C. R. Acad. Sci. Paris, t. 316, Série I, p. 351-355, 1993

Analyse mathématique/*Mathematical Analysis*

Isometries on rearrangement-invariant spaces

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Abstract – We prove that if X is a real rearrangement-invariant function space on [0, 1], which is not isometrically isomorphic to L_2, then every surjective isometry $T:X\to X$ is of the form $Tf(s)=a(s)f(\sigma(s))$ for a nonvanishing Borel function a and an invertible Borel map $\sigma:[0,1]\to[0,1]$. If X is not equal to L_p, up to renorming, for some $1\leqq p\leqq\infty$ then in addition $|a|=1$ a.e. and σ must be measure-preserving.

Isométries sur les espaces invariants par réarrangement

Résumé – *Soit* X *un espace de Banach de fonctions réelles sur* [0, 1] *invariant par réarrangement et tel que* X *ne sot pas isométrique à* L_2. *Nous montrons que toute isométrie surjective* $T:X\to X$ *est de la forme :* $Tf(s)=a(s)f(\sigma(s))$ *où* $a:[0,1]\to\mathbf{R}$ *et* $\sigma:[0,1]\to[0,1]$ *sont des applications boréliennes. Si* X *n'est pas égale à* L_p *(après renormage) pour un* $p\geqq 1$, *alors* $|a|=1$ *p.p. et* σ *conserve la mesure.*

Version française abrégée – On dénote par λ la mesure de Lebesgue sur [0, 1]. Pour tout ce qui concerne les espaces de Banach de fonctions invariants par réarrangement nous renvoyons le lecteur à Lindenstrauss-Tzafriri [13]. Le résultat principal de cet article est le suivant :

THÉORÈME 1. – *Soit* X *un espace de Banach de fonctions réelles invariant par réarrangement. Supposons que* X *ne soit pas isométrique à* $L_2[0,1]$. *Soit* $T:X\to X$ *une isométrie surjective. Alors*

(1) *Il existe une fonction borélienne* $a:[0,1]\to\mathbf{R}$ *et une application* $\sigma:[0,1]\to[0,1]$ *telles que pour tout ensemble borélien* B, *on ait* $\lambda(\sigma^{-1}(B))=0$ *si et seulement si* $\lambda(B)=0$ *et* $Tf(s)=a(s)f(\sigma(s))$ *p.p pour tout* $f\in X$.

(2) *Si* X *n'est pas égal à* $L_p(1\leqq p\leqq\infty)$ *(après renormage), alors* $|a|=1$ *p.p et* σ *conserve la mesure* λ.

La première partie du théorème 1 est connue dans le cas des espaces de Banach de fonctions à valeurs complexes et est due à Zaidenberg [19]. La deuxième partie demeure vraie dans le cas complexe et est apparemment nouvelle même dans le cas complexe.

L'étude des isométries sur les espaces classiques a été introduite par Banach (*voir* [1] p. 175). En 1958, Lamperti a caractérisé les isométries de L_p pour $p\neq 2$. Au début des années 60, Lumer (*voir* [14]) a développé une nouvelle méthode pour étudier les isométries sur les espaces d'Orlicz réflexifs complexes. Sa méthode a été adaptée plus tard par Zaidenberg pour décrire les isométries sur les espaces r.i. complexes en général (*voir* [19], [20]). La méthode de Lumer ne s'applique pas au cas réel. La plupart des résultats obtenus dans le cas réel utilisent des propriétés géométriques de l'espace considéré (*voir* [5]).

Voici les idées principales de la preuve :

On dira qu'un opérateur $T:X\to X$ est *élémentaire* s'il existe deux applications boréliennes $a:[0,1]\to\mathbf{R}$ et $\sigma:[0,1]\to[0,1]$ telles que pour tout $f\in X$, $Tf=af\circ\sigma$.

