

Flat Embeddings of the Möbius Strip in \mathbb{R}^3

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Abstract

This is a paper originally circulated in preprint form in 1984. The paper contains two main results: a theorem that gives sufficient conditions for a space curve to be the axis of a ruled developable Möbius strip in \mathbb{R}^3 and a new formula for the total torsion of a spherical space curve. After the paper was written, in 1984, a referee pointed out that part of the first result—the existence of a flat embedding of the Möbius strip in \mathbb{R}^3 —had been obtained by W. Wunderlich in 1962. While for this reason the paper was not published, it has since been cited several times, for example, by G. Schwarz, *The dark side of the Möbius strip*, *Amer. Math. Monthly*, **97** (1990), 890–897; T. Randrup and P. Røgen, *Sides of the Möbius strip*, *Arch. Math.* **66** (1996), 511–521 and T. Randrup and P. Røgen, *Gauss-Bonnet's theorem and closed Frenet frames*, *Geom. Dedicata* **73** (3)(1998), 295–315. The paper is published here, with minor corrections, to preserve its original contents.

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1 Introduction

If the ends of a strip are joined to form a cylinder the Gauss curvature of the resulting surface is zero. If the ends of the strip are joined after a half twist a Möbius strip is formed. Does the resulting surface have zero Gauss curvature? It is perhaps a surprising fact that the standard embedding of the Möbius strip in \mathbb{R}^3 , i.e., the ruled surface generated by an interval which undergoes a rotation through 180° while its midpoint traverses a circle has negative Gauss curvature everywhere [1, p. 172]. However, after some reflection it is easy to see there are C^∞ embeddings of the Möbius strip with zero curvature. Figure 1 shows such an embedding which lies on the surface of a cylinder which is C^∞ flat to a portion of a plane.

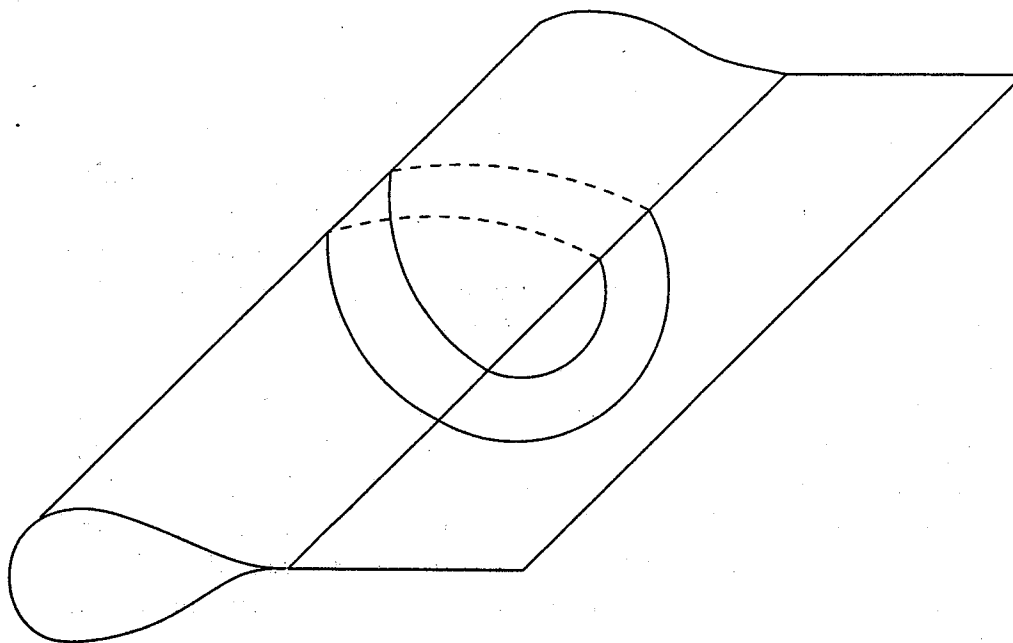


Figure 1:

