

FK-SPACES CONTAINING c_0

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1. Introduction. We denote by ω the vector space of all sequences of complex numbers topologized by means of coordinatewise convergence. An *FK-space* E is a vector subspace of ω endowed with a complete metrizable locally convex topology under which the natural inclusion map $E \rightarrow \omega$ is continuous. *FK-spaces* have proved to be a useful tool in function theory [78] and, more recently, in the study of bases in Banach spaces [64]. Their most important application, though, to summability theory stems from Zeller's observation [76] that the convergence domain c_A , defined below, of a matrix A is an *FK-space*.

The bounded convergence domain of A , $m \cap c_A$, is also an *FK-space* but has received much less attention from the functional analytic viewpoint. One reason for this is that $m \cap c_A$, as an *FK-space*, is in general nonseparable and has a rather intractable dual space [80; 53]. To circumvent these difficulties Alexiewicz and Orlicz [5] used the theory of two-norm spaces; their path also has its shortcomings, however, in that the Hahn-Banach theorem is generally no longer valid in the two-norm setting. Our approach is to identify the two-norm topology as a certain Mackey topology which enables us to use the powerful duality theory of locally convex spaces. Furthermore, this method is applicable to *FK-spaces* in general and leads to results that should be of interest outside summability theory.

We proceed by introducing notation in Section 2 and listing some of the fundamental results from the theory of locally convex spaces. A new proof of the Orlicz-Pettis theorem on unconditional convergence leads to a slightly stronger result (Theorem 1) which is needed later. We close Section 2 with a brief bibliographical discussion of the theory of mixed topologies and their role in summability theory.

Section 3 brings a detailed account of *FK-spaces* containing c_0 . The main results are the identification of a certain two-norm topology mentioned above (Theorem 5), a completeness theorem for the space l^1 (Theorem 3) from which the celebrated bounded consistency theorem follows, and a refinement of a result of Snyder on conull *FK-spaces* (Theorem 2). This section also contains several other results that should be of interest in themselves.

Section 4 deals with direct applications of the above theorems, and the main result (Theorem 7) yields a unified approach to the work on bounded convergence domains of several authors including Brudno, Copping, Mazur, Meyer-König, Orlicz, Petersen, Wilansky and Zeller. Petersen's theory of uniform summability is shown to be a statement concerning the compact subsets of a convergence

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domain; the corresponding statement for weakly compact subsets gives rise to the new concept of quasi-uniform summability. An example is also given of a complete Mackey space with a weak Schauder basis which is not a Schauder basis.

A new form of the closed graph theorem is given in Section 5 and we begin by applying this result to obtain a characterization of Chang's type O *FK*-spaces [17]. Separability plays a crucial role here, and many known results concerning convergence domains (Agnew [1], Bennett [13], Copping [19], Hahn [28], Orlicz [46] and Zeller [77]) are carried over to arbitrary separable *FK*-spaces. We close by establishing a conjecture of Kadec and Pełczyński [30] concerning unconditional convergence in a Fréchet space possessing a total biorthogonal system.

2. Notation and preliminary results. ω denotes the space of all complex-valued sequences and any vector subspace of ω is called a *sequence space*. A sequence space E with a locally convex topology τ is called a *K-space* if the inclusion mapping $(E, \tau) \rightarrow \omega$ is continuous when ω is endowed with the topology of coordinatewise convergence. If, in addition, τ is complete and metrizable, (E, τ) is called an *FK-space*. An *FK-space* whose topology is normable is a *BK-space*.

The following *BK*-spaces will be important in the sequel.

m , the space of all bounded sequences

c , the space of all convergent sequences

c_0 , the space of all null sequences

l^p , $1 \leq p < \infty$, the space of all absolutely p -summable sequences

As usual, l^1 is replaced simply by l , $\|\cdot\|_l$ denotes the norm on l and $\|\cdot\|_m$ denotes the norm on m , c and c_0 .

Let e denote the sequence of ones $(1, 1, \dots)$, let e^j , $j = 1, 2, \dots$, denote the sequence $(0, \dots, 0, 1, 0, \dots)$ with the one in the j -th position, and let ϕ denote the linear span of the e^j 's. For each $x \in \omega$ and each positive integer n , $P_n(x)$ denotes the sequence $(x_1, x_2, \dots, x_n, 0, \dots)$. If (E, τ) is a *K-space* containing ϕ , then $x \in E$ is said to have *SAK* (*AK*) provided that $P_n(x) \rightarrow x$ weakly (τ) in E ; if each $x \in E$ has *SAK* (*AK*), then (E, τ) is said to be a *SAK* (*AK*)-space.

Throughout this section $\langle E, F \rangle$ will be a separated dual pair of vector spaces [59; 32] and $\sigma(E, F)$ denotes the weak topology on E by F . If \mathcal{A} is a collection of $\sigma(F, E)$ -bounded subsets of F such that $\bigcup \{A : A \in \mathcal{A}\}$ spans F , then $\tau_{\mathcal{A}}$ denotes the topology on E of uniform convergence on the members of \mathcal{A} so that $\tau_{\mathcal{A}}$ is an $\langle E, F \rangle$ -polar topology. In particular, if \mathcal{A} is the collection of all absolutely convex $\sigma(F, E)$ -compact subsets of F , then $\tau_{\mathcal{A}} = \tau(E, F)$, the *Mackey topology* on E by F . Any space carrying a Mackey topology will be called a *Mackey space*. If \mathcal{A} contains all $\sigma(F, E)$ -bounded subsets of F , then $\tau_{\mathcal{A}} = \beta(E, F)$, the *strong topology* on E by F . E' will denote the space of all τ -continuous linear functionals on the topological vector space (E, τ) .

We now list for future reference some of the fundamental results from the theory of locally convex spaces.

PROPOSITION 1. (Grothendieck's completeness theorem [59; 103]) *If τ is a topology on F stronger than $\sigma(F, E)$ and every linear functional on F which is τ -continuous on each $A \in \mathcal{A}$ can be identified with a member of E , then E is τ_α -complete.*

In the next result \mathcal{B} denotes a collection of $\sigma(E, F)$ -bounded subsets of E such that $\cup\{B: B \in \mathcal{B}\}$ spans E .

PROPOSITION 2. (Grothendieck's precompactness theorem [59; 51]) *With the above notation the following conditions are equivalent.*

- (i) *Each $A \in \mathcal{A}$ is τ_α -precompact.*
- (ii) *Each $B \in \mathcal{B}$ is τ_α -precompact.*

The next result will be used frequently throughout the sequel.

PROPOSITION 3. *If $\langle E, F \rangle$ and $\langle G, H \rangle$ are separated dual pairs of vector spaces and T is a linear mapping of E into G , then T is $\sigma(E, F) \rightarrow \sigma(G, H)$ -continuous if and only if T is $\tau(E, F) \rightarrow \tau(G, H)$ -continuous.*

Proof. See [33; 262].

We now give a proof of the Orlicz-Pettis Theorem on unconditional convergence of series in locally convex spaces. Several proofs of this theorem have appeared [27], [38], [42], [55] and [58], but the version given below is slightly sharper than the usual statement and this sharpening will be useful in Section 3.

We shall denote by m_0 the subspace of m consisting of all sequences which take only finitely many values; it is then clear that m_0 is the linear span of all sequences taking only the values zero and one.

LEMMA 1. *The topology $\sigma(l, m_0)$ on l defines the same convergent sequences and compact sets as the $\|\cdot\|_1$ -topology.*

Using Lemma 7 the proof of this result is similar to that of the usual statement that $\sigma(l, m)$ defines the same convergent sequences and compact sets as the norm topology. (See, for example, [67; 451].)

LEMMA 2. *A bounded subset K of l is relatively compact if and only if*

$$\lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} |x_i| = 0 \quad \text{uniformly on } K.$$

This lemma is a particular case of a well-known result on bases in Banach spaces [57; 526]; a direct proof can also be given quite easily.

For a separated dual pair $\langle E, F \rangle$ of vector spaces $\lambda(E, F)$ will denote the topology on E of uniform convergence on the $\sigma(F, E)$ -compact subsets of F . In general $\lambda(E, F)$ is stronger than the Mackey topology $\tau(E, F)$ so that the statement of Theorem 1 is stronger than the usual statement of the Orlicz-Pettis theorem. (See, for example, [58].)

