

Distances between Banach spaces

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Abstract. The main object of the paper is to study the distance between Banach spaces introduced by Kadets. For Banach spaces X and Y , the Kadets distance is defined to be the infimum of the Hausdorff distance $d(B_X, B_Y)$ between the respective closed unit balls over all isometric linear embeddings of X and Y into a common Banach space Z . This is compared with the Gromov-Hausdorff distance which is defined to be the infimum of $d(B_X, B_Y)$ over all isometric embeddings into a common metric space Z . We prove continuity type results for the Kadets distance including a result that shows that this notion of distance has applications to the theory of complex interpolation.

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1. Introduction

The standard notion of distance between two Banach spaces is the *Banach-Mazur distance* which is defined by

$$d_{BM}(X, Y) = \log \inf \{ \|T\| \|T^{-1}\| : T: X \rightarrow Y \text{ is an isomorphism} \}.$$

(It is usual to omit the logarithm, but for consistency we will include it). The Banach-Mazur distance is only finite when X and Y are isomorphic. The main object of this paper is to study a measure of distance we call the Kadets distance and certain related notions of distance. The Kadets distance has natural applications in interpolation theory which we explain.

We recall that if Z is a Banach space and X and Y are closed subspaces of Z the *gap* or *opening* $A(X, Y)$ is defined as the Hausdorff distance between the closed unit balls B_X and B_Y of X and Y i.e.

$$A(X, Y) = \max \left\{ \sup_{y \in B_Y} d(y, B_X), \sup_{x \in B_X} d(x, B_Y) \right\}.$$

If X and Y are arbitrary Banach spaces we define the *Kadets distance*

$$d_K(X, Y) = \inf_{Z, U, V} \Lambda(UX, VY)$$

where the infimum is taken over all Banach spaces Z and all linear isometric embeddings $U: X \rightarrow Z$ and $V: Y \rightarrow Z$.

This distance was apparently introduced by Kadets [15] who proved for example that $\lim_{p \rightarrow 2} d_K(l_p, l_2) = 0$. However the basic idea seems to be implicit in some earlier work of Krein, Krasnoselskii and Milman [20], Brown [5] and Douady [10]. The second author studied the notion in [22] and proved that d_K satisfies the triangle law but that there are non-isomorphic Banach spaces X and Y for which $d_K(X, Y) = 0$ (thus d_K is a "pseudo-metric"). In the same paper there is a completeness result: if (X_n) is a sequence of Banach spaces Cauchy with respect to d_K then there is a Banach space X so that $\lim_{n \rightarrow \infty} d_K(X_n, X) = 0$.

There is a series of papers studying the general problem of the identifying properties which are stable under small perturbations in the Kadets distance (see [2], [5], [10], [20], [22], [23], [24] and [25]). Precisely a property \mathcal{P} is called *stable* if there exists $\varepsilon > 0$ so that if X has \mathcal{P} and $d_K(X, Y) < \varepsilon$ then Y has \mathcal{P} . We refer to the survey article [24] Chapter 6: a partial list of stable properties includes reflexivity, super-reflexivity, B-convexity (nontrivial Rademacher type), the Banach-Saks property, the alternate-signs Banach Saks property and the property of not containing l_1 .

The Kadets distance is clearly related to the notion of Gromov-Hausdorff distance between metric spaces (see [11], [27]; the precise definition is given in Section 2). It is natural to introduce the Gromov-Hausdorff distance between two Banach spaces X and Y as

$$d_{GH}(X, Y) = \inf_{Z, U, V} d(UB_X, VB_Y)$$

where the infimum is taken over all isometries $U: X \rightarrow Z$ and $V: Y \rightarrow Z$ into a common metric space Z (here $d(UB_X, VB_Y)$ is the Hausdorff distance between UB_X and VB_Y .) Thus the Gromov-Hausdorff distance is simply the nonlinear analogue of the Kadets distance. It is not difficult to see that this definition coincides with computing the standard Gromov-Hausdorff distance between the unit balls B_X, B_Y as metric spaces. Let us remark at this point that a related global notion was considered in [4] and [12].

In this paper we first compare these two notions of distance. It is worth pointing out that one must distinguish between the case of complex scalars and real scalars, because there are examples ([3], [17], [31]) of complex Banach spaces which are real-isometric but not even complex-isomorphic. Gromov-Hausdorff distance cannot distinguish complex structures.

We show that (for real scalars) while Gromov-Hausdorff distance is not equivalent to the Kadets distance, the two notions are equivalent if one restricts to Banach

spaces which are nice enough. For example if X is B-convex (i.e. has non-trivial Rademacher type) or if X^* embeds into an \mathcal{L}_1 -space then $d_{GH}(X_n, X) \rightarrow 0$ implies $d_K(X_n, X) \rightarrow 0$. If X is isomorphic to either c_0 or l_∞ then $d_{GH}(X_n, X) \rightarrow 0$ implies $d_{BM}(X_n, X) \rightarrow 0$. On the other hand $d_{GH}(l_p, l_1) \rightarrow 0$ as $p \rightarrow 1$ while $d_K(l_p, l_1) = 1$ if $p > 1$. The precise identification of the class on which the two distances are equivalent is related to the notion of a \mathcal{K} -space introduced in [16] (see [18]) and to the theory of twisted sums.

In fact for real scalars, Gromov-Hausdorff distance is equivalent to a notion of distance analogous to the Kadets distance but allowing the superspace Z to be a quasi-Banach space.

These results are developed in Section 3, after some preliminary results in Section 2. In Section 4, we then apply our techniques to prove a number of continuity-type results for the Kadets metric. For example we show that in an obvious sense the map $X \rightarrow X^*$ is continuous for the Kadets metric, and even more one has $d_K(X^*, Y^*) \leq 2d_K(X, Y)$ for any pair of Banach spaces. We also show that if (X_0, X_1) is a complex Banach couple and $X_\theta = [X_0, X_1]_\theta$ are the spaces obtained by the (Calderón) method of complex interpolation (cf. [6]) then the map $\theta \rightarrow X_\theta$ is continuous for the Kadets distance for $0 < \theta < 1$. This result is closely related to recent work on uniform homeomorphisms between the unit balls of two Banach spaces using complex interpolation methods (cf. [8]). We give precise estimates here and obtain the estimate for $1 < p, q < \infty$,

$$d_K(l_p, l_q) \leq 2 \frac{\sin(\pi|1/p - 1/q|/2)}{\sin(\pi(1/p + 1/q)/2)}$$

which improves earlier estimates (see [15], [22]). We remark that in [23] or [24] (pp. 292, 303) there is a lower estimate $d_K(l_p, l_q) \geq \frac{1}{2}(2^{1/p} - 2^{1/q})$.

Finally in Section 5, we make some remarks on the topology of the pseudo-metric space of all Banach spaces with a given density character with either notion of distance. We point out that results on stability or openness of some property lead automatically to results on complex interpolation spaces, and also show that the continuity results of the previous section lead to new stable or open properties. We identify the component of l_1 for the Kadets distance and raise the question of identifying the components of l_2 and c_0 . We do not know if the set of separable Banach spaces is connected for the Gromov-Hausdorff distance. We also show that if $1 < p \neq 2 < \infty$ the set of spaces isomorphic to l_p is non-separable for both notions of distance.

2. Gromov-Hausdorff distance and Kadets distance

We first recall the notion of Gromov-Hausdorff distance between metric spaces. It will be convenient to expand the definition to include pseudo-metric spaces. We recall that if M is a set, a pseudo-metric on M is a map $d: M \times M \rightarrow [0, \infty)$ which

is symmetric and satisfies the triangle law, the condition $d(x, x) = 0$, but not necessarily the condition $d(x, y) = 0$ implies $x = y$. Suppose A and B are metric spaces (or pseudo-metric spaces) with bounded metrics. We define the Gromov-Hausdorff distance between A and B denoted $d_{gh}(A, B)$ to be the infimum of all $\varepsilon \geq 0$ so that there is a pseudo-metric space M and isometric embeddings $i_A: A \rightarrow M$ and $i_B: B \rightarrow M$ such that the Hausdorff distance $d_M(i_A A, i_B B) \leq \varepsilon$. If \mathcal{M} is any set of metric spaces then d_{gh} defines a pseudo-metric on \mathcal{M} ; note that $d_{gh}(A, B) = 0$ does not necessarily imply that A and B are isometric (unless they are compact).

We will be interested in an alternative formulation. For convenience we denote by $\mathcal{F}(A, B)$ the collection of all pairs (ϕ, ψ) of maps $\phi: A \rightarrow B$ and $\psi: B \rightarrow A$. If (ψ, ϕ) is such pair let $G = G(\phi, \psi)$ be the union of the graphs of ψ, ϕ . We define $D(\phi, \psi)$ to be the supremum of all quantities $\frac{1}{2}|d_A(a_1, a_2) - d_B(b_1, b_2)|$ over all $a_1, a_2 \in A, b_1, b_2 \in B$ with $(a_i, b_i) \in G$ for $i = 1, 2$.

In the special case when ϕ is invertible and $\psi = \phi^{-1}$, $D(\phi, \psi)$ is the supremum of all quantities $\frac{1}{2}|d_B(\phi a_1, \phi a_2) - d_A(a_1, a_2)|$ where $a_1, a_2 \in A$.

Theorem 2.1. *Let A and B bounded metric spaces. Then*

$$d_{gh}(A, B) = \inf_{(\phi, \psi) \in \mathcal{F}} D(\phi, \psi).$$

Proof. First suppose M is a pseudo-metric space and that A and B are isometrically embedded in M . Let ε be the Hausdorff distance between A and B . Then if $\sigma > \varepsilon$ we can define $\phi: A \rightarrow B$ and $\psi: B \rightarrow A$ so that $d_M(a, \phi a), d_M(\psi b, b) < \sigma$ for all $a \in A$ and $b \in B$. Now for $(a_i, b_i) \in G = G(\phi, \psi)$ we clearly have $|d_A(a_1, a_2) - d_B(b_1, b_2)| < 2\sigma$.

To obtain the converse direction suppose (ϕ, ψ) given and that $D(\psi, \phi) = \sigma$. We let $M = A \cup B$ (disjoint union) and define a pseudo-metric d_M as follows. We let d_M coincide with d_A on A and with d_B on B . If $a \in A$ and $b \in B$ then

$$d_M(a, b) = \inf_{(a', b') \in G} (d_A(a, a') + d_B(b', b)) + \sigma.$$

We must check this is a pseudo-metric on $A \cup B$. Suppose $a_1, a_2 \in A$ and $b \in B$. We check that $d_A(a_1, a_2) \leq d_M(a_1, b) + d_M(a_2, b)$. To do this suppose that (α_1, β_1) and $(\alpha_2, \beta_2) \in G$. Then

$$\begin{aligned} d_B(\beta_1, b) + d_B(\beta_2, b) &\geq d_B(\beta_1, \beta_2) \\ &\geq d_A(\alpha_1, \alpha_2) - 2\sigma. \end{aligned}$$

Hence

$$d_A(a_1, \alpha_1) + d_B(\beta_1, b) + d_A(\alpha_2, a_2) + d_B(\beta_2, b) \geq d_A(a_1, a_2) - 2\sigma$$

which establishes our claim.

We can also show that $d_M(a_1, b) \leq d_A(a_1, a_2) + d_M(a_2, b)$. We omit the details which are easy. Arguing symmetrically with A, B interchanged gives the conclusion that d_M is a pseudo-metric. Clearly $d_M(a, \phi a) \leq \sigma$ and $d_M(\psi b, b) \leq \sigma$. This shows that $d_{gh}(A, B) \leq \sigma$.

