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THE THREE SPACE PROBLEM FOR LOCALLY BOUNDED F-SPACES

N.J. Kalton

Abstract

Let X be an F-space, and let Y be a subspace of X of dimension one, with $X/Y \cong \ell_p$ $(0 < p < \infty)$. Provided $p \neq 1$, $X \cong \ell_p$; however if $p = 1$, we construct an example to show that X need not be locally convex.

More generally we show that Y is any closed subspace of X, then if Y is an r-Banach space $(0 < r \leq 1)$ and X/Y is a p-Banach space with $p < r \leq 1$ then X is a p-Banach space; if Y and X/Y are B-convex Banach spaces, then X is a B-convex Banach space. We give conditions on Y and X/Y which imply that Y is complemented in X.

We also show that if X is the containing Banach space of a non-locally convex p-Banach space $(p < 1)$ with separating dual, then X is not B-convex.

1. Introduction

Let X be an F-space (i.e. a complete metric linear space) over the real field. Then ([7]) X is called a $\mathcal{K}$-space if every short exact sequence $0 \to \mathbb{R} \to Z \to X \to 0$ of F-spaces splits. The main aim of this paper is to solve the problem raised in [7] of whether the spaces ℓ_p $(0 < p \leq 1)$ are $\mathcal{K}$-spaces. In fact we show that for $0 < p < \infty$, ℓ_p is a $\mathcal{K}$-space if and only if $p \neq 1$. Thus if X is an F-space whose quotient by a one-dimensional subspace is isomorphic to ℓ_p then for $p \neq 1$, we must have $X \cong \ell_p$; we construct an example to show that the result is false for $p = 1$. This enables us to solve negatively a problem of Stiles [12] by giving a subspace X of ℓ_p $(0 < p < 1)$ such that $\ell_p/X \cong \ell_1$ but X does not have the Hahn–Banach Extension Property.

This problem is studied via the so-called "three-space problem": if $0 \to Y \to Z \to X \to 0$ is a short exact sequence of F-spaces, what properties of Z can be deduced from those of X and Y. It is known that X, Y and Z are Banach spaces, then if both X and Y are reflexive then so is Z; if X and Y are super-reflexive then Z is super-reflexive ([4]); if X and Y are B-convex, then Z is B-convex ([5]). However X and Y can be isomorphic to Hilbert spaces with Z not isomorphic to a Hilbert space ([4]). Of course all these results apply only when it is given that Z is a Banach space.

If X and Y are locally bounded, then so is Z (W. Roelcke, see Theorem 1.1 below). We show in Section 2 that if X and Y are both B-convex Banach spaces, then Z is locally convex (and hence a B-convex Banach space, by Giesy's theorem quoted above). In Section 4 we show that X is a p-Banach space (i.e. a locally p-convex locally bounded F-space) and Y is an r-Banach space where $p < r \leq 1$, then Z is a p-Banach space. We also construct in Section 4 a space Z, such that $Z/\mathbb{R} \cong \ell_1$ but Z is not locally convex, showing these results cannot be improved to allow $p = r = 1$, (this answers a question of S. Dierolf [3]).

In Section 3, we also consider pairs of F-spaces (X, Y) such that every short exact sequence $0 \to Y \to Z \to X \to 0$ splits; we then say that (X, Y) splits. We prove that (ℓ_p, Y) and (L_p, Y) split if Y is an r-Banach space with $0 < p < r \leq 1$. We also give generalizations of these results to Orlicz spaces.

Finally we are able to show, as a by-product of this research, that the containing Banach space of any locally bounded F-space, which is not locally convex, cannot be B-convex (Section 2). This answers a question of J.H. Shapiro, by showing that the containing Banach space of a non-locally convex locally bounded F-space cannot be a Hilbert space.

We close the introduction by proving our first result on the three-space problem. As a consequence of this theorem, our attention in later sections will be restricted to locally bounded spaces. Theorem 1.1 is due to W. Roelcke (cf. [3]).

THEOREM 1.1: *Let X be an F-space and let Y be a closed locally bounded subspace of X; if X/Y is locally bounded, then X is locally bounded.*

PROOF: Let U be a neighbourhood of 0 in X such that $\pi(U)$ and $U \cap Y$ are bounded, where $\pi : X \to X/Y$ is the quotient map. Choose a balanced neighbourhood of 0, V say, such that $V + V \subset U$. Suppose V is unbounded. Then there is a sequence (x_n) in V such that $(\frac{1}{n} x_n)$ is unbounded. Now $\frac{1}{n}\pi(x_n) \to 0$ since $\pi(V)$ is bounded. Hence there are

$y_n \in Y$ such that $\frac{1}{n}x_n + y_n \to 0$. For some N and all $n \geq N$, $\frac{1}{n}x_n + y_n \in V$ and hence $y_n \in V + V \subset U$. Thus (y_n) is bounded. Hence $(\frac{1}{n}x_n)$ is bounded and we have a contradiction.

2. Geometric properties of locally bounded spaces

A quasi-norm on a real vector space X is a real-valued function $x \mapsto \|x\|$ satisfying the conditions:

(1) $\|x\| > 0$, $x \in X$, $x \neq 0$

(2) $\|tx\| = |t| \, \|x\| \quad t \in \mathbb{R}, x \in X$

(3) $\|x + y\| \leq k(\|x\| + \|y\|) \quad x, y \in X$

where k is a constant, which we shall call the modulus of concavity of $\|\cdot\|$. A quasi-normed space is a locally bounded topological vector space if we take the sets ϵU, $\epsilon > 0$ for a base of neighbourhoods of 0 where $U = \{x : \|x\| \leq 1\}$. Conversely any locally bounded topological space can be considered as a quasi-normed space by taking the Minkowski functional of any bounded balanced neighbourhood of 0.

If X is a quasi-normed space and N is a closed subspace of X, then the topology of X/N may be determined by the quasi-norm

$$\|u\| = \inf(\|x\| : \pi(x) = u)$$

where $\pi : X \to X/N$ is the quotient map. If $T : X \to Y$ is a linear map between quasi-normed spaces, T is continuous if and only if

$$\|T\| = \sup(\|Tx\| : \|x\| \leq 1) < \infty.$$

Finally two quasi-norms $|\cdot|$ and $\|\cdot\|$ on X are equivalent if there exist constants $0 < m \leq M < \infty$ such that

$$m|x| \leq M\|x\| \leq M|x| \quad x \in X.$$

If $0 < p \leq 1$, the topology of a locally p-convex locally bounded space may be defined by a quasi-norm $x \mapsto \|x\|$, such that $\|x\|^p$ is a p-norm, i.e.

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \quad x, y \in X.$$

A fundamental result of Aoki and Rolewicz ([1], [10]) asserts that every locally bounded space is locally p-convex for some $p > 0$. We note that in general if a quasi-normed space is locally p-convex there is a constant A such that

$$\|x_1+\cdots+x_n\|^p \le A(\|x_1\|^p+\cdots+\|x_n\|^p) \quad x_1,\ldots,x_n \in X.$$

If X is a locally bounded space, let

$$a_n = a_n(X) = \sup(\|x_1+\cdots+x_n\|;\quad \|x_i\| \le 1, 1 \le i \le n).$$

The following is easily verified.

PROPOSITION 2.1:
(i) $a_{mn} \le a_m a_n$, $m, n \in \mathbb{N}$;
(ii) *if X is locally p-convex then* $\sup n^{-1/p} a_n < \infty$.

We do not know if the converse to (ii) holds. We have only the following partial results. We note that (i) is equivalent to Corollary 5.5.3 of [13].

PROPOSITION 2.2:
(i) *X is locally convex if and only if* $\sup n^{-1} a_n < \infty$;
(ii) *if* $0 < p < 1$, *then* $\lim n^{-1/p} a_n = 0$ *if and only if X is locally r-convex for some $r > p$.*

PROOF: (i) If $a_n \le Cn$ $(n \in \mathbb{N})$, then for $\|u_i\| \le 1$ $(1 \le i \le n)$ and $\alpha_i \ge 0$ $(1 \le i \le n)$ with $\Sigma\, \alpha_i = 1$, define for each $m \in \mathbb{N}$, $\lambda_{i,m}$ to be the largest integer such that $\lambda_{i,m} \le m\alpha_i$

Then

$$\left\|\sum_{i=1}^{n} \lambda_{i,m} u_i\right\| \le C \sum \lambda_{i,m}$$

and hence

$$\left\|\sum_{i=1}^{n} \frac{\lambda_{i,m}}{m} u_i\right\| \le C.$$

Letting $m \to \infty$ we have

$$\left\|\sum \alpha_i u_i\right\| \le Ck$$

so the convex hull of the unit ball is bounded.

(ii) It is clear that if X is locally r-convex for some $r > p$ then $\lim_{n\to\infty} n^{-1/p} a_n = 0$. Conversely suppose $\lim_{n\to\infty} n^{-1/p} a_n = 0$. Then for some m, $m^{-1/p} a_m < 1$, so that for some $r > p$, $m^{-1/r} a_m < 1$. Thus for any n with $m^k < n \leq m^{k+1}$,

$$\begin{aligned} n^{-1/r} a_n &\leq m^{1/r}(m^{-(k+1)/r} a_{m^{k+1}}) \\ &\leq m^{1/r}(m^{-1/r} a_m)^{k+1} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence there exists N, such that for all $n \geq N$

$$n^{-1/r} a_n < \tfrac{1}{2}.$$

Select $q > 0$ by the Aoki–Rolewicz theorem so that X is locally q-convex and hence

$$\|x_1 + \cdots + x_n\|^q \leq A(\|x_1\|^q + \cdots + \|x_n\|^q) \quad x_1, \ldots, x_n \in X.$$

for some constant $A \geq 1$.