THEOREM 1. *If the series $\sum_{n=1}^{\infty} x_n$ of elements of E is such that every subseries $\sum_{k=1}^{\infty} x_{n_k}$ converges in $\sigma(E, F)$, then $\sum_{n=1}^{\infty} x_n$ converges in $\lambda(E, F)$.*

Proof. For $f \in F$ and for every subseries $\sum_{k=1}^{\infty} x_{n_k}$ the series of scalars $\sum_{k=1}^{\infty} \langle x_{n_k}, f \rangle$ converges so that

$$\sum_{n=1}^{\infty} |\langle x_n, f \rangle| < \infty.$$

We define the map $T: F \rightarrow l$ by $Tf = \{\langle x_n, f \rangle\}_{n=1}^{\infty}$. If $\theta = \{\theta_n\}_{n=1}^{\infty}$ is a sequence taking the values zero and one only, then

$$\sum_{n=1}^{\infty} \theta_n (Tf)_n = \sum_{n=1}^{\infty} \theta_n \langle x_n, f \rangle = \left\langle \sum_{n=1}^{\infty} \theta_n x_n, f \right\rangle$$

since $\sum_{n=1}^{\infty} \theta_n x_n$ converges in $\sigma(E, F)$. It follows that T is $\sigma(F, E) \rightarrow \sigma(l, m_0)$ -continuous. Thus, if K is a $\sigma(F, E)$ -compact subset of F , then $T(K)$ is $\sigma(l, m_0)$ -compact and so by Lemma 1, $T(K)$ is $\|\cdot\|_1$ -compact. By Lemma 2

$$\limsup_{n \rightarrow \infty} \sum_{f \in K} \sum_{k=n}^{\infty} |\langle x_k, f \rangle| = 0;$$

therefore $\lim_{n \rightarrow \infty} \sup_{f \in K} \sup_{m \geq n} |\sum_{k=n}^m \langle x_k, f \rangle| = 0$ and we see that $\{\sum_{k=1}^n x_k\}_{n=1}^{\infty}$ is a $\lambda(E, F)$ -Cauchy sequence. $\sum_{n=1}^{\infty} x_n$, however, converges in $\sigma(E, F)$ to x , say, so that $x = \sum_{n=1}^{\infty} x_n$ in $\lambda(E, F)$ since $\lambda(E, F)$ is a polar topology.

We observe that it is possible to replace $\lambda(E, F)$ by the stronger topology of uniform convergence on the $\sigma(F, E)$ -relatively countably compact subsets of F ; the argument of Theorem 1 is modified only by the observation that a relatively countably compact subset of a K -space is relatively compact [26; Theorem 7]. Noting by Proposition 3 that the mapping T defined above is also $\tau(F, E) \rightarrow \tau(l, m_0)$ continuous, it is easy to see that $\lambda(E, F)$ may also be replaced by the (not necessarily comparable) topology of uniform convergence on the $\tau(F, E)$ -precompact sets.

The next result, which is also useful for "stepping up" the convergence of weakly convergent sequences, was first proved by Mazur for normed spaces [35]. The extension to general locally convex spaces can be found in [59; 34].

PROPOSITION 4. *If $\langle E, F \rangle$ is a separated dual pair of vector spaces and A is a convex subset of E , then A is $\sigma(E, F)$ -closed if and only if A is $\tau(E, F)$ -closed.*

We conclude this section with a few remarks concerning spaces with mixed topologies. Let E be a vector space and let τ_1 and τ_2 be two Hausdorff locally convex topologies on E such that $\tau_1 \geq \tau_2$. For a sequence $\{x_n\}_{n=0}^{\infty}$ of elements of E we write $x_n \rightarrow x_0(\gamma)$ whenever $x_n \rightarrow x_0(\tau_2)$ and $\{x_n; n = 1, 2, \dots\}$ is τ_1 -bounded. This notion was introduced by Fichtenholz in [24] for certain concrete Banach spaces and was generalized by Alexiewicz in [2] to arbitrary Fréchet spaces. The general theory of such spaces was studied extensively in the papers [3], [7], [8], [9] and [61]; the related theory of Saks spaces was developed at roughly the same time in the papers [44], [45], [46], [47] and [72]. Similar ideas in a more general setting were studied in [6], [25], [49], [73] and [74] and applications to summability can be found in [4], [5], [46], [50], [62], [65], [66] and [71]. It was

first shown by Wiweger [72] that in certain cases there is a locally convex topology on E giving the same convergent sequences as γ . Following Persson [49] we shall call the strongest locally convex topology with this property the *mixed topology* and denote it by $\bar{\gamma}$. One of our main results in Section 3 will be to identify an important mixed topology. This identification allows us to use the powerful duality theory of locally convex spaces and to establish rather quickly many of the deep results concerning bounded convergence domains.

3. Structure theorems for FK -spaces containing c_0 . The results of this section are concerned with FK -spaces containing c_0 and we begin by characterizing this class.

PROPOSITION 5. *Let E be an FK -space containing ϕ . Then the following conditions are equivalent.*

- (i) $E \supseteq c_0$.
- (ii) $\sum_{i=1}^{\infty} |f(e'_i)| < \infty$ for each $f \in E'$.

Proof. (i) \Rightarrow (ii). The inclusion mapping $c_0 \rightarrow E$ is continuous by the closed graph theorem (See [76; Theorem 4.5].) so that each $f \in E'$ is continuous on c_0 .

(ii) \Rightarrow (i). It follows from (ii) that the natural injection $\phi \rightarrow E$ is $\sigma(\phi, l) \rightarrow \sigma(E, E')$ -continuous. By Proposition 3 the injection is also $\tau(\phi, l) \rightarrow \tau(E, E')$ -continuous and so can be continuously extended to the completion of $(\phi, \tau(\phi, l))$. Now $\tau(c_0, l)|_{\phi}$, being the restriction of a metrizable topology, is metrizable and so clearly must coincide with $\tau(\phi, l)$; thus the completion of $(\phi, \tau(\phi, l))$ is just $(c_0, \tau(c_0, l))$. It is easy to see that the extension to c_0 is still the natural injection so that $c_0 \subseteq E$.

This result has also been given by Snyder and Wilansky as Corollary 6 to Theorem 1 of [66].

From now on we shall always assume that (E, τ) denotes an FK -space containing c_0 . We define the set $W_E(S_E)$ to be $\{x \in E : x \text{ has SAK (AK) in } E\}$; when the dependence upon E is not in doubt we replace $W_E(S_E)$ by $W(S)$. The next result, which will be used frequently in the sequel, shows that W and S cannot be too small.

PROPOSITION 6. $W \supseteq S \supseteq c_0$.

Proof. That $W \supseteq S$ follows immediately from the definition. To show $S \supseteq c_0$ note that, as in the proof of Proposition 5, the inclusion mapping $c_0 \rightarrow E$ is continuous; each $x \in c_0$ has AK in c_0 and so has AK in E .

If $\{x^{(n)}\}_{n=1}^{\infty}$ is a sequence of points in $E \cap m$, we write $x^{(n)} \rightarrow x(\gamma)$ whenever $x^{(n)} \rightarrow x(\tau)$ and $\sup_n \|x^{(n)}\|_{\infty} < \infty$. We do not yet identify γ as a two-norm convergence or as a mixed topology in the sense of [3] or [49].

LEMMA 3. *If $x^{(n)} \rightarrow x(\gamma)$, then $x^{(n)} \rightarrow x(\sigma(m, l))$.*

Proof. The set $B_r = \{y \in m : \|y\|_{\infty} \leq r\}$ is $\sigma(m, l)$ -compact since $\sigma(m, l)$ is the weak $*$ topology on m considered as the dual of l . Therefore $\sigma(m, l)$ coincides

with the weaker Hausdorff topology $\sigma(m, \phi)$ on B_r . If $x^{(n)} \rightarrow x(\gamma)$, then $x^{(n)} \rightarrow x(\sigma(m, \phi))$ and $x^{(n)} \in B_r$ for some r so that $x^{(n)} \rightarrow x(\sigma(m, l))$.

