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A REMARK ON A PROBLEM OF KLEE

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This paper treats a property of topological vector spaces first studied by Klee [6]. X is said to have the *Klee property* if there are two (not necessarily Hausdorff) vector topologies on X, say τ_1 and τ_2, such that the quasi-norm topology is the supremum of τ_1 and τ_2 and such that (X, τ_1) has trivial dual while the Hausdorff quotient of (X, τ_2) is nearly convex, i.e. has a separating dual. Klee raised the question of whether every topological vector space has the Klee property.

In this paper we will only consider the case when X is a separable quasi-Banach space. In this context the problem has recently been considered in [2] and [7]. In [7], the problem was considered for the special case when X is a twisted sum of a one-dimensional space and a Banach space, so that there is subspace L of X with $\dim L = 1$ and X/L locally convex; it was shown that X then has the Klee property if the quotient map is not strictly singular. Then in [2] a twisted sum X of a one-dimensional space and ℓ_1 was constructed so that the quotient map is strictly singular and X fails to have the Klee property. Thus Klee's question has a negative answer. The aim of this paper is to completely characterize the class of separable quasi-Banach spaces with the Klee property. Using this characterization we give a much more elementary counter-example to Klee's question.

Given a quasi-Banach space X, the dual of X is denoted by X^*. We define the *kernel* of X to be the linear subspace $\{x : x^*(x) = 0\ \forall x^* \in X^*\}$.

Now we state our theorem:

THEOREM. *Let X be a separable quasi-Banach space, with kernel E. Then X fails to have the Klee property if and only if E has infinite codimension and the quotient map $\pi : X \to X/E$ is strictly singular.*

Proof. For the "if" part, suppose that E has infinite codimension, π is strictly singular and that X has the Klee property. In this case the closure

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of $\{0\}$ for the topology τ_2 must include the kernel E, so that on E the topology τ_1 must coincide with the quasi-norm topology. By a standard construction, there is a vector topology $\tau_3 \leq \tau_1$ which is pseudo-metrizable and coincides with the quasi-norm topology on E. Let F be the closure of $\{0\}$ for this topology.

Note that $F \cap E = \{0\}$ and that $E + F$ is τ_3-closed and hence also closed for the original topology. This implies that π restricted to F is an isomorphism, and so if F is of infinite dimension, we have a contradiction. If F is of finite dimension, then since X^* separates the points of F we can write $X = X_0 \oplus F$, and τ_3 is Hausdorff on X_0. Suppose τ_3 coincides with the original topology on X_0. Then X_0 has trivial dual and so $X_0 \subset E$. This implies that E is of finite codimension, and so contradicts our assumption. Thus τ_3 is strictly weaker than the original topology on X_0.

By a theorem of Aoki–Rolewicz [4], we can assume that the quasi-norm is p-subadditive for some $0 < p < 1$. Now let $|\cdot|$ denote an F-norm defining τ_3 on X. There exists $\delta > 0$ so that $|x| \leq \delta$ and $x \in E$ imply $\|x\|^p \leq 1/2$. We can also choose $0 < \eta < 2^{-1/p}$ so that x in X and $\|x\| \leq \eta$ together imply that $|x| < \delta/2$. Also, there exists a sequence (x_n) in X_0 so that $|x_n| \leq 2^{-n}$ and $\|x\| = 1$. Now by [4] (p. 69, Theorem 4.7) we can pass to a subsequence, also labelled (x_n), which is strongly regular and M-basic in X, i.e. so that for some M we have $\max |a_k| \leq M\|\sum_{k=1}^{\infty} a_k x_k\|$ for all finitely nonzero sequences $(a_k)_{k=1}^{\infty}$. Now pick n_0 so large that $(M+1)2^{-n_0} < \delta/2$.

Let F_0 be the linear span of $(x_n)_{n>n_0}$; we show that π is an isomorphism on F_0. Indeed, suppose $e \in E$ and $(a_n)_{n>n_0}$ is finitely nonzero with $\|e + \sum_{k>n_0} a_k x_k\| < \eta$ but $\|\sum_{k>n_0} a_k x_k\| = 1$. Then $|e + \sum_{k>n_0} a_k x_k| \leq \delta/2$. Further, $1 + \max_{k>n_0} |a_k| \leq M + 1$. Hence $|e| \leq \delta$ and $\|e\|^p \leq 1/2$; this implies $\|e + \sum_{k>n_0} a_k x_k\| \geq 1/2$, which gives a contradiction. Hence the map π is an isomorphism on F_0, and this contradicts our hypothesis.

