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The Geometry of L_0

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Abstract. Suppose that we have the unit Euclidean ball in $\mathbb{R}^n$ and construct new bodies using three operations — linear transformations, closure in the radial metric, and multiplicative summation defined by $\|x\|_{K+_0L} = \sqrt{\|x\|_K\|x\|_L}$. We prove that in dimension 3 this procedure gives all origin-symmetric convex bodies, while this is no longer true in dimensions 4 and higher. We introduce the concept of embedding of a normed space in L_0 that naturally extends the corresponding properties of L_p-spaces with $p \neq 0$, and show that the procedure described above gives exactly the unit balls of subspaces of L_0 in every dimension. We provide Fourier analytic and geometric characterizations of spaces embedding in L_0, and prove several facts confirming the place of L_0 in the scale of L_p-spaces.

1 Introduction

Suppose that we have the unit Euclidean ball in $\mathbb{R}^n$ and are allowed to construct new bodies using three operations — linear tranformations, multiplicative summation, and closure in the radial metric. The *multiplicative sum* $K +_0 L$ of star bodies K and L is defined by

$$\|x\|_{K+_0L} = \sqrt{\|x\|_K\|x\|_L}, \tag{1}$$

where $\|x\|_K = \min\{a \geq 0 : x \in aK\}$ is the Minkowski functional of a star body K. What class of bodies do we get from the unit ball by means of these three operations?

We are going to prove that in dimension $n = 3$ we get all origin-symmetric convex bodies, while in dimension 4 and higher this is no longer the case. However, the class of bodies that we get in arbitrary dimension also has a clear interpretation. We introduce the concept of embedding in L_0 and show that the bodies that we get by means of these three operations are exactly the unit balls of spaces that embed in L_0.

The idea of this interpretation comes from a similar result for L_p-spaces with $p \in [-1, 1]$, $p \neq 0$. Namely, if we replace the multiplicative summation by p-summation

$$\|x\|_{K+_pL} = (\|x\|_K^p + \|x\|_L^p)^{1/p}, \tag{2}$$

then we get the unit balls of all spaces that embed in L_p. The case $p = 1$ is well known (see [G2, Corollary 4.1.12]), and the unit balls of subspaces of L_1 have a clear geometric meaning; these are the polar projection bodies (see [B]). On the other hand, it was proved by Goodey and Weil [GW] that if $p = -1$ (this case corresponds to

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the radial summation), then we get the class of intersection bodies in $\mathbb{R}^n$. As shown in [K4], intersection bodies are the unit balls of spaces that embed in L_{-1}. The concept of embedding in $L_p,\ p < 0$ was introduced in [K3] as an analytic extension of the same property for $p > 0$, see [KK2] for related results. The result of Goodey and Weil can easily be extended to $p \in (-1, 1),\ p \neq 0$. Note that this construction provides a continuous (except for $p = 0$) path from polar projection bodies to intersection bodies, which is important for understanding the duality between projections and sections of convex bodies. One of the goals of this article is to fill the gap in this scheme at $p = 0$ and to better understand the geometry of this intermediate case.

Another interesting similarity of our result with other values of p is that for $p = 1$ the procedure defined above gives all origin-symmetric convex bodies only in dimension 2. This follows from a result of Schneider [S] that every origin-symmetric convex body is a polar projection body only in dimension 2. When $p = -1$, we get all origin-symmetric convex bodies only in dimensions 4 and lower, because by results from [G1, Z, GKS], only in these dimensions is every origin-symmetric convex body an intersection body. The transition between the dimensions 2 and 3 in the case $p = 1$ and the transition between the dimensions 4 and 5 in the case $p = -1$ directly correspond to the transition between the affirmative and negative answers in the Shephard and Busemann–Petty problems, respectively. It would be interesting to find a similar geometric result corresponding to the transition between dimensions 3 and 4 in the case $p = 0$. We refer the reader to [K5, Ch. 6] for more details and the history of the connection between convex geometry and the theory of L_p-spaces.

2 The Definition of Embedding in L_0.

A compact set K in $\mathbb{R}^n$ is called an origin-symmetric star body if every straight line passing through the origin crosses the boundary of K at exactly two points, the boundary is continuous, and the origin is an interior point of K. We denote by $(\mathbb{R}^n, \|\cdot\|_K)$ the Euclidean space equipped with the Minkowski functional of the body K. Clearly, $(\mathbb{R}^n, \|\cdot\|_K)$ is a normed space if and only if the body K is convex. Throughout the paper, we write $(\mathbb{R}^n, \|\cdot\|)$ to mean that $\|\cdot\|$ is the Minkowski functional of some origin-symmetric star body.

A well-known result of P. Lévy, see [BL, p. 189] or [K5, Section 6.1], is that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p > 0$, if and only if there exists a finite Borel measure μ on the unit sphere so that for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p \, d\mu(\xi). \tag{3}$$

On the other hand, the definition of embedding in L_p with $p < 0$ from [K3] implies that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds into L_p, $p \in (-n, 0)$, if and only if there exists a finite symmetric measure μ on the sphere S^{n-1} so that for every test function ϕ,

$$\int_{\mathbb{R}^n} \|x\|^p \phi(x) \, dx = \int_{S^{n-1}} d\mu(\xi) \int_{\mathbb{R}} |t|^{-p-1} \widehat{\phi}(t\xi) \, dt. \tag{4}$$

Both representations (3) and (4) are invariant with respect to p-summation. This gives an idea of defining embedding in L_0 by means of a representation that is invariant with respect to multiplicative summation. Note that the multiplicative summation is the limiting case of p-summation as $p \to 0$.

Definition 2.1 We say that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_0 if there exist a finite Borel measure μ on the sphere S^{n-1} and a constant $C \in \mathbb{R}$ so that for every $x \in \mathbb{R}^n$,

$$\ln \|x\| = \int_{S^{n-1}} \ln |(x,\xi)|\, d\mu(\xi) + C. \tag{5}$$

While similar to (3) and (4), this definition has its unique features. First, the measure μ must be a probability measure on S^{n-1}. In fact, put $x = ky$, $k > 0$ in (5). Then

$$\ln k + \ln \|y\| = \int_{S^{n-1}} \ln k \;\, d\mu(\xi) + \int_{S^{n-1}} \ln |(y,\xi)|\, d\mu(\xi) + C,$$

and, again by (5) with $x = y$, we get $\ln k = \int_{S^{n-1}} \ln k \;\, d\mu(\xi)$, so $\int_{S^{n-1}} \; d\mu(\xi) = 1$.

Secondly, the constant C depends on the norm and can be computed precisely. In order to compute this constant, integrate the equality (5) over the uniform measure on the unit sphere. We get

$$\begin{aligned} C \cdot |S^{n-1}| &= \int_{S^{n-1}} \ln \|x\|\, dx - \int_{S^{n-1}} \int_{S^{n-1}} \ln |(x,\theta)|\, d\mu(\theta) dx \\ &= \int_{S^{n-1}} \ln \|x\|\, dx - \int_{S^{n-1}} \int_{S^{n-1}} \ln |(x,\theta)|\, dx\, d\mu(\theta) \\ &= \int_{S^{n-1}} \ln \|x\|\, dx - \int_{S^{n-1}} \ln |(x,\theta)|\, dx, \end{aligned}$$

since $\int_{S^{n-1}} \ln |(x,\theta)|\, dx$ is rotationally invariant and, therefore, is a constant for $\theta \in S^{n-1}$, and μ is a probability measure.