The next result is a refinement of a theorem on conull FK -spaces due to Snyder [65; Theorem 6]; the proof given here is quicker than that of [65].

THEOREM 2. *The following conditions are equivalent for each $x \in E$.*

- (i) $x \in W \cap m$.
- (ii) There is a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of points of ϕ with $\|x^{(n)}\|_{\infty} \leq \|x\|_{\infty}$ and $x^{(n)} \rightarrow x(\tau)$.
- (iii) There is a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of points of c_0 with $x^{(n)} \rightarrow x(\gamma)$.

Proof. (i) \Rightarrow (ii). If $x \in W \cap m$, then x belongs to the weak closure of the convex hull C of the set $\{P_n(x) : n = 1, 2, \dots\}$. Since τ is metrizable, it follows from Proposition 4 that there is a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of points of C with $x^{(n)} \rightarrow x(\tau)$; it is clear that $\|x^{(n)}\|_{\infty} \leq \|x\|_{\infty}$ for each n .

(ii) \Rightarrow (iii). This is immediate from the definition of γ -convergence.

(iii) \Rightarrow (i). If (iii) holds, then $f(x) = \lim_{n \rightarrow \infty} f(x^{(n)})$ for each $f \in E'$. By Proposition 6, $f(x^{(n)}) = \sum_{j=1}^{\infty} x_j^{(n)} f(e^j)$ and by Lemma 3 and Proposition 5, $\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} x_j^{(n)} f(e^j) = \sum_{j=1}^{\infty} x_j f(e^j)$. It follows that $f(x) = \sum_{j=1}^{\infty} x_j f(e^j)$ for each $f \in E'$ so that $x \in W$; (iii) obviously implies $x \in m$ so that (i) holds.

We remark that the assumption that $E \supseteq c_0$ is not necessary for the proof of (i) \Rightarrow (ii); it is however possible to give a counterexample to the implication (iii) \Rightarrow (i) without this assumption. To see this let $E = \{x \in \omega : \sup_j |x_{2j} - x_{2j-1}| < \infty\}$; then E is an FK -space when topologized by means of the seminorms $x \rightarrow |x_j|$, $j = 1, 2, \dots$, and $\|x\| = \sup_j |x_{2j} - x_{2j-1}|$. It is clear that $P_{2j}(e) \rightarrow e$ so that $P_{2j}(e) \rightarrow e(\gamma)$, but $\sup_j \|P_{2j}(e)\| = \infty$ so that $e \notin W \cap m$.

We are now able to identify γ as a two-norm convergence on $W \cap m$; we recall that γ will be such provided that τ is weaker than the $\|\cdot\|_{\infty}$ -topology on $W \cap m$ and the set $W \cap B_1 = \{x \in W \cap m : \|x\|_{\infty} \leq 1\}$ is closed in $(W \cap m, \tau)$.

PROPOSITION 7. *τ is weaker than the $\|\cdot\|_{\infty}$ -topology on $W \cap m$ and γ is a two-norm convergence.*

Proof. Let p be a τ -continuous seminorm on E ; then by the continuity of the injection $c_0 \rightarrow E$ it follows that

$$p(x) \leq M \|x\|_{\infty} \text{ for each } x \in c_0,$$

where M is a constant independent of x . If $x \in W \cap m$, then by Theorem 2 there is a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of points of ϕ with $x^{(n)} \rightarrow (\tau)$ and $\|x^{(n)}\|_{\infty} \leq \|x\|_{\infty}$. Thus

$$p(x) = \lim_{n \rightarrow \infty} p(x^{(n)}) \leq M \sup_n \|x^{(n)}\|_{\infty} \leq M \|x\|_{\infty}$$

so that p is $\|\cdot\|_{\infty}$ -continuous on $W \cap m$.

Since E is a K -space, the set $B_1 \cap W$ is τ -closed in $W \cap m$ so that γ is a two-norm convergence on $W \cap m$.

We shall not need any properties of mixed topologies in order to establish the main results of this section; γ -convergence, however, will later be identified with the sequential convergence of a particular topology on $W \cap m$.

For the remainder of this section we shall be considering the dual pair $\langle l, W \cap m \rangle$.

PROPOSITION 8. *The strong topology $\beta(W \cap m, l)$ is the $\|\cdot\|_\infty$ -topology.*

Proof. $\sigma(l, c_0)$ and $\sigma(l, m)$ define the same bounded subsets of l as the $\|\cdot\|_1$ -topology. Thus by Proposition 6, $\sigma(l, W \cap m)$ defines the same bounded sets as $\|\cdot\|_1$ so that $\beta(W \cap m, l)$ is just the $\|\cdot\|_\infty$ -topology on $W \cap m$.

LEMMA 4. *If $\{x^{(n)}\}_{n=1}^\infty$ is a sequence of points of ϕ such that $x^{(n)} \rightarrow x(\gamma)$, then $x^{(n)} \rightarrow x(\lambda(W \cap m, l))$.*

Proof. If the hypotheses of the lemma are satisfied, then $x \in W \cap m$ by Theorem 2. Thus, if the result is false, there is a sequence $\{x^{(n)}\}_{n=1}^\infty$ of elements of ϕ with $x^{(n)} \rightarrow x(\gamma)$ but such that for some $\lambda(W \cap m, l)$ -neighborhood U of x , $x^{(n)} \notin U$, $n = 1, 2, \dots$. In this case no subsequence of $\{x^{(n)}\}_{n=1}^\infty$ converges to x in $\lambda(W \cap m, l)$. Let $\sup_n \|x^{(n)}\|_\infty = M$.

For each j , $x_j^{(n)} \rightarrow x_j$ and so there is an increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers such that

$$|x_j - x_j^{(n_k)}| \leq \frac{1}{k}, \quad 1 \leq j \leq k, \quad k = 1, 2, \dots$$

Define the sequence $\{z^{(k)}\}_{k=1}^\infty$ of elements of ϕ as follows.

$$z^{(0)} = 0$$

and

$$z_i^{(k)} = \begin{cases} x_i & \text{if } 1 \leq j \leq k \\ x_j^{(n_k)} & \text{if } j > k \end{cases} \quad k = 1, 2, \dots$$

Then

$$\|z^{(k)} - x^{(n_k)}\|_\infty \leq \frac{1}{k}, \quad k = 1, 2, \dots,$$

so that

$$z^{(k)} - x^{(n_k)} \rightarrow 0(\beta(W \cap m, l))$$

by Proposition 8. Since $\beta(W \cap m, l) \geq \lambda(W \cap m, l)$, it follows that no subsequence of $\{z^{(k)}\}_{k=1}^\infty$ converges to x in $\lambda(W \cap m, l)$. By Proposition 7

$$z^{(k)} - x^{(n_k)} \rightarrow 0(\tau)$$

so that

$$z^{(k)} \rightarrow x(\tau);$$

since $\sup_k \|z^{(k)}\|_\infty \leq M$, it follows that

$$z^{(k)} \rightarrow x(\gamma).$$

Now suppose that $\{p_n\}_{n=1}^\infty$ is an increasing sequence of seminorms defining the topology τ on E . We construct inductively an increasing sequence $\{m_k\}_{k=0}^\infty$ of integers as follows. Let $m_0 = 0$ and suppose that m_1, \dots, m_k have been determined; choose m_{k+1} so that the following hold.

$$(i) \quad z_i^{(m_k)} = 0 \text{ for } i \geq m_{k+1}.$$

$$(ii) \quad p_{k+1}(z^{(r)} - z^{(s)}) \leq \frac{1}{2^k} \text{ if } r \geq s \geq m_{k+1}.$$

Now let $u^{(k)} = z^{(m_k)} - z^{(m_{k-1})}$ for $k = 1, 2, \dots$; then

$$\sum_{k=1}^\infty p_n(u^{(k)}) < \infty, \quad n = 1, 2, \dots$$

Since (E, τ) is complete, it follows that for any subsequence $\{v^{(k)}\}_{k=1}^\infty$ of $\{u^{(k)}\}_{k=1}^\infty$, $\sum_{k=1}^\infty v^{(k)}$ converges in (E, τ) ; furthermore, if $m_i \leq i < m_{i+1}$,

$$\begin{aligned} \left| \sum_{k=1}^n v_i^{(k)} \right| &\leq \sum_{k=1}^\infty |u_i^{(k)}| = \sum_{k=1}^\infty |z_i^{(m_k)} - z_i^{(m_{k-1})}| \\ &= |z_i^{(m_j)}| + |z_i^{(m_{j+1})} - z_i^{(m_j)}| \leq 3M. \end{aligned}$$

Therefore $\sum_{k=1}^\infty v^{(k)}$ converges in γ and by Theorem 2, $\sum_{k=1}^\infty v^{(k)} \in W \cap m$. Moreover, by Lemma 3, $\sum_{k=1}^\infty v^{(k)}$ converges in $\sigma(W \cap m, l)$ for every subseries $\{v^{(k)}\}$ of $\{u^{(k)}\}$; and so by Theorem 1, $\sum_{k=1}^\infty u^{(k)}$ converges in $\lambda(W \cap m, l)$. Again by Lemma 3, $\sum_{i=1}^k u^{(i)} = z^{(m_k)} \rightarrow x$ in $\sigma(W \cap m, l)$ and it follows that $z^{(m_k)} \rightarrow x(\lambda(W \cap m, l))$, which contradicts the construction of $z^{(k)}$.