If X and Y are Banach spaces we define the Gromov-Hausdorff distance $d_{gh}(X, Y)$ to be the Gromov-Hausdorff distance between their closed unit balls B_X, B_Y , i.e. $d_{GH}(X, Y) = d_{gh}(B_X, B_Y)$. Equivalently $d_{GH}(X, Y)$ is the infimum of the Hausdorff distance $d(B_X, B_Y)$ over all isometric embeddings of X, Y into a common metric space M . To establish this last comment, suppose d is any metric on the formal union $B_X \cup B_Y$, which coincides with the respective norm-distances on B_X and B_Y . We can extend d to $X \cup Y$ by defining d again to coincide with the norm-distance on each of X and Y and for $x \in X, y \in Y$,

$$d(x, y) = \inf_{u \in B_X, v \in B_Y} \{ \|x - u\|_X + d(u, v) + \|y - v\|_Y \}.$$

We leave the details to the reader.

Let us note here that our definition applies to both real and complex Banach spaces, but there are complex Banach spaces which are real-isometric and not even complex-isomorphic ([3], [17], [31]). In view of this, Gromov-Hausdorff distance is most natural for the category of real Banach spaces.

Corollary 2.2. *If X and Y are Banach spaces and $d_{GH}(X, Y) < \sigma$ then there exist maps $\phi: B_X \rightarrow B_Y$ and $\psi: B_Y \rightarrow B_X$ such that:*

$$\| \|x - \psi(y)\|_X - \|y - \phi(x)\|_Y \| < 2\sigma$$

whenever $x \in B_X, y \in B_Y$.

The Kadets distance $d_K(X, Y)$ is defined to be the infimum of the gap $A(X, Y) (= d(B_X, B_Y))$ over all linear isometric embeddings of X and Y into a common Banach space Z . Here our definition can equally be applied to the real or complex case. The Kadets distance is again a pseudo-metric on any set of Banach spaces (see [15], [22]). We clearly have the inequality $d_{GH}(X, Y) \leq d_K(X, Y)$.

We now give a similar formulation of the Kadets distance. Let $\mathcal{F}_h(X, Y)$ be the set of all pairs of homogeneous maps $\Phi: X \rightarrow Y$ and $\Psi: Y \rightarrow X$ such that $\|\Phi(x)\|_Y \leq \|x\|_X$ and $\|\Psi(y)\|_X \leq \|y\|_Y$ for all $x \in X$ and $y \in Y$. We define $\Delta = \Delta(\Phi, \Psi)$ to be the least constant such that

$$\left\| \left\| \sum_{i=1}^m x_i - \sum_{j=1}^n \Psi(y_j) \right\|_X - \left\| \sum_{i=1}^m \Phi(x_i) - \sum_{j=1}^n y_j \right\|_Y \right\| \leq \Delta \left(\sum_{i=1}^m \|x_i\|_X + \sum_{j=1}^n \|y_j\|_Y \right)$$

for all $x_1, \dots, x_m \in X$ and $y_1, \dots, y_n \in Y$.

In the special case when $\Psi = \Phi^{-1}$ (and hence Φ is norm-preserving) notice Δ is the least constant such that

$$\left| \left\| \sum_{i=1}^m \Phi(x_i) \right\|_Y - \left\| \sum_{i=1}^m x_i \right\|_X \right| \leq \Delta \sum_{i=1}^m \|x_i\|_X.$$

Theorem 2.3. *If X, Y are Banach spaces then*

$$d_K(X, Y) = \inf_{(\Phi, \Psi) \in \mathcal{F}_h} \Delta(\Psi, \Phi).$$

Proof. If X and Y are isometrically embedded in Z and $d(B_X, B_Y) < \sigma$ we can construct $(\Phi, \Psi) \in \mathcal{F}_h$ with $\|x - \Phi(x)\|_Z \leq \sigma \|x\|_X$ and $\|y - \Psi(y)\|_Y \leq \sigma \|y\|_Y$ whenever $x \in X$ and $y \in Y$. It is then trivial that $\Delta(\Phi, \Psi) \leq \sigma$.

For the converse direction, let us suppose that $(\Phi, \Psi) \in \mathcal{F}_h(X, Y)$ are given and that $\Delta(\Phi, \Psi) = \sigma$. We will define Z to be the direct sum $X \oplus Y$ equipped with an equivalent norm. Precisely we define

$$\|(u, v)\|_Z = \inf \left\{ \|x_0\|_X + \|y_0\|_Y + \sigma \left(\sum_{i=1}^m \|x_i\|_X + \sum_{j=1}^n \|y_j\|_Y \right) \right\}$$

where the infimum is taken over all $\{x_i\}_{i=0}^m$ in X and $\{y_j\}_{j=0}^n$ in Y such that

$$u = x_0 + \sum_{i=1}^m x_i + \sum_{j=1}^n \Psi(y_j)$$

and

$$v = y_0 + \sum_{i=1}^m \Phi(x_i) + \sum_{j=1}^n y_j.$$

We show that $\|(u, 0)\|_Z = \|u\|_X$ (and then it follows similarly that $\|(0, v)\|_Z = \|v\|_Y$). Indeed suppose

$$u = x_0 + \sum_{i=1}^m x_i + \sum_{j=1}^n \Psi(y_j)$$

and

$$0 = y_0 + \sum_{i=1}^m \Phi(x_i) + \sum_{j=1}^n y_j.$$

It follows from our condition that

$$\left\| \sum_{i=1}^m x_i + \sum_{j=1}^n \Psi(y_j) \right\|_X \leq \|y_0\|_Y + \sigma \left(\sum_{i=1}^m \|x_i\|_X + \sum_{j=1}^n \|y_j\|_Y \right).$$

Hence

$$\|u\|_X \leq \|x_0\|_X + \|y_0\|_Y + \sigma \left(\sum_{i=1}^m \|x_i\|_X + \sum_{j=1}^n \|y_j\|_Y \right).$$

This shows that $\|(u, 0)\|_Z = \|u\|_X$.

Clearly the construction yields that $\Delta(X, Y) \leq \sigma$ since $\|(x, \Phi(x))\|_Z \leq \sigma\|x\|_X$ and $\|(\Psi(y), y)\|_Z \leq \sigma\|y\|_Y$. \square

There is an amusing way to interpret $\Delta(\Phi, \Psi)$. Form the vector sequence spaces $l_1(X)$ and $l_1(Y)$ and define $\tilde{\Phi}((x_n))$ and $\tilde{\Psi}((y_n)) = (\Psi(y_n))$. We can consider the pair $(\tilde{\Phi}, \tilde{\Psi}) \in \mathcal{F}_h(l_1(X), l_1(Y))$. Then consider the unit ball $B_{l_1(X)}$ with the pseudo-metric induced by the seminorm $|(x_n)|_X = \|\sum_{n=1}^{\infty} x_n\|_X$; similarly consider $B_{l_1(Y)}$ with the seminorm $|(y_n)|_Y$ defined in the analogous way.

Then $\Delta(\Phi, \Psi) = D(\tilde{\Phi}, \tilde{\Psi})$ for the unit balls of $l_1(X)$ and $l_1(Y)$ equipped with the pseudo-metrics induced by these seminorms. We leave the details to the reader.

It is natural to ask for a theorem of the type of Theorem 2.3 but with Φ a bijection and $\Psi = \Phi^{-1}$. This can be done at the cost of a constant in the calculations.

Theorem 2.4. *Let X and Y be a Banach spaces with $d_K(X, Y) < \sigma < \frac{1}{6}$. Then there is a norm-preserving bijection $\Omega : X \rightarrow Y$ such that if $x_1, \dots, x_n \in X$ then*

$$\left| \left\| \sum_{i=1}^n \Omega(x_i) \right\|_Y - \left\| \sum_{i=1}^n x_i \right\|_X \right| \leq 14\sigma \sum_{i=1}^n \|x_i\|_X.$$

Proof. First we observe that if $d_K(X, Y) < 1/2$ then X and Y have the same density character (cf. [20] or [24] 6.23). We will suppose that X and Y are simultaneously embedded into a common Banach space Z with $d(B_X, B_Y) = \delta < \sigma$. Let us pick a maximal collection of vectors $(x_i : i \in I)$ in S_X such that if $|a| = 1$ and $i \neq j$ then $\|x_i - ax_j\|_X > 4\sigma$. Then there exist vectors $(y_i : i \in I)$ in S_Y with $\|x_i - y_i\|_Z < 2\delta$. If $y \in S_Y$ then there exists $x \in S_X$ with $\|y - x\|_Z < 2\delta$ and so there exists $i \in I$ and $|a| = 1$ such that $\|y - ay_i\|_Z < 8\sigma$. On the other hand if $i \neq j$ and $|a| = 1$ then $\|y_i - ay_j\|_Z > 4(\sigma - \delta)$.

We can now partition S_X into sets $(A_i : i \in I)$ such that if $x \in A_i$ and $|a| = 1$ then $ax \in A$ and further that $\|x - x_i\|_X \leq 2\sigma$ implies $x \in A_i$ while $x \in A_i$ implies that there exists a with $|a| = 1$ and $\|x - ax_i\|_X \leq 4\sigma$. We then define A'_i by taking one representative u from each set $\{ax : |a| = 1\}$ contained in A_i with the property $\|u - x_i\|_X \leq 4\sigma$.

In the same way we can partition S_Y into sets $(B_i : i \in I)$ such that if $y \in B_i$ and $|a| = 1$ then $ay \in B_i$ and further that $\|y - y_i\|_X \leq 2(\sigma - \delta)$ implies $y \in B_i$ while $y \in B_i$ implies that there exists a with $|a| = 1$ and $\|y - ay_i\|_X \leq 8\sigma$. We then define B'_i by taking one representative v from each set $\{ay : |a| = 1\}$ contained in B_i with the property $\|v - y_i\|_X \leq 8\sigma$.

It is easy to see that the sets A'_i, B'_i have the same cardinality for each i and so we can define a bijection $\Omega: S_X \rightarrow S_Y$ such that $\Omega(A'_i) = B'_i$, $\Omega(ax) = a\Omega(x)$ when $|a| = 1$. If $x \in S_X$ then there exists $i \in I$ and $|a| = 1$ with $ax \in A'_i$. Thus $\Omega(ax) \in B'_i$ and so $\|x - \Omega(x)\|_Z \leq 14\sigma$.

The result now follows immediately. \square

Let us give one immediate application of Theorem 2.3.

Theorem 2.5. *Let X and Y be real Banach spaces and suppose Z is a metric linear space equipped with an invariant metric d_Z . Suppose X and Y are linearly and isometrically embedded into Z . Then $d_K(X, Y) \leq d_Z(B_X, B_Y)$.*

Remark. Thus it would make no difference in the definition of d_K to allow Z to be a metric linear space instead of a Banach space.

Remark. The proof given below can be extended to complex spaces, if we further assume that the metric d_Z is invariant under multiplication by $e^{i\theta}$ for $0 < \theta < 2\pi$. For the real case invariance under multiplication by -1 follows from additive invariance since $d_Z(-x, -y) = d_Z(x - y, 0) = d_Z(x, y)$.

Proof. Suppose $\sigma > d_Z(B_X, B_Y)$. Then we can define $\Phi: S_X \rightarrow B_Y$ and $\Psi: S_Y \rightarrow B_X$ such that $d_Z(x, \Phi(x)) \leq \sigma$ and $d_Z(\Psi(y), y) \leq \sigma$ for all x, y . Since the metric is translation-invariant it is clear that we can suppose $\Phi(-x) = -\Phi(x)$ and $\Phi(-y) = \Phi(y)$. We then extend Φ and Ψ to be homogenous.