Now we turn to the converse. By the theorem of Aoki–Rolewicz we can assume that the quasi-norm is p-subadditive for some $0 < p < 1$. Suppose π is an isomorphism on some infinite-dimensional closed subspace F. Then F has separating dual and hence contains a subspace with a basis. We therefore assume that F has a normalized basis $(f_n)_{n=1}^{\infty}$, and that K is a constant so large that $e \in E$ and $f \in F$ imply that $\max(\|e\|, \|f\|) \leq K\|e + f\|$ and so that if (a_n) is finitely nonzero then $\max |a_n| \leq K\|\sum_{n=1}^{\infty} a_n f_n\|$.

Now let $(x_n)_{n=1}^{\infty}$ be a sequence whose linear span is dense in X, chosen in such a way that $M_n^{p-1}\|x_n\|^p = 2^{-(n+4)}$, where each M_n is a positive integer. We also require that for each positive integer m and each x_n, $mx_n = \alpha x_j$ for some $0 \leq \alpha \leq 1$ and some positive integer j. Let $N_n = M_n - M_{n-1}$, $M_0 = 0$. Let $a_n = 2^{(n+4)/p} N_n K^2$. Define V as the absolutely p-convex hull of the set $\{a_n f_k + x_n : \ M_{n-1} + 1 \leq k \leq M_n, \ 1 \leq n < \infty\}$. We let L be the closed linear span of the vectors $\{\sum_{k=M_{n-1}+1}^{M_n} f_k\}$.

We now consider the set $L + V + U_X$, where U_X is the open unit ball of X. This is an open absolutely p-convex set and generates a p-convex semi-quasi-norm $|\cdot|$ on X. We will show that $|\cdot|$ generates the original topology on E; more precisely, we will show that if $e \in E$ and $e \in L + V + U_X$, then $\|e\| \le 2^{1/p}K$.

Indeed, assume $e \in E \cap (L + V + U_X)$. Then there exists $y \in U_X$ so that $e - y \in L + V$. It follows that there exist finitely nonzero sequences $(b_k)_{k=1}^\infty$ and $(c_n)_{n=1}^\infty$ so that $\sum_{k=1}^\infty |b_k|^p \le 1$ and $e - y = z_1 + z_2$, where

$$z_1 = \sum_{n=1}^\infty \sum_{k=N_{n-1}+1}^{N_n} (b_k - c_n)a_n f_k, \qquad z_2 = \sum_{n=1}^\infty \Big(\sum_{k=N_{n-1}+1}^{N_n} b_k \Big) x_n.$$

Let $\beta_n = \sum_{k=N_{n-1}+1}^{N_n} b_k$. Then

$$\|y + z_2\|^p \le 1 + \sum_{n=1}^\infty |\beta_n|^p \|x_n\|^p = A^p,$$

say. It follows that

$$\|z_1\| \le K\|e - z_1\| \le KA.$$

Thus

$$\max_n \max_{M_{n-1}+1 \le k \le M_n} |b_k - c_n| a_n \le K^2 A,$$

so

$$c_n^p \le K^{2p} A^p a_n^{-p} + |b_k|^p \qquad (M_{n-1} + 1 \le k \le M_n).$$

Adding, and using $\sum |b_k|^p \le 1$, we obtain

$$N_n c_n^p \le \sum_{k=N_{n-1}+1}^{N_n} |b_k|^p + a_n^{-p} K^{2p} A^p \le 1 + N_n a_n^{-p} K^{2p} A^p.$$

Hence,

$$c_n^p \le K^{2p} A^p a_n^{-p} + N_n^{-1} \le 2N_n^{-1} \max(N_n K^{2p} A^p a_n^{-p}, 1).$$

Taking pth roots,

$$c_n \le 2^{1/p} N_n^{-1/p} \max(1, N_n^{1/p} a_n^{-1} K^2 A).$$

It now follows that

$$|\beta_n| \le 2^{1/p} N_n^{1-1/p} \max(1, N_n^{1/p} a_n^{-1} K^2 A) + N_n K^2 a_n^{-1} A.$$

This implies that

$$|\beta_n|^p \le 3N_n^p K^{2p} a_n^{-p} A^p + 3N_n^{p-1}.$$

We finally arrive at the inequality

$$A^p \le 1 + A^p \sum_{n=1}^\infty (3N_n^p K^{2p} a_n^{-p} + 3N_n^{p-1}) \|x_n\|^p \le 1 + \frac{1}{2} A^p,$$

which implies $A \le 2^{1/p}$ and hence $\|e\| \le K\|e - z_1\| \le KA \le 2^{1/p}K$, as desired.