LEMMA 5. *Let Ψ be a linear functional on $W \cap m$ such that $\Psi(x^{(n)}) \rightarrow \Psi(x)$ whenever $\{x^{(n)}\}_{n=1}^\infty$ is a sequence of elements of ϕ with $x^{(n)} \rightarrow x(\gamma)$. Then there exists a sequence $f = \{f_i\}_{i=1}^\infty \in l$ such that*

$$\Psi(x) = \sum_{i=1}^\infty f_i x_i \quad \text{for each } x \in W \cap m.$$

Proof. $P_n(x) \rightarrow x(\gamma)$ for each $x \in c_0$ so that $\sum_{i=1}^\infty |f_i| < \infty$ where $f_i = \Psi(e^i)$. If $x \in W \cap m$, then by Theorem 2 there exists a sequence $\{x^{(n)}\}_{n=1}^\infty$ of points of ϕ with $x^{(n)} \rightarrow x(\gamma)$. By Lemma 3

$$\Psi(x) = \lim_{n \rightarrow \infty} \Psi(x^{(n)}) = \lim_{n \rightarrow \infty} \sum_{i=1}^\infty f_i x_i^{(n)} = \sum_{i=1}^\infty f_i x_i.$$

THEOREM 3. *l is $\sigma(l, W \cap m)$ -sequentially complete.*

Proof. Let ρ denote the topology on l of uniform convergence on the $\lambda(W \cap m, l)$ -precompact sets; we show first that (l, ρ) is complete. To do this let Ψ be a linear functional on $W \cap m$ whose restrictions to $\lambda(W \cap m, l)$ -precompact sets

are $\lambda(W \cap m, l)$ -continuous. By Lemma 4, $\Psi(x^{(n)}) \rightarrow \Psi(x)$ whenever $\{x^{(n)}\}_{n=1}^{\infty}$ is a sequence of points of ϕ with $x^{(n)} \rightarrow x(\gamma)$ and so by Lemma 5, $\Psi(x) = \sum_{i=1}^{\infty} f_i x_i$ for some $f \in l$. It follows from Grothendieck's completeness theorem (Proposition 1) that l is ρ -complete.

Now let K be a $\sigma(l, W \cap m)$ -compact subset of l ; by Grothendieck's precompactness theorem (Proposition 2) K is ρ -precompact. It follows from the first part that K is ρ -relatively compact so that ρ and $\sigma(l, W \cap m)$ coincide on the ρ -closure of K . Thus ρ and $\sigma(l, W \cap m)$ have the same convergent sequences and by [68; Proposition 1.4] they also have the same Cauchy sequences. It follows that $(l, \sigma(l, W \cap m))$ is sequentially complete.

THEOREM 4. l is $\sigma(l, S \cap m)$ -sequentially complete.

Proof. Following Garling [26; 1015–1016] we consider the space $E_p = \{x \in E: \{P_n(x)\}_{n=1}^{\infty} \text{ is bounded in } E\}$. E_p can be topologized so that it becomes an FK-space and we have $W_{E_p} = S_{E_p} = S_E$. The result now follows readily from Theorem 3.

We now fulfill a promise made earlier by identifying the mixed topology $\bar{\gamma}$ as a Mackey topology.

THEOREM 5. The following topologies are identical on $W \cap m$.

- (i) $\bar{\gamma}$
- (ii) $\tau(W \cap m, l)$
- (iii) $\lambda(W \cap m, l)$

Proof. We first observe that $\tau(W \cap m, l) \leq \lambda(W \cap m, l)$.

If Ψ is a $\bar{\gamma}$ -continuous linear functional on $W \cap m$, then by Lemma 5, $\Psi(x) = \sum_{i=1}^{\infty} f_i x_i$ for some $f \in l$ so that $\bar{\gamma} \leq \tau(W \cap m, l)$. We complete the proof by showing that $\bar{\gamma}$ has more convergent sequences than $\lambda(W \cap m, l)$ from which it follows that $\lambda(W \cap m, l) \leq \bar{\gamma}$.

To do this suppose there exists a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ of points of $W \cap m$ such that $x^{(n)} \rightarrow x(\gamma)$ while there is an absolutely convex $\lambda(W \cap m, l)$ -neighborhood U of zero with $x^{(n)} \notin x + U$, $n = 1, 2, \dots$. By Theorem 2 there exist sequences $\{x^{(n,m)}\}_{m=1}^{\infty}$ with $x^{(n,m)} \in \phi$ such that

$$\sup_m \|x^{(n,m)}\|_{\infty} \leq \|x^{(n)}\|_{\infty}$$

and

$$\lim_{m \rightarrow \infty} x^{(n,m)} = x^{(n)}(\gamma).$$

Then by Lemma 4, $\lim_{m \rightarrow \infty} x^{(n,m)} = x^{(n)}(\lambda(W \cap m, l))$ for each n .

Let d be any metric on E giving the topology τ ; then there exists an increasing sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers such that for each n the following hold.

$$(i) \quad x^{(n)} - x^{(n,m_n)} \in \frac{1}{2}U$$

$$(ii) \quad d(x^{(n)}, x^{(n,m_n)}) \leq \frac{1}{n}$$

Now $d(x, x^{(n,m)}) \rightarrow 0$ since $d(x, x^{(n)}) \rightarrow 0$ and also $\sup_n \|x^{(n,m)}\|_\infty \leq \sup_{n,m} \|x^{(n,m)}\|_\infty \leq \sup_n \|x^{(n)}\|_\infty < \infty$ so that $\lim_{n \rightarrow \infty} x^{(n,m)} = x(\gamma)$. Since $x^{(n,m)} \in \phi$, it follows from Lemma 4 that $\lim_{n \rightarrow \infty} x^{(n,m)} = x(\lambda(W \cap m, l))$. In particular, there exists an integer k such that

$$x - x^{(k,m)} \in \frac{1}{2}U.$$

Then $x^{(k)} \in x + U$ and the initial assumption is contradicted.

The next result will be useful when we come to consider uniform summability in Section 4.

COROLLARY.

- (a) A subset K of $W \cap m$ is $\tau(W \cap m, l)$ -relatively compact if and only if it is τ -relatively compact and $\|\cdot\|_\infty$ -bounded.
- (b) A subset K of $W \cap m$ is $\sigma(W \cap m, l)$ -relatively compact if and only if it is $\sigma(E, E')$ -relatively compact and $\|\cdot\|_\infty$ -bounded.

Proof. (a). This follows from a familiar property of mixed topologies [49] which says that $\tau(W \cap m, l)$ and τ must coincide on the $\|\cdot\|_\infty$ -bounded subsets of $W \cap m$.

(b). The proof follows as in (a) using [49; Corollary 2.1].

We close this section by giving two more results concerning the topology $\tau(W \cap m, l)$.

PROPOSITION 9. $(W \cap m, \tau(W \cap m, l))$ is complete.

Proof. If $\Psi \in E'$, then the restriction of Ψ to $W \cap m$ is clearly $\tilde{\gamma}$ -continuous and so by Theorem 5 the inclusion mapping $W \cap m \rightarrow E$ is $\sigma(W \cap m, l) \rightarrow \sigma(E, E')$ -continuous. By Proposition 3 the inclusion is also $\tau(W \cap m, l) \rightarrow \tau$ continuous.