To compute the latter integral, use the well-known formula (see [K5, Section 6.4])

$$\int_{S^{n-1}} |(x,\theta)|^p\, dx = \frac{2\pi^{(n-1)/2}\Gamma((p+1)/2)}{\Gamma((n+p)/2)}.$$

Differentiating with respect to p and letting $p = 0$, we get

$$\int_{S^{n-1}} \ln |(x,\theta)|\, dx = \pi^{(n-1)/2}\left[\frac{\Gamma'(1/2)}{\Gamma(n/2)} - \sqrt{\pi}\frac{\Gamma'(n/2)}{\Gamma^2(n/2)}\right].$$

Note that $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, so

$$C = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \ln \|x\|\, dx - \frac{1}{2\sqrt{\pi}}\Gamma'(1/2) + \frac{1}{2}\frac{\Gamma'(n/2)}{\Gamma(n/2)}.$$

Let us remark that Definition 2.1 is equivalent to the following. A finite-dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|)$ embeds into L_0 if and only if there is a probability space (Ω, μ) and a linear map $T\colon X \to \mathcal{M}(\Omega, \mu)$ (where $\mathcal{M}(\Omega, \mu)$ denotes the space of μ-measurable functions on Ω) such that

$$\int_\Omega \ln |Tx(\omega)|\, d\mu(\omega) < \infty, \quad x \in X$$

and

$$\ln \|x\| = \int_\Omega \ln |Tx(\omega)|\, d\mu(\omega), \quad x \in X.$$

Indeed, if such an operator T exists, we can write it in the form

$$Tx(\omega) = h(\omega)(x, \xi(\omega)), \quad x \in X,$$

where $h\colon \Omega \to \mathbb{R}^+$ and $\xi\colon \Omega \to S^{n-1}$ are measurable. Then for each $\omega \in \Omega$,

$$\int_{S^{n-1}} \ln |(x, \xi(\omega))|\, dx > -\infty,$$

so that it follows for some $x \in S^{n-1}$, $\omega \to \ln |(x, \xi(\omega)|$ is μ-integrable. Hence, so is $\ln h$, and furthermore, $\ln \|x\| = \int \ln h(\omega)\, d\mu(\omega) + \int \ln |(x, \xi(\omega))| d\mu(\omega)$. Now we can induce a probability measure μ' on S^{n-1} by $\mu'(B) = \mu\{\omega : \xi(\omega) \in B\}$, and we have the same situation as Definition 2.1.

On the other hand, if X satisfies Definition 2.1, we may take $\Omega = S^{n-1}$, and μ is a probability measure. If we define $Tx(\xi) = e^C(x, \xi)$, then $T\colon X \to \mathcal{M}(S^{n-1}, \mu)$ satisfies our conditions.

One advantage of this viewpoint is that we can make sense of the statement that an infinite-dimensional Banach space embeds into L_0.

3 A Fourier Analytic Characterization of Subspaces of L_0

As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ (test functions), and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$.

We say that a distribution is positive (negative) outside of the origin in $\mathbb{R}^n$ if it assumes non-negative (non-positive) values on non-negative Schwartz's test functions with compact support outside of the origin.

The Fourier transform of a distribution f is defined by $\langle \widehat{f}, \phi \rangle = \langle f, \widehat{\phi} \rangle$ for every test function ϕ.

Let ϕ be an integrable function on $\mathbb{R}^n$ that is also integrable on hyperplanes, let $\xi \in S^{n-1}$, and let $t \in \mathbb{R}^n$. Then

$$\mathcal{R}\phi(\xi; t) = \int_{(x,\xi)=t} \phi(x)\, dx$$

is the *Radon transform of ϕ in the direction ξ at the point t*. A simple connection between the Fourier and Radon transforms is that for every fixed $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\widehat{\phi}(s\xi) = (\mathcal{R}\phi(\xi; t))^\wedge(s) \quad \forall s \in \mathbb{R}, \tag{6}$$

where in the right-hand side we have the Fourier transform of the function $t \mapsto \mathcal{R}\phi(\xi; t)$, see [K5, Lemma 2.11].

The fact that the Fourier transform is useful in the study of subspaces of L_p has been known for a long time. A well-known result of P. Levy is that a finite dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p, $0 < p \le 2$ if and only if $\exp(-\|\cdot\|^p)$ is a positive definite function on $\mathbb{R}^n$. It was proved in [K2] that a space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p, $p > 0$, $p \notin 2\mathbb{N}$ if and only if the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ (in the sense of distributions) is a positive distribution outside of the origin. For $-n < p < 0$, a similar fact was proved in [K3]: a space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p if and only if the Fourier transform of $\|\cdot\|^p$ is a positive distribution in the whole $\mathbb{R}^n$. These characterizations have proved to be useful in the study of subspaces of L_p and intersection bodies, see [K5, Ch. 6]. In this section we prove a similar characterization of spaces that embed in L_0.

Theorem 3.1 *Let K be an origin-symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if the Fourier transform of $\ln\|x\|_K$ is a negative distribution outside of the origin in $\mathbb{R}^n$.*

Proof First, assume that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. Let ϕ be a non-negative even test function with compact support outside of the origin. By the definition of embedding in L_0, formula (6) (note that $\widehat{\widehat{\phi}} = (2\pi)^n\phi$ for even ϕ) and the Fubini theorem,

$$
\begin{aligned}
(7)\quad \langle (\ln\|x\|)^\wedge, \phi\rangle &= \langle \ln\|x\|, \widehat{\phi}(x)\rangle \\
&= \int_{S^{n-1}}\int_{\mathbb{R}^n} \ln|(x,\xi)|\widehat{\phi}(x)\,dx\,d\mu(\xi) + C\int_{\mathbb{R}^n}\widehat{\phi}(x)\,dx \\
&= \int_{S^{n-1}}\left\langle \ln|t|, \int_{(x,\xi)=t}\widehat{\phi}(x)\,dx\right\rangle d\mu(\xi) \\
&= (2\pi)^{-1}\int_{S^{n-1}}\left\langle (\ln|t|)^\wedge(z), \left(\int_{(x,\xi)=t}\widehat{\phi}(x)\,dx\right)^\wedge(z)\right\rangle d\mu(\xi) \\
&= (2\pi)^{n-1}\int_{S^{n-1}}\int_{\mathbb{R}}(\ln|t|)^\wedge(z)\phi(z\xi)\,dz d\mu(\xi),
\end{aligned}
$$

since $\int_{\mathbb{R}^n}\widehat{\phi}(x)\,dx = (2\pi)^n\phi(0) = 0$. Now, the formula for the Fourier transform of $\ln|t|$ from [GS, p.362] implies that

$$(8)\qquad (\ln|t|)^\wedge(z) = -\pi|z|^{-1} < 0$$

outside of the origin, so (7) is negative (recall that ϕ is non-negative with support outside of the origin). This means that $(\ln\|x\|)^\wedge$ is a negative distribution.

To prove the other direction, note that by [K5, Section 2.6], a distribution that is positive outside of the origin coincides with a finite Borel measure on every set of the form

$$A \times (a,b) = \{x \in \mathbb{R}^n : x = t\theta, t \in (a,b), \theta \in A\},$$

where A is an open subset of S^{n-1} and $0 < a < b < \infty$.