Soit $e_i^n=\chi_{((i-1)/2^n,\, i/2^n)}$ et $X_n=[e_i^n:1\leqq i\leqq 2^n]$. Nous introduisons une propriété (P) plus faible que celle de la convexité stricte et telle que pour un espace r.i. X, X ou X′ ait la propriété (P). On dit que X a la propriété (P) si on a $\|e_1^1+te_2^1\|_X>\|e_1^1\|_X$ pour $t>0$.

Note présentée par Gustave CHOQUET.

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Pour un espace de Banach arbitraire Z on dit qu'un élément $u \in Z$ *est un élément de Flinn* s'il existe une projection $P : Z \to [u]$ telle que $\| Id - P \| = 1$. Si $X \neq L_2$ est un espace r.i., $u \in X$ est un élément de Flinn pour X si et seulement si $u = 0$. Cependant, on peut étudier les éléments de Flinn de X_n pour chaque n, et ainsi démontrer le théorème suivant :

THÉORÈME 2. — *Soit* X *un espace* r.i. *réel sur* [0, 1] *avec la propriété* (P). *Alors, si* $0 < p \leq 1$, *il existe une constante* $C_p = C_p(X)$ *telle que, si* $T : X \to X$ *est une isométrie surjective, il existe des applications boréliennes* $\sigma_n : [0, 1] \to [0, 1]$, *telles que* $\sigma_m(s) \neq \sigma_n(s)$ *si* $m \neq n$, *et des fonctions réelles* a_n *sur* [0, 1] *telles que*

$$\left(\int_0^1 \sum_{n=1}^{\infty} |a_n(s)|^p \, ds \right)^{1/p} \leq C_p$$

et

$$T f(s) = \sum_{n=1}^{\infty} a_n(s) f(\sigma_n s) \text{ p. p.}$$

pour tout $f \in L_\infty [0, 1]$.

Ce théorème utilise la théorie de la représentation d'opérateurs par des mesures aléatoires ([8], [17], [18]). Maintenant, on peut considérer des produits d'isométries pour démontrer que $C_p \leq 1$, pour tout $0 < p \leq 1$, et on peut en déduire que T est élémentaire si l'espace X a la propriété (P). Le cas où X' a la propriété (P) est analogue.

Pour la preuve du théorème 1 (2) on a besoin de la proposition suivante (*voir* [9]) :

PROPOSITION 3. — *Soit* X *un espace* r. i. *sur* [0, 1] *et soit* $T : X \to X$ *un opérateur élémentaire. Alors, si* $p_X \leq r \leq q_X$ (*où* p_X *et* q_X *sont les indices de Boyd pour* X), *on a* $T : L_r \to L_r$ *et* $\| T \|_r \leq \| T \|_X$.

We denote the Lebesgue measure on [0, 1] by λ and use the term rearrangement-invariant Banach function space in the sense of Lindenstrauss-Tzafriri [13]. Our main result is the following:

THEOREM 1. — *Let* X *be a* (*real*) *rearrangement-invariant Banach function space on* [0, 1]. *Suppose* X *is not* (*isometrically*) *equal to* $L_2[0, 1]$. *Let* $T : X \to X$ *be a surjective isometry. Then*

(1) *There exists a nonvanishing Borel function* $a : [0, 1] \to \mathbf{R}$ *and an invertible Borel map* $\sigma : [0, 1] \to [0, 1]$ *such that, for any Borel set* $B \subset [0, 1]$, *we have* $\lambda(\sigma^{-1} B) = 0$ *if and only if* $\lambda(B) = 0$ *and so that* $T f(s) = a(s) f(\sigma(s))$ *a. e. for any* $f \in X$.

(2) *If* X *is not equal to* L_p *for some* $1 \leq p \leq \infty$ *up to renorming, then* $|a| = 1$ a. e. *and* σ *is measure-preserving.*

The first part of this theorem is known for the case of spaces of complex functions and is due to Zaidenberg ([19], [20]). The second part also holds in both the real and complex case, and is apparently new, even in the complex case.