If $\|u_i\| \leq 1$, $1 \leq i \leq \ell$, $\alpha_i \geq 0$, $1 \leq i \leq \ell$ and $\sum \alpha_i^r = 1$, we shall show that

$$\left\| \sum_{i=1}^{\ell} \alpha_i u_i \right\| \leq N^{1/q} A\{1 - (\tfrac{1}{2})^q\}^{-1/q} \quad (*)$$

and hence X is locally r-convex.

We prove $(*)$ by induction on ℓ; it is trivially true for $\ell = 1$. Now suppose the result is true for $\ell - 1$ where $\ell \geq 2$. For $k \geq 0$, let $\sigma_k = \{i; 2^{-k} \geq \alpha_i \geq 2^{-(k+1)}\}$. If $|\sigma_k| \leq N$ for all k, then

$$\begin{aligned} \left\| \sum_{i=1}^{\ell} \alpha_i u_i \right\| &= \left\| \sum_{k=0}^{\infty} \sum_{i \in \sigma_k} \alpha_i u_i \right\| \\ &\leq A\left(\sum_{k=0}^{\infty} \sum_{i \in \sigma_k} |\alpha_i|^q \right)^{1/q} \\ &\leq N^{1/q} A\{1 - (\tfrac{1}{2})^q\}^{-1/q} \end{aligned}$$

as required.

If $|\sigma_k| > N$ for some k, let $v = \sum_{i \in \sigma_k} \alpha_i u_i$. Then

$$\begin{aligned} \|v\| &\leq \tfrac{1}{2} |\sigma_k|^{1/r} 2^{-k} \\ &\leq \left(\sum_{i \in \sigma_k} \alpha_i^r \right)^{1/r}. \end{aligned}$$

Hence we may apply the inductive hypothesis to deduce that

$$\left\| \sum_{j\neq k} \sum_{i\in\sigma_k} \alpha_i u_i + \left(\sum_{i\in\sigma_k} \alpha_i^r\right)^{1/r} \left(\sum_{i\in\sigma_k} \alpha_i^r\right)^{-1/r} v \right\| \leq N^{1/q} A\{1-(\tfrac{1}{2})^q\}^{-1/q}.$$

REMARK: We do not know if X is locally p-convex if and only if $\sup n^{-1/p} a_n < \infty$ for $p<1$.

Next we define a sequence $b_n = b_n(X)$ by

$$b_n = \sup_{\|x_i\|\leq 1} \min_{\epsilon_i=\pm 1} \left\| \sum_{i=1}^n \epsilon_i x_i \right\|.$$

REMARK: A Banach space is B-convex if and only if $\lim n^{-1} b_n = 0$ ([2], [5]).

PROPOSITION 2.3: (i) $b_{mn} \leq b_m b_n \quad m, n \in \mathbb{N}$
(ii) $b_n \leq a_n \qquad n \in \mathbb{N}$

PROOF: (i) If $(x_{ij}: 1 \leq i \leq m,\ 1 \leq j \leq n)$ is a set of mn vectors there exist signs $\theta_{ij} = \pm 1$ such that for each i,

$$\left\| \sum_{j=1}^n \theta_{ij} x_{ij} \right\| \leq b_m$$

and signs $\eta_i = \pm 1$ such that

$$\left\| \sum_{i=1}^m \eta_i \sum_{j=1}^n \theta_{ij} x_j \right\| \leq b_m b_n.$$

Hence taking $\epsilon_{ij} = \eta_i \theta_{ij}$

$$\left\| \sum_{i,j} \epsilon_{ij} x_{ij} \right\| \leq b_m b_n.$$

(ii) is trivial.

LEMMA 2.4: *If* $\lim_{n\to\infty} n^{-1/p} b_n = 0$ *where* $0 < p \leq 1$ *then* $\sup_n n^{-1/p} a_n < \infty$.

PROOF: First we observe that the quasi-norm on X may be replaced by the Aoki–Rolewicz theorem by an equivalent quasi-norm satisfying

$$\|x+y\|^r \leq \|x\|^r + \|y\|^r \quad x, y \in X.$$

Clearly it is enough to prove the lemma for this equivalent quasi-norm. Let $\{x_1 \dots x_{2n}\}$ be a set of $2n$-vectors with $\|x_i\| \leq 1$. For some choice of signs $\epsilon_i = \pm 1$, $1 \leq i \leq 2n$ we have

$$\left\| \sum_{i=1}^{2n} \epsilon_i x_i \right\| \leq b_{2n}.$$

Let $A = \{i\colon \epsilon_i = +1\}$ and $B = \{i\colon \epsilon_i = -1\}$. Without loss of generality we suppose $|A| \leq n$. Then

$$\sum_{i=1}^{2n} x_i = 2 \sum_{i \in A} x_i - \sum_{i=1}^{2n} \epsilon_i x_i$$

and hence

$$\begin{aligned} \left\| \sum_{i=1}^{2n} x_i \right\|^r &\leq 2^r \left\| \sum_{i \in A} x_i \right\|^r + b_{2n}^r \\ &\leq 2^r a_n^r + b_{2n}^r \end{aligned}$$

and so

$$a_{2n}^r \leq 2^r a_n^r + b_{2n}^r.$$

Hence

$$(2n)^{-r/p} a_{2n}^r \leq n^{-r/p} a_n^r + (2n)^{-r/p} b_{2n}^r.$$

Now let $\alpha_n = (2^{-n})^{1/p} a_{2^n}$ and $\beta_n = (2^{-n})^{1/p} b_{2^n}$.

$$\alpha_{n+1}^r - \alpha_n^r \leq \beta_{n+1}^r.$$

As (b_n) is submultiplicative and monotone increasing, there exists $q > p$ such that $\sup n^{-1/q} b_n = C < \infty$ (cf. the proof of Proposition 2.2), and thus

$$\beta_n \leq C 2^{-n(1/p - 1/q)}.$$

Hence $\Sigma \beta_n^r < \infty$ and so (α_n) is bounded. It follows easily that $\sup_n n^{-1/p} a_n < \infty$.

THEOREM 2.5: (i) *For $p < 1$, the following are equivalent*

(a) $\lim n^{-1/p} a_n = 0$,

(b) $\lim n^{-1/p} b_n = 0$,

(c) *X is locally r-convex for some $r > p$.*

(ii) $\lim n^{-1} b_n = 0$ *if and only if X is isomorphic to a B-convex normed space.*

PROOF: (i) Clearly (a)$\Rightarrow$(b). For (b)$\Rightarrow$(a), note that since (b_n) is submultiplicative $\lim n^{-1/r}b_n = 0$ for some $r > p$. Hence $\sup n^{-1/r}a_n < \infty$ and hence $\lim n^{-1/r}a_n = 0$. (a)$\Leftrightarrow$(c) is Proposition 2.2(ii).

(ii) If $\lim n^{-1}b_n = 0$ then $\sup n^{-1}a_n < \infty$ and hence X is locally convex. X is then B-convex by definition. The converse is trivial.

We shall not be further interested in the condition $\lim n^{-1/p}b_n = 0$ for $p < 1$, which was considered only for its analogy to B-convexity. Our main interest in Theorem 2.5 is that $\lim n^{-1}b_n = 0$ implies local convexity. We now turn to the "three space problem" and give our first result.

THEOREM 2.6: *Let X be an F-space with a closed subspace Y such that X/Y and Y are both isomorphic to B-convex Banach spaces. Then X is isomorphic to a B-convex Banach space.*

REMARK: If we assume X is a Banach space, this is due to Giesy [5]. Our main interest here is that the hypotheses force X to be locally convex; we shall see later (Theorem 4.10) that the assumption on Y may be weakened for this conclusion.

PROOF: Our proof closely follows the proof of Theorem 1 of [4].

First we observe that X is locally bounded and hence may be quasi-normed. We suppose that the quasi-norm has modulus of concavity k.

Next, for any locally bounded space Z define $c_n(Z)$ to be the least constant such that

$$\frac{1}{2^n}\sum_{\epsilon_i=\pm 1}\left\|\sum_{i=1}^n \epsilon_i x_i\right\|^2 \leq nc_n^2\sum_{i=1}^n \|x_i\|^2$$

As observed in [4] it is known that if Z is a Banach space, then Z is B-convex if and only if $c_n(Z)\to 0$. We observe that if Z is isomorphic to a B-convex Banach space then $c_n(Z)\to 0$ (since changing to an equivalent quasi-norm does not affect this phenomenon). Thus in our case $c_n(X/Y)\to 0$ and $c_n(Y)\to 0$.