Denote by $\mu = -(\ln\|x\|)^\wedge$. This distribution coincides with a finite Borel measure on each set $A \times (a,b)$, as above, so for any test function ϕ supported outside of the origin

$$\langle -(\ln\|x\|)^\wedge, \phi\rangle = \langle \mu, \phi\rangle = \int_{\mathbb{R}^n} \phi(x)\, d\mu(x). \tag{9}$$

Now for every test function ϕ with support outside of the origin and $t > 0$, we have $(\phi(x/t))^\wedge(z) = t^n\widehat{\phi}(tz)$, so

$$\begin{aligned}\langle \mu(x), \phi(x/t)\rangle &= -\langle (\ln\|x\|)^\wedge(x), \phi(x/t)\rangle \\ &= -\int_{\mathbb{R}^n} \ln\|z\|\widehat{\phi}(tz)t^n\, dz \\ &= -\int_{\mathbb{R}^n} \widehat{\phi}(\tilde{x}) \ln\|\frac{1}{t}\tilde{x}\|\, d\tilde{x} \\ &= -\int_{\mathbb{R}^n} \widehat{\phi}(\tilde{x}) \ln\|\tilde{x}\|\, d\tilde{x} + \ln|t| \int_{\mathbb{R}^n} \widehat{\phi}(\tilde{x})\, d\tilde{x} \\ &= -\int_{\mathbb{R}^n} \widehat{\phi}(\tilde{x}) \ln\|\tilde{x}\|\, d\tilde{x} \\ &= \langle \mu(x), \phi(x)\rangle.\end{aligned} \tag{10}$$

Let $\chi_{A\times(a,b)}$ be the indicator of the set $A \times (a,b)$. Approximating $\chi_{A\times(a,b)}$ by test functions and using (10), we get for any $(a,b) \subset (0,\infty)$ and $A \subset S^{n-1}$

$$\begin{aligned}\mu(A \times (a,b)) &= \int_{\mathbb{R}^n} \chi_{A\times(a,b)}(x)\, d\mu(x) = \int_{\mathbb{R}^n} \chi_{A\times(1,b/a)}(x/a)\, d\mu(x) \\ &= \int_{\mathbb{R}^n} \chi_{A\times(1,b/a)}(x)\, d\mu(x) \\ &= \mu(A \times (1, b/a)).\end{aligned}$$

Applying this formula n times,

$$\mu(A \times (1, a^n)) = n\mu(A \times (1,a)) \tag{11}$$

for $n \in \mathbb{N}$. Moreover, we can extend formula (11) to $n \in \mathbb{R}$. So, for any $a \in (0,\infty)$ and $A \subset S^{n-1}$,

$$\mu(A \times [1,a]) = \mu(A \times [1, e^{\ln a}]) = \ln a \cdot \mu(A \times [1,e]).$$

Now for every $(a,b) \subset (0,\infty)$ and $A \subset S^{n-1}$, we have

$$\begin{aligned}\mu(A \times (a,b)) &= \mu(A \times (1, b/a)) \\ &= \ln\Big(\frac{b}{a}\Big)\, \mu(A \times (1,e)) \\ &= (\ln(b) - \ln(a))\mu(A \times (1,e)).\end{aligned}$$

Define a measure μ_0 on S^{n-1} by

$$\mu_0(A) = \frac{\mu(A \times (a,b))}{(\ln(b) - \ln(a))} = \mu(A \times (1,e))$$

for every Borel set $A \subset S^{n-1}$. We have

$$\begin{aligned}(12)\qquad \int_{S^{n-1}} d\mu_0(\theta) \int_0^\infty |t|^{-1}\chi_{A\times(a,b)}(t\theta)\,dt &= (\ln(b) - \ln(a))\mu_0(A)\\ &= \mu(A \times (a,b))\\ &= \int_{\mathbb{R}^n} \chi_{A\times(a,b)}(x)\,d\mu(x).\end{aligned}$$

Therefore, for an arbitrary even test function ϕ supported outside of the origin,

$$(13)\quad \langle \mu, \phi\rangle = \int_{S^{n-1}} d\mu_0(\theta) \int_0^\infty |t|^{-1}\phi(t\theta) \ = \frac{1}{2}\int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} |t|^{-1}\phi(t\theta)\,dt,$$

since A, a, b are arbitrary in (12).

Using $\mu = -(\ln\|x\|)^\wedge$, we get

$$\langle (\ln\|x\|)^\wedge(\xi), \phi\rangle = -\frac{1}{2}\int_{S^{n-1}} d\mu_0(\theta)\int_{\mathbb{R}} |t|^{-1}\phi(t\theta)\,dt.$$

Define a new measure $\tilde{\mu}_0 = (2\pi)^n\mu_0$. By (8), (13) and the connection between the Fourier and Radon transforms

$$\begin{aligned}\langle \ln\|x\|, \widehat{\phi}(x)\rangle &= -\frac{1}{2(2\pi)^n}\int_{S^{n-1}} d\tilde{\mu}_0(\theta)\int_{\mathbb{R}} |t|^{-1}\phi(t\theta)\,dt\\ &= \int_{S^{n-1}} \langle \ln|z|, \mathcal{R}\widehat{\phi}(\theta; z)\rangle\, d\tilde{\mu}_0(\theta)\\ &= \int_{S^{n-1}} d\tilde{\mu}_0(\theta)\int_{\mathbb{R}} \ln|z| \Big(\int_{(x,\theta)=z} \widehat{\phi}(x)\,dx\Big)\, dz\\ &= \int_{S^{n-1}} d\tilde{\mu}_0(\theta)\int_{\mathbb{R}^n} \ln|(x,\theta).|\widehat{\phi}(x)\,dx.\end{aligned}$$

Thus, we have proved that for any even test function ϕ supported outside of the origin

$$\langle (\ln\|x\|)^\wedge, \phi\rangle = \Big\langle \Big(\int_{S^{n-1}} \ln|(x,\theta)|\,d\tilde{\mu}_0(\theta)\Big)^\wedge, \phi\Big\rangle.$$

Therefore the distributions $\ln\|x\|$ and $\int_{S^{n-1}} \ln|(x,\theta)|\,d\tilde{\mu}_0(\theta)$ can differ only by a polynomial. Clearly, this polynomial cannot contain terms homogeneous of degree different from zero, so it is a constant. ■

Remark 3.2 Let K be an infinitely smooth body. From the proof of the previous theorem it follows that the measure μ from Definition 2.1 is equal to the restriction of the Fourier transform of $\ln \|x\|_K$ to the sphere. In the next section we are going to prove that this is a function, therefore

$$d\mu(\xi) = -\frac{1}{(2\pi)^n} \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi.$$

In particular, since μ is a probability measure, for any infinitely smooth body K we get

$$-\frac{1}{(2\pi)^n} \int_{S^{n-1}} \left(\ln \|x\|_K\right)^\wedge (\xi) d\xi = 1.$$

4 A Geometric Characterization of Subspaces of L_0.

Let K be an origin-symmetric star body in $\mathbb{R}^n$. The function $\rho_K(x) = \|x\|_K^{-1}$ is called the *radial function* of K. If $x \in S^{n-1}$, then $\rho_K(x)$ is the distance from the origin to the boundary of K in the direction of x.