Now if $x_1, \dots, x_m \in S_X$ and $y_1, \dots, y_n \in S_Y$ then

$$d_Z\left(\sum_{i=1}^m x_i + \sum_{j=1}^n \Psi(y_j), \sum_{i=1}^m \Phi(x_i) + \sum_{j=1}^n y_j\right) \leq (m+n)\sigma.$$

It follows that since $\|x\|_X = d_Z(x, 0)$ for $x \in X$ and $\|y\|_Y = d_Z(y, 0)$ for $y \in Y$ we have:

$$\left\| \left\| \sum_{i=1}^m x_i + \sum_{j=1}^n \Psi(y_j) \right\|_X - \left\| \sum_{i=1}^m \Phi(x_i) + \sum_{j=1}^n y_j \right\|_Y \right\| \leq (m+n)\sigma.$$

From this it follows easily that if $r_1, \dots, r_m, s_1, \dots, s_n$ are integers that

$$\left\| \left\| \sum_{i=1}^m r_i x_i + \sum_{j=1}^n \Psi(s_j y_j) \right\|_X - \left\| \sum_{i=1}^m \Phi(r_i x_i) + \sum_{j=1}^n s_j y_j \right\|_Y \right\| \leq \left(\sum_{i=1}^m r_i + \sum_{j=1}^n s_j \right) \sigma.$$

Clearly the same inequality then holds for $r_1, \dots, r_m, s_1, \dots, s_n$ rational and then even real by a density argument. This implies that $\Delta(\Phi, \Psi) \leq \sigma$. \square

Now suppose $0 < r < 1$. We recall that an r -norm on a real or complex vector space X is a map $x \rightarrow \|x\|_X$ such that:

- (1) $\|x\|_X > 0$ if $x \neq 0$,
- (2) $\|\alpha x\|_X = |\alpha| \|x\|_X$ for $\alpha \in \mathbb{K}$, $x \in X$,
- (3) $\|x_1 + x_2\|_X^r \leq \|x_1\|_X^r + \|x_2\|_X^r$ for $x_1, x_2 \in X$.

Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Notice that in [18] this is called an r -subadditive quasi-norm. If the metric $d(x_1, x_2) = \|x_1 - x_2\|_X^r$ makes X complete we say that X is an r -normed quasi-Banach space.

Now suppose X and Y are Banach spaces (so that X and Y are also r -normed quasi-Banach spaces for any $0 < r < 1$). We define $d_r(X, Y)$ to be the infimum of $d(B_X, B_Y)$ over all linear isometric embeddings of X and Y into an r -normed quasi-Banach space Z . Note here that the r -norm does not induce a metric on Z so this is not covered by the preceding theorem. There is, however an analogue of Theorem 2.3 for this situation. If $(\Phi, \Psi) \in \mathcal{F}_h$ then we define $\Delta_r(\Phi, \Psi)$ to be the least constant Δ_r such that we have for any $x_1, \dots, x_m \in X$ and $y_1, \dots, y_n \in Y$ that

$$\left\| \left\| \sum_{i=1}^m x_i - \sum_{j=1}^n \Psi(y_j) \right\|_X^r - \left\| \sum_{i=1}^m \Phi(x_i) - \sum_{j=1}^n y_j \right\|_Y^r \right\| \leq \Delta_r^r \left(\sum_{i=1}^m \|x_i\|_X^r + \sum_{j=1}^n \|y_j\|_Y^r \right).$$

In an exactly analogous fashion we may prove:

Theorem 2.6. *If X and Y are Banach spaces then*

$$d_r(X, Y) = \inf_{(\Phi, \Psi) \in \mathcal{F}_h} \Delta_r(\Psi, \Phi).$$

3. Comparison of metrics

Proposition 3.1. *Suppose X and Y are Banach spaces and $(\Phi, \Psi) \in \mathcal{F}_h(X, Y)$. Let $\sigma = \Delta(\Phi, \Psi)$. Then*

- (1) *Given $y \in Y$ there exists $x \in X$ with $\|x\|_X \leq \|y\|_Y$ and $\|y - \Phi(x)\|_Y \leq 2\sigma \|y\|_Y$*
- (2) *If $x \in X$ then $\|\Phi(x)\|_Y \geq (1 - \sigma)\|x\|_X$.*
- (3) *If $x_1, \dots, x_n \in X$ and $\sum_{k=1}^n x_k = 0$ then*

$$\left\| \sum_{k=1}^n \Phi(x_k) \right\|_Y \leq \sigma \sum_{k=1}^n \|x_k\|_X.$$

Proof. (1) Just take $x = \Psi(y)$. Then $\|x - \Psi(y)\|_X = 0$ so that

$$\|y - \Phi(x)\|_Y \leq \sigma(\|x\|_X + \|y\|_Y) \leq 2\sigma \|y\|_Y.$$

- (2) and (3) are immediate from the definition of $\Delta(\Phi, \Psi)$. \square

In a very similar way we can establish:

Proposition 3.2. *Suppose $0 < r < 1$. Suppose X and Y are Banach spaces and $(\Phi, \Psi) \in \mathcal{F}_h(X, Y)$. Let $\sigma = \Delta_r(\Phi, \Psi)$. Then*

- (1) *Given $y \in Y$ there exists $x \in X$ with $\|x\|_X \leq \|y\|_Y$ and $\|y - \Phi(x)\|_Y \leq 2^{1/r} \sigma \|y\|_Y$*
- (2) *If $x \in X$ then $\|\Phi(x)\|_Y \geq (1 - \sigma^r)^{1/r} \|x\|_X$.*
- (3) *If $x_1, \dots, x_n \in X$ and $\sum_{k=1}^n x_k = 0$ then*

$$\left\| \sum_{k=1}^n \Phi(x_k) \right\|_Y \leq \sigma \left(\sum_{k=1}^n \|x_k\|_X^r \right)^{1/r}.$$

Our interest in Propositions 3.1 and 3.2 is to establish a converse.

Proposition 3.3. *Let X and Y be Banach spaces and suppose $\Phi: X \rightarrow Y$ is a homogeneous map satisfying $\frac{1}{2} \|x\|_X \leq \|\Phi(x)\|_Y \leq \|x\|_X$ such that for a constant $0 \leq \sigma < 1$ we have:*

- (1) *Given $y \in Y$ there exists $x \in X$ with $\|x\|_X \leq \|y\|_Y$ and $\|y - \Phi(x)\|_Y \leq \sigma \|y\|_Y$*
- (2) *If $x_1, x_2, x_3 \in X$ and $\sum_{k=1}^3 x_k = 0$ then*

$$\left\| \sum_{k=1}^3 \Phi(x_k) \right\|_Y \leq \sigma \sum_{k=1}^3 \|x_k\|_X.$$

Then if $\Psi: Y \rightarrow X$ is a homogeneous map satisfying $\|\Psi(y)\|_X \leq \|y\|_Y$ and $\|y - \Phi(\Psi(y))\|_Y \leq \sigma \|y\|_Y$ (whose existence is guaranteed by (1)) we have

$$|\|x - \Psi(y)\|_X - \|y - \Phi(x)\|_Y| \leq 6\sigma (\|x\|_X + \|y\|_Y).$$

Furthermore for each $0 < r < 1$ there is a universal constant $C = C(r)$ such that $\Delta_r(\Phi, \Psi) \leq C\sigma$.

- (3) *If $x_1, \dots, x_n \in X$ and $\sum_{k=1}^n x_k = 0$ then*

$$\left\| \sum_{k=1}^n \Phi(x_k) \right\|_Y \leq \sigma \sum_{k=1}^n \|x_k\|_X,$$

then $\Delta(\Phi, \Psi) \leq 20\sigma$.

Proof. Suppose $x \in X$ and $y \in Y$. Then

$$\|\Phi(x) - \Phi(\Psi(y)) - \Phi(x - \Psi(y))\|_Y \leq 2\sigma (\|x\|_X + \|y\|_Y).$$

Hence

$$\|\Phi(x) - y - \Phi(x - \Psi(y))\|_Y \leq 3\sigma(\|x\|_X + \|y\|_Y).$$

It follows that

$$\|\Phi(x) - y\|_Y \leq \|x - \Psi(y)\|_X + 3\sigma(\|x\|_X + \|y\|_Y)$$

and

$$\|x - \Psi(y)\|_X \leq \|\Phi(x) - y\|_Y + 6\sigma(\|x\|_X + \|y\|_Y).$$

This proves the first part of the Proposition.

Notice that the same proof yields that for $0 < r < 1$,

$$\| \|x - \Psi(y)\|_X^r - \|y - \Phi(x)\|_Y^r \leq 6^r \sigma^r (\|x\|_X + \|y\|_Y)^r.$$

Now suppose $0 < r \leq 1$ and that there is a constant τ so that for any $x_1, \dots, x_n \in X$ with $\sum_{i=1}^n x_i = 0$ we have

$$(3.1) \quad \left\| \sum_{i=1}^n \Phi(x_i) \right\|_Y \leq \tau \left(\sum_{i=1}^n \|x_i\|_X^r \right)^{1/r}.$$

First suppose $y_1, \dots, y_n \in Y$ and $\sum_{i=1}^n y_i = 0$. Let $v = \sum_{i=1}^n \Psi(y_i)$. Then

$$\left\| \sum_{i=1}^n \Phi(\Psi(y_i)) - \Phi(v) \right\|_Y \leq 2^{1/r} \tau \left(\sum_{i=1}^n \|y_i\|_Y^r \right)^{1/r}.$$

Thus

$$\left\| \sum_{i=1}^n y_i - \Phi(v) \right\|_Y \leq (2^{1/r} \tau + \sigma) \left(\sum_{i=1}^n \|y_i\|_Y^r \right)^{1/r}.$$

It follows that

$$\|v\|_X \leq 2\|\Phi(v)\|_Y \leq (2^{1+1/r}\tau + 2\sigma) \left(\sum_{i=1}^n \|y_i\|_Y^r \right)^{1/r}.$$

Now suppose that $x_1, \dots, x_m \in X$ and $y_1, \dots, y_n \in Y$. Let $u = \sum_{i=1}^m x_i$ and $v = \sum_{j=1}^n y_j$. Then

$$\| \|u - \Psi(v)\|_X^r - \|v - \Phi(u)\|_Y^r \leq 6^r \sigma^r \left(\sum_{i=1}^m \|x_i\|_X^r + \sum_{j=1}^n \|y_j\|_Y^r \right).$$

However we also have:

$$\left\| \Psi(v) - \sum_{j=1}^n \Psi(y_j) \right\|_Y^r \leq 2(2^{1+1/r}\tau + 2\sigma)^r \sum_{j=1}^n \|y_j\|_Y^r$$

and

$$\left\| \Phi(u) - \sum_{i=1}^m \Phi(x_i) \right\|_X^r \leq 2\tau^r \sum_{i=1}^m \|x_i\|_X^r.$$

Combining these we have

$$\Delta_r(\Phi, \Psi) \leq (6^r \sigma^r + 2\tau^r + 2(2^{1+1/r}\tau + 2\sigma)^r)^{1/r}.$$

The final part of the Proposition is then immediate taking $r = 1$ and $\tau = \sigma$.

For the remaining case we observe that for $0 < r < 1$ (2) implies that (3.1) holds with a constant $\tau = C\sigma$ where C depends on r . This is a well-known calculation first observed in [16] (see also [18] p. 91). Suppose x_1, \dots, x_n are nonzero in X and $x^* \in X^*$ with $\|x^*\|_{X^*} = 1$. Define a linear map $T: l_r^n \rightarrow X$ by $Te_i = x_i/\|x_i\|_X$; then $\|T\| \leq 1$ and (2) implies that the functional $F(\xi) = x^*(\Phi T(\xi))$ satisfies

$$|F(\xi_1 + \xi_2) - F(\xi_1) - F(\xi_2)| \leq 2\sigma(\|\xi_1\|_r + \|\xi_2\|_r).$$

Appealing to Lemma 5.8 of [18] we have

$$\left| x^* \left(\Phi \left(\sum_{i=1}^n x_i \right) \right) - \sum_{i=1}^n x^*(\Phi(x_i)) \right| \leq 2 \left(\sum_{i=1}^n (2/i)^{1/r} \right) \sigma \left(\sum_{i=1}^n \|x_i\|_X^r \right)^{1/r}.$$

This establishes the Proposition, since $\sum_{i=1}^{\infty} (2/i)^{1/r} < \infty$. \square

Lemma 3.4. *Suppose $\varepsilon > 0$. Then there exists $\sigma = \sigma(\varepsilon) > 0$ so that if X and Y are real Banach spaces and $(\phi, \psi) \in \mathcal{F}(B_X, B_Y)$ with $D(\phi, \psi) < \sigma$ then there exists a homogeneous map $\Phi: X \rightarrow Y$ such that $\frac{1}{2}\|x\|_X \leq \|\Phi(x)\|_Y \leq \|x\|_X$ for $x \in X$ and:*

1. *Given $y \in Y$ there exists $x \in X$ with $\|x\|_X \leq \|y\|_Y$ and $\|y - \Phi(x)\|_Y \leq \varepsilon\|y\|_Y$*
2. *If $x_1, x_2, x_3 \in X$ and $\sum_{k=1}^3 x_k = 0$ then*

$$\left\| \sum_{k=1}^3 \Phi(x_k) \right\|_Y \leq \varepsilon \left(\sum_{k=1}^3 \|x_k\|_X \right).$$

Proof. Suppose first $D(\phi, \psi) = \sigma$. Let $\phi(0) = y \in B_Y$. Then there exists $v \in B_Y$ with $\|y - v\|_Y = 1 + \|y\|_Y$. Thus $\|(\psi(v))\|_X \geq 1 + \|y\|_Y - 2\sigma$. It follows that $\|\phi(0)\|_Y \leq 2\sigma$. From this it follows similarly that for any $x \in B_X$ we have $\|x\|_X - 4\sigma \leq \|\phi(x)\|_Y \leq \|x\|_X + 4\sigma$.