Now let $(x_{ij}: 1\leq i\leq m, 1\leq j\leq n)$ be a set of mn vectors in X. Denote by θ_{ij} $(1\leq i\leq m, 1\leq j\leq n)$ the first mn Rademacher functions on $[0, 1]$,

$$\frac{1}{2^{mn}}\sum_{\epsilon_{ij}=\pm 1}\left\|\sum_{i,j}\epsilon_{ij}x_{ij}\right\|^2 = \int_0^1\left\|\sum_{i=1}^m\sum_{j=1}^n \theta_{ij}(t)x_{ij}\right\|^2 dt$$

Denote by $\varphi_1 \dots \varphi_m$ the first m Rademacher functions on $[0, 1]$. By symmetry

$$\int_0^1 \left\| \sum_{i=1}^m \sum_{j=1}^n \theta_{ij}(t) x_{ij} \right\|^2 dt = \int_0^1 \int_0^1 \left\| \sum_{i=1}^m \sum_{j=1}^n \varphi_i(s) \theta_{ij}(t) x_{ij} \right\|^2 ds dt$$
$$= \int_0^1 \int_0^1 \left\| \sum_{i=1}^m \varphi_i(s) u_i(t) \right\|^2 ds dt$$

where

$$u_i(t) = \sum_{j=1}^n \theta_{ij}(t) x_{ij} \quad 1 \le i \le m$$

Let

$$A(t) = \left\{ \int_0^1 \left\| \sum_{i=1}^m \varphi_i(s) u_i(t) \right\|^2 ds \right\}^{1/2} \quad 0 \le t \le 1$$

Since u_i is a simple X-valued function on $[0, 1]$ we may choose a simple Y-valued function v_i such that

$$\|u_i(t) + v_i(t)\| \le 2\|\pi(u_i(t))\|$$

where $\pi: X \to X/Y$ is the quotient map. Then

$$\|v_i(t)\| \le k(\|u_i(t)\| + 2\|\pi(u_i(t))\|)$$
$$\le 3k\|u_i(t)\|$$

Now

$$\left\| \sum_{i=1}^m \varphi_i(s) u_i(t) \right\| \le k\left(\left\| \sum_{i=1}^m \varphi_i(s)(u_i(t) + v_i(t)) \right\| + \left\| \sum_{i=1}^m \varphi_i(s) v_i(t) \right\| \right)$$

Hence

$$A(t) \le k\left\{ \int_0^1 \left\| \sum_{i=1}^m \varphi_i(s)(u_i(t) + v_i(t)) \right\|^2 ds \right\}^{1/2} + k\left\{ \int_0^1 \left\| \sum_{i=1}^m \varphi_i(s) v_i(t) \right\|^2 ds \right\}^{1/2}$$
$$\le k\sqrt{m}\left[c_m(X)\left(\sum_{i=1}^m \|u_i(t) + v_i(t)\|^2 \right)^{1/2} + c_m(Y)\left(\sum_{i=1}^m \|v_i(t)\|^2 \right)^{1/2} \right]$$
$$\le k\sqrt{m}\left[2c_m(X)\left(\sum_{i=1}^m \|\pi u_i(t)\|^2 \right)^{1/2} + 3kc_m(Y)\left(\sum_{i=1}^m \|u_i(t)\|^2 \right)^{1/2} \right].$$

Thus

$$\left\{ \int_0^1 A(t)^2 dt \right\}^{1/2} \le k\sqrt{m}\left[2c_m(\mathrm{X})\left(\sum_{i=1}^m \int_0^1 \|\pi u_i(t)\|^2 dt \right)^{1/2} \right.$$

$$+ 3kc_m(Y)\left(\sum_{i=1}^m \int_0^1 \|u_i(t)\|^2 dt\right)^{1/2}\Bigg]$$

$$\leq k\sqrt{mn}\Bigg[2c_m(X)c_n(X/Y)\left(\sum_{i=1}^m \sum_{j=1}^n \|\pi(x_{ij})\|^2\right)^{1/2}$$

$$+ 3kc_m(Y)c_n(X)\left(\sum_{i=1}^m \sum_{j=1}^n \|x_{ij}\|^2\right)^{1/2}\Bigg].$$

Thus

$$c_{mn}(X) \leq k[2c_m(X)c_n(X/Y) + 3kc_m(Y)c_n(X)].$$

In particular

$$c_{m^2}(X) \leq [2kc_m(X/Y) + 3k^2c_m(Y)]c_m(X).$$

As $\lim_{m\to\infty}[2kc_m(X/Y) + 3k^2c_m(Y)] = 0$, we conclude that for some large enough m, $c_m(X) = \alpha < 1$.

Hence for any $x_1 \ldots x_m$, $\|x_i\| \leq 1$

$$\frac{1}{2^m}\sum_{\epsilon_i+\pm 1}\left\|\sum \epsilon_i x_i\right\|^2 \leq \alpha^2 m^2.$$

Thus for some $\epsilon_i = \pm 1$

$$\left\|\sum \epsilon_i x_i\right\| \leq \alpha m$$

and so $b_m \leq \alpha m$. As (b_m) is submultiplicative $\lim m^{-1}b_m = 0$, and hence X is a B-convex Banach space.

Remark: (i) We shall show later that it is not true that if X/Y and Y are isomorphic to Banach spaces then X is isomorphic to a Banach space.

(ii) Results for p-Banach spaces ($p < 1$) can be obtained by these methods, but we will obtain better results in this case in the next section by different techniques.

We conclude this section by showing that the containing Banach space of a non-locally convex locally bounded space is never B-convex.

THEOREM 2.7: *Let X be a locally bounded F-space and T a B-convex Banach space. Suppose $T: X \to Y$ is a continuous linear operator. Suppose U is the unit ball of X and that $\overline{\mathrm{co}}\, T(U)$ is a neighbourhood of 0 in Y. Then T is an open mapping of X onto Y.*

PROOF: For each $n \geq 1$, let $U_n = \{\frac{1}{n}(u_1 + \cdots + u_n): u_i \in U, 1 \leq i \leq n\}$. Then each U_n is bounded and $\bigcup_{n\geq 1} T(U_n)$ is dense in co $T(U)$. If $x \in U_n$, $x = \frac{1}{n}(u_1 + \cdots + u_n)$, then for some choice of signs $\epsilon_i = \pm 1$

$$\|\epsilon_1 u_1 + \cdots + \epsilon_n u_n\| \leq b_n \|T\|.$$

where $b_n = b_n(Y)$. Hence for some $k \leq \frac{1}{2}n$, and a set A of $\{1, 2, \ldots, n\}$ with $|A| = k$

$$\left\| nx - 2 \sum_{i \in A} u_i \right\| \leq 2b_n \|T\|.$$

Then

$$T(U_n) \subset \left(2\|T\| \frac{b_n}{n}\right) B + \bigcup_{k \leq n/2} T(U_k)$$

where B is the unit ball of Y.

As $b_n/n \to 0$, there exists $p > 0$ such that $b_n \leq Cn^{1-p}$, $n \in \mathbb{N}$ for some $C < \infty$. Now let $V_n = \bigcup_{k \leq 2^n} U_k$. Then

$$T(V_{n+1}) \subset (2C\|T\| 2^{-np}) B + T(V_n)$$

and so for $m > n$

$$T(V_m) \subset T(V_n) + \left(2C\|T\| \frac{2^{-np}}{1 - 2^{-p}}\right) B.$$

Choose $\epsilon > 0$ so that $\epsilon B \subset \overline{\mathrm{co}}\, T(U)$ and n so that $2C\|T\|2^{-np}(1 - 2^{-p})^{-1} \leq \epsilon/4$. Then

$$\begin{aligned} \overline{\mathrm{co}}\, T(U) &\subset T\left(\bigcup_{m=1}^{\infty} V_m\right) + \frac{\epsilon}{4} B \\ &\subset T(V_n) + \frac{\epsilon}{2} B \\ &\subset T(V_n) + \tfrac{1}{2} \overline{\mathrm{co}}\, T(U). \end{aligned}$$

Let $W_m = V_n + \frac{1}{2} V_n + \cdots + (1/2^m) V_n$, $m \geq 1$. Then

$$\overline{\mathrm{co}}\, T(U) \subset T(W_m) + \frac{1}{2^{m+1}} \overline{\mathrm{co}}\, T(U)$$

and hence

$$\overline{\text{co}}\, T(U) \subset \overline{T(W)}$$

where $W = \cup(W_m : m \geq 1)$.

However W is bounded. This is because V_n is bounded, and for $r > 0$, there is a constant $A < \infty$, such that

$$\|x_i + \cdots + x_n\| \leq A(\|x_1\|^r + \cdots + \|x_n\|^r)^{1/r}$$

for $x_1 \dots x_n \in X$. Thus if $w \in W$

$$\|w\| \leq A\left(\sum_{m=1}^{\infty} 2^{-mr}\right)^{1/r} \sup_{v \in V_n} \|v\|.$$

Thus the map T is almost open, and it follows from the Open Mapping Theorem that T is open and surjective.

If X is an F-space whose dual separates points, then the Mackey topology on X is the finest locally convex topology on X consistent with the original topology. This topology is metrizable and the completion of X in the Mackey topology is called the containing Fréchet space $\hat{X}$ of X. If X is locally bounded then $\hat{X}$ is the containing Banach space of X.

THEOREM 2.8: *If X is a locally bounded F-space whose dual separates points and whose containing Banach space is B-convex, then X is locally convex (and hence $X = \hat{X}$).*

PROOF: The natural identity map $i : X \to \hat{X}$ satisfies the conditions of Theorem 2.7.

REMARKS: This resolves a question of J.H. Shapiro (private communication): is it possible to have a locally bounded non-locally convex F-space whose containing Banach space is a Hilbert space? The above theorem shows that any locally bounded F-space whose containing Banach space is a Hilbert space is locally convex (and hence a Hilbert space). It is possible to give examples of non-locally convex locally bounded F-spaces whose containing Banach spaces are reflexive; see the 'pseudo-reflexive' spaces constructed in [8] (these were based on a suggestion of A. Pełczynski). The author does not know whether c_0 can be the containing Banach space of a non-locally convex locally bounded space.

We conclude by restating Theorem 2.7 in pure Banach space terms. Denote by $\mathrm{co}_p\, U$ the p-convex hull of the set U.

THEOREM 2.9: *Suppose X is a B-convex Banach space and U is a balanced bounded subset of X such that $\overline{\mathrm{co}}\, U$ is a neighbourhood of 0. Then for any $0 < p < 1$, $\overline{\mathrm{co}}_p\, U$ is a neighbourhood of 0.*

PROOF: Let V be the linear span of $\overline{\mathrm{co}}_p\, U$; then V is a locally bounded F-space with unit ball $\overline{\mathrm{co}}_p\, U$. Apply Theorem 2.7 to the identity map $i: V \to X$.