The *radial metric* on the set of all origin-symmetric star bodies is defined by

$$\rho(K, L) = \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|.$$

Let $\xi \in S^{n-1}$ and $(x, \xi) = t$ be the hyperplane orthogonal to ξ at the distance t from the origin. Define the *parallel section function* of a star body K in the direction of ξ by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{(x, \xi) = t\}), \quad t \in \mathbb{R}.$$

Let f be an integrable continuous function on $\mathbb{R}$, m-times continuously differentiable in some neighborhood of zero, $m \in \mathbb{N}$. For a number $q \in (m-1, m)$, the *fractional derivative* of the order q of the function f at zero is defined as follows [K5, Section 2.5]:

$$f^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} \Big(f(t) - f(0) - t f'(0) - \cdots - \frac{t^{m-1}}{(m-1)!} f^{(m-1)}(0) \Big)\, dt.$$

Note that fractional derivatives of integer orders coincide with usual derivatives up to a sign:

$$f^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} f(t)|_{t=0}.$$

If K has an infinitely smooth boundary, then the function $A_{K,\xi}(t)$ is an infinitely differentiable function of t in some neighborhood of zero, and, as was shown in [GKS], the fractional derivatives of $A_{K,\xi}(t)$ can be computed in terms of the Fourier transform of the Minkowski functional raised to certain powers. Namely, for $q \in \mathbb{C}$, $q \neq n-1$,

$$A_{K,\xi}^{(q)}(0) = \frac{\cos \frac{q\pi}{2}}{\pi(n-q-1)} \left(\|x\|_K^{-n+q+1} \right)^\wedge (\xi), \tag{14}$$

and, in particular, $\left(\|x\|_K^{-n+q+1}\right)^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$. Here we extend $A^{(q)}_{K,\xi}(0)$ from the sphere to the whole of $\mathbb{R}^n$ as a homogeneous function of the variable ξ of degree $-q-1$. Note that $\langle A^{(q)}_{K,\xi}(0), \phi\rangle$ is an analytic function of q for any fixed test function ϕ.

Since the right-hand side of formula (14) is not defined for $q = n-1$, in our next Theorem we use a limiting argument to extend this formula to the case $q = n-1$.

Let $\mathcal{D}$ be an open set in $\mathbb{R}^n$, f, g two distributions. We say that $f = g$ on $\mathcal{D}$ if $\langle f, \phi\rangle = \langle g, \phi\rangle$ for any test function ϕ with compact support in $\mathcal{D}$.

Theorem 4.1 *Let K be an infinitely smooth origin-symmetric star body in $\mathbb{R}^n$. Extend $A^{(n-1)}_{K,\xi}(0)$ to a homogeneous function of degree $-n$ of the variable $\xi \in \mathbb{R}^n \setminus \{0\}$. Then $(\ln \|\cdot\|_K)^\wedge$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ and*

$$A^{(n-1)}_{K,\xi}(0) = -\frac{\cos(\pi(n-1)/2)}{\pi}(\ln \|\cdot\|_K)^\wedge(\xi), \tag{15}$$

as distributions (of the variable ξ) acting on test functions with compact support outside of the origin. In particular, for $\xi \in \mathbb{R}^n \setminus \{0\}$,

(i) *if n is odd,*

$$(\ln \|x\|_K)^\wedge(\xi) = (-1)^{(n+1)/2}\pi A^{(n-1)}_{K,\xi}(0),$$

(ii) *if n is even,*

$$(\ln \|x\|_K)^\wedge(\xi) = a_n \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \cdots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n}\, dz,$$

where $a_n = 2(-1)^{n/2+1}(n-1)!$

Proof Let us start with the case where n is odd. Let ϕ be a test function supported outside of the origin.

Using formula (14) for q close to $n-1$, we have

$$\begin{aligned}
\langle A^{(q)}_{K,\xi}(0), \phi(\xi)\rangle &= \frac{\cos(\pi q/2)}{\pi(n-q-1)}\langle(\|x\|^{-n+q+1})^\wedge(\xi), \phi(\xi)\rangle \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)}\langle\|x\|^{-n+q+1}, \widehat{\phi}(x)\rangle \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)}\int_{\mathbb{R}^n}\|x\|^{-n+q+1}\widehat{\phi}(x)\, dx \\
&= \frac{\cos(\pi q/2)}{\pi(n-q-1)}\int_{\mathbb{R}^n}(\|x\|^{-n+q+1}-1)\widehat{\phi}(x)\, dx \\
&\quad + \frac{\cos(\pi q/2)}{\pi(n-q-1)}\int_{\mathbb{R}^n}\widehat{\phi}(x)\, dx \\
&= \frac{\cos(\pi q/2)}{\pi}\int_{\mathbb{R}^n}\frac{\|x\|^{-n+q+1}-1}{n-q-1}\widehat{\phi}(x)\, dx,
\end{aligned}$$

since $\int_{\mathbb{R}^n} \widehat{\phi}(x)dx = (2\pi)^n \phi(0) = 0$. Taking the limit of both sides as $q \to n-1$, we get

$$\langle A_{K,\xi}^{(n-1)}(0), \phi(\xi)\rangle = \Big\langle -\frac{\cos(\pi(n-1)/2)}{\pi} (\ln \|x\|)^\wedge(\xi), \phi(\xi)\Big\rangle,$$

since

$$\lim_{q\to n-1} \int_{\mathbb{R}^n} \frac{\|x\|^{-n+q+1} - 1}{n-q-1} \widehat{\phi}(x)\, dx = -\int_{\mathbb{R}^n} \ln \|x\| \widehat{\phi}(x)\, dx = \langle -(\ln \|x\|)^\wedge(\xi), \phi(\xi)\rangle.$$

When n is odd the formula of (i) follows immediately.

When n is even, both sides of (15) are equal to zero, and we repeat the reasoning from [GKS, Theorem 1]. Divide both sides of (14) by $\cos(\frac{\pi q}{2})$,

$$\Big\langle \frac{(\|x\|_K^{-n+q+1})^\wedge(\xi)}{(n-q-1)}, \phi(\xi)\Big\rangle = \pi \Big\langle \frac{A_{K,\xi}^{(q)}(0)}{\cos\frac{\pi q}{2}}, \phi(\xi)\Big\rangle,$$

and take the limit of both sides when $q \to n-1$.

We have already proved that

$$\lim_{q\to n-1} \Big\langle \frac{(\|x\|_K^{-n+q+1})^\wedge(\xi)}{(n-q-1)}, \phi(\xi)\Big\rangle = \langle -(\ln \|x\|)^\wedge(\xi), \phi(\xi)\rangle$$

for any test function ϕ supported outside of the origin.

To compute the limit of

$$\frac{A_{K,\xi}^{(q)}(0)}{\cos\frac{q\pi}{2}},$$

we use the definition of fractional derivatives in exactly the same way as was done in [GKS, Theorem 1]:

$$\lim_{q\to n-1} \Gamma(-q) A_{K,\xi}^{(q)}(0) = \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \cdots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n}\, dz$$

and

$$\lim_{q\to n-1} \Gamma(-q) \sin\frac{(q+1)\pi}{2} = \frac{\pi}{2}(-1)^{n/2} \frac{1}{(n-1)!}.$$

Combining these two formulas we get the formula in statement (ii) of the theorem. ∎

The following is an immediate application of Theorem 4.1.