Now let us assume the conclusion of the Lemma is false.

Indeed if this is so we can find a sequence of pairs of Banach spaces (X_n, Y_n) and pairs of functions (ϕ_n, ψ_n) with $D(\phi_n, \psi_n) < 1/n^2$ but such that for every homogeneous map $\Phi: X_n \rightarrow Y_n$ with $\frac{1}{2}\|x\|_{X_n} \leq \|\Phi(x)\|_{Y_n}$ either (1) or (2) fails.

We will in particular define $\Phi_n : X \rightarrow Y$ to be a homogenous map such that either $\Phi_n(x) = (1 + 4/n)^{-1} \phi_n(x)$ or $\Phi_n(x) = -(1 + 4/n)^{-1} \phi_n(-x)$ when $\|x\|_X = 1/n$. As long as $n \geq 20$ we then have $\frac{1}{2} \|x\|_{X_n} \leq \|\Phi_n(x)\|_{Y_n} \leq \|x\|_{X_n}$.

Now let \mathcal{U} be a nonprincipal ultrafilter on \mathbb{N} . We form the ultraproducts $X_{\mathcal{U}} = \prod_{\mathcal{U}} (X_n)$ and $Y_{\mathcal{U}} = \prod_{\mathcal{U}} (Y_n)$. Thus $X_{\mathcal{U}}$ can be realized as the Hausdorff quotient of the seminormed space $l_{\infty}(X_n)$ with the seminorm $\|\mathbf{x}\|_{X_{\mathcal{U}}} = \lim_{n \in \mathcal{U}} \|x_n\|_{X_n}$ where $\mathbf{x} = (x_n)$. We refer to [13] or [9] for details about ultraproducts.

Define a map $\Omega : X_{\mathcal{U}} \rightarrow Y_{\mathcal{U}}$ by

$$\Omega(\mathbf{x}) = \left(n \phi_n \left(\frac{1}{n} x_n \right) \right)_{n=1}^{\infty}$$

(where we defined $\phi_n(x) = 0$ if $\|x\|_{X_n} > 1$). We may argue that Ω is well-defined since

$$\frac{1}{n} \|x_n - u_n\|_{X_n} - \frac{2}{n^2} \leq \left\| \phi_n \left(\frac{1}{n} x_n \right) - \phi_n \left(\frac{1}{n} u_n \right) \right\|_{Y_n} \leq \frac{1}{n} \|x_n - u_n\|_{X_n} + \frac{2}{n^2}$$

if $\|x_n\|_{X_n}, \|u_n\|_{X_n} \leq n$. The same inequality implies that Ω is an isometry.

If $\mathbf{y} = (y_n) \in l_{\infty}(Y_n)$ we can define $x_n = n \psi_n(y_n/n)$ for all but finitely many n . Then $\|\phi_n(x_n/n) - y_n/n\|_{Y_n} \leq 2/n^2$. It follows that Ω is onto.

We also note that since $\|\phi_n(0)\|_{Y_n} \leq 2/n^2$ we have $\Omega(0) = 0$. It follows from the Mazur-Ulam theorem ([21], [29]) that Ω is linear.

Now notice that if $\|x_n\|_{X_n} = 1$ then $\Omega(\mathbf{x}) = (\Phi_n(x_n))$ (as elements of the ultraproduct). Since Φ_n is homogeneous this means that for every $\mathbf{x} \in X_{\mathcal{U}}$ we have $\Omega(\mathbf{x}) = (\Phi_n(x_n))$.

Since (1) or (2) fails for every $n \geq 20$ we can assume that for some set $P \in \mathcal{U}$ we have (1) failing for every $n \in P$ or (2) failing for every $n \in P$. If (1) fails there exists $\mathbf{y} = (y_n)$ with $y_n \in S_{Y_n}$ such that if $n \in P$ then $\|y_n - \Phi_n(x_n)\| \geq \varepsilon$ whenever $\|x_n\|_{X_n} \leq 1$. This contradicts the fact that Ω is an onto isometry.

Similarly if (2) fails then there exists $\mathbf{x}_k = (x_{kn})$ for $k = 1, 2, 3$ with $\sum_{k=1}^3 \|x_{kn}\|_{X_n} = 1$, $\sum_{k=1}^3 x_{kn} = 0$ and $\|\sum_{k=1}^3 \Phi_n(x_{kn})\|_{Y_n} \geq \varepsilon$ for $n \in P$. This contradicts the linearity of Ω . \square

It is perhaps worth recording a result implicit in this argument:

Proposition 3.5. *Suppose (X_n) and (Y_n) are two sequences of real Banach spaces for which $\lim_{n \rightarrow \infty} d_{GH}(X_n, Y_n) = 0$. Then for any non-principal ultrafilter \mathcal{U} on \mathbb{N} the ultraproducts $\prod_{\mathcal{U}} (X_n)$ and $\prod_{\mathcal{U}} (Y_n)$ are isometric.*

Now we are able to establish that for $0 < r < 1$ we have equivalence for the metrics d_{GH} and d_r .

Theorem 3.6. *Suppose $0 < r < 1$. Then there is a function $f = f_r : (0, 1] \rightarrow (0, 1]$ with $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0$ such that for every pair of real Banach spaces X and Y we have*

$$2^{-2/r+1}d_{GH}(X, Y) \leq d_r(X, Y) \leq f(d_{GH}(X, Y)).$$

Remark. It would be interesting to find an explicit function f satisfying the conditions of the theorem.

Proof. In fact the upper estimate follows immediately from Proposition 3.3 and Lemma 3.4.

For the lower estimate we begin by noticing that if $(\Phi, \Psi) \in \mathcal{F}_h(X, Y)$ and $\Delta_r(X, Y) = \sigma$ then for $x_1, \dots, x_m \in X$ and $y_1, \dots, y_n \in Y$ we have, by putting $u = \sum_{i=1}^m x_i - \sum_{j=1}^n \Psi(y_j)$,

$$\left\| \sum_{i=1}^m \Phi(x_i) - \sum_{j=1}^n y_j - \Phi(u) \right\|_Y \leq 2^{1/r} \sigma \left(\sum_{i=1}^m \|x_i\|_X^r + \sum_{j=1}^n \|y_j\|_Y^r \right)^{1/r}.$$

Since $\|\Phi(u)\|_Y \leq \|u\|_X$ this implies that

$$\left\| \sum_{i=1}^m \Phi(x_i) - \sum_{j=1}^n y_j \right\|_Y \leq \|u\|_X + 2^{1/r} \sigma \left(\sum_{i=1}^m \|x_i\|_X^r + \sum_{j=1}^n \|y_j\|_Y^r \right)^{1/r}.$$

By using this and the symmetrical calculation with X, Y interchanged we obtain

$$\begin{aligned} & \left\| \sum_{i=1}^m x_i - \sum_{j=1}^n \Psi(y_j) \right\|_X - \left\| \sum_{i=1}^m \Phi(x_i) - \sum_{j=1}^n y_j \right\|_Y \\ & \leq 2^{1/r} \sigma \left(\sum_{i=1}^m \|x_i\|_X^r + \sum_{j=1}^n \|y_j\|_Y^r \right)^{1/r}. \end{aligned}$$

If we let ϕ and ψ be the restrictions of Φ and Ψ to the respective unit balls and apply the above inequality for $m+n \leq 2$ we obtain that $D(\phi, \psi) \leq 2^{2/r-1} \sigma$. \square

We now recall the definition of \mathcal{H} -space from [16] or [18]. We say that a Banach space X is a \mathcal{H} -space if there is a constant κ (we denote the best such constant by $\kappa(X)$) such that whenever $f: X \rightarrow \mathbb{R}$ is a homogeneous function satisfying

$$|f(x_1 + x_2) - f(x_1) - f(x_2)| \leq \|x_1\|_X + \|x_2\|_X$$

there is a linear functional $g: X \rightarrow \mathbb{R}$ with $|f(x) - g(x)| \leq \kappa \|x\|_X$ for all $x \in X$. It is also natural to consider the notion of a \mathcal{H}_0 -space. We say that a Banach space X is a \mathcal{H}_0 -space if there is a constant κ_0 (we denote the best such constant by $\kappa_0(X)$) such that whenever $f: X \rightarrow \mathbb{R}$ is a homogeneous function, which is bounded on B_X and satisfies

$$|f(x_1 + x_2) - f(x_1) - f(x_2)| \leq \|x_1\|_X + \|x_2\|_X$$

there is a linear functional $x^* \in X^*$ with $|f(x) - x^*(x)| \leq \kappa_0 \|x\|_X$ for all $x \in X$. Clearly $\kappa_0(X) \leq \kappa(X)$. If X has the Bounded Approximation Property it may be shown that X is a \mathcal{H}_0 -space if and only if X is a \mathcal{H} -space. In general however this equivalence is not known.

It is known that a Banach space X is a \mathcal{H} -space if X has nontrivial type ([16]) or if X^* is isomorphic to a subspace of L_1 (see [19]).

Theorem 3.7. *Suppose $0 < r < 1$. Then there is a constant $C = C(r)$ so that if X is a \mathcal{H}_0 -space then for any Banach space Y , $d_K(X, Y) \leq C \kappa_0(X) d_r(X, Y)$.*

In particular if X is a real \mathcal{H}_0 -space then for any sequence of Banach spaces (X_n) we have $\lim_{n \rightarrow \infty} d_{GH}(X_n, X) = 0$ implies $\lim_{n \rightarrow \infty} d_K(X_n, X) = 0$.

Remark. Notice that this theorem applies if X is super-reflexive or if X is isomorphic to c_0 or $C(K)$ for some compact Hausdorff space. As we note later in the case when X is isomorphic to c_0 or l_∞ one can show that $\lim_{n \rightarrow \infty} d_{BM}(X_n, X) = 0$ whenever $\lim_{n \rightarrow \infty} d_{GH}(X_n, X) = 0$ (again for real spaces only).

Proof. Suppose $(\Phi, \Psi) \in \mathcal{F}_h(X, Y)$, and $\sigma = \Delta_r(\Phi, \Psi)$. Then for $x_1, x_2 \in X$ we have

$$\|\Phi(x_1 + x_2) - \Phi(x_1) - \Phi(x_2)\|_Y \leq 2^{2/r-1} \sigma (\|x_1\|_X + \|x_2\|_X).$$

If $\|y^*\|_{Y^*} = 1$ then there is a linear map $g: X \rightarrow \mathbb{R}$ such that

$$|y^*(\Phi(x)) - g(x)| \leq 2^{2/r-1} \sigma \kappa_0(X) \|x\|_X.$$

Now if $\sum_{i=1}^n x_i = 0$ we obtain

$$\left| \sum_{i=1}^n y^*(\Phi(x_i)) \right| \leq 2^{2/r-1} \sigma \kappa_0(X) \sum_{i=1}^n \|x_i\|_X.$$

This in turn implies that

$$\left\| \sum_{i=1}^n \Phi(x_i) \right\| \leq 2^{2/r-1} \sigma \kappa_0(X) \sum_{i=1}^n \|x_i\|_X.$$

The result follows on appealing to Proposition 3.3. \square

Example. We now show by example that the Kadets and Gromov-Hausdorff distance are not equivalent for general Banach spaces. To see this we show that $\lim_{p \rightarrow 1} d_{GH}(l_p, l_1) = 0$. It is known (cf. [22] or [24]) that $d_K(l_1, l_p) = 1$ for all $p > 1$.

