsilon_i x_i \|^2 d\lambda \right)^{1/2}$$

for $x_1, \ldots, x_N \in X$. We let $T_2(u) = \sup_N T_2^{(N)}(u)$ be the type 2 constant of u and $C_2(u) = \sup_N C_2^{(N)}(u)$ be the cotype two constant of u. In the case when $u = I_X$ we let $T_2(I_X) = T_2(X)$ and $C_2(I_X) = C_2(X)$ the type two and cotype two constants of X.

Finally we let $K^{(N)}(u)$ be the least constant so that if $f \in L_2(D_N, X)$ then

$$\| \sum_{i=1}^N (\int \epsilon_i u \circ f d\lambda) \epsilon_i \|_{L_2(D_N, Y)} \leq K^{(N)}(u) \|f\|_{L_2(D_N, X)}.$$

We then let $K(u) = \sup_N K^{(N)}(u)$. If $u = I_X$ then $K(I_X) = K(X)$ is the K-convexity constant of X.

We will need the following estimate.

Lemma 1. *If $0 < r < 1$ then there is a constant $C = C(r)$ so that for any r-normed space X, we have $K(X) \leq Cd_X^\phi(1 + \log d_X)$, where $\phi = (\frac{1}{r} - 1)/(\frac{1}{r} - \frac{1}{2})$.*

Proof. It is clear that $K(X) \leq \delta_X K(\hat{X})$. Now we have $\delta_X \leq Cd_X^\phi$ where $C = C(r)$ by Lemma 3 of [5]. We also have, by a result of Pisier ([16], [18]) that $K(\hat{X}) \leq C(1 + \log d_{\hat{X}})$. It remains to observe that $d_{\hat{X}} \leq d_X$ since any operator $u : X \to H$ where H is a Hilbert space factorizes through the Banach envelope of X with preservation of norm.∎

We next discuss some aspects of Lions-Peetre interpolation (see [1] or [2]). We will only need to interpolate between pairs of equivalent quasi-norms on a fixed quasi-Banach space. Let us suppose that X is an r-Banach space for quasi-norm $\| \ \|_0$ and that $\| \ \|_1$ is an equivalent r-norm on X; we write $X_j = (X, \| \ \|_j)$ for $j = 0, 1$. Let

$$K_s(t, x) = \inf\{(\|x_0\|_0^s + t^s\|x_1\|_1^s)^{1/s} : x = x_0 + x_1\}$$

where $r \leq s < \infty$. Then K_s is an r-norm on X. We define

$$\|x\|_{\theta,2} = (\theta(1 - \theta))^{1/2} \left(\int_0^\infty \frac{K_2(t, x)^2}{t^{1+2\theta}} dt \right)^{1/2}$$

for $0 < \theta < 1$. We write $X_{\theta,2} = (X, \| \ \|_{\theta,2}) = (X_0, X_1)_{\theta,2}$.

We will need some well-known observations.

Lemma 2. *There exists a $C = C(r)$ so that if $\| \ \|_0 = \| \ \|_1$ then for any $x \in X$ we have $C^{-1}\|x\|_0 \leq \|x\|_{\theta,2} \leq C\|x\|_0$.*

Proof. This follows from the simple observation that $K_r(t, x) = \min(1, t)\|x\|_0$ and that $2^{1/2-1/r}K_r \leq K_2 \leq K_r$.∎

Lemma 3. *There exists $C = C(r)$ so that if $N \in \mathbf{N}$ then $(\ell_2^N(X_0), \ell_2^N(X_1))_{\theta,2}$ is isometrically isomorphic to $\ell_2^N(X_{\theta,2})$.*

Proof. This follows from the routine estimate

$$K_2(t, (x_1, \dots, x_N)) = (\sum_{i=1}^N K_2(t, x_i)^2)^{1/2}$$
∎

The following lemma is standard.

Lemma 4. *Suppose (X_0, X_1) are as above and that (Y_0, Y_1) is a similar pair of r-normings of a quasi-Banach space Y. Let $u : X \to Y$ be a bounded linear operator. Then*

$$\|u\|_{X_{\theta,2} \to Y_{\theta,2}} \leq \|u\|_{X_0 \to Y_0}^{1-\theta} \|u\|_{X_1 \to Y_1}^{\theta}.$$

We now combine these results to give a criterion for convexity of the interpolated space. For convenience we will drop the subscript 2 and write X_θ. We also recall the definition of equal norms type p for $p \leq 2$. We let $\hat{T}_p(X)$ be the least constant so that for any N and any $x_1, \dots, x_N \in X$ we have

$$\|\sum_{i=1}^N \epsilon_i x_i\|_{L_p(D_N, X)} \leq \hat{T}_p(X) N^{1/p} \max_{1 \leq i \leq N} \|x_i\|.$$