Let $\{x^{(\alpha)}\}$ be a $\tau(W \cap m, l)$ -Cauchy net in $W \cap m$; then $\{x^{(\alpha)}\}$ is τ -Cauchy in E so that $x = \lim_\alpha x^{(\alpha)}$ exists in (E, τ) . If $f \in l$, then the set $\{P_n(f) : n = 1, 2, \dots\}$ is $\sigma(l, W \cap m)$ -relatively compact, i.e., $\tau(W \cap m, l)$ -equicontinuous by Theorem 5. Therefore

$$\lim_\alpha \lim_n \sum_{i=1}^n f_i x_i^{(\alpha)} = \lim_n \lim_\alpha \sum_{i=1}^n f_i x_i^{(\alpha)},$$

that is,

$$(*) \quad \lim_\alpha \sum_{i=1}^\infty f_i x_i^{(\alpha)} = \sum_{i=1}^\infty f_i x_i.$$

Thus $\sum_{i=1}^\infty f_i x_i$ converges for each $f \in l$ so that $x \in m$. If $\Psi \in E'$, then from the remarks made above $\{\Psi(e^{(i)})\}_{i=1}^\infty \in l$ and

$$\lim_\alpha \sum_{i=1}^\infty \Psi(e^{(i)}) x_i^{(\alpha)} = \lim_\alpha \Psi(x^{(\alpha)}) = \Psi(x)$$

so that $\Psi(x) = \sum_{i=1}^\infty \Psi(e^{(i)}) x_i$ and $x \in W$. By (*), $x^{(\alpha)} \rightarrow x(\sigma(W \cap m, l))$ and it follows that $x^{(\alpha)} \rightarrow x(\tau(W \cap m, l))$.

PROPOSITION 10. *The following conditions are equivalent.*

- (i) c_0 is closed in (E, τ) .
- (ii) $W = c_0$.
- (iii) $W \cap m = c_0$.
- (iv) $\tau(W \cap m, l)$ is metrizable.
- (v) $\tau(W \cap m, l)$ is bornological.
- (vi) $\tau(W \cap m, l)$ is quasi-barrelled.
- (vii) $\tau(W \cap m, l)$ is barrelled.
- (viii) l is $\sigma(l, W \cap m)$ -quasi-complete.

Proof. (i) \Rightarrow (ii). Clearly $W \subseteq \bar{\phi}$, the closure of ϕ in (E, τ) . If (i) holds, then $\bar{\phi} \subseteq c_0$ so that $W \subseteq c_0$; (ii) now follows from Proposition 6.

(ii) \Rightarrow (iii). This is immediate.

(iii) \Rightarrow (iv). If $W \cap m = c_0$, then $\tau(W \cap m, l) = \tau(c_0, l)$ which is just the $\|\cdot\|_\infty$ -topology on c_0 .

(iv) \Rightarrow (v) \Rightarrow (vi). These implications are true in any locally convex space [33].

(vi) \Rightarrow (vii). This follows from Proposition 9 since any complete quasi-barrelled space must be barrelled. (See [33; 368].)

(vii) \Leftrightarrow (i). If $\tau(W \cap m, l)$ is barrelled, then $\tau(W \cap m, l) = \beta(W \cap m, l)$ which is the $\|\cdot\|_\infty$ -topology by Proposition 8. Thus by Theorem 5 the mixed topology $\bar{\gamma}$ defines the same convergent sequences on c_0 as the $\|\cdot\|_\infty$ -topology. If τ is strictly weaker than $\|\cdot\|_\infty$ on c_0 , then there is a sequence $\{x^{(n)}\}_{n=1}^\infty$ of points of c_0 with $\|x^{(n)}\|_\infty = 1$ and $x^{(n)} \rightarrow 0(\tau)$, i.e., $x^{(n)} \rightarrow 0(\gamma)$ but $\|x^{(n)}\|_\infty \not\rightarrow 0$. It follows that τ is equivalent to the $\|\cdot\|_\infty$ -topology on c_0 so that c_0 is closed in (E, τ) .

(vii) \Leftrightarrow (viii). This is a special case of [33; §23.6 (4)].

Finally, we remark that it follows at once from Proposition 10 that l is never $\sigma(l, W \cap m)$ -complete.

4. Applications. If E and F are arbitrary FK -spaces, it is easy to see that $E + F$ is also an FK -space when given the direct sum topology. The following elementary observation enables us to obtain a result of Meyer-König and Zeller.

LEMMA 6. *Let E and F be arbitrary FK -spaces. Then $E \cap F$ is closed in F whenever E is closed in $E + F$.*

Proof. By the closed graph theorem the inclusion $F \rightarrow E + F$ is continuous so that if $\overline{E \cap F}^F$ denotes the closure of $E \cap F$ in F , we have $\overline{E \cap F}^F \subseteq \overline{E \cap F^{E+F}} \cap F \subseteq E \cap F$.

The next result was used by Meyer-König and Zeller [39], [40] to establish "high indices theorems" in summability. (See also [14].)

THEOREM 6. *Let E be an arbitrary FK -space with $c_0 \cap E$ not closed in E . Then E contains a bounded divergent sequence x .*

Proof. Since $c_0 \cap E$ is not closed in E , it follows from Lemma 6 that c_0 is not closed in $c_0 + E$. By Proposition 10, c_0 is a proper subset of $W_{c_0+E} \cap m$ from which it follows that E contains a bounded divergent sequence.

We note that if E contains c_0 , then x can be chosen in W_E .

Our next application concerns Schauder bases. An example of a locally convex space with a weak Schauder basis which is not a Schauder basis has been given by Dubinsky and Retherford [22]. Their space, however, is of countable dimension and possesses a weak Schauder basis which is not a Hamel basis; it follows from the results of [31] that their space is not sequentially complete. We give here an example of a complete Mackey space with a weak Schauder basis which is not a Schauder basis.

Example. Let E be the BK -space of all sequences x such that $\sup_i |x_i - x_{i+1}| < \infty$ under the norm $\|x\| = |x_1| + \sup_i |x_i - x_{i+1}|$. If $\{x^{(n)}\}_{n=1}^\infty$ is defined as

$$x_i^{(n)} = \begin{cases} 1 - j/n & \text{if } 1 \leq j \leq n \\ 0 & \text{if } j > n, \end{cases}$$

then clearly $\|x^{(n)}\|_\infty \leq 1$, $\|e - x^{(n)}\| \leq 1/n$ and $x^{(n)} \notin \phi$. It follows from Theorem 2 that $e \in W \cap m$. Suppose that $P_n(e) \rightarrow e(\tau(W \cap m, l))$; then by Theorem 5, $P_n(e) \rightarrow e(\gamma)$ so that $\|e - P_n(e)\| \rightarrow 0$. However $\|e - P_n(e)\| = 1$ for every n so that $P_n(e) \not\rightarrow e(\tau(W \cap m, l))$. Therefore $\{e^i\}_{i=1}^\infty$ is not a $\tau(W \cap m, l)$ -basis for $W \cap m$; Proposition 9 shows that $(W \cap m, \tau(W \cap m, l))$ is complete and $\{e^i\}_{i=1}^\infty$ is clearly a $\sigma(W \cap m, l)$ -Schauder basis.

We now turn our attention to summability theory. We shall be concerned with matrix transformations $y = Ax$, where $x = \{x_i\}_{i=1}^\infty$ and $y = \{y_i\}_{i=1}^\infty$ are complex-valued sequences, $A = \{a_{ij}\}_{i,j=1}^\infty$ is an infinite matrix with complex coefficients, and

$$y_i = \sum_{j=1}^{\infty} a_{ij}x_j, \quad i = 1, 2, \dots$$

Of particular interest will be the spaces $c_A = \{x \in \omega : Ax \text{ exists and } Ax \in c\}$, the convergence domain of A , and $(c_0)_A = \{x \in \omega : Ax \text{ exists and } Ax \in c_0\}$, the null convergence domain of A . Zeller [76] has shown that both these spaces may be topologized so that they become FK -spaces, and unless otherwise specified it will always be assumed that they carry their unique FK -topologies. W (or W_A if the dependence upon A is in doubt) will denote the set of all elements that have SAK in c_A . It will always be assumed that $c_0 \subseteq c_A$; in particular this means that

$$\lim_{i \rightarrow \infty} a_{ij} = a_j \quad \text{exists,} \quad j = 1, 2, \dots$$

It is known [77; Theorem 5.6] that for $x \in m$, $x \in W$ if and only if

$$\lim_A x = \lim_{i \rightarrow \infty} \sum_{j=1}^{\infty} a_{ij}x_j = \sum_{j=1}^{\infty} a_jx_j.$$

THEOREM 7. Let A be a matrix mapping c_0 into c and let B be such that $W_A \cap m \subseteq c_B$. Then the following statements are valid.