We consider the Mazur map $\phi: B_{l_p} \rightarrow B_{l_1}$ which is defined by

$$\phi(\xi) = (\text{sgn } \xi_n |\xi_n|^p)_{n=1}^\infty$$

for $\zeta = (\zeta_n)$. Let $\psi = \phi^{-1}$ and then we compute $D(\phi, \psi)$. We need only estimate

$$|||\zeta - \eta||_{l_p} - \|\phi(\zeta) - \phi(\eta)\|_l|$$

over all $\zeta, \eta \in B_{l_p}$. Clearly

$$|||\zeta - \eta||_{l_p} - \|\zeta - \eta\|_{l_p}^p \leq 2^p - 2.$$

For any $a, b \in \mathbb{R}$ we have the estimate

$$||a - b|^p - |\operatorname{sgn} a|a|^p - \operatorname{sgn} b|b|^p|| \leq (2^{p-1} - 1)(|a|^p + |b|^p).$$

To see this note that if $s, t \geq 0$ we have

$$s^p + t^p \leq (s + t)^p \leq 2^{p-1}(s^p + t^p).$$

This implies that if $0 \leq s \leq t$ then

$$(t - s)^p \leq t^p - s^p \leq (t - s)^p + (2^{p-1} - 1)(s^p + (t - s)^p).$$

The required inequality now follows by considering cases.

Now by summing we obtain

$$|||\zeta - \eta||_{l_p}^p - \|\phi(\zeta) - \phi(\eta)\|_{l_1} \leq 2^p - 2.$$

This implies that $D(\phi, \psi) \leq 2(2^p - 2)$. Hence $d_{GH}(l_1, l_p) \leq 2(2^p - 2) \rightarrow 0$ as $p \rightarrow 1$. \square

4. Continuity of certain maps for the Kadets metric

In this section, we will establish a number of continuity-type results for the Kadets distance. Thus for example Theorem 4.3 below can be interpreted as saying that the map $X \rightarrow X^*$ is continuous for the Kadets distance.

Theorem 4.1. *Let Z be a Banach space and let E, F be closed subspaces of Z . Then*

$$d_K(Z/E, Z/F) \leq 2\Lambda(E, F).$$

Remark. A somewhat different version of this result can be found in the unpublished manuscript [14].

Proof. Let $X = Z/E$ and $Y = Z/F$. Suppose $\sigma > \Lambda(E, F)$. Let q_X and q_Y be the respective quotient maps. If $\theta > 1$ we can define a homogeneous map $f_X : X \rightarrow Z$ such that $q_X f_X = I_X$ and $\|f_X x\|_Z \leq \theta \|x\|_X$. We make a similar definition

of f_Y . Finally define $\Phi: X \rightarrow Y$ by $\Phi(x) = \theta^{-1}q_Y f_X(x)$ and $\Psi: Y \rightarrow X$ by $\Psi(y) = \theta^{-1}q_X f_Y(y)$.

Suppose $x_1, \dots, x_m \in X$ and $y_1, \dots, y_n \in Y$. Let $u = \sum_{i=1}^m x_i + \sum_{j=1}^n \Psi(y_j)$. Then $f_X(u) - \sum_{i=1}^m f_X(x_i) - \theta^{-1} \sum_{j=1}^n f_Y(y_j) \in E$. It follows that there exists $z \in F$ with

$$\begin{aligned} & \left\| f_X(u) - \sum_{i=1}^m f_X(x_i) - \theta^{-1} \sum_{j=1}^n f_Y(y_j) - z \right\|_Z \\ & \leq \sigma \left(\theta \left(\|u\|_X + \sum_{i=1}^m \|x_i\|_X \right) + \sum_{j=1}^n \|y_j\|_Y \right). \end{aligned}$$

Since $\|u\|_X \leq \sum_{i=1}^m \|x_i\|_X + \sum_{j=1}^n \|y_j\|_Y$ this implies that

$$\left\| \theta \Phi(u) - \theta \sum_{i=1}^m \Phi(x_i) - \theta^{-1} \sum_{j=1}^n y_j \right\|_Y \leq 2\sigma\theta \left(\sum_{i=1}^m \|x_i\|_X + \sum_{j=1}^n \|y_j\|_Y \right).$$

Note $\|\Phi(u)\|_Y \leq \|u\|_X$ so that if $v = \sum_{i=1}^m \Phi(x_i) + \sum_{j=1}^n y_j$ we obtain that

$$\|v\|_Y \leq \|u\|_X + (2\sigma + (1 - \theta^{-2})) \left(\sum_{i=1}^m \|x_i\|_X + \sum_{j=1}^n \|y_j\|_Y \right).$$

Combined with the corresponding converse inequality this implies that

$$\Delta(\Phi, \Psi) \leq 2\sigma + (1 - \theta^{-2}).$$

The result now follows. \square

It is interesting to note that there is a converse to this result.

Theorem 4.2. *Suppose X and Y are Banach spaces, and suppose $\sigma > d_K(X, Y)$. Then there is a Banach space Z with closed subspaces E, F such that Z/E is isometric to X , Z/F is isometric to Y and $\Delta(E, F) \leq \sigma$. Furthermore E is isometric to Y and F is isometric to X .*

Proof. Suppose that $(\Phi, \Psi) \in \mathcal{F}_h(X, Y)$ with $\Delta(\Phi, \Psi) \leq \sigma$. We apply the construction of Theorem 2.3. As there, we define Z to be algebraically $X \oplus Y$ and then define the norm by

$$\|(u, v)\|_Z = \inf \left(\|x_0\|_X + \|y_0\|_Y + \sigma \sum_{i=1}^m \|x_i\|_X + \sigma \sum_{j=1}^n \|y_j\|_Y \right)$$

where the infimum is taken over all $x_0, \dots, x_m \in X$ and $y_0, \dots, y_n \in Y$ such that $u = x_0 + \sum_{j=1}^m x_j + \sum_{j=1}^n \Psi(y_j)$ and $v = y_0 + \sum_{j=1}^m \Phi(x_j) + \sum_{j=1}^n y_j$.

Now we have $\Lambda(X, Y) \leq \sigma$, where X is identified with $X \oplus \{0\}$ and Y with $\{0\} \oplus Y$. Let us compute Z/Y : this is easily seen to be isometric to X since

$$\inf_{y \in Y} \|(x, y)\|_Z = \|(x, \Phi(x))\|_Z = \sigma \|x\|_X.$$

Similarly Z/X is isometric to Y . \square

Remark. If X and Y are separable then so is Z and there is a quotient map of l_1 onto Z . It may then be verified that if $\sigma > d_K(X, Y)$, X and Y can be represented as l_1/E and l_1/F where $\Lambda(E, F) \leq 2\sigma$.

Theorem 4.3. *Suppose X and Y are Banach spaces. Then $d_K(X^*, Y^*) \leq 2d_K(X, Y)$.*

Remark. The reverse inequality (i.e. $d_K(X, Y) \leq cd_K(X^*, Y^*)$) is not valid. In order to see this consider the case $X^* = Y^* = l_1$, $X = c_0$, $Y = \text{predual of } l_1$, non-isomorphic to c_0 . Proposition 5.3 of [24] imply that $d_K(X, Y) \geq 1/2$. On the other hand $d_K(X^*, Y^*) = 0$.

Proof. Suppose $\sigma > d_K(X, Y)$. Then as in Theorem 4.2, there is a Banach space Z and closed subspaces E, F with $\Lambda(E, F) \leq \sigma$ and $Z/E = X$, $Z/F = Y$. We then have $d_K(X^*, Y^*) \leq \Lambda(E^\perp, F^\perp)$. We now use the fact that $\Lambda(E^\perp, F^\perp) \leq 2\Lambda(E, F)$ which is well-known (cf. Theorem 3.4(d) and 3.13 of [24]); we will provide a direct proof. If $z^* \in E^\perp$ and $\|z^*\|_{Z^*} = 1$ then for $f \in F$ we have $|z^*(f)| \leq \sigma \|f\|_Z$. By the Hahn-Banach theorem there exists $u^* \in Z^*$ with $\|u^*\|_{Z^*} \leq \sigma$ and $z^* - u^* \in F^\perp$. Then $\|z^* - u^*\|_{Z^*} \leq 1 + \sigma$ and so $d(z^*, B_{F^\perp}) \leq 2\sigma$. It follows by symmetry that $\Lambda(E^\perp, F^\perp) \leq 2\sigma$. \square

Theorem 4.4. *If X, Y are Banach spaces then we have*

- (1) $d_K(X^{**}, Y^{**}) \leq d_K(X, Y)$
- (2) For any fixed ultrafilter \mathcal{U} on \mathbb{N} , $d_K(X_{\mathcal{U}}, Y_{\mathcal{U}}) \leq d_K(X, Y)$
- (3) $d_K(X^{**}/X, Y^{**}/Y) \leq 8d_K(X, Y)$.

Proof. For (1) observe simply that X and Y are embedded into a common Banach space Z then X^{**}, Y^{**} can be identified with $X^{\perp\perp}, Y^{\perp\perp}$ in Z^{**} and $\Lambda(X^{\perp\perp}, Y^{\perp\perp}) \leq \Lambda(X, Y)$. For (2) we may use a similar argument with $Z_{\mathcal{U}}$. For (3) we use 4.2. If $\sigma > d_K(X, Y)$ there is a Banach space Z with closed subspaces E, F with $\Lambda(E, F) \leq \sigma$ and such that Z/E is isometric to X and Z/F is isometric to Y . Furthermore from the construction we have $Z = E + F$. Consider Z^{**} ; then it is easy to verify that $\Lambda(E^{\perp\perp}, F^{\perp\perp}) \leq \sigma$. Let $Q_X: Z \rightarrow X$ and $Q_Y: Z \rightarrow Y$ be the quotient maps. Now suppose $z^{**} \in Z + F^{\perp\perp}$ with $\|z^{**}\|_{Z^{**}} = 1$ and that $\varepsilon > 0$. Then $Q_Y^{**} z^{**} \in X \subset X^{**}$ so that there exists $z \in Z$ with $\|z\|_Z \leq (1 + \varepsilon)$ and $Q_Y z = Q_Y^{**} z^{**}$.

Thus $z^{**} - z \in F^{\perp\perp}$. Now pick $e^{**} \in E^{\perp\perp}$ with $\|z^{**} - z - e^{**}\|_{Z^{**}} \leq (2 + \varepsilon)\sigma$. Thus $d(z^{**}, E^{\perp\perp} + Z) \leq 2\sigma$. This and the similar inequality with E, F reversed leads to the estimate $\Lambda(Z + F^{\perp\perp}, E^{\perp\perp} + Z) \leq 4\sigma$. Now note that $Z^{**}/(Z + F^{\perp\perp})$ is isometric to Y^{**}/Y and $Z^{**}/(Z + E^{\perp\perp})$ is isometric to X^{**}/X . Now Theorem 4.1 gives the result. \square

We now turn to complex interpolation of Banach spaces. We present an approach which encompasses several such situations.

Let E be a complex Banach space and \mathcal{U} is an open subset of the complex plane \mathbb{C} which is conformally equivalent to the unit disk \mathcal{D} . We define an *interpolation field* \mathcal{X} to be a vector space of E -valued analytic functions $f: \mathcal{U} \rightarrow E$ equipped with a norm $\|\cdot\|_{\mathcal{X}}$ such that \mathcal{X} is a Banach space and such that the following two conditions hold:

- (1) If $\varphi: \mathcal{U} \rightarrow \mathcal{D}$ is a conformal equivalence then we have $f \in \mathcal{X}$ if and only if $\varphi f \in \mathcal{X}$ and $\|f\|_{\mathcal{X}} = \|\varphi f\|_{\mathcal{X}}$.
- (2) Each evaluation $f \rightarrow f(w)$ for $w \in \mathcal{U}$ is bounded from \mathcal{X} to E .