Lemma 5. *Suppose (X_0, X_1) are as above. Suppose $1 \leq a < \infty$ and $0 < \theta < r/(2-r)$. There is a constant $C = C(a, \theta, r)$ so that if $T_2(X_0) \leq a$, then $\delta_{X_\theta} \leq C$.*

Proof. Consider the map $u : \ell_2^N(X) \to L_2(\Omega_N, X)$ defined by $u((x_i)_{i=1}^N) = \sum_{i=1}^N \epsilon_i x_i$. Then $\|u\|_{X_0} \leq a$. Since $\| \ \|_1$ is an r-norm it follows from Holder's inequality that $\|u\|_{X_1} \leq N^{1/r-1/2}$. Hence $\|u\|_{X_\theta} \leq a^{1-\theta} N^{\theta(1/r-1/2)}$. Now $\theta(1/r - 1/2) = 1/2 - \phi$ where $\phi > 0$. Assume $x_1, \dots, x_N \in X$. Then

$$\left(\int \|\sum_{i=1}^N \epsilon_i x_i\|_\theta^2 d\lambda \right)^{1/2} \leq a^{1-\theta} N^{1-\phi} \max_{1 \leq i \leq N} \|x_i\|_\theta.$$

This means that $\hat{T}_p(X_\theta) \leq a^{1-\theta}$ where $p = (1-\phi)^{-1} > 1$. Applying Lemma 2 of [5] we get the lemma. ∎

We are now in position to prove the generalization of Pisier's abstract Grothendieck theorem.

Theorem 6. *Let X, Y be quasi-Banach spaces so that X^* and Y have cotype 2. Then there is a constant C so that if $u : X \to Y$ is a strongly approximable operator, then $\gamma_2(u) \leq C\|u\|$.*

Proof. We may suppose that both X and Y are r-normed. Consider first an operator $u : X \to Y$ such that $\|u\| = 1$ and $\gamma_2(u) < \infty$. Then there

is a Hilbert space Z and a factorization $u = vw$ where $w : X \to Z$ and $v : Z \to Y$ satisfy $\|v\| \le 2\gamma_2(u)$ and $\|w\| \le 1$.

We will let $Z = Z_0$ and define Z_1 by the quasi-norm $\|z\|_1 = \max(\|z\|_0, \|v(z)\|_Y)$. Then $T_2(Z_0) = 1$; so we pick $0 < \theta < r/(2 - r)$ depending on r and deduce an estimate $\delta_{Z_\theta} \le C = C(r)$.

Now consider the map $\tilde{w}_N : L_2(D_N, X) \to \ell_2^N(Z)$ defined by

$$\tilde{w}_N(f) = \left(\int w \circ f \epsilon_i d\lambda \right)_{i=1}^N .$$

Clearly for the Euclidean norm $\| \|_0$ we have $\|\tilde{w}_N\|_0 \le 1$.

We now consider $\| \|_1$. It is routine to see that $C_2(Z_1) \le 1 + C_2(Y) \le 2C_2(Y)$. We also clearly have the estimate $\|z\|_0 \le \|z\|_1 \le 2\gamma_2(u)\|z\|_0$ so that $d_{Z_1} \le 2\gamma_2$. From this and Lemma 1 we can obtain an estimate $K(Z_1) \le C(\gamma_2(u))^\phi (1 + \log \gamma_2(u))$ where $C = C(r)$ and $\phi = 2(1-r)/(2-r)$. To simplify our estimate we replace this by $K(Z_1) \le C(\gamma_2(u))^\tau$ where τ depends only on r and $\phi < \tau < 1$. These estimates combine to give

$$\|\tilde{w}_N\|_1 \le C(\gamma_2(u))^\tau C_2(Y).$$

Interpolation now yields

$$\|\tilde{w}_N\|_\theta \le C(\gamma_2(u))^{\tau\theta} C_2(Y)^\theta.$$

Now consider $w^* : Z_\theta^* \to X^*$. By taking adjoints of $\tilde{w}_N$ and observing that $L_2(D_N, X)^*$ can be identified with $L_2(D_N, X^*)$ in the standard way we see that we have an estimate

$$T_2(w^* : Z_\theta^* \to X^*) \le C(\gamma_2(u))^{\tau\theta} C_2(Y)^\theta.$$

It follows immediately from Maurey's extension of Kwapien's theorem [12], [18] Theorem 3.4 that

$$\gamma_2(w^* : Z_\theta^* \to X^*) \le C(\gamma_2(u))^{\tau\theta} C_2(Y)^\theta C_2(X^*).$$