Corollary 4.2 *Let K be an infinitely smooth body in $\mathbb{R}^n$. Then*

(i) *if n is odd, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if for every $\xi \in S^{n-1}$*

$$(-1)^{(n-1)/2} A_{K,\xi}^{(n-1)}(0) \geq 0;$$

(ii) *if n is even, $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if for every $\xi \in S^{n-1}$,*

$$(-1)^{(n+2)/2}\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - \cdots - A_\xi^{n-2}(z)\frac{z^{n-2}}{(n-2)!}}{z^n}\,dz \geq 0.$$

Corollary 4.3 *Every 3-dimensional normed space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0.*

Proof The unit ball K of a normed space is an origin-symmetric convex body. First assume that K is infinitely smooth. By Brunn's theorem the central section of a convex body has maximal volume among all sections perpendicular to a given direction. Therefore, for any ξ the function $A_{K,\xi}(t)$ attains its maximum at $t = 0$, hence $A''_{K,\xi}(0) \leq 0$. So, by Theorem 4.1, for smooth convex bodies in $\mathbb{R}^3$ the distribution $-(\ln\|x\|)^\wedge$ is positive outside of the origin, and our result follows from Theorem 3.1. For general convex bodies the result follows from the facts that any convex body can be approximated by smooth convex bodies and that positive definiteness is preserved under limits. In fact, let $\{K_i\}$ be a sequence of infinitely smooth convex bodies that approach K in the radial metric. Then for any non-negative test function ϕ supported outside of the origin we have

$$-\int_{\mathbb{R}^n} \ln\|x\|_{K_i}\widehat{\phi}(x)\,dx = \big\langle -\ln\|x\|_{K_i}, \widehat{\phi}(x)\big\rangle = \big\langle -(\ln\|x\|_{K_i})^\wedge(\xi), \phi(\xi)\big\rangle \geq 0.$$

Since K_i approximate K, there is a constant $C > 0$, such that

$$|\ln\|x\|_{K_i}| \leq C + |\ln|x|_2|\,,$$

therefore the functions $|\ln\|x\|_{K_i}\widehat{\phi}(x)|$ are majorized by an integrable function $(C + |\ln|x|_2|)|\widehat{\phi}(x)|$, and by the Lebesgue dominated convergence theorem we get

$$\begin{aligned} -\lim_{i\to\infty}\int_{\mathbb{R}^n} \ln\|x\|_{K_i}\widehat{\phi}(x)\,dx &= -\int_{\mathbb{R}^n} \ln\|x\|_K\widehat{\phi}(x)\,dx \\ &= \langle -(\ln\|x\|_K)^\wedge(\xi), \phi(\xi)\rangle \geq 0 \end{aligned}$$

■

Our next result shows that that the previous statement is no longer true in $\mathbb{R}^n$, $n \geq 4$.

Theorem 4.4 *There exists an origin-symmetric convex body K in $\mathbb{R}^n$, $n \geq 4$, so that the space $(\mathbb{R}^n, \|\cdot\|_K)$ does not embed in L_0.*

Proof It is enough to construct a convex body for which the distribution $-(\ln\|x\|)^\wedge$ is not positive. The construction will be similar to that from [GKS].

Define $f_N(x) = (1 - x^2 - Nx^4)^{1/3}$. Let $a_N > 0$ be such that $f_N(a_N) = 0$ and $f_N(x) > 0$ on the interval $(0, a_N)$. Define a body K in $\mathbb{R}^4$ by

$$K = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 \in [-a_N, a_N] \text{ and } \sqrt{x_1^2 + x_2^2 + x_3^2} \leq f_N(x_4)\}.$$

The body K is strictly convex and infinitely smooth. By Theorem 4.1,

$$-(\ln \|x\|_K)^\wedge(\xi) = 12 \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2}}{z^4}\, dz.$$

The function $A_{K,\xi}$ can easily be computed:

$$A_{K,\xi}(x) = \frac{4\pi}{3}(1 - x^2 - Nx^4).$$

We have

$$\int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2}}{z^4}\, dz = \frac{4\pi}{3}\Big(-Na_N + \frac{1}{a_N} - \frac{1}{3a_N^3}\Big).$$

The latter is negative for N large enough, because $N^{1/4} \cdot a_N \to 1$ as $N \to \infty$. ■

5 Addition in L_0

It is clear from the definition that the class of bodies K for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 is closed with respect to multiplicative summation, *i.e.,* if two spaces $(\mathbb{R}^n, \|\cdot\|_{K_1})$ and $(\mathbb{R}^n, \|\cdot\|_{K_2})$ embed in L_0 and $K = K_1 +_0 K_2$, then $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. In this section we are going to prove that the unit ball of every space $(\mathbb{R}^n, \|\cdot\|_K)$ that embeds in L_0 can be obtained from the Euclidean ball by means of multiplicative summation, linear transformations and closure in the radial metric, *i.e.,* it can be approximated in the radial metric by multiplicative sums of ellipsoids.

Consider the set of bodies K for which $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. As mentioned above, this set is closed with respect to multiplicative summation; also, from the proof of Corollary 4.3 it follows that this set is closed with respect to limits in the radial metric. Let us show that it is closed with respect to linear transformations. Suppose that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0. By Theorem 3.1 $\big(\ln \|x\|_K\big)^\wedge$ is a negative distribution outside of the origin. Let T be an invertible linear transformation in $\mathbb{R}^n$. Then for any non-negative test function ϕ with support outside of the origin, we have

$$\begin{aligned}
\langle \big(\ln \|Tx\|_K\big)^\wedge, \phi\rangle &= \langle \ln \|Tx\|_K, \widehat{\phi}(x)\rangle \\
&= \int_{\mathbb{R}^n} \ln \|Tx\|_K \widehat{\phi}(x)\, dx \\
&= |\det T|^{-1} \int_{\mathbb{R}^n} \ln \|x\|_K \widehat{\phi}(T^{-1}x)\, dx \\
&= \int_{\mathbb{R}^n} \ln \|x\|_K \big(\phi(T^* y)\big)^\wedge (x)\, dx \\
&= \big\langle \ln \|x\|_K, \big(\phi(T^* y)\big)^\wedge (x)\big\rangle, \\
&= \big\langle \big(\ln \|x\|_K\big)^\wedge (y), \phi(T^* y)\big\rangle \le 0.
\end{aligned}$$

So $(\ln \|Tx\|_K)^\wedge$ is a negative distribution outside of the origin. By Theorem 3.1, $(\mathbb{R}^n, \|\cdot\|_{TK})$ embeds in L_0. Moreover, if $(\ln \|x\|)^\wedge$ is a function, then

$$(\ln \|Tx\|)^\wedge(y) = |\det T|^{-1}(\ln \|x\|)^\wedge((T^*)^{-1}y). \tag{16}$$

To prove the main result of this section, we need a few lemmas. For a fixed $x \in S^{n-1}$, let $E_{a,b}(x)$ be an ellipsoid with the norm

$$\|\theta\|_{E_{a,b}(x)} = \left(\frac{(x,\theta)^2}{a^2} + \frac{1-(x,\theta)^2}{b^2}\right)^{1/2}, \quad \text{for } \theta \in S^{n-1}.$$

Lemma 5.1 *For all $\theta \in S^{n-1}$,*

$$(\ln \|\xi\|_{E_{a,b}(x)})^\wedge_\xi(\theta) = -\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{a^{n-1}b}\|\theta\|^{-n}_{E_{b,a}(x)}.$$

Proof For $-n < \lambda < 0$, the following formula holds (see [GS, p. 192]):

$$(|x|_2^\lambda)^\wedge(\xi) = 2^{\lambda+n}\pi^{n/2}\frac{\Gamma((\lambda+n)/2)}{\Gamma(-\lambda/2)}|\xi|_2^{-\lambda-n}.$$