If \mathcal{X} is an interpolation field we define for each $w \in \mathcal{U}$ the Banach space X_w to be the set of $x \in E$ such that there exists $f \in \mathcal{X}$ with $f(w) = x$ equipped with the quotient norm $\|x\|_w = \inf\{\|f\|_{\mathcal{X}} : f(w) = x\}$.

This definition is easily seen to encompass the standard definitions of complex interpolation spaces in the literature by taking \mathcal{U} to be the open strip $\mathcal{S} = \{w : 0 < \Re w < 1\}$. It also covers the interpolation method introduced by Coifman, Cwikel, Rochberg, Sagher and Weiss in [7].

Our main result is that the map $w \rightarrow X_w$ is continuous for the Kadets metric. To state our result we introduce the pseudo-hyperbolic metric on \mathcal{U} . If $\zeta, \eta \in \mathcal{U}$ let $h(\zeta, \eta)$ be the pseudo-hyperbolic distance on \mathcal{U} defined by $h(\zeta, \eta) = |\varphi(\eta)| (= h(\eta, \zeta))$ where $\varphi: \mathcal{U} \rightarrow \mathcal{D}$ is a conformal equivalence with $\varphi(\zeta) = 0$.

Theorem 4.5. *Let \mathcal{U} be an open subset of the complex plane which is conformally equivalent to the unit disk, and let \mathcal{F} be an interpolation field on \mathcal{U} . Then if $\zeta, \eta \in \mathcal{U}$,*

$$d_{\mathcal{K}}(X_{\zeta}, X_{\eta}) \leq 2h(\zeta, \eta).$$

Proof. The argument is very similar to that of Theorem 4.1. Suppose $\theta > 1$. We define a homogenous map $\phi_{\zeta}: X_{\zeta} \rightarrow \mathcal{X}$ with the property that $\phi_{\zeta}(x; \zeta) = x$ and $\|\phi_{\zeta}(x)\|_{\mathcal{X}} \leq \theta\|x\|_{\zeta}$. Define $\Phi: X_{\zeta} \rightarrow X_{\eta}$ by $\Phi(x) = \theta^{-1}\phi_{\zeta}(x; \eta)$ and similarly let $\Psi: X_{\eta} \rightarrow X_{\zeta}$ be defined by $\Psi(y) = \theta^{-1}\phi_{\eta}(y; \zeta)$.

Now suppose $x_1, \dots, x_m \in X_{\zeta}$ and $y_1, \dots, y_n \in X_{\eta}$. Let $u = \sum_{i=1}^m x_i + \sum_{j=1}^n \Psi(y_j)$ and $v = \sum_{i=1}^m \Phi(x_i) + \sum_{j=1}^n y_j$. We note that

$$\phi_\xi(u; \zeta) - \sum_{i=1}^m \phi_\xi(x_i; \zeta) - \theta^{-1} \sum_{j=1}^n \phi_\eta(y_j; \zeta) = 0.$$

Let φ be a conformal map of \mathcal{U} onto \mathcal{D} with $\varphi(\zeta) = 0$. Then there exists $f \in \mathcal{X}$ with

$$\phi_\xi(u) - \sum_{i=1}^m \phi_\xi(x_i) - \theta^{-1} \sum_{j=1}^n \phi_\eta(y_j) = \varphi f.$$

Using the fact that $\|f\|_{\mathcal{X}} = \|\varphi f\|_{\mathcal{X}}$, we obtain

$$\left\| \theta \Phi(u) - \theta \sum_{i=1}^m \Phi(x_i) - \theta^{-1} \sum_{j=1}^n y_j \right\|_{\eta} \leq 2\theta |\varphi(\eta)| \left(\sum_{i=1}^m \|x_i\|_{\xi} + \sum_{j=1}^n \|y_j\|_{\eta} \right).$$

It follows that

$$\|v\|_{\eta} \leq \|u\|_{\xi} + (2h(\zeta, \eta) + (1 - \theta^{-2})) \left(\sum_{i=1}^m \|x_i\|_{\xi} + \sum_{j=1}^n \|y_j\|_{\eta} \right).$$

With the symmetrical inequality this leads to

$$\Delta(\Phi, \Psi) \leq 2h(\zeta, \eta) + (1 - \theta^{-2})$$

and the result follows. \square

We can apply these results to standard interpolation couples. Let X_0, X_1 be a Banach couple and let $X_\theta = [X_0, X_1]_\theta$ be the standard complex interpolation space obtained by the Calderon method.

Corollary 4.6. *If $0 < \theta < \phi < 1$ then*

$$d_K(X_\theta, X_\phi) \leq 2 \frac{\sin(\pi(\phi - \theta)/2)}{\sin(\pi(\phi + \theta)/2)}.$$

Proof. Define

$$\varphi(z) = \frac{\sin(\pi(z - \theta)/2)}{\sin(\pi(z + \theta)/2)}.$$

Then φ is a conformal mapping of the strip $\{z : 0 < \Re z < 1\}$ onto the open unit disk with $\varphi(\theta) = 0$. \square

Corollary 4.7. *If $1 < p < q < \infty$ then for the (complex) spaces l_p, l_q we have:*

$$d_K(l_p, l_q) \leq 2 \frac{\sin(\pi(1/p - 1/q)/2)}{\sin(\pi(1/p + 1/q)/2)}.$$

Remark. In fact Corollary 4.7 holds for the corresponding real spaces. One way to see this is note that the pair (Φ, Ψ) of Theorem 4.4 can be constructed even in the case $\theta = 1$ and then map real sequences to real sequences. \square

Remark. The estimates of Corollary 4.6 improves on previous estimates ([15], [22]). The best known lower estimate ([23]) is

$$d_K(l_p, l_q) \geq 2^{1/p-1} - 2^{1/q-1}.$$

Remark. We can also consider complex interpolation of quasi-Banach spaces. If we fix $0 < r < 1$ and allow E in our definition to be an r -Banach space it is easy to show that the map $\xi \rightarrow X_\xi$ is continuous for the pseudo-metric d_r . By interpolation between l_r when $r < 1$ and l_2 one can then see that $\lim_{p \rightarrow 1} d_{GH}(l_p, l_1) = 0$ as verified directly in the previous section.

5. The Kadets and Gromov-Hausdorff topologies

Let \aleph be any arbitrary but fixed cardinal. Then we may consider the pseudo-metric space \mathcal{B}_\aleph of all Banach spaces with density character at most \aleph with the Kadets or Gromov-Hausdorff pseudo-metrics. We note that there are examples of non-isomorphic Banach spaces for which $d_K(X, Y) = 0$. An easy way to construct examples is to fix $1 < s < \infty$ and take a sequence $1 < p_n < \infty$ with $\lim_{n \rightarrow \infty} p_n = s$, but $p_n \neq s$ for all n . Then $X = l_s \oplus_2 l_2(l_{p_n})$ and $Y = l_2(l_{p_n})$ satisfy $d_K(X, Y) = 0$. This follows easily from the estimate from Corollary 4.6,

$$d_K(l_2(l_{p_n}), l_2(l_{r_n})) \leq 2 \sup_n \frac{|\sin(\pi(p_n^{-1} - r_n^{-1})/2)|}{\sin(\pi(p_n^{-1} + r_n^{-1})/2)}.$$

Clearly these spaces are non-isomorphic. If one takes $s = 1$ then one gets an example where $d_{GH}(X, Y) = 0$ but $d_K(X, Y) = 1$ since l_1 embeds in X but not Y .

We will be interested in this section in the topology of the (pseudo-)metric spaces \mathcal{B}_\aleph with the Kadets or Gromov-Hausdorff distances (we will sometimes use the term metric with the understanding that the spaces actually considered are the Hausdorff quotients). We consider a fixed cardinal \aleph to avoid certain set-theoretic problems; the collection of all Banach spaces fails to be a set. The most interesting choice of \aleph is of course \aleph_0 the set of separable Banach spaces; however in computing duals, biduals etc. it is necessary to consider larger cardinals. We observe that each set \mathcal{B}_\aleph is clopen (closed and open) in any larger \mathcal{B}_\aleph .

Let \mathcal{P} be a property of Banach spaces; then for each cardinal \aleph we may consider the set $\mathcal{P} = \mathcal{P}_\aleph$ of all $X \in \mathcal{B}_\aleph$ with property \mathcal{P} , so that we can think of \mathcal{P} as a set. We will say (following [24]) that \mathcal{P} is *stable* if there exists a fixed $\alpha > 0$ so that $X \in \mathcal{P}$ and $d_K(X, Y) < \alpha$ imply $Y \in \mathcal{P}$. We will say that \mathcal{P} is respectively *clopen*, *open*, *closed* if (for each \aleph) the set \mathcal{P} is respectively clopen, open or closed for the Kadets pseudo-metric. Obviously a stable property is clopen; also the negation of a stable or clopen property is also stable or clopen.

There are many known examples of stable properties. Let us list some:

- (1) X is separable.
- (2) X does not contain l_1 .
- (3) X is reflexive.
- (4) X is super-reflexive.
- (5) X has nontrivial type.

See [22], [23] and [24], where other stable properties are also discussed. We do not know of any examples of clopen properties which are not stable.

The following Proposition is trivial from Theorem 4.3 and Theorem 4.4. Notice that this proposition was known for stable properties (cf. [24], [2]) and for open properties only under some restrictions ([24], [2]).

Proposition 5.1. *Suppose \mathcal{P} is a stable (respectively clopen, open, closed) property; then the properties $\mathcal{P}^* = \{X: X^* \in \mathcal{P}\}$, $\mathcal{P}^{co} = \{X: X^{**} \in \mathcal{P}\}$ and $\mathcal{P}^{\mathcal{U}} = \{X: X_{\mathcal{U}} \in \mathcal{P}\}$ for some fixed ultrafilter \mathcal{U} on \mathbb{N} are also stable (respectively, clopen, open, closed).*

Note that for (1) and (2) above this leads to new stable properties:

$$\{X: X^* \text{ is separable}\} \text{ and } \{X: X^* \text{ does not contain } l_1\}.$$

Let us also mention some examples of open properties:

- (6) X is isomorphic to l_1 .
- (7) X is isomorphic to l_∞ .
- (8) X is injective.
- (9) X is isomorphic to c_0 .

We refer again to [22], [23] and [24]. In fact, in each example it is easy to show additionally that the Kadets distance defines a topology equivalent on the set to the Banach-Mazur distance, i.e. $\lim_{n \rightarrow \infty} d_K(X_n, X) = 0$ if and only if

$\lim_{n \rightarrow \infty} d_{BM}(X_n, X) = 0$. Notice that we also have that the following are open properties by Proposition 5.1:

(10) X^* is isomorphic to l_1 .

(11) X is a \mathcal{L}_1 -space.