By duality this gives the same estimate for $\gamma_2(w : \hat{X} \to \hat{Z}_\theta)$. Our previous estimate on δ_{Z_θ} gives that the norm of the identity map $I : \hat{Z}_\theta \to Z_\theta$ is bounded by some $C = C(r)$. Since $I_X : X \to \hat{X}$ has norm one, we have:

$$\gamma_2(w : X \to Z_\theta) \le C(\gamma_2(u))^{\tau\theta} C_2(Y)^\theta C_2(X^*).$$

Now $\|v\|_0 \leq 2\gamma_2(u)$ and $\|v\|_1 \leq 1$ by construction. By interpolation we have $\|v\|_\theta \leq C\gamma_2(u)^{1-\theta}$. Now by factoring through Z_θ we obtain

$$\gamma_2(u) \leq C(\gamma_2(u))^{1-\theta+\tau\theta}C_2(Y)^\theta C_2(X^*)$$

and so

$$\gamma_2(u) \leq C(C_2(Y))^{1/(1-\tau)}(C_2(X^*))^{1/(1-\tau)\theta}.$$

Thus we conclude that if $\gamma_2(u) < \infty$ then $\gamma_2(u) \leq C\|u\|$ where C is a constant depending only on X, Y. The remainder of the argument is standard. Let $\mathcal{J}$ be the subspace of $\mathcal{L}(X,Y)$ of all operators for which $\gamma_2(u) < \infty$. Then $\mathcal{J}$ contains all finite-rank operators. We show it is closed under pointwise convergence of bounded nets. Let (u_α) be a bounded net in $\mathcal{J}$ converging pointwise to u. Then $\sup \gamma_2(u_\alpha) = B < \infty$. For each α there is a Euclidean seminorm (i.e. a seminorm obeying the parallelogram law) $\| \ \|_\alpha$, on X satisfying $\|u_\alpha(x)\| \leq \|x\|_\alpha \leq B\|x\|$ for $x \in X$. By a straightforward compactness argument there is a Euclidean seminorm $\| \ \|_E$ on X satisfying $\|u(x)\| \leq \|x\|_E \leq B\|x\|$ for $x \in X$, i.e. $u \in \mathcal{J}$. ∎

We will next prove a similar result for factorization through a Banach space. We recall that a quasi-Banach space has cotype q where $q \geq 2$ if there is a constant C so that for every N and all $x_1, \dots, x_N \in X$ we have:

$$\left(\sum_{i=1}^N \|x_i\|^q\right)^{1/q} \leq C\|\sum_{i=1}^N \epsilon_i x_i\|_{L_q(D_N,X)}.$$

Lemma 7. *Let X be a Banach space so that X^* is isomorphic to a subspace of an L_1-space, and suppose Y is a quasi-Banach space of cotype $q < \infty$. Then there is a constant $C = C(X,Y)$ so that if $u : X \to Y$ is a bounded operator then there is a Banach space Z with $T_2(Z) \leq C$ and a factorization $u = vw$ where $w : X \to Z$ and $v : Z \to Y$ satisfy $\|v\|\|w\| \leq C\|u\|$.*

Proof. By assumption, there is a compact Hausdorff space Ω and an open mapping $q : C(\Omega) \to X^{**}$. It follows that there is a constant C_0 so that if E is a finite-dimensional subspace of X there is a finite rank operator $t_E : C(\Omega) \to X^{**}$ with $\|t_E\| < C_0$ and if $x \in E$ with $\|x\| = 1$ there exists $f \in C(\Omega)$ with $\|f\| < 2$ and $t_E f = x$. It follows from the Principle of Local Reflexivity (cf. [22] p.76) that we can suppose that t_E has range in X.