Moe Hirsch has asked if there is an *analytic* embedding of the Möbius strip with zero Gauss curvature. Actually this question has a rather long history [8]. M. Sadowsky [5, 6] found a developable Möbius strip M which is constructed from portions of plane and cylindrical surfaces such that M has minimal deformation energy E among ruled Möbius strip surfaces with a single half twist. Here E is defined to be

$$\int_M \chi^2 dA$$

where χ denotes the principal normal curvature perpendicular to the generator of the ruled surface and dA denotes the element of area. Later, W. Wunderlich [8] described a flat embedding of the Möbius strip as a portion of a developable ruled algebraic surface (defined by polynomials of degree 39) in 3-dimensional real projective space. This surface is obtained as the envelope of the rectifying planes along a curve in projective space parametrized by polynomials of degree 6. This curve is given in homogeneous coordinates by

$$\begin{aligned} x_0 &= \frac{1}{2}(1 + (\delta t + \epsilon t^3)^2) \\ x_1 &= \alpha t + \beta t^3 + \gamma t^5 \\ x_2 &= \delta t + \epsilon t^3 \\ x_3 &= -c \end{aligned}$$

where a convenient choice of the coefficients as explained in [8] is

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{3}, \quad \gamma = \frac{1}{6}, \quad \delta = \frac{2}{3}, \quad \epsilon = \frac{1}{3}, \quad c = \frac{4}{5}.$$

Of course, in \mathbb{R}^3 the curve is represented (using the projection $x_0 : x_1 : x_2 : x_3 = 1 : x : y : (z - c)$) by the rational functions

$$\begin{aligned} x(t) &= \frac{\alpha t + \beta t^3 + \gamma t^5}{\frac{1}{2}(1 + (\delta t + \epsilon t^3)^2)} \\ y(t) &= \frac{\delta t + \epsilon t^3}{\frac{1}{2}(1 + (\delta t + \epsilon t^3)^2)} \\ z(t) &= c - \frac{c}{\frac{1}{2}(1 + (\delta t + \epsilon t^3)^2)}. \end{aligned}$$

Since these equations do not parametrize a closed space curve they must be interpreted to include the parameter values $t = \pm\infty$ which correspond to

the point in space with coordinates $(0, 0, c)$. Actually this leads to a slight technical difficulty (which our analysis avoids) since the curvature of the space curve vanishes when $t = \pm\infty$. a similar interpretation must be made for the family of rectifying planes along the space curve.

In this paper we also answer Hirsch's question affirmatively; we prove there exists an analytic flat (Gauss curvature zero) Möbius strip in \mathbb{R}^3 . The embedding of the Möbius strip which we construct is a subset of an analytic ruled developable surface where the ruling is parametrized by a circle. Our embedding is defined by a function of the form $F(s, u) = \alpha(s) + u\omega(s)$ where α , called an axis of the ruled surface, is an analytic periodic function which parametrizes a closed space curve with nonzero curvature everywhere transverse to the ruling lines generated by the analytic curve of unit vectors given by ω . In addition to this result we show that a smooth axis of a smooth ruled developable Möbius strip cannot be a plane curve and any nonplanar closed curve with nonzero curvature is the axis of some ruled developable Möbius strip. Moreover, if α is a closed space curve with nonzero curvature which is not planar then α is an axis of a ruled developable Möbius strip whose ruling is everywhere orthogonal to α if and only if the total torsion of α is an odd multiple of π .

The authors wish to thank the referee for pointing out reference [8] and suggesting many improvements of the original version of this paper.

2 An Embedding Theorem

In this section we will prove the following theorem: There exists an analytic flat Möbius strip embedded in \mathbb{R}^3 . To this end we discuss some notation and basic facts about space curves and ruled surfaces which will be used throughout this paper.