- (i) \lim_B is $\sigma(W_A \cap m, l)$ -continuous on $W_A \cap m$.
- (ii) $\lim_B x = \sum_{i=1}^{\infty} b_i x_i$ for each $x \in W_A \cap m$.
- (iii) $W_A \cap m \subseteq W_B$.
- (iv) The inclusion mapping $W_A \cap m \rightarrow c_B$ is $\sigma(W_A \cap m, l) \rightarrow \sigma(c_B, c'_B)$ -continuous.
- (v) The inclusion mapping $W_A \cap m \rightarrow c_B$ is $\tau(W_A \cap m, l) \rightarrow \tau(c_B, c'_B)$ -continuous.

Proof. (i). Let $\rho^{(i)}$ denote the i -th row of the matrix B . By Proposition 6 the hypotheses of the theorem imply that $c_0 \subseteq c_B$ so that $\rho^{(i)} \in l$, $i = 1, 2, \dots$. Since $W_A \cap m \subseteq c_B$, it follows that $\{\rho^{(i)}\}_{i=1}^{\infty}$ is a $\sigma(l, W_A \cap m)$ -Cauchy sequence and so by Theorem 3 there exists $\rho \in l$ such that $\rho^{(i)} \rightarrow \rho$ in $\sigma(l, W_A \cap m)$. Thus $\lim_B x = \lim_{i \rightarrow \infty} \langle \rho^{(i)}, x \rangle = \langle \rho, x \rangle$ and (i) is established.

(ii). As in (i) b_k , the k -th column limit of B , equals $\lim_B e^{(k)} = \sum_{i=1}^{\infty} \rho_i e_i^{(k)} = \rho_k$, $k = 1, 2, \dots$, so that (ii) follows.

(iii). This follows immediately from (ii) and the remarks concerning W made just before the statement of the theorem.

(iv). Zeller [76; Theorem 5.2] has shown that each $f \in c_B$ has a representation of the form

$$f(x) = \alpha \lim_B x + \sum_{i=1}^{\infty} t_i (Bx)_i + \sum_{i=1}^{\infty} s_i x_i$$

for suitable $s, t \in l$ and scalar α . If $x \in m$, it is easy to check that $t(Bx) = (tB)x$ and that $tB \in l$. It follows from (i) that f is $\sigma(W_A \cap m, l)$ -continuous on $W_A \cap m$ and (iv) is established.

(v). This follows at once from (iv) by using Proposition 3.

Following Wilansky [69] and Snyder [65], we call a matrix *conull* if $e \in W$ and *coregular* if $e \notin W$. It is known [65], [77] that if A is conull and $c_A \subseteq c_B$, then B is also conull. However, the following deeper result [19], [70] is available though no functional analytic proof has yet been given in the literature.

THEOREM 8. Let A be a conull matrix and let B be any matrix with $c_A \cap m \subseteq c_B$; then B is conull.

Proof. This follows at once from Theorem 7 (iii) and the definition of conullity given above.

Theorem 8 is carried over to *FK*-spaces in the next section (Theorem 17).

THEOREM 9. Let A and B be matrices which map c_0 into c_0 . If $(c_0)_A \cap m \subseteq c_B$, then $(c_0)_A \cap m \subseteq (c_0)_B$.

Proof. The hypotheses imply that $a_j = b_j = 0$ for all j ; so by the remarks made just before Theorem 7 we have $(c_0)_A \cap m = W_A \cap m$ and $(c_0)_B \cap m = W_B \cap m$. By Theorem 7 (iii), $W_A \cap m \subseteq W_B$ and the result follows at once.

For a subset X of ω and matrices A and B we say that A and B are *consistent* on X if $X \subseteq c_A \cap c_B$ and $\lim_A x = \lim_B x$ for each $x \in X$. If A is consistent with the identity matrix on c , we say that A is *regular*.

THEOREM 10. *Let A and B be coregular matrices with $c_A \cap m \subseteq c_B$. If A and B are consistent on c , they are consistent on $c_A \cap m$.*

Proof. This follows at once from Theorem 9 by using the fact that for coregular matrices we have $c_A \cap m = (W_A \cap m) \oplus \text{lin span } \{e\}$.

We note here that Theorem 10 is false for conull matrices. (See [18; 212] for a counterexample.)

The next result is the celebrated bounded consistency theorem first stated by Mazur and Orlicz in [36] and first proved by Brudno in [16]. Subsequent proofs can be found in the papers [11], [23], [37] and [46] with generalizations to double sequences, vector-valued methods, continuous methods, etc. in [4], [5], [20], [50], [51], [63] and [75].

COROLLARY. *Let A and B be regular matrices with $c_A \cap m \subseteq c_B$. Then A and B are consistent on $c_A \cap m$.*

The next result is established in [12; Theorems 2 and 3].

THEOREM 11. *Let A be any matrix (not necessarily mapping c_0 into c) and let K be a bounded subset of c_A . Then K is (weakly) relatively compact if and only if the following two conditions are satisfied.*

- (i) $\sum_{i=n}^{\infty} a_{i,j} x_i \rightarrow 0$ as $n \rightarrow \infty$ (quasi-) uniformly on K , $j = 1, 2, \dots$.
- (ii) $\sum_{i=1}^{\infty} a_{i,j} x_i \rightarrow \lim_A x$ as $j \rightarrow \infty$ (quasi-) uniformly on K .

THEOREM 12. *Let A be any matrix mapping c_0 into c and let B be such that $W_A \cap m \subseteq c_B$. If K is a $\|\cdot\|_{\infty}$ -bounded subset of $W_A \cap m$ with*

$$\sum_{i=1}^{\infty} a_{i,j} x_i \rightarrow \lim_A x \quad \text{as } j \rightarrow \infty \quad \text{(quasi-) uniformly on } K,$$

then

$$\sum_{i=1}^{\infty} b_{i,j} x_i \rightarrow \lim_B x \quad \text{as } j \rightarrow \infty \quad \text{(quasi-) uniformly on } K.$$

Proof. The uniform case. Since A maps c_0 into c and K is $\|\cdot\|_{\infty}$ -bounded, it follows that condition (i) of Theorem 11 is satisfied. By Proposition 7, K is bounded in c_A and so by Theorem 11 must be relatively compact there. By part (a) of the corollary to Theorem 5, K is $\tau(W_A \cap m, l)$ -relatively compact and Theorem 7 (v) shows that K is relatively compact in c_B ; the assertion now follows by again appealing to Theorem 11.

The quasi-uniform case. The proof follows similarly.

The uniform version of the next result has been given by Petersen [52], [53], [54].

COROLLARY. Let A and B be regular matrices such that $c_A \cap m \subseteq c_B$. If K is a $\|\cdot\|_\infty$ -bounded subset with $\lim_A x = 0$ (quasi-) uniformly on K , then $\lim_B x = 0$ (quasi-) uniformly on K .

5. A closed graph theorem and some further applications. Throughout this section we use the following closed graph theorem which is established in [32].

THEOREM 13. Let E be a Mackey space such that $(E', \sigma(E', E))$ is sequentially complete and let F be a separable Fréchet space. Then any linear mapping $E \rightarrow F$ with closed graph must be continuous.

The class of type 0 FK -spaces was introduced by Chang in [17]. An FK -space containing c_0 (we extend his definition slightly) is said to be of type 0 if $E \cap m = W_E \cap m$. We now give a characterization of all separable type 0 FK -spaces.