Let us add to the list the following simple further open properties:

Properties 5.2. *The properties “ X is isomorphic to a subspace of c_0 ” and “ X is isomorphic to a subspace of l_∞ ” are open.*

Proof. We prove only the former assertion. Assume that X is a Banach space with Banach-Mazur distance less than λ to a subspace of c_0 . Suppose $d_K(X, Y) < 1/(2\lambda + 1)$. Then Y is separable. There is a separable Banach space Z which contains X and Y isometrically so that $\sigma = A(X, Y) < 1/(2\lambda + 1)$. Let $T: X \rightarrow c_0$ be a linear operator with $\|x\|_X \leq \|Tx\|_{c_0} \leq \lambda\|x\|_X$. Then T may be extended to an operator $T: Z \rightarrow c_0$ with $\|T\| \leq 2\lambda$ (cf. [30]). If $y \in Y$ then there exists $x \in X$ with $\|x\|_X \leq \|y\|_Y$ and $\|x - y\|_Z \leq \sigma\|y\|_Y$. Then

$$\|Ty\|_{c_0} \geq \|Tx\|_{c_0} - 2\lambda\sigma\|y\|_Y \geq (1 - \sigma - 2\lambda\sigma)\|y\|_Y.$$

Thus Y is also isomorphic to a subspace of c_0 . \square

Theorem 5.3. *The property “ $\kappa_0(X) < \infty$ ” is open.*

Proof. Suppose X is a Banach space with $\kappa_0 = \kappa_0(X) < \infty$. Suppose $d_K(X, Y) < \sigma$, where $(14 + 56\kappa_0)\sigma < 1$. Then there exists a bijective norm-preserving homogeneous map $\Omega: X \rightarrow Y$ such that for $x_1, \dots, x_n \in X$ we have

$$\left| \left\| \sum_{k=1}^n \Omega(x_k) \right\|_Y - \left\| \sum_{k=1}^n x_k \right\|_X \right| \leq 14\sigma \sum_{k=1}^n \|x_k\|_X.$$

Now suppose $f: Y \rightarrow \mathbb{R}$ is a homogeneous map which is bounded on B_Y and such that $|f(y_1 + y_2) - f(y_1) - f(y_2)| \leq \|y_1\|_Y + \|y_2\|_Y$. Since f is bounded a weak* compactness argument shows the existence of a best approximation $y^* \in Y^*$ so that $\sup_{y \in B_Y} |f(y) - y^*(y)| = M$ is minimized.

Let $\varphi = f - y^*$. Suppose $x_1, x_2 \in X$. Then $\|\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2)\|_Y \leq 28\sigma(\|x_1\|_X + \|x_2\|_X)$. Hence

$$|\varphi(\Omega(x_1 + x_2)) - \varphi(\Omega(x_1) + \Omega(x_2))| \leq (28\sigma M + 2)(\|x_1\|_X + \|x_2\|_X).$$

Now

$$|\varphi(\Omega(x_1) + \Omega(x_2)) - \varphi(\Omega(x_1)) - \varphi(\Omega(x_2))| \leq \|x_1\|_X + \|x_2\|_X$$

so that we conclude that

$$|\varphi(\Omega(x_1 + x_2)) - \varphi(\Omega(x_1)) - \varphi(\Omega(x_2))| \leq (28\sigma M + 3)(\|x_1\|_X + \|x_2\|_X).$$

It follows that there exists $x^* \in X^*$ so that

$$|\varphi(\Omega(x)) - x^*(x)| \leq \kappa_0(28\sigma M + 3)\|x\|_X.$$

Now suppose $y_1, \dots, y_n \in Y$ and $\sum_{k=1}^n y_k = 0$. Let $x = \sum_{k=1}^n \Omega^{-1}(y_k)$. Then $\|x\|_X \leq 14\sigma \sum_{k=1}^n \|y_k\|_Y$.

Notice that

$$\begin{aligned} \left| \sum_{k=1}^n \varphi(y_k) - \varphi(\Omega(x)) \right| &= \left| \sum_{k=1}^n \varphi(\Omega^{-1}y_k) - \varphi(\Omega(x)) - x^* \left(\sum_{k=1}^n \Omega^{-1}(y_k) - x \right) \right| \\ &\leq \kappa_0(3 + 28\sigma M)(1 + 14\sigma) \sum_{k=1}^n \|y_k\|_Y. \end{aligned}$$

Now

$$|\varphi(\Omega(x))| \leq M\|x\|_X \leq 14\sigma M \sum_{k=1}^n \|y_k\|_Y$$

so that we finally obtain

$$\left| \sum_{k=1}^n \varphi(y_k) \right| \leq C \sum_{k=1}^n \|y_k\|_Y$$

where

$$C = (14\sigma M + \kappa_0(6 + 56\sigma M)).$$

Now define a sublinear functional on Y by

$$p(y) = \inf \left(\sum_{k=1}^n \varphi(y_k) + C \sum_{k=1}^n \|y_k\|_Y : \sum_{k=1}^n y_k = y \right).$$

Notice that if $\sum_{k=1}^n y_k = y$ then

$$\sum_{k=1}^n \varphi(y_k) + C \sum_{k=1}^n \|y_k\|_Y \geq \varphi(y) - C\|y\|_Y$$

so that p is well-defined. Let h be any linear functional on Y so that $h(y) \leq p(y)$ for all y . Then $h(y) \leq \varphi(y) + C\|y\|_Y$ and by applying to $-y$ we have $h(y) \geq \varphi(y) - C\|y\|_Y$. Thus $|h(y) - \varphi(y)| \leq C\|y\|_Y$.

Now considering $f - y^* - h$ we see that we must have $M \leq C$ and so if $(14 + 56\kappa_0)\sigma < 1$ we have

$$M \leq \frac{6\kappa_0}{1 - 14\sigma - 56\sigma\kappa_0}.$$

and this gives an estimate for $\kappa_0(Y)$. \square

The notions of stable and open properties are most naturally applied to interpolation scales.

Proposition 5.4. *Suppose (X_0, X_1) is a Banach couple and that $X_\theta = [X_0, X_1]_\theta$. If \mathcal{P} is a stable property and there exists $0 < \theta < 1$ so that X_θ has property \mathcal{P} then X_ϕ has \mathcal{P} for every $0 < \phi < 1$.*

Proof. This is immediate from Corollary 4.6. \square

Remark. This can be applied to each of the table properties listed above. This yields a number of results, many of which are certainly known to specialists. For the example the case of reflexivity can be deduced from Calderón's original paper [6]. However we feel this general framework for such a result has some interest.

Proposition 5.5. *Suppose (X_0, X_1) is a Banach couple and that $X_\theta = [X_0, X_1]_\theta$. If \mathcal{P} is an open property and there exists $0 < \theta < 1$ so that X_θ has property \mathcal{P} then there exists $\varepsilon > 0$ so that X_ϕ has \mathcal{P} for every ϕ with $|\phi - \theta| < \varepsilon$.*

Here we single out three special cases which seem to be new and of some interest:

Proposition 5.6. *Suppose (X_0, X_1) is a Banach couple and that $X_\theta = [X_0, X_1]_\theta$. Suppose there exists $0 < \theta < 1$ so that X_θ is isomorphic to c_0 (respectively isomorphic to a subspace of c_0 , resp. isomorphic to l_1) then there exists $\varepsilon > 0$ so that X_ϕ is isomorphic to c_0 (respectively isomorphic to a subspace of c_0 , resp. isomorphic to l_1) for every ϕ with $|\phi - \theta| < \varepsilon$.*

Remarks. Of course, Propositions 5.4–5.6 apply to general interpolation fields.

Proposition 5.4 suggests it is of interest to make the following definition. Let X be an arbitrary Banach space. We will say that the (Kadets) component, \mathcal{C}_X of X is the intersection of all clopen properties containing X . Clearly if (X_0, X_1) is a Banach couple then for all $0 < \theta, \phi < 1$ we have that X_ϕ is in the component of X_θ .

We first state some elementary properties of components.

Proposition 5.7. *Let X be an arbitrary Banach space. Then:*

- (1) $Y \in \mathcal{C}_X$ if and only if $\mathcal{C}_Y = \mathcal{C}_X$.
- (2) For every $Y \in \mathcal{C}_X$ the density character $\text{dens } X = \text{dens } Y$ and $\text{dens } X^* = \text{dens } Y^*$.

(3) If $Y \in \mathcal{C}_X$ and Y_1 is isomorphic to Y then $Y_1 \in \mathcal{C}_X$.

(4) If Y is an arbitrary Banach space then for any subspace E we have $Y/E \oplus E \in \mathcal{C}_X$ if and only if $Y \in \mathcal{C}_X$.

Remark. Note that (3) allows us not to specify any special norm on the spaces in (4).

Proof. (1) is elementary. For (2) note that the sets $\{Y: \text{dens } Y = \text{dens } X\}$ and $\{Y: \text{dens } Y^* = \text{dens } X^*\}$ are both clopen.

For (3) observe as in Proposition 6.7 of [24] that there is a family of isomorphic copies of Y , Y_t for $0 \leq t \leq 1$, say, so that $t \rightarrow Y_t$ is continuous for the Kadets distance, with $Y_0 = Y$.

For (4) we use Lemma 5.9 of [24]. Let $Z = Y \oplus_1 Y/E$ and let $Q: Y \rightarrow Y/E$ be the quotient map. For any $t \in \mathbb{R}$ with $t \neq 0$ we let $G_t = \{(ty, Qy) : y \in Y\}$. Then it is easy to show that $\lim_{s \rightarrow t} \Lambda(G_s, G_t) = 0$ for any $t \in \mathbb{R}$. If $t = 0$ we define $G_0 = E \oplus_1 Y/E$. Then using Lemma 5.9 of [24] we also have $\lim_{s \rightarrow 0} \Lambda(G_s, G_0) = 0$. Thus the map $t \rightarrow G_t$ is continuous for the Kadets distance. However G_t is isomorphic to Y for all $t \neq 0$ while $G_0 = E \oplus_1 Y/E$. \square

Proposition 5.8. We have $X \in \mathcal{C}_{l_1}$ if and only if X is separable and contains a copy of l_1 .

Proof. Since the set of X which is both separable and contains a copy of l_1 is stable inclusion is immediate. Conversely suppose X is any separable Banach space containing a subspace E isomorphic to l_1 . Then X is in the same component as $l_1 \oplus_1 X/E$. Let F be kernel of a quotient map from l_1 onto X/E . Let $G = l_1(F) \oplus_1 l_1(X/E) \in \mathcal{C}_{l_1}$. Since the map $Y \rightarrow Y \oplus_1 X/E$ is trivially seen to be continuous for the Kadets metric the set $\{Y \oplus_1 X/E; Y \in \mathcal{C}_{l_1}\}$ is connected and meets \mathcal{C}_{l_1} since $G \oplus_1 (X/E)$ is isomorphic to G . Thus $l_1 \oplus_1 X/E \in \mathcal{C}_{l_1}$ and so $X \in \mathcal{C}_{l_1}$. \square

Problem 1. What is the component of c_0 ? Similar techniques to the above Proposition show that this contains all infinite-dimensional subspaces of c_0 . Proposition 5.7 (4) does not help to give any other examples, since being a subspace of c_0 is a three-space property ([1]).

Problem 2. What is the component of l_2 ? This is contained in the stable set of all separable super-reflexive spaces. We do not know if it coincides with this set. This is related to Pisier's notion of θ -Hilbertian spaces ([28]); any space which is θ -Hilbertian for $\theta > 0$ belongs to \mathcal{C}_{l_2} .

One can obviously introduce the notion of a Gromov-Hausdorff component in an analogous fashion. Clearly the Gromov-Hausdorff component of X , \mathcal{G}_X contains \mathcal{C}_X . Since the map $p \rightarrow l_p$ is continuous for the Gromov-Hausdorff distance for $p \in [1, \infty)$ we have that $\mathcal{G}_{l_1} = \mathcal{G}_{l_2}$. It seems quite possible that this will correspond with the collection of all separable Banach spaces so we ask:

Problem 3. Is the set of all separable Banach spaces connected for the Gromov-Hausdorff distance? We are unable to decide if c_0 is in \mathcal{G}_{l_2} so it would be interesting to identify \mathcal{G}_{c_0} .

Let us take this opportunity to make a few remarks about the Gromov-Hausdorff distance.

Proposition 5.9. *The following properties are open for the Gromov-Hausdorff distance:*

- (1) X is isomorphic to c_0 .
- (2) X is isomorphic to l_∞ ,
- (3) $\kappa_0(X) < \infty$
- (4) X has nontrivial type.
- (5) X has nontrivial cotype.
- (6) X^* has nontrivial type.
- (7) X^* has nontrivial cotype.

Furthermore, on the sets defined by properties (1) and (2) Gromov-Hausdorff distance defines a topology equivalent to the Banach-Mazur distance.

Proof. (1) and (2) follow almost immediately from Theorem 3.7, and the remarks above. In a similar way (3) follows from Theorem 3.7 and Theorem 5.3. Then (4) follows since if X has nontrivial type then $\kappa_0(X) < \infty$ again by using Theorem 3.7.