Let α denote a closed space curve with no self intersections parametrized by arc length. We assume α is periodic with period T and that the curvature of α never vanishes. Then, as usual, we define

$$t = \alpha', \quad n = \frac{\alpha''}{|\alpha''|} \quad \text{and} \quad b = t \times n.$$

The Frenet frame $\langle t, n, b \rangle$ along α satisfies the cross product relations

$$b = t \times n, \quad t \times b = -n, \quad n \times b = t$$

and the Frenet equations

$$\begin{aligned}t' &= \kappa n \\n' &= -\kappa t - \tau b \\b' &= \tau n\end{aligned}$$

where $\kappa > 0$ is the curvature and τ (with the sign convention of [1, p. 181]) is the torsion of α .

We intend to construct our embedding of the Möbius strip as a portion of a ruled surface parametrized by a function of the form

$$F(s, u) = \alpha(s) + u\omega(s)$$

where α is a space curve satisfying the conditions stated above, ω is a curve of unit vectors, $s \in R$ and $|u| < \epsilon$ for some $\epsilon > 0$. Here the ruling of the surface is the family of straight lines generated by the directrix ω . Since F is to be an embedding F must parametrize a regular surface, i.e., there must be a nonvanishing normal vector at each point of the surface. In our notation the normal vector is given by $(t + u\omega') \times \omega$. Thus, since the image of α is compact the condition $t(s) \times \omega(s) \neq 0$ for all s insures there is an $\epsilon > 0$ such that the strip $\{(s, u) | s \in R \text{ and } |u| < \epsilon\}$ is mapped to a regular surface by F . A closed curve which satisfies this condition is transversal to all the lines in the ruling and is called an axis of the ruled surface. Finally, the ruled surface will be flat (have Gauss curvature identically zero) if and only if it is developable [1, p. 194], i.e., $\langle t, \omega \times \omega' \rangle = 0$. If $\omega = pt + qn + rb$ we compute

$$\langle t, \omega \times \omega' \rangle = qr' - rq' - \tau(q^2 + r^2) - rp\kappa.$$

Hence, F parametrizes a flat surface if and only if

$$qr' - rq' = \tau(q^2 + r^2) + rp\kappa.$$

If we view (q, r) as coordinates in a plane then the equation may be interpreted as

$$\theta' = p\kappa \sin \theta + \tau$$

where θ is the angle $(q(s), r(s))$ makes with the positive r axis. In this interpretation (q, r) are the coordinates of the projection of ω into the (normal) plane spanned by n and b at each point along α . Since the Frenet frame is periodic of period T , ω will generate a Möbius strip provided $\omega(s+T) = -\omega(s)$ which is the same as saying $\theta(s+T) = \theta(s) + (2n+1)\pi$. This is our basic geometric idea; it leads to the following proposition.

Proposition 2.1. *Assume α is a smooth (resp. analytic) closed space curve with nonvanishing curvature κ , no self intersections and torsion τ . If there is a smooth (resp. analytic) function θ satisfying $\theta(s+T) = \theta(s) + (2n+1)\pi$ for some integer n and some real number $T > 0$ such that the function*

$$f(s) = \frac{\theta' - \tau}{\sin \theta}$$

is smooth (resp. analytic), then α is the axis of a regular developable embedding of the Möbius strip in \mathbb{R}^3 which is smooth (resp. analytic).

Proof. We assume $\alpha(s+T) = \alpha(s)$, $|\alpha'(s)| = 1$ and $\alpha''(s) \neq 0$. We will construct $\omega(s) = pt + qn + rb$ such that (1) ω is smooth, (2) $|\omega| = 1$, (3) $t \times \omega \neq 0$, (4) $\langle t, \omega \times \omega' \rangle = 0$ and (5) $\omega(s+T) = -\omega(s)$. If these conditions are satisfied it is clear that for some $\epsilon > 0$ the function $F(s, u) = \alpha(s) + u\omega(s)$ will parametrize a regular developable Möbius strip for $s \in R$ and $|u| < \epsilon$.