This shows that $L+V+U_X$ intersects E in a bounded set and $|\cdot|$ induces the original topology on E. However, for each n,

$$x_n = \frac{1}{N_n} \sum_{k=M_{n-1}+1}^{M_n} (a_n f_k + x_n) - \frac{a_n}{N_n} \sum_{k=M_{n-1}+1}^{M_n} f_k$$

is in the convex hull of $L+V$. Hence, by assumption on (x_n), mx_n is in the convex hull of $L+V$ as well for all m in N, and hence $(X, |\cdot|)$ has trivial dual.

Now note that X with the quasi-norm $d(x, E)$ is nearly convex. We finally show that the original topology on X is the supremum of the $|\cdot|$-topology and the topology induced by $d(x, E)$. Suppose $d(x_n, E) \to 0$ and $|x_n| \to 0$. Then there exist $e_n \in E$ so that $\|x_n - e_n\| \to 0$ and so $|x_n - e_n| \to 0$. Hence $|e_n| \to 0$, and hence $\|e_n\| \to 0$, which implies $\|x_n\| \to 0$.

Remark. The main theorem of [7] is an immediate consequence of our theorem. Indeed, assume $X = R \oplus_F Y$ is a twisted sum and that the quasi-linear map F on the separable normed space Y splits on an infinite-dimensional subspace. Then it is bounded on a further infinite-dimensional subspace, so the quotient map $\pi : X \to X/E = X/R$ is not strictly singular.

Remark. One special case, which is sometimes applicable, is that X has the Klee property if it has an infinite-dimensional locally convex subspace with the Hahn–Banach Extension Property (cf. [4]). Indeed, in these circumstances, there is a locally convex subspace Z with $\dim Z = \infty$, so the Banach envelope seminorm is equivalent to the original quasi-norm on Z; it then follows rapidly that the quotient map $\pi : X \to X/E$ is an isomorphism on Z.

For a particular case of this, let (A_n) be a sequence of pairwise disjoint measurable subsets of $(0, 1)$, of positive measure. Let (f_n) be a sequence of measurable functions, with f_n supported on A_n and each f_n having the distribution of $t \to 1/t$, for small t. Let F be the closed linear span of (f_n) in weak L_1. Then F has the Hahn–Banach Extension Property in weak L_1 and so if X is any separable subspace of weak L_1 containing F then X has the Klee property.

Example. Finally, we construct an elementary counter-example to Klee's problem, using much less technical arguments than [2]. We use the twisted sum of Hilbert spaces, Z_2, introduced in [3] (see alternative treatments in [1] and [5]). To define this it will be convenient to consider the space c_{00} of all finitely nonzero sequences as a dense subspace of ℓ_2 and consider the map $\Omega : c_{00} \to \ell_2$ given by

$$\Omega(\xi)(k) = \xi(k) \log(\|\xi\|_2 / |\xi(k)|),$$

where as usual the right-hand side is interpreted as zero if $\xi(k) = 0$. Then $\Omega(\alpha\xi) = \alpha\Omega(\xi)$ for $\alpha \in \mathbb{R}$ and

$$\|\Omega(\xi + \eta) - \Omega(\xi) - \Omega(\eta)\|_2 \le C(\|\xi\|_2 + \|\eta\|_2)$$

for a suitable absolute constant C. Now $Z_2 = \ell_2 \oplus_\Omega \ell_2$ is the completion of $c_{00} \oplus \ell_2$ under the quasi-norm

$$\|(\xi, \eta)\| = \|\xi - \Omega(\eta)\|_2 + \|\eta\|_2.$$

Now (cf. [3]) the map $(\xi, \eta) \to \eta$ extends to a quotient map from Z_2 onto ℓ_2 which is strictly singular. More precisely, if F is any infinite-dimensional subspace of c_{00} then the completion of $\ell_2 \oplus_\Omega F$ contains an isometric copy of Z_2 (this is essentially Theorem 6.5 of [3], or see [1]). In particular, this subspace is never of cotype 2.

Now to construct our example, embed ℓ_2 into L_p, where $p < 1$. Then $L_p \oplus_\Omega \ell_2 = X$ has its kernel E isomorphic to L_p and $X/E \sim \ell_2$. If the quotient map is not strictly singular then there is an infinite-dimensional subspace F of c_{00} such that the completion of $L_p \oplus_\Omega F$ is linearly isomorphic to $L_p \oplus \ell_2$ and hence has cotype 2. Then $\ell_2 \oplus_\Omega F$ is also cotype 2, and this is impossible as we have seen.

It follows from our main theorem that the space we have constructed fails the Klee property.

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