3. Lifting theorems

In this section, we shall approach the three space problem from a different direction. We shall say that an ordered pair of F-spaces (X, Y) *splits* if every short exact sequence of F-spaces $0 \to Y \to Z \to X \to 0$ splits, i.e. if whenever Z is an F-space containing Y such that $Z/Y \cong X$ then Y is complemented in Z and so $Z \cong X \oplus Y$.

THEOREM 3.1: *The following are equivalent*:

(i) *(X, Y) splits*;

(ii) *whenever Z is an F-space containing Y and $T: X \to Z/Y$ is a linear operator, there is a linear operator $\tilde{T}: X \to Z$ such that $\pi\tilde{T} = T$, where $\pi: Z \to Z/Y$ is the quotient map.*

PROOF: Of course (ii)$\Rightarrow$(i) is trivial. Conversely suppose (X, Y) splits and $T: X \to Z/Y$ is a linear operator. Let $G \subset Y \oplus Z$ be the set of (x, z) such that $Tx = \pi z$; then G contains a subspace $Y_0 = \{(0, y): y \in Y\}$ isomorphic to Y. Clearly the map $q_1: G \to X$ given by $q_1(x, y) = x$ is a surjection with kernel Y_0. Hence there is a lifting $S: X \to G$ such that $q_1 \circ S = id_X$. Define $\tilde{T} = q_2 \circ S$ where $q_2(x, y) = y$. Then $\pi\tilde{T}(x) = \pi q_2 S(x) = \pi(z)$ where $(x, z) \in G$. Thus $\pi\tilde{T}(x) = Tx$ as required.

REMARKS: If Y is a locally bounded space, then the pair (L_0, Y) splits where $L_0 = L_0(0, 1)$ is the space of measurable functions on $(0, 1)$ (see [7]). It follows also from results in [7] that (ω, Y) splits, where ω is the space of all sequences.

If X and Y are p-Banach spaces, then let us say that (X, Y) *p-splits* if every short exact sequence $0 \to Y \to Z \to X \to 0$, with Z a

p-Banach space, splits. Clearly (ℓ_p, Y) p-splits for any p-Banach space Y, while (L_p, Y) p-splits provided Y is locally r-convex for some $r > p$ or Y is a pseudo-dual space (see [7]). We remark also that (ℓ_2, ℓ_2) does not 1-split ([4]).

Suppose X and Y are quasi-normed spaces. We denote by $\Lambda(X, Y)$ the spaces of all maps $f: X \to Y$ satisfying

(i) $f(\lambda x) = \lambda f(x)$, $\lambda \in \mathbb{R}$, $x \in X$

(ii) $\|f(x+y) - f(x) - f(y)\| \le M(\|x\| + \|y\|)$, $x, y \in X$

where M is a constant, independent of x and y. We denote by $\Delta(f)$ the least constant M in (ii).

Lemma 3.2: *For given X and Y, there exists $r > 0$ and $A < \infty$ such that for any $f \in \Lambda(X, Y)$ and $x_1, \ldots, x_n \in X$*

$$\left\| f\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n f(x_i) \right\| \le A\Delta(f)\left\{\sum_{i=1}^n \|x_i\|^r\right\}^{1/r}$$

Proof: There exists $0 < p < 1$ such that Y may be given an equivalent quasi-norm $|\cdot|$ satisfying

$$|y_1 + y_2|^p \le |y_1|^p + |y_2|^p \quad y_1, y_2 \in Y$$

and such that

$$\|y\| \le |y| \le C\|y\| \quad y \in Y$$

for some $C < \infty$. Then

$$\begin{aligned} |f(x_1 + x_2) - f(x_1) - f(x_2)| &\le C\Delta(f)(|x_1| + |x_2|) \quad x_1, x_2 \in X \\ &\le C\Delta(f)(|x_1|^p + |x_2|^p)^{1/p}. \end{aligned}$$

By induction we have for $n = 2, 3 \ldots$

$$\left| f\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n f(x_i) \right| \le C\Delta(f)\left(\sum_{i=1}^n i|x_i|^p\right)^{1/p} \qquad (*)$$

Indeed $(*)$ is trivial for $n = 2$. Now suppose it is known for $n - 1$. Then

$$\left| f\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n f(x_i) \right|^p \le$$

$$\leq \left|f\left(\sum_{i=2}^{n} x_i\right) - \sum_{i=2}^{n} f(x_i)\right|^p + \left|f\left(\sum_{i=1}^{n} x_i\right) - f\left(\sum_{i=2}^{n} x_i\right) - f(x_1)\right|^p$$

$$\leq C^p \Delta^p(f)\left[\sum_{i=2}^{n} (i-1)|x_i|^p + \left|\sum_{i=2}^{n} x_i\right|^p + |x_1|^p\right]$$

$$\leq C^p \Delta^p(f)\left[\sum_{i=1}^{n} i|x_i|^p\right].$$

Now let $r = p/2$. For $x_1 \dots x_n \in X$ with $|x_1| \geq |x_2| \dots \geq |x_n|$,

$$\left|f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i)\right| \leq C\Delta(f)\left(\sum_{i=1}^{n} i|x_i|^p\right)^{1/p}$$

$$\leq C\Delta(f)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i|^r |x_j|^r\right)^{1/p}$$

$$= C\Delta(f)\left(\sum_{i=1}^{n} |x_i|^r\right)^{1/r}$$

By re-ordering, this inequality holds for all $x_1 \dots x_n$, and hence

$$\left\|f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i)\right\| \leq C^2\Delta(f)\left(\sum_{i=1}^{n} \|x_i\|^r\right)^{1/r}$$

PROPOSITION 3.3: *Let X and Y be complete quasi-normed spaces and let X_0 be a dense subspace of X. Then the following conditions are equivalent*:

(i) *(X, Y) splits*;

(ii) *if $f \in \Lambda(X_0, Y)$ there exists a linear map $h : X_0 \to Y$ and $L < \infty$,*

$$\|f(x) - h(x)\| \leq L\|x\| \quad x \in X_0$$

(iii) *there is a constant $B < \infty$ such that for any $f \in \Lambda(X_0, Y)$ there exists a linear map $h : X_0 \to Y$ with*

$$\|f(x) - h(x)\| \leq B\Delta(f)\|x\|. \quad x \in X_0$$

PROOF: (i)$\Rightarrow$(ii). Let $f \in \Lambda(X_0, Y)$. Choose $r > 0$ sufficiently small and a constant $C < \infty$ such that for $x_1, \dots, x_n \in X$, $y_1, \dots, y_n \in Y$

$$\|x_1 + \cdots + x_n\| \leq C\left(\sum \|x_i\|^r\right)^{1/r}$$

$$\|y_1 + \cdots + y_n\| \le C\Big(\sum \|y_i\|^r\Big)^{1/r}$$

$$\Big\|f\Big(\sum_{i=1}^n x_i\Big) - \sum_{i=1}^n f(x_i)\Big\| \le C\Big(\sum \|x_i\|^r\Big)^{1/r}$$

(The existence of such r, C follows from Lemma 3.2 and the Aoki–Rolewicz Theorem.) Let $Z_0 = X_0 \oplus Y$ (as a vector space) and define for $(x, y) \in Z_0$

$$\|(x, y)\| = \inf\Big\{\sum_{i=1}^m \|x_i\|^r + \sum_{i=1}^n \|y_i\|^r\Big\}^{1/r}$$

where the infimum is taken over all m, n and $x_1 \ldots x_m \in X_0$, $y_1 \ldots y_n \in Y$ such that

$$x = \sum_{i=1}^m x_i$$

and

$$y = \sum_{i=1}^m f(x_i) + \sum_{i=1}^n y_i.$$

We observe:

(1) $$\|(x_1 + x_2, y_1 + y_2)\| \le (\|(x_1, y_1)\|^r + \|(x_2, y_2)\|^r)^{1/r}$$

$$x_1, x_2 \in X_0, \quad y_1, y_2 \in Y.$$

(2) $$\|(0, y)\| \le \|y\| \quad y \in Y.$$

If

$$\sum_{i=1}^n x_i = 0$$

and

$$\sum_{i=1}^m f(x_i) + \sum_{i=1}^n y_i = y$$

then

$$\|y\|^r \le C^r\Big(\sum_{i=1}^n \|y_i\|^r + \Big\|\sum_{i=1}^m f(x_i)\Big\|^r\Big)$$

but

$$\Big\|\sum_{i=1}^m f(x_i)\Big\| = \Big\|\sum_{i=1}^m f(x_i) - f(0)\Big\| \le C\Big(\sum \|x_i\|^r\Big)^{1/r}$$

Hence

$$\|y\|^r \le C^2\Big(\sum_{i=1}^n \|y_i\|^r + \sum_{i=1}^m \|x_i\|^r\Big)^{1/r}$$

Thus

$$(3) \qquad \|(0, y)\| \geq C^{-2}\|y\|.$$

We also have

$$(4) \qquad \|(x, y)\| \geq C^{-1}\|x\| \quad x \in X_0, \quad y \in Y.$$

$$(5) \qquad \|(x, f(x))\| \leq \|x\|.$$

(1), (3) and (4) together imply that $\|\cdot\|$ is a quasi-norm on Z_0. Let Z be the completion of Z_0. If we let $P : Z_0 \to X_0$ by $P(x, y) = x$, then P is continuous by (4) and hence may be extended to an operator $\bar{P} : Z \to X$. Condition (5) implies that P is open and hence $\bar{P}$ is open and surjective.

Let $N \subset Z_0$ be the space $\{0\} \times Y$; by (2) and (3) N is isomorphic to Y and hence is closed in Z. Suppose $z \in Z$ and $\bar{P}z = 0$; there exist $(x_n, y_n) \in Z_0$ such that $(x_n, y_n) \to z$ and $x_n \to 0$. Then $(0, y_n - f(x_n)) \to z$ so that $z \in N$. Hence $\ker \bar{P} = N \cong Y$.