THEOREM 14. Let E be a separable FK -space containing c_0 . Then E is of type 0 if and only if l is $\sigma(l, E \cap m)$ -sequentially complete.

Proof. If E is of type 0, then l is $\sigma(l, E \cap m)$ -sequentially complete by Theorem 3.

On the other hand, suppose that l is $\sigma(l, E \cap m)$ -sequentially complete. The inclusion mapping $(E \cap m, \tau(E \cap m, l)) \rightarrow E$ clearly has closed graph and so must be continuous by Theorem 13. By Proposition 3 the inclusion mapping is $\sigma(E \cap m, l) \rightarrow \sigma(E, E')$ -continuous, from which fact the theorem follows.

Following [13] we say that a K -space (E, τ) containing ϕ is a *wedge space* provided that $e^j \rightarrow 0(\tau)$ as $j \rightarrow \infty$.

THEOREM 15. Let E be a wedge FK -space containing c_0 and let F be a separable FK -space such that $E \cap m \subseteq F$; then F is a wedge space.

Proof. $e^j \rightarrow 0(\tau)$ and $\sup_j \|e^j\|_\infty < \infty$ so that $(W \cap m, \tau(W \cap m, l))$ is a wedge space by Theorem 5. The assertion now follows since the inclusion $(W \cap m, \tau(W \cap m, l)) \rightarrow F$ is continuous by Theorems 3 and 13.

It is known that the convergence domain of any matrix has to be separable [37; Theorem 1.4.1], [12; Corollary 1 to Theorem 5]; our next result extends Theorem 7 (iii) to arbitrary separable FK -spaces.

THEOREM 16. Let E be an FK -space containing c_0 and let F be a separable FK -space with $W_E \cap m \subseteq F$; then $W_E \cap m \subseteq W_F$.

Proof. As in the proof of Theorem 14, the inclusion mapping $W_E \cap m \rightarrow F$ is $\sigma(W_E \cap m, l) \rightarrow \sigma(F, F')$ -continuous and the result follows at once.

In a similar fashion Theorem 8 can be carried over to separable FK -spaces. Following Snyder [65] we call an FK -space E containing c *conull* provided that $e \in W$.

THEOREM 17. Let E be a conull FK -space and let F be a separable FK -space with $E \cap m \subseteq F$; then F is conull.

THEOREM 18. *Let E be an FK AK-space containing c_0 and let F be a separable FK-space with $E \cap m \subseteq F$; then $E \cap m \subseteq S_F$.*

Proof. If $x \in E \cap m$, then $P_n(x) \rightarrow x(\tau)$ and $\sup_n \|P_n(x)\|_\infty < \infty$ so that $(E \cap m, \tau(E \cap m, l))$ is an AK-space by Theorem 5. The assertion now follows from Theorems 4 and 13.

THEOREM 19. *Let E be a separable FK-space containing m ; then $m \subseteq S_E$ and E is a conull wedge space.*

Proof. This follows at once by putting $E = \omega$ in Theorem 18.

We next turn our attention to some theorems in summability.

THEOREM 20. *Let E be an FK-space containing c_0 and let A be a matrix mapping c_0 into $c(c_0)$. If $A(W_E \cap m)$ is separable in m , then $A(W_E \cap m) \subseteq c(c_0)$.*

Proof. Let F denote the $\|\cdot\|_\infty$ -closure of $A(W_E \cap m)$ in m ; then F is a separable BK-space. Since A maps c_0 into c , each of its rows belongs to l and it follows that $A: (W_E \cap m, \tau(W_E \cap m, l)) \rightarrow (F, \|\cdot\|_\infty)$ has closed graph; by Theorems 3 and 13, A is continuous. $\{e^j: j = 1, 2, \dots\}$ is fundamental in $(W_E \cap m, \tau(W_E \cap m, l))$ by Proposition 4 so that $A(W_E \cap m) \subseteq \text{lin span } \{A(e^j)\} \subseteq c(c_0)$.

THEOREM 21. *Let E be an FK-space containing c_0 and let A be a matrix mapping c_0 into $c(c_0)$. Then either $W_E \cap m \subseteq c_A((c_0)_A)$ or $W_E \cap m$ is nonseparable in m_A .*

Proof. If $W_E \cap m$ is separable in m_A , then $A(W_E \cap m)$, being the continuous image [76; Theorem 4.4] of a separable space, is separable in m and the result follows from Theorem 20.

The next result has been obtained by Agnew [1] and Zeller [77] in the case when E is a convergence domain. (See also [14].)

THEOREM 22. *Let E be an FK-space containing c_0 ; then either $W \cap m = c_0$ or $W \cap m$ is nonseparable in m .*

Proof. This follows at once from Proposition 6 by taking A to be the identity matrix in Theorem 20.

The next result was originally obtained by Orlicz [46] by using inductive limits of Saks spaces.

THEOREM 23. *Let $A^n, n = 1, 2, \dots$, be matrices mapping c_0 into c_0 and let A map c_0 into $c(c_0)$; then $A(\bigcap_{n=1}^{\infty} (c_0)_{A^n} \cap m)$ is a subset of $c(c_0)$ or is nonseparable in m .*

Proof. Noting that $W_{A^n} \cap m = (c_0)_{A^n} \cap m, n = 1, 2, \dots$, the result follows by putting $E = \bigcap_{n=1}^{\infty} (c_0)_{A^n} \cap m$ in Theorem 20. (E is an FK-space by [76; Theorem 4.7].)

COROLLARY. *Let $A^n, n = 1, 2, \dots$, be as in Theorem 23; then $\bigcap_{n=1}^{\infty} (c_0)_{A^n} \cap m$ either is a subset of c_0 or is nonseparable in m .*

Proof. Take A to be the identity matrix in Theorem 23.

Finally, we close this section with a few results concerning FK -spaces containing m . We begin by noting that $(m, \tau(m, l))$ is complete and semi-Montel, i.e., every $\sigma(m, l)$ -bounded subset of m is a $\tau(m, l)$ -relatively compact AK -space. (See [33; §30].)

THEOREM 24. *If E is a separable FK -space, then E contains m if and only if E contains m_0 ; if either of these conditions is satisfied, the inclusion mapping $m \rightarrow E$ is compact and $m \subseteq S_E$.*

Proof. By Lemma 1, l is $\sigma(l, m_0)$ -sequentially complete. If $E \supseteq m_0$, the inclusion mapping $(m_0, \tau(m_0, l)) \rightarrow E$ clearly has closed graph and so must be continuous by Theorem 13. Thus we can extend it to the completion $(m, \tau(m, l))$ of $(m_0, \tau(m_0, l))$ and it is easy to check that this extension is still the identity mapping so that $m \subseteq E$. The last part follows at once from the remarks made just before the theorem.

In the forthcoming paper [15] we show that the separability condition on E can be dropped from the first part of Theorem 24; this naturally prompts the following question.

Is every FK -space the intersection of all the separable FK -spaces containing it?

As long ago as 1922, Hahn [28; Theorem V(c)] made the observation that a matrix limits every bounded sequence if and only if it limits every sequence of zeros and ones. He further noticed [28; Theorem VII(c)] that if this is so and if $x \in m$, then $\lim_A x = \sum_{i=1}^{\infty} a_{i,j} x_i$. Observing that the convergence domain of any matrix has to be separable [37], [12], we note that Hahn's result is an immediate corollary to Theorem 24 by putting $E = c_A$. Indeed, if $x \in m$, we can assert the stronger condition that

$$\limsup_{n \rightarrow \infty} \left| \sum_{i=1}^{\infty} a_{i,j} x_i \right| = 0.$$

For further applications of Theorem 24 to summability the reader is referred to [15] where known results [34], [48], [56], [60] and [79] concerning the absolute p -summability domains are improved.

We now give a result related to Theorem 24 in which the FK -space E need not contain c_0 ; first we need the following elementary lemma.

LEMMA 7. *Let c_1, c_2, \dots, c_n be fixed complex numbers; then*

$$\sup_{J \in \mathfrak{A}} \left| \sum_{i \in J} c_i \right| \geq \frac{1}{4} \sum_{i=1}^n |c_i|,$$

where \mathfrak{A} denotes the family of all subsets of $\{1, 2, \dots, n\}$.