If θ exists satisfying the hypothesis define $p = f(\kappa^2 + f^2)^{1/2}$. Then, $p(s+T) = -p(s)$, $|p(s)| < 1$ and p is smooth. Since $|p| < 1$, $R(s) = (1 - p^2)^{1/2}$ is smooth as are q, r defined by $q = R \cos \theta$ and $r = R \sin \theta$. With these definitions we have

$$(1) \quad \omega = pt + qn + rb \text{ is smooth,}$$

$$(2) \quad |\omega|^2 = p^2 + q^2 + r^2 = f^2(\kappa^2 + f^2)^{-1} + 1 - p^2 = 1,$$

$$(3) \quad |t \times \omega|^2 = |qb - rn|^2 = 1 - p^2 \neq 0,$$

$$(4)$$

$$\begin{aligned} r p \kappa + \tau(q^2 + p^2) &= R p \kappa \sin \theta + \tau R^2 \\ &= R^2(p R^{-1} \kappa \sin \theta + \tau) \\ &= R^2(p(1 - p^2)^{-1/2} \kappa \sin \theta + \tau) \\ &= R^2(f \sin \theta + \tau) \\ &= R^2 \theta' = q r' - r q' \end{aligned}$$

and

$$(5) \quad \omega(s+T) = -\omega(s)$$

as required. □

Theorem 2.2. *There exists an analytic flat Möbius strip embedded in \mathbb{R}^3 .*

Proof. We will construct an analytic closed space curve α of length T with nonzero curvature, no self intersections and total torsion π , i.e., $\alpha : R \rightarrow \mathbb{R}^3$ is parametrized by arc length, satisfies $\alpha(s + T) = \alpha(s)$ and

$$\int_0^T \tau(s) ds = \pi.$$

Then, we define

$$\theta(s) = \int_0^s \tau(\sigma) d\sigma$$

and observe that

$$\theta(s + T) = \theta(s) + \pi$$

and

$$\theta'(s) = \tau(s)$$

for all $s \in R$. Thus

$$f(s) = \frac{\theta' - \tau}{\sin \theta} \equiv 0$$

is analytic and the theorem will follow from Proposition 2.1.

To construct α we observe that a curve on the surface of a round cylinder of form $\beta(\theta) = (a \cos \theta, a \sin \theta, v(\theta))$ with $v(\theta + 2\pi) = v(\theta)$ automatically has nonvanishing curvature and no self intersections. Set $\theta = \rho(s)$ so $\alpha = \beta(\rho(s))$ is parametrized by arc length and assume the total length of α is T . Also, as a notational convenience let $w = v'(\theta)$. Then, the total torsion of α is

$$\begin{aligned} \int_0^T \tau(\rho(s)) ds &= - \int_0^{2\pi} \frac{\langle \beta' \times \beta'', \beta''' \rangle}{|\beta' \times \beta''|^2 \rho'(\rho^{-1}(\theta))} d\theta \\ &= - \int_0^{2\pi} \frac{(w'' + w)(a^2 + w^2)^{1/2}}{a^2 + w^2 + (w')^2} d\theta \\ &= - \int_0^{2\pi} \frac{(w'' + w)(a^2 + w^2)^{1/2}}{a^2 + w^2 + (w')^2} - \frac{d}{d\theta} \tan^{-1} \frac{w'}{(a^2 + w^2)^{1/2}} d\theta \\ &= - \int_0^{2\pi} \frac{w}{(a^2 + w^2)^{1/2}} d\theta. \end{aligned}$$

□

Thus, the proof of the Theorem will be complete after we prove

Proposition 2.3. *There is an analytic 2π -periodic function v such that*

$$\int_0^{2\pi} \frac{v'}{(a^2 + (v')^2)^{1/2}} d\theta = I$$

if and only if $I \in (-2\pi, 2\pi)$.