Now, by assumption (i), there is a linear operator $T : X \to Z$ such that $\bar{P}T = id_X$. For $x \in X_0$, $\bar{P}^{-1}\{x\} \subset Z_0$ so that we may write $Tx = (x, h(x))$ where $h : X_0 \to Y$ is linear. Then

$$\begin{aligned} \|h(x) - f(x)\| &\leq C^2\|(x, h(x)) - (x, f(x))\| \\ &\leq C^2(\|Tx\|^r + \|x\|^r)^{1/4} \\ &\leq L\|x\| \quad x \in X \end{aligned}$$

where $L = C^2(\|T\|^r + 1)^{1/r}$.

(ii)$\Rightarrow$(iii). Let $(e_\alpha : \alpha \in \mathcal{A})$ be a Hamel basis of X_0, and let $\Lambda_0(X_0, Y)$ be the subspace of $\Lambda(X, Y)$ of all f such that $f(e_\alpha) = 0$, $\alpha \in \mathcal{A}$. Then $\Lambda_0(X_0, Y)$ is a vector space, under pointwise addition, and Δ is a quasi-norm on $\Lambda_0(X_0, Y)$. If $x \in X_0$, then $x = \Sigma\, \xi_\alpha e_\alpha$ for some finitely non-zero (ξ_α), and hence for $f \in \Lambda_0(X_0, Y)$

$$\|f(x)\| \leq A\Big(\sum |\xi_\alpha|^r \|e_\alpha\|^r\Big)^{1/r} \Delta(f)$$

where r, A are chosen as in Lemma 3.2. Hence the evaluation maps $f \to f(x)$ are Δ-continuous. If (f_n) is a Δ-Cauchy sequence in $\Lambda_0(X, Y)$, then $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X_0$ and it is easy to show that $\Delta(f - f_n) \to 0$. Hence (Λ_0, Δ) is complete.

We also define for $f \in \Lambda_0$

$$\|f\|^* = \inf_h \sup_{\|x\| \le 1} \|f(x) - h(x)\|$$

where the infimum is taken over all linear h. By assumption $\|f\|^* < \infty$ for $f \in \Lambda_0$, and it is clear that $\|\cdot\|^*$ is also a quasi-norm on $\Lambda_0(X_0, Y)$. Clearly also $\Delta(f) \le C\|f\|^*$, for some constant C (depending on the modulus of concavity of the quasi-norm in Y).

We shall show that $(\Lambda_0, \|\cdot\|^*)$ is also complete. To do this it is enough to show that if $\|f_n\|^* \le 2^{-n}$ then Σf_n converges. We observe that there is a constant M such that if $y_n \in Y$ and $\|y_n\| \le 2^{-n}$ ($n = 1, 2, \ldots$) then Σy_n converges and

$$\left\| \sum y_n \right\| \le M,$$

(cf. the proof of Theorem 2.7).

The series Σf_n converges to some f in (Λ_0, Δ). Choose linear maps h_n so that

$$\|f_n(x) - h_n(x)\| \le 2^{1-n}\|x\| \quad x \in X_0.$$

Then

$$\|h_n(e_\alpha)\| \le 2^{1-n}\|e_\alpha\| \quad \alpha \in \mathcal{A}$$

and hence Σh_n converges pointwise to a linear map h. Now

$$\begin{aligned} &\left\| f(x) - \sum_{i=1}^{n} f_i(x) - h(x) + \sum_{i=1}^{n} h_i(x) \right\| \\ &\quad = \left\| \sum_{i=n+1}^{\infty} (f_i(x) - h_i(x)) \right\| \\ &\quad \le M 2^{1-n}\|x\|, \end{aligned}$$

and hence $\|f - \Sigma_{i=1}^n f_i\|^* \to 0$.

Now by the closed graph theorem, there is a constant $B < \infty$ such that $\|f\|^* \le \frac{1}{2}B\Delta(f)$, $f \in \Lambda_0$. For general $f \in \Lambda(X, Y)$, define a linear map

$$g\left(\sum \xi_\alpha e_\alpha\right) = \sum \xi_\alpha f(e_\alpha).$$

Then $f-g \in \Lambda_0$ so that there exists a linear h such that

$$\begin{aligned}\|f(x)-g(x)-h(x)\| &\leq B\Delta(f-g)\\ &= B\Delta(f).\end{aligned}$$

Since $g+h$ is linear the result is proved.

(iii)$\Rightarrow$(ii) Trivial.

(iii)$\Rightarrow$(i) Suppose Z is a complete quasi-normed space and $S:Z\to X$ is a surjective operator such that $S^{-1}(0)\cong Y$. Let $T:S^{-1}(0)\to Y$ be an isomorphism.

Let $\rho:X\to Z$ be any, not necessarily continuous, linear map, such that $\rho S(x)=x$, $x\in X$. Since S is onto, there is a $C<\infty$ such that for any $x\in X$, there exists $z\in Z$ with $Sz=x$ and $\|z\|\leq C\|x\|$. Let $\sigma:X\to Z$ be a map satisfying $\sigma S(x)=x$, $x\in X$, $\sigma(\lambda x)=\lambda\sigma(x)$, $\lambda\in\mathbb{R}$, $x\in X$ and $\|\sigma(x)\|\leq C\|x\|$. Now consider the map $f:X\to Y$ defined by

$$f(x)=T(\sigma(x)-\rho(x)).$$

Then

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &= \|T(\sigma(x+y)-\sigma(x)-\sigma(y))\|\\ &\leq Ck^2\|T\|(\|x+y\|+\|x\|+\|y\|)\\ &\leq 2Ck^3\|T\|(\|x\|+\|y\|)\end{aligned}$$

where k is the modulus of concavity of the norm on Y.

Thus there is a linear map $h:X_0\to Y$ such that

$$\|f(x)-h(x)\|\leq L\|x\| \quad x\in X_0.$$

Now define $R:X_0\to Z$ by

$$Rx=\rho(x)+T^{-1}h(x)$$

R is clearly linear and

$$\begin{aligned}Rx &= \rho(x)+T^{-1}f(x)+T^{-1}(h(x)-f(x))\\ &= \sigma(x)+T^{-1}(h(x)-f(x))\end{aligned}$$

so that

$$\|Rx\|\leq k(\|\sigma(x)\|+\|T^{-1}\|\,\|h(x)-f(x)\|)$$

Hence

$$\|Rx\|\leq k(C+\|T^{-1}\|L)\|x\|$$

Thus R is continuous and extends to a map $\bar{R}: X \to Z$. Clearly $S\bar{R} = id_X$ so that Z splits.

Let $(u_i: i \in I)$, with I some index set, be an unconditional Schauder basis of a quasi-normed space X. We shall say that $(u_i: i \in I)$ is *p-concave* if there is a constant $C < \infty$, such that whenever $I_1, \ldots, I_n$ are disjoint subsets of I, then

$$\left(\sum_{j=1}^{n} \left\| \sum_{i \in I_j} t_i u_i \right\|^p\right)^{1/p} \leq C \left\| \sum_{j=1}^{n} \sum_{i \in I_j} t_i u_i \right\|$$

for all $(t_i: i \in I)$. The best constant C in this equation will be called the *degree* (of p-concavity) of $(u_i: i \in I)$.

LEMMA 3.4: *Let Y be a locally r-convex quasi-normed space and let $A < \infty$ be a constant such that*

$$\|y_1 + \cdots + y_n\| \leq A(\|y_1\|^r + \cdots + \|y_n\|^r)^{1/r}$$

for $y_1, \ldots, y_n \in Y$. If $p < r$, and X is a quasi-normed space with a p-concave basis (u_i) with degree C, then for $f \in \Lambda(X, Y)$

$$\left\| f\left(\sum_{i \in I} t_i u_i\right) - \sum_{i \in I} t_i f(u_i) \right\| \leq AC\Delta(f) \left(\sum_{k=1}^{\infty} \left(\frac{2}{k}\right)^{r/p}\right)^{1/r} \left\| \sum_{i \in I} t_i u_i \right\|$$

for $(t_i: i \in I)$ finitely non-zero.

PROOF: For $y \in Y$ define

$$|y| = \inf\left\{\left(\sum_{i=1}^{n} \|y_i\|^r\right)^{1/r} : \sum_{i=1}^{n} y_i = y, n \in \mathbb{N}\right\}$$

Then $|y| \leq \|y\| \leq A|y|$, and $|y|^r$ is an r-norm on Y. We shall establish first that if $\dim X = m < \infty$ and $f \in \Lambda(X, Y)$

$$\left| f\left(\sum_{i \in I} t_i u_i\right) - \sum_{i \in I} t_i f(u_i) \right| \leq C\Delta(f) \left[\sum_{k=1}^{m} \left(\frac{2}{k}\right)^{r/p}\right]^{1/r} \left\| \sum_{i \in I} t_i u_i \right\|, \quad (\dagger)$$

for all $(t_i: i \in I)$; of course in this case $|I| = m$.