The study of isometries on classical function spaces goes back to Banach [1] who proved that the isometries of L_p [0, 1] are disjointness-preserving when $p \neq 2$ (*see* [1], p. 175). In 1958 Lamperti [12] characterized the isometries on L_p. In early 60's Lumer [14] studied isometries on complex reflexive Orlicz spaces and developed a technique which was later used by Zaidenberg ([19], [20]) in 1977 to describe isometries on general complex r. i. spaces. However, Lumer's method cannot be applied for real spaces. Most of the studies treating the real case use geometric properties of particular function spaces;

see e. g. [2], [3], [4], [7]. For a fuller discussion of the existing literature we refer the reader to the forthcoming survey of Fleming and Jamison [5].

The proof of Theorem 1 follows a line of reasoning which is remotelly related to the Lumer technique. We must restrict ourselves (as do Lumer and Zaidenberg) to surjective isometries of r. i. spaces on [0, 1] (it is not clear how to extend it to r.i. spaces on $[0, \infty)$). However our results do apply equally to separable and nonseparable r. i. spaces on [0, 1]. Full details of our work will appear in [10].

We say that an operator $T: X \to X$ is *elementary* if there is a Borel function a and a Borel map $\sigma: [0, 1] \to [0, 1]$ such that $Tf(s) = a(s) f(\sigma(s))$ a. e. for every $f \in X$. The main objective is to show that every surjective isometry is elementary. It may first be shown (using arguments of [6] for the nonseparable case) that every surjective isometry has an adjoint $T': X' \to X'$ where X' is the Köthe dual of X and this adjoint is also an isometry. We then show that if T is a surjective isometry on an r. i. space X then T is elementary if and only if $(T^{-1})': X' \to X'$ is elementary. Also it is a general fact that if T is invertible and elementary, then there exists a nonvanishing Borel function $a: [0, 1] \to \mathbf{R}$ and an invertible Borel map $\sigma: [0, 1] \to [0, 1]$ such that, for any Borel set $B \subset [0, 1]$, we have $\lambda(\sigma^{-1} B) = 0$ if and only if $\lambda(B) = 0$ and so that $Tf(s) = a(s) f(\sigma(s))$ a. e. for any $f \in X$. So it now remains to prove that T elementary.

We introduce an idea which is a real analogue of the notion of hermitian elements [11]. First if X is a real Banach space we say that an operator $T: X \to X$ is *numerically positive* (Rosenthal [16]) if $x^*(Tx) \geqq 0$ whenever $x^*(x) = \|x\| = \|x^*\| = 1$. In the case when T is a projection, T is numerically positive if and only if $\|I - T\| = 1$. Based on ideas of P. H. Flinn [16] we say that $u \in X$ is a *Flinn element* if there is a numerically positive projection $P: X \to [u]$. The set of Flinn elements will be denoted $\mathcal{F}(X)$. We establish first:

Theorem 2. — *Suppose* μ *is nonatomic and suppose* X *is an order-continuous Köthe function space on* (Ω, μ). *Then* $u \in X$ *is a Flinn element if and only if there is a nonnegative function* $w \in L_0(\mu)$ *with* supp ω = supp u = B, *so that:*

(*a*) *If* $x \in X(B)$ (i. e. $x(s) = 0$ *for* $s \notin B$) *then* $\|x\| = \left(\int |x|^2 w\, d\mu\right)^{1/2}$. *and*

(*b*) *If* $v \in X(\Omega \setminus B)$ *and* $x, y \in X(B)$ *satisfy* $\|x\| = \|y\|$ *then* $\|v + x\| = \|v + y\|$.

Corollary 3. — *Suppose* X *is a separable* r. i. *space on* [0, 1]. *If* $\mathcal{F}(X) \neq \{0\}$ *then* $X = L_2[0,1]$.

Suppose N is a natural number. Let $e_i^N = \chi_{((i-1)2^{-N}, i 2^{-N})}$ for $1 \leqq i \leqq 2^N$ and $X_N = [e_i^N : 1 \leqq i \leqq 2^N]$. We say that an r. i. space X has property (P) if for every $t > 0$,

$$\|e_1^1\|_X < \|e_1^1 + t e_2^1\|_X.$$

We note that property (P) is weaker than strict convexity and either X or X′ has (P). We next prove an asymptotic version of Corollary 3.