We now form an ultraproduct of X and Y. Let I be the the collection of all finite-dimensional subspaces E of X and let $\mathcal{U}$ be an ultrafilter on I containing all sets of the form $\{E : E \supset F\}$ where F is a fixed finite-dimensional subspace. Consider the space $X_{\mathcal{U}}$ defined to be the quotient of $\ell_\infty(I; X)$ by the subspace $c_{\mathcal{U},0}(I; X)$ of all (x_E) such that $\lim_{\mathcal{U}} x_E = 0$; $X_{\mathcal{U}}$ is thus the space of (equivalence classes of) (x_E) normed by $\lim_{\mathcal{U}} \|x_E\|$. We regard X as a subspace of $X_{\mathcal{U}}$ by identifying x with the constant function $x_E = x$ for all E. We similarly introduce $Y_{\mathcal{U}}$ and note that $Y_{\mathcal{U}}$ has cotype q with the same constant as Y. We extend $u : X \to Y$ to $u_1 : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ by setting $u_1((x_E)_{E\in I}) = (u(x_E)_{E\in I})$. Let us also introduce the operator $t : C(\Omega) \to X_{\mathcal{U}}$ by putting $t(f) = (t_E(f))_{E\in I}$. Consider $u_1 t : C(\Omega) \to Y_{\mathcal{U}}$. By Theorem 4.1 of [10] there is a regular probability measure μ on Ω so that if $p = q + 1$ then $\|u_1 t(f)\| \leq C\|u\|(\int |f|^p d\mu)^{1/p}$. Here C depends on X, Y but not on u. This implies that $u_1 t = v_1 j$ where $j : C(\Omega) \to L_p(\mu)$ is the canonical injection and $\|v_1\| \leq C\|u\|$. Let $N = v_1^{-1}(0)$ and form the quotient $Z_1 = L_p/N$; let π be the quotient map. For each $x \in X$ there exists $f \in C(\Omega)$ with $t(f) = x$; this follows from the choice of ultrafilter. Then $w(x) = \pi j(f)$ is uniquely determined independent of f. Furthermore f can be chosen so that $\|f\| \leq 2\|x\|$ so that $\|w\| \leq 2$. Let Z be the closure of the range of w. Then clearly $v_1(\pi^{-1}(Z)) \subset Y$ so that we can define $v : Z \to Y$ with $\|v\| \leq C$ and $vw = u$. Finally $T_2(Z) \leq T_2(Z_1) \leq T_2(L_p)$ is bounded by a constant depending only on q.■

Theorem 8. *Suppose X is a quasi-Banach space such that X^* is isomorphic to a subspace of an L_1-space, and Y is a quasi-Banach space of cotype $q < \infty$. There is a constant C so that if $u : X \to Y$ is a strongly approximable operator then $\delta(u) \leq C\|u\|$.*

Proof. Let us assume that X, Y are both r-normed. Suppose first that $u : X \to Y$ satisfies $\|u\| = 1$ and $\delta(u) < \infty$. We show that $\delta(u) \leq C$ where C depends only on X, Y. Since $\delta(u) < \infty$ we can factor u through $\hat{X}$ and apply Lemma 7 to the induced operator $\hat{X} \to Y$. Thus u can be factored through a Banach space Z satisfying $T_2(Z) \leq C$ where $C = C(X, Y)$ so that $u = vw$ where $w : X \to Z$ with $\|w\| = 1$ and $v : Z \to Y$ with $\|v\| \leq C\delta(u)$. Let $Z = Z_0$ and introduce Z_1 by setting $\|z\|_1 = \max(\|z\|, \|v(z)\|)$. If we pick $\theta < r/(2 - r)$ then $\delta_{Z_\theta} \leq C$ where again C depends only on X, Y. Hence

$$\delta(u) \leq C\|v\|_{Z_\theta \to Y}\|w\|_{X \to Z_\theta}.$$

By interpolation this yields

$$\delta(u) \leq C(\delta(u))^{1-\theta},$$

and hence $\delta(u) \leq C$. The remainder of the proof is similar to that of Theorem 6.■

3. Applications to Banach envelopes and Sidon sets.

It is proved in [7] that if X is a natural quasi-Banach space (i.e. a space isomorphic to a subspace of a space $\ell_\infty(I; L_p(\mu_i)))$ with the strong approximation property (i.e. the identity on X is strongly approximable) and if Y is any subspace of X such that Y^* has cotype $q < \infty$ then Y is locally convex. We present now two variations on this theme.

Let us say that a quasi-Banach space X is (isometrically) subordinate to a quasi-Banach space Y if X is (isometrically) isomorphic to a closed subspace of a space $\ell_\infty(I; Y)$ for some index set I. Thus a separable space X is natural if it is subordinate to $L_p[0, 1]$ for some $0 < p < 1$.

Theorem 9. *Let Z be a quasi-Banach space and let X be subordinate to Z. Assume that either X or Z has the strong approximation property. Let Y be any subspace of X. Then*
(1) If Z has cotype 2 and if Y^* has cotype 2 then Y is locally convex.
(2) If Z has cotype $q < \infty$ and Y^* is isomorphic to a subspace of an L_1-space, then Y is locally convex.