THEOREM 25. *If E is a separable FK -space containing ϕ and $E + c_0 \supseteq m_0$, then $E \supseteq m$.*

Proof. The hypotheses imply that $(E \cap m) + c_0 \supseteq m_0$ so that $E \cap m$ is

$\|\cdot\|_\infty$ -dense in m . We first show that $\sigma(l, E \cap m)$ and the $\|\cdot\|_1$ -topology on l have the same convergent sequences.

If not, then we may choose a sequence $\{a^{(n)}\}_{n=1}^\infty$ of points of l such that

$$(**) \quad a^{(n)} \rightarrow 0(\sigma(l, E \cap m)) \text{ yet } \inf_n \|a^{(n)}\|_1 = 1.$$

We construct inductively two sequences $\{m_k\}_{k=1}^\infty$ and $\{n_k\}_{k=1}^\infty$ of positive integers to satisfy $m_1 < n_1 < m_2 < \dots$. Let $m_0 = n_0 = 0$ and suppose that m_1, \dots, m_k and n_1, \dots, n_{k-1} have already been chosen. Choose $n_k > m_k$ so that

$$(***) \quad \sum_{j=1}^{m_k} |a_j^{(n_k)}| < \frac{1}{k}, \quad k = 1, 2, \dots$$

This is possible by (**) since $E \cap m$ contains ϕ . If $k < 13$, choose $m_{k+1} > n_k$ arbitrarily; otherwise choose $m_{k+1} > n_k$ so that

$$\sum_{j=m_k+1}^{m_{k+1}} |a_j^{(n_k)}| > \frac{12}{13} \|a^{(n_k)}\|_1, \quad k = 13, 14, \dots,$$

which is certainly possible by (**) and (***). Note also that from (**) and (***) we have

$$\sum_{j=m_{k+1}+1}^\infty |a_j^{(n_k)}| < \frac{1}{13} \|a^{(n_k)}\|_1, \quad k = 13, 14, \dots$$

Now choose $x \in m_0$ as follows. For $m_k < j < m_{k+1}$ choose $x_j = 0$ or 1 so that using Lemma 7

$$\left| \sum_{j=m_k+1}^{m_{k+1}} x_j a_j^{(n_k)} \right| \geq \frac{1}{4} \sum_{j=m_k+1}^{m_{k+1}} |a_j^{(n_k)}|.$$

Now $x = y + z$ for suitable $y \in E \cap m$ and $z \in c_0$; choose k so large that $|z_j| < \frac{1}{13}$ whenever $j \geq m_k$. Then we have

$$\begin{aligned} |\langle y, a^{(n_k)} \rangle| &= \left| \sum_{j=1}^\infty y_j a_j^{(n_k)} \right| \geq \left| \sum_{j=m_k+1}^{m_{k+1}} x_j a_j^{(n_k)} \right| - \sum_{j=1}^{m_k} |x_j a_j^{(n_k)}| - \sum_{j=m_k+1}^\infty |x_j a_j^{(n_k)}| \\ &\quad - \sum_{j=1}^{m_k} |z_j a_j^{(n_k)}| - \sum_{j=m_k+1}^\infty |z_j a_j^{(n_k)}| \\ &\geq \frac{1}{4} \frac{12}{13} \|a^{(n_k)}\|_1 - \frac{1}{k} - \frac{\|a^{(n_k)}\|_1}{13} - \frac{\|z\|_\infty}{k} - \frac{\|a^{(n_k)}\|_1}{13} \end{aligned}$$

whenever $k > 13$ so that

$$\liminf_{k \rightarrow \infty} |\langle y, a^{(n_k)} \rangle| \geq \frac{1}{13} > 0.$$

Consequently $a^{(n)} \rightarrow 0(\sigma(l, E \cap m))$, and we have shown that $\sigma(l, E \cap m)$ and $\|\cdot\|_1$ define the same convergent sequences in l .

It follows that $\sigma(l, E \cap m)$ and $\|\cdot\|_1$ define the same Cauchy sequences [68; Proposition 1.4] and the same compact sets. Thus l is $\sigma(l, E \cap m)$ -sequentially complete and $\tau(E \cap m, l) = \tau(m, l)|_{E \cap m}$. The inclusion mapping $(E \cap m,$

$\tau(E \cap m, l) \rightarrow E$ clearly has closed graph and so by Theorem 13 must be continuous. Extending to the completion $(m, \tau(m, l))$ of $(E \cap m, \tau(E \cap m, l))$, it is easy to check that we still have the identity mapping so that $m \subseteq E$.

Finally, we give an application of Theorem 13 to biorthogonal systems. We recall that a *biorthogonal system* $\{\langle x_i, f_i \rangle\}_{i=1}^{\infty}$ for the locally convex space E is a sequence $\{x_i\}_{i=1}^{\infty}$ of elements of E together with a sequence $\{f_i\}_{i=1}^{\infty}$ of elements of E' such that

$$\langle x_i, f_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad i, j = 1, 2, \dots$$

The system $\{\langle x_i, f_i \rangle\}_{i=1}^{\infty}$ is called *total* if for $x \in E$, $\langle x, f_j \rangle = 0$ for all j implies that $x = 0$. The following result was established by Kadec [29].

Let E be a separable Banach space with a total biorthogonal system $\{\langle x_i, f_i \rangle\}_{i=1}^{\infty}$ such that the linear subspace of E' spanned by $\{f_i\}_{i=1}^{\infty}$ has positive characteristic. Then for $x \in E$ the following conditions are equivalent.

(i) $\sum_{i=1}^{\infty} f_i(x)x_i$ converges unconditionally to x .

(ii) Given $\lambda \in m_0$ there exists $y \in E$ such that $f_i(y) = \lambda_i f_i(x)$, $i = 1, 2, \dots$.

Mitiagin [41] asked if Kadec's result remains true when the restriction on the characteristic is dropped. That this is indeed the case, even for Fréchet spaces, was conjectured by Kadec and Pełczyński [30]; they only proved this however when m_0 is replaced by m in (ii). More recently Davis, Dean and Singer [21] have shown that Mitiagin's question has an affirmative answer provided that (i) and (ii) hold for every $x \in E$. Our next result establishes the conjecture of Kadec and Pełczyński.

THEOREM 26. *Let E be a separable Fréchet space with a total biorthogonal sequence $\{\langle x_i, f_i \rangle\}_{i=1}^{\infty}$; then the following conditions are equivalent for each $x \in E$.*

(i) $\sum_{i=1}^{\infty} f_i(x)x_i$ converges unconditionally to x .

(ii) Given $\lambda \in m_0$ there exists $y \in E$ such that $f_i(y) = \lambda_i f_i(x)$, $i = 1, 2, \dots$.

Proof. (i) \Rightarrow (ii). By [43; Theorem 1] $\sum_{i=1}^{\infty} f_i(x)x_i$ is subseries convergent so that (ii) follows.

(ii) \Rightarrow (i). With N denoting the positive integers let $N_0 = \{k \in N : f_k(x) = 0\}$ and let $N \setminus N_0 = \{n_1, n_2, \dots\}$ which we may assume to be infinite. We consider the subspace $E_0 = \{y \in E : f_k(y) = 0 \text{ for each } k \in N_0\}$ which is closed in E and we define $T: E_0 \rightarrow \omega$ by

$$(Ty)_k = \frac{f_{n_k}(y)}{f_{n_k}(x)}, \quad k = 1, 2, \dots$$

Then T is one-to-one and $T(E_0)$ is a separable FK-space in the induced topology. Furthermore, since (ii) holds, $T(E_0) \supset m_0$ so that $S_{T(E_0)} \supset m$ by Theorem 24, i.e., $\sum_{k=1}^{\infty} f_{n_k}(x)x_k$ converges unconditionally.

We note here that in a recent manuscript [10] Bachelis and Rosenthal have also answered Mitiagin's question; their results, however, pertain only to Banach spaces and so are not as general as ours.

Finally, we observe that from Theorems 24 and 25 it follows that conditions (i) and (ii) of Theorem 26 are equivalent to the following condition.

(iii). Given $\lambda \in m_0$ there exist $y \in E$ and $\mu \in c_0$ such that $f_i(y) = (\lambda_i - \mu_i)f_i(x)$, $i = 1, 2, \dots$.

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