Proposition 4. — *Suppose* X *is an* r. i. *space on* [0, 1] *such that* $X \neq L_2$ *and* X′ *has* (P). *Then for any* $0 < p < \infty$ *there is a constant* $A_p = A_p(X)$ *so that for every* $n \in N$ *and every* $u = \sum_{i=1}^{2^n} a_i e_i^n \in \mathcal{F}(X_n)$ *we have*

$$\left(\sum_{i=1}^{2^n} |a_i|^p\right)^{1/p} \leqq A_p \max_{1 \leqq i \leqq 2^n} |a_i|.$$

Now suppose $T: X \to X$ is an isometry. We show that for a.e. $\omega \in [0, 1]$ an element $F_n(\omega) = \sum_{i=1}^{2^n} T e_i^n(\omega) e_i^n$ is Flinn in X_n' for large enough n, and hence using Proposition 4 and the theory of random measure representations ([8], [17], [18]) we conclude the following:

THEOREM 5. — *Let* X *be an* r.i. *space on* [0, 1] *with property* (P), *and such that* $X \neq L_2$ (*isometrically*). *Then for any surjective isometry* $T: X \to X$ *there is a sequence of Borel maps* $\sigma_n: [0, 1] \to [0, 1]$ *and Borel functions* a_n *on* [0, 1] *so that* $|a_n(s)| \geqq |a_{n+1}(s)|$ a.e. *for every* n *and* $\sigma_m(s) \neq \sigma_n(s)$ *whenever* $m \neq n$ *and* $s \in [0, 1]$ *and for which*

$$T f(s) = \sum_{n=1}^{\infty} a_n(s) f(\sigma_n(s))$$

for any $f \in L_\infty$.

Moreover for any $0 < p \leqq 1$ *there is a constant* C_p *depending only on* X *such that*

$$\left(\int_0^1 \sum_{n=1}^{\infty} |a_n(s)|^p \, ds\right)^{1/p} = C_{p,T} \leqq C_p.$$

To establish Theorem 1 (1) we use a probabilistic technique. We consider isometries of the form $S(\tau) = TV_\tau T$, where $V_\tau f = f \circ \tau$ for a measure preserving map τ from a certain family.

We conclude that for every $\varepsilon > 0$ there is τ such that

$$C_{p,S(\tau)} \geqq C_{p,T}^2 - \varepsilon.$$

As T was arbitrary $C_p \leqq 1$ for all $0 < p \leqq 1$ and this quickly proves that T is elementary.

An important step in the proof of Theorem 1 (2) is the following (*cf.* [9]):

PROPOSITION 6. — *Let* X *be an* r.i. *space and suppose* $T: X \to X$ *is an elementary operator. Suppose* $p_X \leqq r \leqq q_X$, *where* p_X *and* q_X *are the Boyd indices of* X. *Then* T *is bounded on* $L_r[0, 1]$ *and* $\|T\|_{L_r} \leqq \|T\|_X$.

Remarks. — Theorem 1 (2) can be cast as a statement about maximal norms (*cf.* [15], [11]). A Banach space X has a *maximal norm* if no equivalent norm has a strictly bigger group of invertible isometries. Theorem 1 (2) shows immediately that any r.i. space on [0, 1] which is not isomorphic to $L_p[0, 1]$ has a maximal norm; Rolewicz [15] showed that the spaces $L_p[0, 1]$ have maximal norms. However if X is isomorphic but not isometric to L_p its norm cannot be maximal; this follow rather easily from Proposition 6 and the almost transitivity of the norm in L_p.

We wish to thank Jim Jamison and Anna Kaminska for bringing this question to our attention and for providing copies of both [5] and a translation of [20].

Note remise le 7 décembre 1992, acceptée le 16 décembre 1992.

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