Dividing both sides by λ, using the formula $x\Gamma(x) = \Gamma(1+x)$, and sending $\lambda \to 0$, we get

$$(\ln |x|_2)^\wedge(\xi) = -2^{n-1}\pi^{n/2}\Gamma(n/2)|\xi|_2^{-n}$$

as distributions outside of the origin. Note that, by rotation, it is enough to prove the lemma for the ellipsoids $E_{a,b}(x)$ with $x = (0,0,\dots,0,1)$:

$$\|\xi\|_{E_{a,b}(x)} = \left(\frac{\xi_n^2}{a^2} + \frac{\xi_1^2+\cdots+\xi_{n-1}^2}{b^2}\right)^{1/2}.$$

Since this norm can be obtained from the Euclidean norm by an obvious linear transformation, one can use formula (16) to get

$$\begin{aligned}(\ln \|\xi\|_{E_{a,b}(x)})^\wedge_\xi(\theta) &= -2^{n-1}\pi^{n/2}\Gamma(n/2)ab^{n-1}\|\theta\|^{-n}_{E_{1/a,1/b}(x)}\\ &= -\frac{2^{n-1}\pi^{n/2}\Gamma(n/2)}{a^{n-1}b}\|\theta\|^{-n}_{E_{b,a}(x)}.\end{aligned}$$

∎

Lemma 5.2 *Let K be a star body, then $\ln \|x\|_K$ can be approximated in the space $C(S^{n-1})$ by the functions of the form*

$$f_{a,b}(x) = \frac{1}{|S^{n-1}|a^{n-1}b}\int_{S^{n-1}} \ln \|\theta\|_K \|\theta\|^{-n}_{E_{b,a}(x)}\, d\theta, \tag{17}$$

as $a \to 0$ and b is fixed.

Proof The proof is similar to that of [GW, Lemma 2]. First, note that the space $\mathbb{R}^n$ with the Euclidean norm embeds in L_0, so $(\mathbb{R}^n, \|\cdot\|_E)$ embeds in L_0 for any ellipsoid E with center at the origin. Therefore, by Remark 3.2 and Lemma 5.1 we get

$$\int_{S^{n-1}} \frac{1}{|S^{n-1}|a^{n-1}b} \|\theta\|_{E_{b,a}(x)}^{-n}\, d\theta = 1$$

for all values of a and b. From now on b will be fixed.

We have

$$\begin{aligned}
\Big|\ln \|x\|_K - &\frac{1}{|S^{n-1}|a^{n-1}b} \int_{S^{n-1}} \ln \|\theta\|_K \|\theta\|_{E_{b,a}(x)}^{-n}\, d\theta\Big| \\
&\le \frac{1}{|S^{n-1}|a^{n-1}b} \int_{S^{n-1}} \big|\ln \|x\|_K - \ln \|\theta\|_K\big|\, \|\theta\|_{E_{b,a}(x)}^{-n}\, d\theta \\
&= \frac{1}{|S^{n-1}|a^{n-1}b} \int_{|(x,\theta)|\ge\delta} \big|\ln \|x\|_K - \ln \|\theta\|_K\big|\, \|\theta\|_{E_{b,a}(x)}^{-n}\, d\theta \\
&\quad + \frac{1}{|S^{n-1}|a^{n-1}b} \int_{|(x,\theta)|<\delta} \big|\ln \|x\|_K - \ln \|\theta\|_K\big|\, \|\theta\|_{E_{b,a}(x)}^{-n}\, d\theta \\
&= I_1 + I_2.
\end{aligned}$$

For the first integral I_1, use the uniform continuity of $\ln \|x\|_K$ on the sphere. For any given $\epsilon > 0$ there exists $\delta \in (0,1)$, δ close to 1, so that $|(x,\theta)| \ge \delta$ implies $|\ln \|x\|_K - \ln \|\theta\|_K| < \epsilon/2$. Therefore,

$$\begin{aligned}
I_1 &= \frac{1}{|S^{n-1}|a^{n-1}b} \int_{|(x,\theta)|\ge\delta} \big|\ln \|x\|_K - \ln \|\theta\|_K\big|\, \|\theta\|_{E_{a,b}(x)}^{-n}\, d\theta \\
&\le \frac{\epsilon}{2}\Big[\frac{1}{|S^{n-1}|a^{n-1}b} \int_{|(x,\theta)|\ge\delta} \|\theta\|_{E_{a,b}(x)}^{-n}\, d\theta\Big] \le \frac{\epsilon}{2}.
\end{aligned}$$

Now fix δ chosen above and estimate the integral I_2 as follows:

$$\begin{aligned}
I_2 &= \frac{1}{|S^{n-1}|a^{n-1}b} \int_{|(x,\theta)|<\delta} \big|\ln \|x\|_K - \ln \|\theta\|_K\big|\, \|\theta\|_{E_{b,a}(x)}^{-n}\, d\theta \\
&\le \frac{C(n,b,K)}{a^{n-1}} \int_{|(x,\theta)|<\delta} \|\theta\|_{E_{b,a}(x)}^{-n}\, d\theta,
\end{aligned}$$

where

$$C(n,b,K) = \frac{2\max_{S^{n-1}} |\ln \|x\|_K|}{|S^{n-1}|b}.$$

For the latter integral we use an elementary formula (see [K5, Section 6.4]):

$$\int_{|(x,\theta)|<\delta} f((x,\theta))\, d\theta = |S^{n-2}| \int_{-\delta}^{\delta} (1-t^2)^{(n-3)/2} f(t)\, dt, \quad \text{for } x \in S^{n-1}.$$

Now,

$$\begin{aligned}
I_2 &\le \frac{C(n,b,K)|S^{n-2}|}{a^{n-1}} \int_{-\delta}^{\delta} (1-t^2)^{(n-3)/2} \Big(\frac{t^2}{b^2} + \frac{1-t^2}{a^2} \Big)^{-n/2} dt \\
&\le \frac{C(n,b,K)|S^{n-2}|}{a^{n-1}} \int_{-\delta}^{\delta} (1-t^2)^{(n-3)/2} \Big(\frac{1-t^2}{a^2} \Big)^{-n/2} dt \\
&= a \cdot C(n,b,K)|S^{n-2}| \int_{-\delta}^{\delta} (1-t^2)^{-3/2}\, dt \\
&\le a \cdot C(n,b,K)|S^{n-2}| \frac{2\delta}{(1-\delta^2)^{3/2}}.
\end{aligned}$$

Now we can choose a so small that $I_2 \le \epsilon/2$. ■

Lemma 5.3 *If μ is a probability measure on S^{n-1} and $a, b > 0$, then the function*

$$f(x) = \int_{S^{n-1}} \ln \|\xi\|_{E_{a,b}(x)}\, d\mu(\xi)$$

can be approximated in $C(S^{n-1})$ by the sums of the form

$$\sum_{i=1}^{m} \frac{1}{p_i} \ln \|x\|_{E_i},$$

where $E_1, \dots, E_m$ are ellipsoids and $1/p_1 + \cdots + 1/p_m = 1$.

Proof Let $\sigma > 0$ be a small number and choose a finite covering of the sphere by spherical σ-balls $B_\sigma(\eta_i) = \{\eta \in S^{n-1} : |\eta - \eta_i| < \sigma\}$, $\eta_i \in S^{n-1}$, $i = 1, \dots, m = m(\delta)$. Define

$$\begin{aligned}
&\widetilde{B}_\sigma(\xi_1) = B_\sigma(\xi_1), \\
&\widetilde{B}_\sigma(\xi_i) = B_\sigma(\xi_i) \setminus \bigcup_{j=1}^{i-1} B_\sigma(\xi_j), \quad \text{for } i = 2, \dots, m.
\end{aligned}$$

Let $1/p_i = \mu(\widetilde{B}_\sigma(\xi_i))$. Clearly, $1/p_1 + \cdots + 1/p_m = 1$.