Proof. The proofs are essentially identical. We therefore prove only (2). Let $j : Y \to \ell_\infty(Z)$ be the inclusion map. Then since j factors through X it is strongly approximable, under either hypothesis. Let $\pi_i : \ell_\infty(Z) \to Z$ be the co-ordinate map. Then by Theorem 8, we have $\delta(\pi_i j) \leq C$ for some constant C depending only on Y. Thus for $y \in Y$, $\|y\| = \sup_i \|\pi_i j y\| \leq C\|y\|_{\hat{Y}}$.■

Let us now give an application. Suppose G is a compact abelian group with normalized Haar measure μ_G, and suppose Γ is the dual group. We recall that a subset $E \subset \Gamma$ is a Sidon set if for every $\epsilon_\gamma = \pm 1$ there exists $\nu \in \mathcal{M}(G)$ whose Fourier transform satisfies $\hat{\nu}(\gamma) = \epsilon_\gamma$.

In [9] the first author introduced the property $\mathcal{C}_p(X)$ for a subset E of Γ where $0 < p \leq \infty$. We say that E has $\mathcal{C}_p(X)$ if there is a constant M

so that for any $\gamma_1, \ldots, \gamma_n \in E$ and any $x_1, \ldots, x_n \in X$ we have

$$M^{-1} \Big\| \sum_{k=1}^{n} x_k \epsilon_k \Big\|_{L_p(D_n, X)} \leq \Big\| \sum_{k=1}^{n} x_k \gamma_k \Big\|_{L_p(G, X)} \leq M \Big\| \sum_{k=1}^{n} x_k \epsilon_k \Big\|_{L_p(D_n, X)}$$

where $\epsilon_1, \ldots, \epsilon_n$ are the Rademacher functions on D_n. Let us say that a quasi-Banach space X is *Sidon-regular* if every Sidon set E has property $\mathcal{C}_p(X)$ for every $0 < p \leq \infty$. It is a well-known result of Pisier [15] that every Banach space is Sidon-regular. By way of contrast, in [9] an example of a quasi-Banach space which is not Sidon-regular is constructed. However, every natural space is Sidon-regular. The above results enable us to extend this to a wider class of spaces.

Theorem 10. *Let X be a quasi-Banach space of cotype $q < \infty$. Then every quasi-Banach space which is subordinate to X, (and, in particular, X itself) is Sidon-regular.*

Proof. This is very similar to the proof of Theorem 4 in [9]. Suppose G is a compact Abelian group and E is a Sidon subset of Γ. Let $s = \min(p, 2)$. Let $Z = L_s(G, X)$. Then Z also has cotype q. To see this we need first to observe that the Kahane-Khintchine inequality holds in an arbitrary quasi-Banach space (Theorem 2.1 of [6]) so that there is a constant C depending only on X so that if $x_1, \ldots, x_n \in X$ then

$$\left(\int \Big\| \sum_{k=1}^{n} \epsilon_k x_k \Big\|^q d\lambda \right)^{1/q} \leq C \left(\int \Big\| \sum_{k=1}^{n} \epsilon_k x_k \Big\|^s d\lambda \right)^{1/s}.$$

Now if $f_1, \ldots, f_n \in Z$ then, for constants C_1, C_2, C_3 depending only on X,

$$\left(\sum_{k=1}^{n} \|f_k\|^q \right)^{1/q} \leq C_1 \left(\int_G \Big(\sum_{k=1}^{n} \|f_k(t)\|_X^q \Big)^{s/q} d\mu_G(t) \right)^{1/s}$$

$$\leq C_2 \left(\int_G \left(\int_{D_n} \Big\| \sum_{i=1}^{n} \epsilon_k f_k(t) \Big\|_X^q d\lambda \right)^{s/q} d\mu_G(t) \right)^{1/s}$$

$$\leq C_3 \left(\int_{D_n} \int_G \Big\| \sum_{k=1}^{n} \epsilon_k f_k(t) \Big\|_X^s d\mu_G(t) d\lambda \right)^{1/s}$$