Equation (†) is trivial for $m = 1$; we complete the proof by induction. Suppose the result proved for $1 \leq m < n$, and suppose

$\dim X = n$. Suppose $f \in \Lambda(X, Y)$ and $x \in X$, where

$$x = \sum_{i \in I} t_i u_i$$

then

$$C^p \|x\|^p \geq \sum_{i \in I} |\xi_i|^p \|u_i\|^p.$$

Hence for some $j, k \in I$, since $|I| = n$,

$$|t_j|^p \|u_j\|^p + |t_k|^p \|u_k\|^p \leq \frac{2C^p \|x\|^p}{n}.$$

Now consider the subspace X_0 spanned by $\{u_i: i \neq j, k\}$ and $t_j u_j + t_k u_k$. Then $\dim X_0 = n - 1$ and X_0 has p-concave basis, $\{u_i, i \neq j, k\} \cup \{t_j u_j + t_k u_k\}$. Hence

$$\left| f(x) - \sum_{i \neq j, k} t_i f(u_i) - f(t_j u_j + t_k u_k) \right| \leq C\Delta(f) \left(\sum_{k=1}^{n-1} \left(\frac{2}{k} \right)^{r/p} \right)^{1/r} \|x\|.$$

However

$$\begin{aligned} |f(t_j u_j + t_k u_k) - t_j f(u_j) - t_k f(u_k)| &\leq \|f(t_j u_j + t_k u_k) - t_j f(u_j) - t_k f(u_k)\| \\ &\leq \Delta(f)(\|t_j u_j\| + \|t_k u_k\|) \\ &\leq \Delta(f)(\|t_j u_j\|^p + \|t_k u_k\|^p)^{1/p} \\ &\leq C\Delta(f) \left(\frac{2}{n} \right)^{1/p} \|x\|. \end{aligned}$$

Hence

$$\left| f(x) - \sum_{i \in I} t_i f(u_i) \right|^r \leq C^r \Delta^r(f) \|x\|^r \left(\sum_{k=1}^{n-1} \left(\frac{2}{k} \right)^{r/p} + \left(\frac{2}{n} \right)^{r/p} \right)$$

and (†) is proved.

Hence if $\dim X = n$

$$\left\| f(x) - \sum t_i f(u_i) \right\| \leq AC\Delta(f) \left(\sum_{k=1}^{n} \left(\frac{2}{k} \right)^{r/p} \right)^{1/r} \|x\|$$

and the lemma follows trivially, since if $\dim X = \infty$, then each $\Sigma\, t_i u_i$ with $(t_i: i \in I)$ finitely non-zero belongs to the subspace spanned by $(u_i: t_i \neq 0)$.

THEOREM 3.5: *Let X be a locally bounded F-space with a p-concave unconditional basis, and let Y be a locally r-convex locally bounded F-space. Then (X, Y) splits, provided $p < r \leq 1$.*

In particular (ℓ_p, Y) splits for any locally r-convex locally bounded F-space Y, where $r > p$.

PROOF: Combine Proposition 3.3 and Lemma 3.4.

Let F be an Orlicz function on $[0, \infty)$ i.e. a continuous, non-decreasing function such that $F(0) = 0$ and $F(x) > 0$ for $x > 0$. F is said to satisfy the Δ_2-condition at 0 (respectively at ∞) if

$$\sup_{0<x\leq 1} \frac{F(2x)}{F(x)} < \infty$$

$$\left(\text{respectively } \sup_{1\leq x<\infty} \frac{F(2x)}{F(x)} < \infty\right).$$

The Orlicz sequence space ℓ_F is the space of all sequences $x = (x_n)$ such that $\Sigma\, F(|tx_n|) < \infty$ for some $t > 0$. This is an F-space with a base of neighbourhoods of 0 of the form $rB_F(\epsilon)$, $r > 0$, $\epsilon > 0$ where

$$B_F(\epsilon) = \left\{x : \sum F(|x_n|) \leq \epsilon\right\}$$

Similarly $L_F = L_F(0, 1)$ is the space of measurable functions $x = x(s)$ on $(0, 1)$ such that

$$\int_0^1 \mathrm{F}(|tx(s)|)ds < \infty$$

for some $t > 0$ with a base of neighbourhoods of the form $rB_F(\epsilon)$, $r > 0$, $\epsilon > 0$, where

$$B_F(\epsilon) = \left\{x : \int_0^1 F(|x(s)|)ds \leq \epsilon\right\}.$$

Let

$$\alpha_F = \sup\left\{p \geq 0 : \sup_{0<t,x\leq 1} \frac{F(tx)}{t^p F(x)} < \infty\right\}$$

$$\alpha_F^\infty = \sup\left\{p \geq 0 : \sup_{1\leq t,x<\infty} \frac{t^p F(x)}{F(tx)} < \infty\right\}$$

$$\beta_F = \inf\left\{p \geq 0 \colon \sup_{0<t,x\leq 1} \frac{t^p F(x)}{F(tx)} < \infty\right\}$$

$$\beta_F^\infty = \inf\left\{p \geq 0 \colon \sup_{1\leq t,x<\infty} \frac{F(tx)}{t^p F(x)} < \infty\right\}$$

Then $0 \leq \alpha_F \leq \beta_F \leq \infty$ and $0 \leq \alpha_F^\infty \leq \beta_F^\infty \leq \infty$. ℓ_F is locally bounded if and only if $\alpha_F > 0$ and L_F is locally bounded if and only if $\alpha_F^\infty > 0$; F satisfies the Δ_2-condition at 0 (respectively at ∞) if and only if $\beta_F < \infty$ (respectively $\beta_F^\infty < \infty$) (see [6], [11] and [13]).

If ℓ_F or L_F is locally bounded, we may define the quasi-norm $x \mapsto \|x\|$ so that

$$\sum_{n=1}^\infty F\left(\frac{|x_n|}{\|x\|}\right) = 1$$

or

$$\int_0^1 F\left(\frac{|x(s)|}{\|x\|}\right) ds = 1.$$

THEOREM 3.6: *Let F be an Orlicz function and let Y be an r-Banach space. Then*

(i) *if $0 < \alpha_F \leq \beta_F < r$, (ℓ_F, Y) splits;*

(ii) *if $0 < \alpha_F^\infty \leq \beta_F^\infty < r$, (L_F, Y) splits.*

PROOF: (i) Select p so that $\beta_F < p < r$. The unit vectors (e_n) form a basis of ℓ_F (this follows from the Δ_2-conditions at 0). We show the basis is p-concave. We clearly have that there exists a constant $A < \infty$ so that

$$t^p F(x) \leq AF(tx) \quad 0 \leq t \leq 1$$
$$0 \leq F(x) \leq 1.$$

Now suppose $u_1 \dots u_k \in \ell_F$ have disjoint support and that $v = u_1 + \cdots + u_k$. Then if $u_i = (u_i(j))_{j=1}^\infty$

$$\begin{aligned}\sum_{i=1}^k \|u_i\|^p &= \|v\|^p \sum_{i=1}^k \frac{\|u_i\|^p}{\|v\|^p} \sum_{j=1}^\infty F\left(\frac{|u_i(j)|}{\|u_i\|}\right) \\ &\leq A\|v\|^p \sum_{i=1}^k \sum_{j=1}^\infty F\left(\frac{|u_i(j)|}{\|v\|}\right) \\ &\leq A\|v\|^p.\end{aligned}$$

(ii) Select p so that $\beta_F^\infty < p < r$; then there is a constant $A < \infty$ such that

$$F(tx) \le At^p F(x) \quad 1 \le t < \infty$$
$$1 \le F(x) < \infty.$$

Now suppose $u_1, \ldots, u_k$ are functions with disjoint support in L_F, and let $v = u_1 + \cdots + u_k$. Let $K = \{s: F(\|v\|^{-1}|v(s)|) \ge 1\}$. For each i let $M_i = K \cap \operatorname{supp} u_i$ and $M_i' = ((0,1)\backslash K) \cap \operatorname{supp} u_i$. Then

$$\begin{aligned} 1 &= \int_{M_i} f\left(\frac{|u_i(s)|}{\|u_i\|}\right) ds + \int_{M_i'} F\left(\frac{|u_i(s)|}{\|u_i\|}\right) ds \\ &\le A \frac{\|v\|^p}{\|u_i\|^p} \left[\int_{M_i} F\left(\frac{|u_i(s)|}{\|v\|}\right) ds + \mu(M_i')\right] \end{aligned}$$

where μ is a Lebesgue measure on $(0,1)$. Thus

$$\sum_{i=1}^k \|u_i\|^p \le A\|v\|^p \left[\sum_{i=1}^k \int_{M_i} F\left(\frac{|u_i(s)|}{\|v\|}\right) ds + 1\right]$$

but

$$\sum_{i=1}^k \int_{M_i} F\left(\frac{|u_i(s)|}{\|v\|}\right) ds = \int_K F\left(\frac{|u_i(s)|}{\|v\|}\right) ds \le 1.$$

Hence

$$\sum_{i=1}^k \|u_i\|^p \le 2A\|v\|^p.$$

For each $n = 1, 2, \ldots$, let E_n be the 2^n-dimensional subspace of L_F spanned by the characteristic function χ_n^k of the interval $((k-1)2^{-n}, k.2^{-n})$ for $k = 1, 2, \ldots, 2^n$. Let $E = \cup E_n$; then E is dense in L_F (this follows from the Δ_2-condition at ∞). Suppose $f \in \Lambda(E, Y)$. As each E_n has a p-concave basis of degree $(2A)^{1/p}$, there is a constant $C < \infty$ such that for each n, there exists a linear map $h_n: E_n \to Y$ with

$$\|F(x) - h_n(x)\| \le C\|x\|.$$

We shall show that $\lim_{n\to\infty} h_n(x)$ exists for $x \in E$. As Y is locally r-convex there is a constant $B < \infty$ such that

$$\|y_1 + \cdots + y_n\| \le B(\|y_1\|^r + \cdots + \|y_n\|^r)^{1/r}$$

for $y_1 \ldots y_n \in Y$.

Consider $\chi_n^k \in E_n$, and suppose $\ell \ge m \ge n$. Then χ_n^k splits into 2^{m-n}

disjoint characteristic functions of sets of measure 2^{-m} in E_M. For each such set K,

$$F(\|\chi_K\|^{-1}) = 2^m$$

and hence $2^m \leq \|\chi_K\|^{-p}\alpha$, where $F(\alpha) = 1$.