Let $\rho(E_{a,b}(\xi), x)$ be the radial function of the ellipsoid $E_{a,b}(\xi)$, that is,

$$\rho(E_{a,b}(\xi), x) = \|x\|_{E_{a,b}(\xi)}^{-1}.$$

Note that $\rho(E_{a,b}(\xi), x) = \rho(E_{a,b}(x), \xi)$, therefore

$$|\rho(E_{a,b}(\xi), x) - \rho(E_{a,b}(\theta), x)| \le C_{a,b}|\xi - \theta|,$$

with a constant $C_{a,b}$ that depends on a and b. Also note that since we consider a close to zero and b fixed, we may assume $a \le \rho(E_{a,b}(\xi), x) \le b$, $x \in S^{n-1}$. Then

$$\begin{aligned}
&\Big| \int_{S^{n-1}} \ln\rho(E_{a,b}(\xi), x)\, d\mu(\xi) - \sum_{i=1}^{m} \frac{1}{p_i} \ln \rho(E_{a,b}(\xi_i), x) \Big| \\
&\qquad = \Big| \sum_{i=1}^{m} \Big(\int_{\widetilde{B}_\sigma(\xi_i)} \ln \rho(E_{a,b}(\xi), x)\, d\mu(\xi) - \int_{\widetilde{B}_\sigma(\xi_i)} \ln \rho(E_{a,b}(\xi_i), x)\, d\mu(\xi) \Big) \Big| \\
&\qquad \le \sum_{i=1}^{m} \int_{\widetilde{B}_\sigma(\xi_i)} \Big| \ln \frac{\rho(E_{a,b}(\xi), x)}{\rho(E_{a,b}(\xi_i), x)} \Big|\, d\mu(\xi) \\
&\qquad \le \sum_{i=1}^{m} \int_{\widetilde{B}_\sigma(\xi_i)} \Big| \ln \frac{\rho(E_{a,b}(\xi_i), x) + [\rho(E_{a,b}(\xi), x) - \rho(E_{a,b}(\xi_i), x)]}{\rho(E_{a,b}(\xi_i), x)} \Big|\, d\mu(\xi) \\
&\qquad \le \sum_{i=1}^{m} \int_{\widetilde{B}_\sigma(\xi_i)} | \ln(1 \pm C'_{a,b} |\xi - \xi_i|) |\, d\mu(\xi) \\
&\qquad \le | \ln(1 \pm C'_{a,b} \sigma) |,
\end{aligned}$$

and the result follows since σ is arbitrarily small. ∎

Now we are ready to prove the following.

Theorem 5.4 *Let K be an origin-symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 if and only if $\|x\|_K$ is the limit (in the radial metric) of finite products $\|x\|_{E_1}^{1/p_1} \cdots \|x\|_{E_m}^{1/p_m}$, where $E_1, \dots, E_m$ are ellipsoids and $1/p_1 + \cdots + 1/p_m = 1$.*

Proof The "if" part is a consequence of the fact that L_0 is closed with respect to the three operations as discussed above. The proof of the "only if" part easily follows from the lemmas we have proved. Suppose that $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0 with the corresponding probability measure μ on S^{n-1} and constant C. By Remark 3.2, $(\mathbb{R}^n, \|\cdot\|_{E_{a,b}(x)})$ embeds in L_0 with the measure $-\frac{1}{(2\pi)^n} (\ln \|x\|_E)^\wedge(\theta) d\theta$ and some constant $C_{E_{a,b}}$. Note, this constant does not depend on x. We have

$$\begin{aligned}
&\int_{S^{n-1}} \ln\|\xi\|_{E_{a,b}(x)}\, d\mu(\xi) \\
&\qquad = \int_{S^{n-1}} \int_{S^{n-1}} \ln |(\xi, \theta)| \Big(-\frac{1}{(2\pi)^n} \Big) (\ln \|x\|_{E_{a,b}(x)})^\wedge(\theta)\, d\theta d\mu(\xi) + C_{E_{a,b}} \\
&\qquad = \int_{S^{n-1}} \Big[\int_{S^{n-1}} \ln |(\xi, \theta)|\, d\mu(\xi) + C_K \Big] \Big(-\frac{1}{(2\pi)^n} \Big) (\ln \|x\|_{E_{a,b}(x)})^\wedge(\theta)\, d\theta \\
&\qquad\quad + C_{E_{a,b}} - C_K \\
&\qquad = \int_{S^{n-1}} \ln \|\theta\|_K \Big(-\frac{1}{(2\pi)^n} \Big) (\ln \|x\|_{E_{a,b}(x)})^\wedge(\theta)\, d\theta + C_{E_{a,b}} - C_K
\end{aligned}$$

$$= \int_{S^{n-1}} \ln \|\theta\|_K \Big(-\frac{1}{(2\pi)^n} \Big) (\ln \|x\|_{E_{a,b}(x)})^\wedge(\theta)\, d\theta + C_{E_{a,b}} - C_K$$
$$= \frac{1}{|S^{n-1}|a^{n-1}b} \int_{S^{n-1}} \ln \|\theta\|_K \|\theta\|^{-n}_{E_{b,a}(x)}\, d\theta + C_{E_{a,b}} - C_K.$$

In Lemma 5.2 we proved that $\ln \|x\|_K$ can be uniformly approximated by the integrals of the form

$$\frac{1}{|S^{n-1}|a^{n-1}b} \int_{S^{n-1}} \ln \|\theta\|_K \|\theta\|^{-n}_{E_{b,a}(x)}\, d\theta,$$

as $a \to 0$. Therefore, using the previous calculations, one can see that $\ln \|x\|_K$ can be uniformly approximated by $\int_{S^{n-1}} \ln \|\xi\|_{E_{a,b}(x)} d\mu(\xi) + C'$. Hence, by Lemma 5.3, $\ln \|x\|_K$ can be uniformly approximated by the sums

$$\sum_{i=1}^{m} \frac{1}{p_i} \ln \|x\|_{E_i} + C'.$$

Replacing E_1 by another ellipsoid E_1' given by $\|x\|^{1/p_1}_{E_1'} = e^{C'} \|x\|^{1/p_1}_{E_1}$, we get the statement of the theorem. ■

Corollary 5.5 *Any convex body in $\mathbb{R}^3$ can be obtained from the Euclidean unit ball by means of three operations: linear transformations, multiplicative addition, and closure in the radial metric.*

Proof As was proved in Theorem 5.4, any convex body can be approximated by the finite products of the type $\|x\|^{1/p_1}_{E_1} \cdots \|x\|^{1/p_m}_{E_m}$. Since any number $1/p$ can be approximated by the sums

$$\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_k}},$$

the result follows. ■

A proof similar to that of Theorem 5.4 can be used to show that the previous theorem holds for p-summation with $-1 < p < 1$, $p \neq 0$, in place of the multiplicative summation.

Theorem 5.6 *Let K be an origin-symmetric star body in $\mathbb{R}^n$. The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p, $-1 < p < 1$, $p \neq 0$ if and only if $\|x\|^p_K$ is the limit (in the radial topology) of finite sums $\|x\|^p_{E_1} + \cdots + \|x\|^p_{E_m}$, where $E_1, \ldots, E_m$ are ellipsoids.*

6 Confirming the Place of L_0 in the Scale of L_p-Spaces.

In this section we establish the relations between embedding in L_0 and in L_p with $p \neq 0$, which confirm the place of L_0 between L_p with $p > 0$ and $p < 0$. We are going to use the following result from [K3, Theorem 1].