$$\leq C_3 \Big\| \sum_{k=1}^{n} \epsilon_k f_k \Big\|_{L_q(D_n, Z)}.$$

Now let E_n be any sequence of finite subsets of E. Let $\mathcal{P}_{E_n}(Y)$ be the space of Y-valued polynomials $\sum_{\gamma \in E_n} y_\gamma \gamma$ equipped with the $L_p(G, Y)$ quasi-norm. Then $\mathcal{P}_{E_n}(Y)$ is isometrically subordinate to Z. Next equip the finite-dimensional space $\ell_\infty(E_n)$ of all bounded functions $h : E_n \to \mathbf{C}$ with the quasi-norm of the operator $T_h : \mathcal{P}_{E_n} \to \mathcal{P}_{E_n}$ given by $T_h(\sum y_\gamma \gamma) = \sum h(\gamma) y_\gamma \gamma$. Let us denote this space $\mathcal{M}_n$. Then $\mathcal{M}_n$ is isometrically subordinate to Z. Thus the product $c_0(\mathcal{M}_n)$ is isometrically subordinate to Z and has the strong approximation property. However, as in Theorem 4 of [9] the assumption that E is a Sidon set shows that we have a constant C depending only the Sidon constant of E so that the envelope norm on $\mathcal{M}_n$ satisfies $\|h\|_\infty \leq \|h\|_{\hat{\mathcal{M}}_n} \leq C\|h\|_\infty$. Thus the envelope of $c_0(\mathcal{M}_n)$ is isomorphic to c_0 and Theorem 9(ii) applies to give that this space is locally convex so that for some uniform constant C' we have for every n, $\|h\|_{\mathcal{M}_n} \leq C'\|h\|_\infty$. As in [9] Theorem 4 this implies that E has property $\mathcal{C}_p(Y)$.∎

Remarks. The above theorem applies to L_p/H_p when $p < 1$ and to the Schatten ideals S_p when $p < 1$, since these spaces have cotype 2 by recent results of Pisier [20] and Xu [23]. These spaces are known not to be natural; S_p is A-convex (i.e. has an equivalent plurisubharmonic quasi-norm) while L_p/H_p is not A-convex (see [8]).

Let us also remark that if $0 < p < 1$ and E is a symmetric p-convex sequence space with the Fatou property then we can define an associated Schatten class S_E (see Gohberg-Krein [4] for the Banach space versions). Precisely if H is a separable Hilbert space and A is a compact operator with singular values $(s_n(A))$ we say $A \in S_E$ if $(s_n(A)) \in E$ and we set $\|A\|_E = \|(s_n(A))\|_E$. It can then be shown that S_E is subordinate to S_p. In fact we define a sequence space F by $\|(t_n)\|_F = \sup\{\|(s_n t_n)\|_p : \|(s_n)\|_E \leq 1\}$ and it can then be shown that $\|A\|_E = \sup\{\|AB\|_p : \|B\|_F \leq 1\}$. This result follows quickly from an inequality of Horn (cf. [4] pp. 48-9) that

$$\sum_{j=1}^k s_j(AB)^p \leq \sum_{j=1}^k s_j(A)^p s_j(B)^p$$

for every k.

4. Quasi-Banach spaces with duals of weak cotype 2.

Let X be a finite-dimensional continuously quasi-normed quasi-Banach space with unit ball B_X. We recall that the *volume-ratio* of X is defined by $\mathrm{vr}(X) = (\mathrm{Vol}\, B_X / \mathrm{Vol}\, \mathcal{E})^{1/n}$ where $\mathcal{E}$ is an ellipsoid of maximal

volume contained in B_X and $n = \dim X$. We define the *outer volume-ratio* of X by $\mathrm{vr}^*(X) = (\mathrm{Vol}\,\mathcal{F}/\mathrm{Vol}\,B_X)^{1/n}$ where $\mathcal{F}$ is an ellipsoid of minimal volume containing B_X. The Santalo inequality ([19], [21]) shows that $\mathrm{vr}^*(X) \geq (\mathrm{Vol}\,\mathcal{B}_{X^*}/\mathrm{Vol}\,\mathcal{F}^0)^{1/n} = \mathrm{vr}(X^*)$. The reverse Santalo inequality of Bourgain and Milman ([3],[19]) shows that, if X is normed, $\mathrm{vr}^*(X) \leq C\mathrm{vr}(X^*)$ so that $\mathrm{vr}^*(X)$ is then equivalent to $\mathrm{vr}(X^*)$. For general quasi-normed spaces the reverse Santalo inequality is not available.