Hence

$$\|\chi_K\| \leq \alpha^{1/p} 2^{-m/p}.$$

Now

$$\begin{aligned}\|h_m(\chi_K) - h_\ell(\chi_K)\| &\leq 2^{1/r} BC \|\chi_K\| \\ &\leq 2^{-m/p+1/r} \alpha^{1/r} BC.\end{aligned}$$

Hence

$$\begin{aligned}\|h_m(\chi_n^k) - h_\ell(\chi_n^k)\| &\leq B^2 C \alpha^{1/p} \,.\, 2^{[(m+1)/r]-m/p} \\ &\to 0 \text{ as } m \to \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} h_n(x)$ exists for $x \in \cup E_n$ and so there exists a linear h on $\cup E_n$ such that

$$\|f(x) - h(x)\| \leq C\|x\|.$$

It follows that (L_F, Y) splits.

4. The three-space problem and $\mathcal{K}$-spaces

As an immediate application of the results of §3, we have

THEOREM 4.1: *Let Z be an F-space with a closed subspace Y, such that Y and Z/Y are locally bounded; suppose Z/Y is locally p-convex and Y is locally r-convex where $p < r \leq 1$. Then Z is a p-Banach space.*

PROOF: That Z is locally bounded is Proposition 1. Since Z/Y is locally p-convex there is a surjection $S: \ell_p(I) \to Z/Y$ for some index set I. Since $(\ell_p(I), Y)$ splits there is a lifting $\tilde{S}: \ell_p(I) \to Z$ such that $\pi\tilde{S} = S$ where $\pi: Z \to Z/Y$ is the quotient map. Let U be the unit ball of $\ell_p(I)$ and V a bounded absolutely r-convex neighbourhood of 0 in Z. Then $\tilde{S}(U) + V$ is absolutely p-convex and bounded in Z. If $z \in Z$ there exists m such that $\pi z \in mS(U)$, and hence $z \in m\tilde{S}(U) + Y$; thus there exists $n \geq m$ such that $z \in n(\tilde{S}(U) + V)$. The set $\tilde{S}(U) + V$ is therefore also absorbing and its closure is a bounded absolutely p-convex neighbourhood of 0 in Z.

REMARKS: Compare Theorem 2.6. We will show by example that the theorem is false if $p = r = 1$.

In [7] an F-space X for which $(X, \mathbb{R})$ splits is called a *$\mathcal{K}$-space.* A $\mathcal{K}_p$-space is defined as a p-Banach space X for which every short exact sequence $0 \to \mathbb{R} \to Z \to X \to 0$ of p-Banach spaces splits. From Theorem 4.1, we have immediately:

THEOREM 4.2: *If $p < 1$, a p-Banach space is a $\mathcal{K}_p$-space if and only if it is a $\mathcal{K}$-space.*

Thus for $p < 1$, the notion of a $\mathcal{K}_p$-space is redundant. For $p = 1$, it is of course trivial – every Banach space is a $\mathcal{K}_1$-space.

THEOREM 4.3: *The following are $\mathcal{K}$-spaces:*
(i) *ℓ_F, where F is an Orlicz function satisfying $0 < \alpha_F \leq \beta_F < 1$;*
(ii) *L_F, where F is an Orlicz function satisfying $0 < \alpha_F^\infty \leq \beta_F^\infty < 1$;*
(iii) *any B-convex Banach space;*
(iv) *L_0 and ω.*

These results follow from Theorem 2.6, Theorem 3.6 and the remarks at the beginning of §3. In particular the spaces ℓ_p, L_p $(p < 1)$ are $\mathcal{K}$-spaces, and (see [7]) any quotient of such a space by a subspace with the Hahn–Banach Extension Property (HBEP).

Our main result in this section is that the space ℓ_1 is not a $\mathcal{K}$-space. Let (e_n) be the usual basis of ℓ_1 and let $E_n = \operatorname{lin}(e_1, \ldots, e_n)$ and $E = \operatorname{lin}(e_1, \ldots, e_n, \ldots) = \cup E_n$. For $x = (x_n) \in E$, with $x_n \geq 0$ for all n define

$$f(x) = \sum_{n=1}^{\infty} \tilde{x}_n \log n$$

where $(\tilde{x}_n)$ is the decreasing re-arrangement of (x_n). We then extend f to E by

$$f(x) = f(x^+) - f(x^-)$$

where $x_n^+ = \max(x_n, 0)$, $x_n^- = \max(-x_n, 0)$.

LEMMA 4.4: *If $x \geq 0$, $y \geq 0$ then*

$$f(x) + f(y) \leq f(x + y) \leq f(x) + f(y) + \log 2(\|x\| + \|y\|)$$

PROOF: For $x \geq 0$

$$f(x) = \min_{\sigma} \sum_{n=1}^{\infty} x_n \log \sigma(n)$$

where σ runs through all permutations of $\mathbb{N}$. Hence

$$f(x+y) = \sum_{n=1}^{\infty} (x_n + y_n) \log \sigma(n)$$

for some σ, and hence

$$\begin{aligned} f(x+y) &= \sum_{n=1}^{\infty} x_n \log \sigma(n) + \sum_{n=1}^{\infty} y_n \log \sigma(n) \\ &\geq f(x) + f(y). \end{aligned}$$

For the opposite inequality consider first the case when x and y have disjoint supports. For each $n \in \mathbb{N}$, there clearly exists $\bar{x}, \bar{y} \in E_n$ which maximize $f(x+y) - f(x) - f(y)$ subject to

(a) $\|x\| + \|y\| = 1$

(b) x, y have disjoint support.

Let $z = \bar{x} + \bar{y}$; we may assume without loss of generality that $z_1 \geq z_2 \geq \cdots \geq z_n$. Thus there are sets M_1 and M_2 such that $M_1 \cap M_2 = \emptyset$, $M_1 \cup M_2 = \{1, 2, \ldots, n\}$ and

$$\begin{aligned} \bar{x}_i &= z_i \quad i \in M_1 \\ &= 0 \quad i \in M_2 \end{aligned}$$

Let $M_1 = \{\pi(1) \ldots \pi(k)\}$, $M_2 = \{\rho(1) \ldots \rho(n-k)\}$ where $\pi(1) < \pi(2) < \cdots < \pi(k)$ and $\rho(1) < \rho(2) < \cdots < \rho(n-k)$.

Clearly z solves the linear programme:

$$\text{Maximize: } \sum_{\ell=1}^{n} u_\ell \log \ell - \sum_{\ell=1}^{k} u_{\pi(\ell)} \log \pi(\ell) - \sum_{\ell=1}^{n-k} u_{\rho(\ell)} \log \rho(\ell)$$

subject to

$$u_1 \geq u_2 \geq \cdots \geq u_n \geq 0$$

and

$$u_i + \cdots + u_n = 1.$$

The extreme points of the feasible set are of the form $u_i = 1/m$, $1 \leq i \leq m$, $u_i = 0$, $m < i$, where $m \leq n$.

At such a point

$$\sum_{\ell=1}^{n} u_\ell \log \ell - \sum_{\ell=1}^{k} u_{\pi(\ell)} \log \pi(\ell) - \sum_{\ell=1}^{n-k} u_{\rho(\ell)} \log \rho(\ell) = \frac{1}{m} \log \binom{m}{r}$$

where $r = |M_1 \cap \{1, 2 \ldots m\}|$. Hence

$$\begin{aligned} f(\bar{x} + \bar{y}) - f(\bar{x}) - f(\bar{y}) &\leq \max_{\substack{1 \leq m \leq n \\ 1 \leq r \leq m}} \frac{1}{m} \log \binom{m}{r} \\ &\leq \log 2 \end{aligned}$$

since $\binom{m}{r} \leq 2^m$ for all r, $1 \leq r \leq m$.

Hence for $x, y \in E_n$, disjoint

$$f(x + y) - f(x) - f(y) \leq (\log 2)(\|x\| + \|y\|).$$

as n is arbitrary we have the result for x, y with disjoint support.

For general $x, y \geq 0$, we proceed by induction on $|M|$ where $M = \{i\colon x_i y_i \neq 0\}$. We have proved the result if $|M| = 0$. Now assume the result proved for $|M| = m - 1$, and suppose that for given x and y, $|M| = m$. Select $j \in M$ and choose k such that $x_k = y_k = 0$. Now define

$$\begin{aligned} x_i^* &= x_i \quad i \neq j, k \\ x_j^* &= 0 \\ x_k^* &= x_j \end{aligned}$$

Then by the inductive hypothesis

$$\begin{aligned} f(x^* + y) &\leq f(x^*) + f(y) + \log 2(\|x\| + \|y\|) \\ &= f(x) + f(y) + \log 2(\|x\| + \|y\|). \end{aligned}$$

Let u and v be the decreasing rearrangements of $x + y$ and $x^* + y$. Suppose $x_j + y_j = u_\ell$, and that $x_j = v_n$ and $y_j = v_N$ where we assume without loss of generality that $n < N$. Clearly $\ell \leq n$, and

$$\begin{aligned} v_i &= u_i && 1 \leq i < \ell \\ v_i &= u_{i+1} && \ell \leq i \leq n \\ v_i &= u_i && n + 1 \leq i \leq N \\ v_i &= u_{i-1} && N < i. \end{aligned}$$

Hence

$$\sum_{i=1}^{\infty} v_i \log i - \sum_{i=1}^{\infty} u_i \log i$$

$$= \sum_{\ell+1}^{n} u_i(\log(i-1) - \log i) + x_j \log n + y_j \log N - (x_j + y_j) \log \ell$$

$$+ \sum_{N}^{\infty} u_i(\log(i+1) - \log i)$$

$$\geq (x_j + y_j) \log \frac{n}{\ell} + \sum_{\ell+1}^{n} (x_j + y_j) \log\left(\frac{i-1}{i}\right)$$

$$\geq 0.$$

Hence $f(x+y) \leq f(x^* + y) \leq f(x) + f(y) + \log 2(\|x\| + \|y\|)$.