Theorem 6.1 *An n-dimensional homogeneous space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-p}, $p \in (0, n)$ if and only if $\|x\|_K^{-p}$ is a positive definite distribution.*

We also use a well-known result of P. Levy (see [BL, p. 189], and [BDK] for the infinite dimensional case).

Theorem 6.2 *A space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_p, $p \in (0, 2]$ if and only if the function $\exp(-\|x\|_K^p)$ is positive definite.*

Now we are ready to prove the following.

Theorem 6.3 *Let K be an origin-symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_0, then it also embeds in L_{-p}, $0 < p < n$.*

Proof By Theorem 5.4, $\|x\|_K$ is the limit of the finite products $\|x\|_{E_1}^{1/p_1} \cdots \|x\|_{E_m}^{1/p_m}$. Consider $\|x\|_K^{-p}$ for $0 < p < n$. It is the limit of the products $\|x\|_{E_1}^{-p/p_1} \cdots \|x\|_{E_m}^{-p/p_m}$. Using the formula

$$\|x\|^{-p} = \frac{2}{\Gamma(p/2)} \int_0^\infty t^{p-1} \exp(-t^2\|x\|^2)\, dt,$$

we get

$$\|x\|_{E_1}^{-p/p_1} \cdots \|x\|_{E_m}^{-p/p_m} = C \int_0^\infty \cdots \int_0^\infty t_1^{p/p_1-1} \cdots t_m^{p/p_m-1} \times \exp(-t_1^2\|x\|_{E_1}^2 - \cdots - t_m^2\|x\|_{E_m}^2)\, dt_1 \cdots dt_m,$$

where

$$C = \frac{2^m}{\Gamma(p/2p_1) \cdots \Gamma(p/2p_m)}.$$

Therefore, for any non-negative test function ϕ we have

$$\begin{aligned} \left\langle (\|x\|_{E_1}^{-p/p_1} \cdots \|x\|_{E_m}^{-p/p_m})^\wedge(\xi), \phi(\xi) \right\rangle &= \left\langle \|x\|_{E_1}^{-p/p_1} \cdots \|x\|_{E_m}^{-p/p_m}, \widehat{\phi}(x) \right\rangle \\ &= C \int_0^\infty \cdots \int_0^\infty t_1^{p/p_1-1} \cdots t_m^{p/p_m-1} \\ &\quad \times \left\langle \exp(-t_1^2\|x\|_{E_1}^2 - \cdots - t_m^2\|x\|_{E_m}^2), \widehat{\phi}(x) \right\rangle dt_1 \cdots dt_m \\ &= C \int_0^\infty \cdots \int_0^\infty t_1^{p/p_1-1} \cdots t_m^{p/p_m-1} \\ &\quad \times \left\langle (\exp(-t_1^2\|x\|_{E_1}^2 - \cdots - t_m^2\|x\|_{E_m}^2))^\wedge(\xi), \phi(\xi) \right\rangle dt_1 \cdots dt_m. \end{aligned}$$

We claim that the latter expression is non-negative. Indeed, $(\mathbb{R}^n, \|x\|_E)$ embeds in L_2 for any ellipsoid, therefore the 2-sum of ellipsoids $t_1^2\|x\|_{E_1}^2 + \cdots + t_m^2\|x\|_{E_m}^2$ embeds in L_2, and hence by Theorem 6.2, the function $\exp(-t_1^2\|x\|_{E_1}^2 - \cdots - t_m^2\|x\|_{E_m}^2)$ is positive definite. Now the fact that $\langle (\|x\|_K^{-p})^\wedge, \phi \rangle \geq 0$ follows by an approximation argument, as in Corollary 4.3. ∎

Theorem 6.4 *Let K be an origin-symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-p} for every $p \in (0,\epsilon)$, then it also embeds in L_0.*

Proof The space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{-p}, so by Theorem 6.1 the distribution $\|x\|^{-p}$ is positive definite. Then for every non-negative test function ϕ supported outside of the origin,

$$\begin{aligned}-\int_{\mathbb{R}^n} \ln\|x\|\widehat{\phi}(x)\,dx &= \lim_{p\to 0}\frac{1}{p}\int_{\mathbb{R}^n}(\|x\|^{-p}-1)\widehat{\phi}(x)\,dx\\ &= \lim_{p\to 0}\frac{1}{p}\int_{\mathbb{R}^n}\|x\|^{-p}\widehat{\phi}(x)\,dx \ge 0.\end{aligned}$$

The result follows from Theorem 3.1. ■

Theorem 6.5 *There are normed spaces that embed in L_0, but do not embed in L_p for $p > 0$.*

Proof As proved above, every 3-dimensional normed space embeds in L_0, hence l_q^3 with $q > 2$ does. On the other hand, l_q^3, $q > 2$ does not embed in L_p for $0 < p \le 2$ (see [K1]). ■

Let us also mention that one can use the approach of [KK1] to produce examples in the same spirit. It follows from [KK1, Proposition 3.5] that $\mathbb{R}\oplus_2\ell_1$ does not embed isometrically into L_p for $p > 0$; hence neither does $\mathbb{R}\oplus_2\ell_1^n$ for large enough n.

Proposition 6.6 *For any $n \in \mathbb{N}$ the space $\mathbb{R}\oplus_2\ell_1^n$ embeds in L_0.*

Proof Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions on some probability space which are independent and 1-stable symmetric, so that $\mathbb{E}(e^{itf_j}) = e^{-|t|}$ (*i.e.*, the f_j have the Cauchy distribution). Then it is clear that

$$\mathbb{E}\ln\Big|\sum_{j=1}^n a_jf_j\Big| = \ln\sum_{j=1}^n |a_j|.$$

Indeed, this follows from the fact that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|x|}{1+x^2}\,dx = 0.$$

On the other hand, if $f = \sum_{j=1}^n a_jf_j$, where $\sum_{j=1}^n |a_j| = 1$, then f has the Cauchy distribution and so has the same distribution as g_1/g_2 where g_1 and g_2 are independent normalized Gaussians. Hence

$$\begin{aligned}\mathbb{E}\ln|a+bf| &= \mathbb{E}(\ln|ag_2+bg_1| - \ln|g_2|)\\ &= \ln(a^2+b^2)^{\frac{1}{2}}.\end{aligned}$$

Now for any $a_0, a_1, \ldots, a_n \in \mathbb{R}$ we have

$$\mathbb{E}|a_0 + \sum_{j=1}^{n} a_j f_j| = \ln\Big(|a_0|^2 + \Big(\sum_{j=1}^{n}|a_j|\Big)^2\Big)^{\frac{1}{2}}.$$

This shows (using the remarks at the end of §2) that $\mathbb{R}\oplus_2 \ell_1^n$ embeds into L_0 for every n. ■

Theorem 6.7 *Let K be an origin-symmetric star body in $\mathbb{R}^n$. If the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{p_0}, $0 < p_0 \le 2$, then it also embeds in L_0.*

Proof Since $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in L_{p_0}, $0 < p_0 \le 2$, by [K3, Theorem 2] it also embeds in L_{-p} for any $p \in (0, n)$ and hence, by Theorem 6.4, it embeds in L_0. ■

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