To establish (5), (6) and (7) we use Proposition 3.5. Indeed it is immediate from 3.5 that the property " l_∞ is finitely representable in X " is closed for the Gromov-Hausdorff distance. Let us establish (6) and (7) by showing that for a fixed Banach space E the condition " E is finitely representable in X^* " is closed for the Gromov-Hausdorff distance. Then we may take $E = l_1$ and $E = l_\infty$.

To show this last statement it suffices to suppose E finite-dimensional. Then if X_n converges to X in Gromov-Hausdorff distance i.e. $d_{GH}(X_n, X) \rightarrow 0$ and E is finitely representable in each X_n^* it is immediate that E is isometric to a subspace of $(\prod_{\mathcal{U}} (X_n))^*$. Hence by Theorem 3.5 E embeds isometrically into the space $X_{\mathcal{U}}^*$. Now $(X^*)_{\mathcal{U}}$ embeds naturally into $X_{\mathcal{U}}^*$ as a norming subspace by the identification

$$\mathbf{x}^*(\mathbf{x}) = \lim_{n \in \mathcal{U}} x_n^*(x_n).$$

The space of operators $\mathcal{L}(E, X_{\mathcal{U}}^*)$ is naturally a dual space of $E \otimes_{\pi} X_{\mathcal{U}}$ and $\mathcal{L}(E, (X^*)_{\mathcal{U}})$ is norming as a subspace. Hence $J: E \rightarrow X_{\mathcal{U}}^*$ is an isometry we can find a net $J_x: E \rightarrow (X^*)_{\mathcal{U}}$ so that $J_x \rightarrow J$ weak* and $\|J_x\| \leq 1$. By the weak* lower-semicontinuity of the norm in $X_{\mathcal{U}}^*$ it follows that for any $\varepsilon > 0$ there exists α with

$(1 - \varepsilon)\|e\|_E \leq \|J_x\|_{(X^*)_{\eta}} \leq \|e\|_E$ for $e \in E$. Hence E is finitely representable in $(X^*)_{\eta}$ and thus also in X^* . \square

Remark. It has been conjectured by the first author that $\kappa_0(X) < \infty$ might be equivalent to the property that X^* has finite cotype.

We conclude the paper by showing that the collection of all separable Banach spaces is not itself separable for either the Gromov-Hausdorff or Kadets distances. More precisely we show:

Theorem 5.10. *Suppose $1 < p < \infty$ and $p \neq 2$. Then the set of Banach spaces X isomorphic to l_p is not separable in either Kadets or Gromov-Hausdorff distances.*

Proof. First notice that since $\kappa_0(l_p) < \infty$ the two pseudo-metrics d_K and d_{GH} define equivalent topologies on the set \mathcal{I}_p of all isomorphic copies of l_p . Secondly since $X \rightarrow X^*$ is a homeomorphism on the set of reflexive spaces for the Kadets pseudo-metric it suffices to consider $1 < p < 2$.

We shall consider spaces $l_p(E_n)$ where each E_n is a finite-dimensional Hilbert space. Each such space is isomorphic to l_p by an old result of Pełczyński [26]. We will prove the following Lemma:

Lemma 5.11. *There exists $\sigma_0 > 0$ so that if (E_n) and (F_n) are two sequences of finite-dimensional Hilbert spaces and $d_K(l_p(E_n), l_p(F_n)) < \sigma_0$ then there is a bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$ so that $\frac{1}{4} \dim E_n \leq \dim F_{\pi(n)} \leq 10 \dim E_n$.*

Proof of the lemma. We will write $X = l_p(E_n)$ and $Y = l_p(F_n)$. A typical element of X will be denoted $\mathbf{x} = (x_n)_{n=1}^{\infty}$ with a similar notation for Y . We also adopt the convention that η will denote a function of σ satisfying $\lim_{\sigma \rightarrow 0} \eta(\sigma) = 0$ which may vary from occurrence to occurrence.

First observe that by Theorem 2.4 there exists a homogeneous, norm-preserving bijection $\Omega : X \rightarrow Y$ with the property that if $\mathbf{x}_1, \dots, \mathbf{x}_n \in X$ then

$$\left| \left\| \sum_{k=1}^n \Omega \mathbf{x}_k \right\|_Y - \left\| \sum_{k=1}^n \mathbf{x}_k \right\|_X \right| \leq 14 \sigma \sum_{k=1}^n \|\mathbf{x}_k\|_X.$$

Suppose $\mathbf{u}, \mathbf{v} \in S_X$ have disjoint supports, i.e. $\|u_n\|_{E_n} \|v_n\|_{E_n} = 0$ for each n . Then $\|\mathbf{u} \pm \mathbf{v}\|_X = 2^{1/p}$ and hence $\|\Omega(\mathbf{u}) \pm \Omega(\mathbf{v})\|_Y \geq 2^{1/p} - 28 \sigma$. Let $\Omega(\mathbf{u}) = \mathbf{y}$ and $\Omega(\mathbf{v}) = \mathbf{z}$. Then we have

$$\begin{aligned}
2^{1/p} - 28\sigma &\leq \left(\frac{1}{2} \|\mathbf{y} + \mathbf{z}\|_Y^2 + \frac{1}{2} \|\mathbf{y} - \mathbf{z}\|_Y^2 \right)^{1/2} \\
&= \left(\left(\sum_{k=1}^{\infty} 2^{-p/2} \|y_k + z_k\|_{F_k}^p \right)^{2/p} + \left(\sum_{k=1}^{\infty} 2^{-p/2} \|y_k - z_k\|_{F_k}^p \right)^{2/p} \right)^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} \left(\frac{1}{2} \|y_k + z_k\|_{F_k}^2 + \frac{1}{2} \|y_k - z_k\|_{F_k}^2 \right)^{p/2} \right)^{1/2} \\
&= \left(\sum_{k=1}^{\infty} (\|y_k\|_{F_k}^2 + \|z_k\|_{F_k}^2)^{p/2} \right)^{1/p} \\
&\leq \left(\sum_{k=1}^{\infty} \max(\|y_k\|_{F_k}, \|z_k\|_{F_k})^p \right)^{1/p-1/2} \left(\sum_{k=1}^{\infty} (\|y_k\|_{F_k}^2 + \|z_k\|_{F_k}^2) \right)^{1/2}.
\end{aligned}$$

This implies an estimate that

$$\left(\sum_{k=1}^{\infty} \max(\|y_k\|_{F_k}, \|z_k\|_{F_k})^p \right)^{1/p} \geq 2^{1/p} - \eta(\sigma)$$

and hence an estimate

$$\left(\sum_{k=1}^{\infty} \min(\|y_k\|_{F_k}, \|z_k\|_{F_k})^p \right)^{1/p} \leq \eta(\sigma).$$

Now suppose $\mathbf{y} \in S_Y$ is supported on exactly one co-ordinate m say. Let $\Omega^{-1}(\mathbf{y}) = \mathbf{x}$. Then we can write $\mathbf{x} = \mathbf{u} + \mathbf{v}$ disjointly where $\|\mathbf{u}\|_X \geq \|\mathbf{v}\|_X$, and $\|\mathbf{u}\|_X^p \leq \|\mathbf{v}\|_X^p + \max\|x_n\|_{E_n}^p$. Let $\mathbf{w} = \Omega(\mathbf{u})$ and $\mathbf{z} = \Omega(\mathbf{v})$. Then since $\|\mathbf{u}\|_X, \|\mathbf{v}\|_X \leq 1$ we obviously have that $\min(\|w_m\|_{F_m}, \|z_m\|_{F_m}) \leq \eta$. Now

$$\|\mathbf{y} - \Omega(\mathbf{u}) - \Omega(\mathbf{v})\|_Y \leq 42\sigma$$

so that

$$\|y_m - w_m - z_m\|_{F_m} \leq 42\sigma$$

and hence

$$\|w_m + z_m\|_{F_m} \geq 1 - 42\sigma.$$

From this we have an estimate $\max(\|w_m\|_{F_m}, \|z_m\|_{F_m}) \geq 1 - \eta$. This in turn means $\|\mathbf{u}\|_X \geq 1 - \eta$ and hence an estimate $\max\|x_n\|_{F_n} \geq 1 - \eta$.

It follows that if σ is small enough there is a unique $n = n(\mathbf{y})$ so that $\|x_n\|_{F_n} \geq 1 - \eta$.

Now suppose that \mathbf{y}_1 and \mathbf{y}_2 are both unit vectors with the same singleton support m . If $n(\mathbf{y}_1) \neq n(\mathbf{y}_2)$ we will have $\|\Omega^{-1}(\mathbf{y}_1) \pm \Omega^{-1}(\mathbf{y}_2)\|_X \geq 2^{1/p} - \eta$ but

$\min(\|\mathbf{y}_1 \pm \mathbf{y}_2\|_Y) \leq 2^{1/2}$. Again for σ small enough this is a contradiction. Hence we conclude that n is a function only of m .

Applying the same technique to X and Y interchanged we obtain two maps $\pi: \mathbb{N} \rightarrow \mathbb{N}$ and $\varrho: \mathbb{N} \rightarrow \mathbb{N}$ such that if $\mathbf{y} = \Omega \mathbf{x}$ and both are unit vectors then:

(a) if \mathbf{x} is supported only at m then $\|y_{\pi(m)}\|_{F_{\pi(m)}} \geq 1 - \eta$ and

(b) if \mathbf{y} is supported only at m then $\|x_{\varrho(m)}\|_{E_{\varrho(m)}} \geq 1 - \eta$.

Now if \mathbf{x} is supported only at m then $\mathbf{y} = \mathbf{z} + \mathbf{w}$ where $\|\mathbf{z}\|_Y \geq 1 - \eta$ and $\|\mathbf{w}\|_Y \leq \eta$ are disjoint and \mathbf{z} is supported only at $\pi(m)$. Thus

$$\|\mathbf{x} - \Omega^{-1}(\mathbf{z}) - \Omega^{-1}(\mathbf{w})\|_X \leq 42\sigma$$

from which it follows that for σ small enough we must have $\varrho(\pi(m)) = m$ and similarly $\pi(\varrho(m)) = m$. Thus π is a bijection.

Pick a maximal subset $(\xi_k)_{k \in J}$ of B_{E_m} with $\|\xi_k - \xi_l\|_{E_m} > \frac{1}{2}$. Then $|J| \leq 5^{\dim E_m}$. Let \mathbf{x}_k be the element of X with zeros everywhere except ξ_k in the m th position. Suppose $\zeta \in S_{F_{\pi(m)}}$; let \mathbf{z} be the similarly defined element of Y with exactly one nonzero element ζ . Let $\mathbf{x} = \Omega^{-1}\mathbf{z}$. Then there exists $k \in J$ with $\|\mathbf{x} - \mathbf{x}_k\|_X \leq \frac{1}{2} + \eta(\sigma)$. Then

$$\|\mathbf{z} - \Omega(\mathbf{x}_k) - \Omega(\mathbf{x} - \mathbf{x}_k)\|_Y \leq 42\sigma.$$

Hence

$$\|\mathbf{z} - \Omega(\mathbf{x}_k)\|_Y \leq \frac{1}{2} + \eta(\sigma).$$

Now let (ψ_k) be the $\pi(m)$ -coordinate of $\Omega(\mathbf{x}_k)$. We have

$$\|\zeta - \psi_k\|_{F_{\pi(m)}} \leq \frac{1}{2} + \eta(\sigma).$$

For σ small enough this implies that $(\psi_k)_{k \in J}$ is a 3/4-net in $B_{F_{\pi(m)}}$ so that

$$|J| \geq (4/3)^{\dim F_{\pi(m)}}.$$

We conclude that $\dim F_{\pi(m)} \leq C \dim E_m$ where $C = \log 5 / \log(4/3) \leq 10$. This proves the lemma. \square

Proof of the theorem. For each infinite subset M of \mathbb{N} form the space $X_M = l_p(l_2^{10 \cdot 2^n})_{n \in M}$. The lemma shows these are uniformly separated. \square

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