LEMMA 4.5: *For any $x, y \in E$*

$$|f(x+y) - f(x) - f(y)| \leq 4 \log 2(\|x\| + \|y\|).$$

PROOF: First we observe that if $x + y \geq 0$, $x \geq 0$ and $y \geq 0$ then

$$|f(x) - f(x+y) - f(-y)| \leq \log 2(\|x+y\| + \|y\|)$$

$$\leq (\log 2)\|x\|$$

so that

$$|f(x+y) - f(x) - f(y)| \leq \log 2(\|x\| + \|y\|).$$

Now suppose in general $z = x + y$. We break $\mathbf{N}$ up into six regions: $M_1 = \{i: x_i \geq 0,\ y_i \geq 0\}$, $M_2 = \{i: x_i \geq 0,\ y_i < 0,\ z_i \geq 0\}$, $M_3 = \{i: x_i \geq 0, y_i < 0,\ z_i < 0\}$, $M_4 = \{i: x_i < 0,\ y_i \geq 0,\ z_i \geq 0\}$, $M_5 = \{i: x_i < 0,\ y_i \geq 0,\ z_i < 0\}$, and $M_6 = \{i: x_i < 0,\ y_i < 0\}$. For $u = x$, y or z let u_i be the restriction of u to M_i. Hence

$$f(z) = f(z_1 + z_2 + z_4) + f(z_3 + z_5 + z_6)$$

by the definition of f and

$$|f(z_1 + z_2 + z_4) - f(z_1) - f(z_2) - f(z_4)| \leq 2 \log 2(\|z_1\| + \|z_2\| + \|z_4\|)$$

$$|f(z_3 + z_5 + z_6) - f(z_3) - f(z_5) - f(z_6)| \leq 2 \log 2(\|z_3\| + \|z_5\| + \|z_6\|).$$

Thus

$$|f(z)-\sum_{i=1}^{6} f(z_i)| \leq 2 \log 2(\|z\|).$$

Similarly

$$|f(x)-\sum_{i=1}^{6} f(x_i)| \leq 2 \log 2\|x\|$$

$$|f(y)-\sum_{i=1}^{6} f(y_i)| \leq 2 \log 2\|y\|.$$

However

$$|f(z_i)-f(x_i)-f(y_i)| \leq \log 2(\|x_i\|+\|y_i\|),$$

and hence

$$\begin{aligned} |f(z)-f(x)-f(y)| &\leq 2 \log 2(\|z\|+\|x\|+\|y\|)+\log 2(\|x\|+\|y\|) \\ &\leq 4 \log 2(\|x\|+\|y\|). \end{aligned}$$

THEOREM 4.6: $(\ell_1, \mathbb{R})$ *does not split, i.e.* ℓ_1 *is not a* $\mathcal{K}$*-space.*

PROOF: Consider $f: E \to \mathbb{R}$. By Lemma 4.5 $f \in \Lambda(E, \mathbb{R})$. Moreover if $h: E \to \mathbb{R}$ is linear and

$$\|h(x)-f(x)\| \leq B\|x\|$$

then $|h(e_i)| \leq B$, so that $|h(x)| \leq B\|x\|$, $x \in E$. However

$$f\left(\frac{1}{n}(e_1+\cdots+e_n)\right)=\frac{1}{n} \log n! \to \infty,$$

and we have a contradiction.

REMARK: Of course if $0 \to \mathbb{R} \to Z \to \ell_1 \to 0$ is an exact sequence then Z is locally p-convex for every $p<1$ (Theorem 4.1).

THEOREM 4.7: *Let* X *be a Banach space, containing a sequence* X_n *of finite-dimensional subspaces such that*

(i) dim $X_n=m(n) \to \infty$;

(ii) *there are linear isomorphisms* $S_n: X_n \to \ell_1^{m(n)}$ *with* $\|S_n\| \leq 1$ *and* $\sup\|S_n^{-1}\|<\infty$;

(iii) *there are projections* $P_n: X \to X_n$ *with* $\|P_n\|=o(\log m(n))$. *Then* X *is not a* $\mathcal{K}$*-space.*

PROOF: Identify $\ell_1^{m(n)}$ with $E_{m(n)} \subset \ell_1$, and consider $f_n(x)=f(S_n P_n x)$

for $x \in X$. Then $f \in \Lambda(X, \mathbb{R})$ and

$$\Delta(f_n) \leq 4 \log 2\|P_n\|.$$

If X is a $\mathcal{K}$-space, there is a constant $L < \infty$ such that for each n, there is a linear map $h_n : X \to \mathbb{R}$ with

$$\|h_n(x) - f_n(x)\| \leq L\|P_n\| \, \|x\|.$$

Now consider $h_n^* : E_{m(n)} \to \mathbb{R}$ defined by

$$h_n^*(x) = h_n(S_n^{-1}x).$$

Then

$$\begin{aligned} |h_n^*(x) - f(x)| &= |h_n(S_n^{-1}x) - f_n(S_n^{-1}x)| \\ &\leq L\|P_n\| \, \|S_n^{-1}\| \, \|x\|. \end{aligned}$$

For the unit vectors e_k, $1 \leq k \leq m(n)$ $f(e_k) = 0$ and hence

$$|h_n^*(e_k)| \leq L\|P_n\| \, \|S_n^{-1}\|.$$

Hence

$$|h_n^*(e_1 + \cdots + e_{m(n)}| \leq L\|P_n\| \, \|S_n^{-1}\| m(n)$$

but

$$f(e_1 + \cdots + e_{m(n)}) = \log m(n)!$$

Thus

$$\log m(n)! \leq 2L\|P_n\| \, \|S^{-1}\| m(n).$$

By Stirling's formula

$$\lim_{n\to\infty} \frac{\log n!}{n \log n} = 1,$$

and hence

$$2L \liminf_{n\to\infty} \frac{\|P_n\| \, \|S_n^{-1}\|}{\log m(n)} \geq 1$$

which contradicts our assumptions.

Example: $\ell_2(\ell_1^{(n)})$ is not a $\mathcal{K}$-space. This space is reflexive, so that not every reflexive Banach space is a $\mathcal{K}$-space.

Stiles [12] asks whether if X is a subspace of ℓ_p with $p < 1$ such that ℓ_p/X is locally convex, does X have the Hahn–Banach Extension Property. The above examples resolve this question negatively, for from results in [7], X has HBEP if and only if ℓ_p/X is a $\mathscr{K}$-space. Thus we have

THEOREM 4.8: (i) *If* $p < 1$ *and* $\ell_p/X \cong \ell_1$ *then* X *does not have the HBEP*;

(ii) *if* $p < 1$, *and* ℓ_p/X *is a* B*-convex Banach space,* X *does have the HBEP.*

We also note:

THEOREM 4.9: ℓ_p $(0 < p < \infty)$ *and* L_P $(0 < p < \infty)$ *are* $\mathscr{K}$*-spaces if and only if* $p \neq 1$.

PROOF: L_p $(p > 1)$ and ℓ_p $(p > 1)$ are B-convex.

The following theorem is a modification of a result of S. Dierolf ([3] Satz 2.4.1). Note that we do not assume X to be locally bounded.

THEOREM 4.10: *If* X *is an* F*-space with a closed locally convex subspace* L *such that* X/L *is a locally convex* $\mathscr{K}$*-space, then* X *is locally convex.*

PROOF: By Theorem 5.2 of [7], L has the HBEP. Hence, if μ denotes the Mackey topology on X, then on L, μ agrees with the original topology. If $x_n \in X$ and $x_n \to 0(\mu)$ then $q(x_n) \to 0$ (where $q: X \to X/L$ is the quotient map) since X/L is locally convex. Hence $x_n = u_n + v_n$, where $u_n \to 0$ and $v_n \in L$. Thus $v_n \to 0(\mu)$ and hence $v_n \to 0$, so that $x_n \to 0$, Thus μ agrees with the original topology.

Note that the above theorem applies when X/L is a B-convex Banach space, which extends Theorem 2.6.

5. Open problems

We collect in this section some questions which arose in the course of the paper.

PROBLEM 1: If X is a locally bounded F-space for which $a_n = O(n^{1/p})$, is X locally p-convex $(p < 1)$ (cf. Proposition 2.2)?

PROBLEM 2: Is there a non-locally convex locally bounded F-space whose containing Banach space is c_0 or ℓ_∞?

PROBLEM 3: In general, give a necessary and sufficient condition for a Banach space to be a containing Banach space of a non-locally convex locally bounded F-space.

PROBLEM 4: Is c_0 or ℓ_∞ a $\mathcal{K}$-space?

PROBLEM 5: Give a necessary and sufficient condition for a Banach space to be a $\mathcal{K}$-space.

PROBLEM 6: Is H_p $(0 < p < 1)$ a $\mathcal{K}$-space?

PROBLEM 7: Does (ℓ_p, ℓ_p) $(p < 1)$ split? This seems unlikely, but the author does not have a counter example. (Added in proof: the author and N.T. Peck have shown that the answer is no).

PROBLEM 8: Does there exist an F-space X and a subspace Y of dimension one such that $X/Y \cong \ell_1$ and such that the quotient map is strictly singular? This is not true in our example.

NOTE: We are grateful to the referee for the information that Theorem 4.6 has independently been obtained by M. Ribe [9]. (Added in proof: a similar example has been found by J.W. Roberts).

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