We recall that a Banach space X is of weak cotype 2 if there exists C so that whenever H is a finite-dimensional Hilbert space with orthonormal basis $(e_1, \ldots, e_n)$ and $u : H \to X$ is a linear operator then $a_k(u) \leq Ck^{-1/2}\ell(u)$ for $1 \leq k \leq n$. Here

$$\ell(u) = (\mathbf{E}(\|\sum_{k=1}^{n} g_k u(e_k)\|^2))^{1/2}$$

(for $g_1, \ldots, g_n$ a sequence of independent normalized Gaussian random variables) and $a_k(u) = \inf\{\|u - v\| : v : H \to X, \text{ rank } v < k\}$. The least such constant C is denoted by $wC_2(X)$. It is known that X is of weak cotype 2 if and only if there exists C so that $\mathrm{vr}(E) \leq C$ for every finite-dimensional subspace of X. See Pisier [19] for details. It follows quickly that X^* is of weak cotype 2 if and only if $\mathrm{vr}^*(E)$ is bounded for all finite-dimensional quotients of X. We prove in this section that the same characterization extends to quasi-Banach spaces.

We will require a preparatory lemma:

Lemma 11. *Let E be an N-dimensional Euclidean space and suppose B is the unit ball of an r-norm on E. Let S be a subspace of E of dimension k. Suppose $1/r = \beta \in \mathbf{N}$. Then*

$$\frac{\mathrm{Vol}\,(B \cap S)\,\mathrm{Vol}\,P_{S^\perp}(B)}{\mathrm{Vol}\,B} \leq \binom{N\beta}{k\beta}$$

where $P_{S^\perp}$ is the orthogonal projection of E onto $S^\perp$.

Proof. We duplicate the argument of Lemma 8.8 of [19] (p. 132). One finds

that Vol $B \geq a$Vol $(B \cap S)$Vol $P_{S^\perp} B$ where

$$
\begin{aligned}
a &= (N - k) \int_0^1 (1 - t^r)^{k/r} t^{N-k-1} dt \\
&= \frac{N - k}{r} \int_0^1 (1 - s)^{k/r} s^{N/r - k/r - 1} ds \\
&= \binom{N\beta}{k\beta}. \blacksquare
\end{aligned}
$$

Lemma 12. *There is a constant C depending only r so that if E is a finite-dimensional r-normed space then $d_E \leq C d_{\hat{E}}^{2/r-1}$, and $\delta_E \leq C d_{\hat{E}}^{2/r-2}$.*

Proof. By Lemma 3 of [5] we have $d_E \leq \delta_E d_{\hat{E}} \leq C d_E^\phi d_{\hat{E}}$ where $\phi = (1/r - 1)/(1/r - 1/2)$. This proves the first part and the second part follows on reapplying Lemma 3 of [5]. $\blacksquare$

Theorem 13. *Let X be a quasi-Banach space and suppose $0 < \alpha < 1$. Then X^* has weak cotype 2 if and only if there is a constant C so that whenever F is a finite-dimensional quotient of X, there exists a quotient E of F with dim $E \geq \alpha$dim F and $d_E \leq C$.*

Proof. Suppose X^* has weak cotype 2. Then if F is a finite-dimensional quotient, $wC_2(F^*) \leq wC_2(X^*)$ and so F^* has a subspace G with dim $G \geq \alpha$dim F and $d_G \leq C$ (where C depends only on X and α). Let $E = F/G^\perp$. Then $d_{\hat{E}} = d_G$ and so the preceding Lemma gives an estimate $d_E \leq C'$ where $C' = C'(\alpha, X)$.

For the converse note that if F is a finite-dimensional subspace of X^* then $\hat{X}/F^\perp$ can be identified with the Banach envelope of $X/(X \cap F^\perp)$. From this it follows that if X has the given property then so must $\hat{X}$ and this leads quickly to the fact that X^* has weak cotype 2 by the results of [13].$\blacksquare$

Proposition 14. *Suppose $0 < r < 1$ and $a \geq 1$; then there is a constant $C = C(a, r)$ so that if E is an N-dimensional r-normed space and $wC_2(E^*) \leq a$ then $vr^*(E) \leq C$.*

Proof. In the argument which follows we use C for a constant which depends only a, r but may vary from line to line. It suffices to establish the result

when $1/r = \beta \in \mathbf{N}$. Let $\mathcal{E}$ be the ellipsoid of minimal volume containing B_E. Using this ellipsoid to introduce an inner-product we can define $\|x\|_{E^*} = \sup_{b \in B_E} |(x, b)|$. Then $\mathcal{E}$ is the ellipsoid of minimal volume containing $B_{\hat{E}} = \operatorname{co} B_E$ and the ellipsoid of maximal volume contained in B_{E^*}.

Now, by imitating the argument of Theorem 8 of [5] we can construct an increasing sequence of subspaces $(W_k)_{k=1}^\infty$ of E with $\dim W_k = N - \sigma_k \geq (1 - 2^{-k})N$ and $B_{E^*} \cap W_k \subset C2^{3k}\mathcal{E}$ for $k \geq 1$. We let $\tau_k = \sigma_k - \sigma_{k+1}$.

It follows from the Hahn-Banach theorem that $\mathcal{E} \supset P_{W_k}B_{\hat{E}} \supset C^{-1}2^{-3k}\mathcal{E} \cap W_k$. Now, identifying $H_k = E/W_k^\perp$ with W_k under the quasinorm with unit ball $P_{W_k}(B_E)$ this implies that $d_{\hat{H}_k} \leq C2^{3k}$. Now from Lemma 12 $\delta_{H_k} \leq C2^{sk}$ for suitable $s > 0$ depending on r. We conclude that $\mathcal{E} \cap W_k \subset C2^{tk}P_{W_k}(B_E)$, where t depends only on r, and $C = C(a, r)$.

Let Z_k is the orthogonal complement of W_k in W_{k+1}. Notice that $\dim Z_k = \tau_k$. Now by Lemma 11, if we set $A_k = P_{W_{k+1}}(B_E) \cap Z_k$,

$$\operatorname{Vol} P_{W_k}(B_E)\operatorname{Vol} A_k \leq \binom{(N - \sigma_{k+1})\beta}{\tau_k \beta}\operatorname{Vol} P_{W_{k+1}}(B_E).$$

Let l be the first index for which $W_l = E$ and so $\sigma_l = 0$. We first estimate

$$\prod_{k=1}^{l-1}\binom{(N - \sigma_{k+1})\beta}{\tau_k \beta} = \frac{(N\beta)!}{((N - \sigma_1)\beta)!(\tau_1\beta)!\ldots(\tau_{l-1}\beta)!}$$

$$= \binom{N\beta}{\sigma_1\beta}\prod_{k=1}^{l-2}\binom{\sigma_k\beta}{\tau_k\beta}$$

$$\leq 2^{\beta(N+\sum\sigma_k)}$$

$$\leq 2^{2\beta N}.$$

Now $\mathcal{E} \cap Z_k \subset C2^{t(k+1)}A_k$ and so $\log_2 \operatorname{Vol} P_{Z_k}(B_E) \geq -Ck\tau_k + \log_2 \operatorname{Vol} \mathcal{E} \cap Z_k$. (If $\tau_k = 0$ we interpret the relative volume as one). Summing we obtain since $\tau_k \leq 2^{-k}N$,

$$\sum_{k=1}^{l-1}\log_2 \operatorname{Vol} A_k \geq -CN + \sum_{k=1}^{l-1}\log_2 \operatorname{Vol} \mathcal{E} \cap Z_k.$$

Thus

$$\log_2 \text{Vol } B_E = \log_2 \text{Vol } P_{W_l}(B_E)$$

$$\geq \log_2 \text{Vol } P_{W_1}(B_E) - CN + \sum_{k=1}^{l-1} \text{Vol } \mathcal{E} \cap Z_k$$

$$\geq -CN + \log_2 \text{Vol } \mathcal{E} \cap W_1 + \log_2 \text{Vol } \mathcal{E} \cap W_1^{\perp}$$

$$\geq -CN + \log_2 \text{Vol } \mathcal{E}.$$

This completes the proof of the Proposition.

Remark. This Proposition can be interpreted as follows. Suppose X is a finite-dimensional normed space so that $wC_2(X^*) \leq a$. Consider the set ∂B_X of extreme points of B_X and form the r-convex hull $A_r = \text{co}_r \partial B_X$. Then although A_r is smaller than B_X it is not too much smaller, for $\text{Vol } B_X / \text{Vol } A_r \leq C^{\dim E}$.

Theorem 15. *Let X be a quasi-Banach space. Then X^* has weak cotype 2 if and only if there is a constant C so that $\text{vr}^*(E) \leq C$ for every finite-dimensional quotient E of X.*

Proof. First suppose $\text{vr}^*(E) \leq C$ for every finite-dimensional quotient E of X. Let F be a finite-dimensional subspace of X^* and consider $E = X/F^{\perp}$. It is easy to see that the envelope norm on E is the quotient norm from the envelope norm on X. Clearly from the definition, $\text{vr}^*(\hat{E}) \leq \text{vr}^*(E) \leq C$. Hence by the Santalo inequality $\text{vr}(F) \leq C$. This shows that X^* has weak cotype 2.

The converse is immediate from Proposition 14. ■

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