Proof. The fact that $I \in (-2\pi, 2\pi)$ follows immediately since

$$\left| \frac{w}{(a^2 + w^2)^{1/2}} \right| < 1.$$

For the construction of a function v such that the integral has value $I \in (-2\pi, 2\pi)$ we define for $\lambda \in (-\infty, \infty)$ (and $w = v'(\theta)$).

$$I(\lambda) = \int_0^{2\pi} \frac{\lambda w}{(a^2 + \lambda^2 w^2)^{1/2}} d\theta$$

and note that I depends continuously on λ with $I(0) = 0$. Since the integrand is bounded Lebesgue's Dominated Convergence Theorem applies and we have

$$\lim_{\lambda \rightarrow \infty} I(\lambda) = \int_0^{2\pi} \text{sgn}(w) d\theta = A - B$$

where A denotes the length of the subset of $[0, 2\pi]$ where $w(\theta) > 0$ and B denotes the length of the subset where $w(\theta) < 0$. Choose a continuous function g defined on $[0, 2\pi]$ which satisfies (1) $\int_0^{2\pi} g d\theta = 0$, (2) $g(0) = 0 = g(2\pi)$, and (3) the length of the set where $g(\theta) > 0$ is $2\pi - \delta$ for $\delta > 0$. Now, for any $\epsilon > 0$ there is a trigonometric polynomial $Q_\epsilon(\theta)$ such that $\|Q_\epsilon - g\| < \epsilon$. Thus, the function w_ϵ defined by

$$w_\epsilon(\theta) = Q_\epsilon(\theta) - \frac{1}{2\pi} \int_0^{2\pi} Q_\epsilon d\theta$$

satisfies

$$\|w_\epsilon - g\| \leq \epsilon + \frac{1}{2\pi} \left| \int_0^{2\pi} (Q_\epsilon - g) d\theta \right| < 2\epsilon.$$

It follows that $w_\epsilon(\theta) > g(\theta) - 2\epsilon$. Thus, for $\epsilon > 0$ sufficiently small $w_\epsilon(\theta) > 0$ on a subset of $[0, 2\pi]$ whose length exceeds $2\pi - 2\delta$. For this ϵ and $w = w_\epsilon$ there is some λ such that $I(\lambda) = 2\pi - 2\delta$. Finally, we define

$$v = \int_0^\theta \lambda w(t) dt.$$

It follows at once from the definition of w_ϵ that v is analytic and 2π -periodic. In this way we obtain any $I \in [0, 2\pi]$. For the negative values of I simply note that $I(-\lambda) = -I(\lambda)$. \square

The embedding of the Möbius strip obtained in Theorem 2.2 was constructed as a parametrized ruled developable surface of form

$$F(s, u) = \alpha(s) + u\omega(s)$$

where the axis α is a smooth embedding of a circle. It is not clear that all flat embeddings of the Möbius strip can be obtained in this way. However, since any flat surface in \mathbb{R}^3 is "locally ruled" [1, sec 5.8] a further investigation of the geometrical properties of embeddings of this type seems reasonable. In general the ruled developable surfaces can be quite complicated although in some sense they are all made up of sections of cones, cylinders and tangent developables [1, p. 195]. It would be interesting to classify the flat surfaces which can contain a Möbius strip. For example an analytic Möbius strip cannot be embedded in a cylinder or a cone (although, taking the most general definition of a cylinder, the figure in the introduction indicates that a smooth Möbius strip can be embedded in a cylinder). Our next theorem shows the axis of a flat Möbius strip cannot be an arbitrary closed curve.

Theorem 2.4. *A smooth axis with nonzero curvature of a smooth ruled developable Möbius strip cannot lie in a plane.*

Proof. Let α be the axis and let

$$F(s, u) = \alpha(s) + u\omega(s)$$

be a parametrization of the Möbius strip (for $s \in R$ and $|u| < \epsilon$ for some $\epsilon > 0$). Then if $\omega = pt + qn + rb$ according to the discussion preceding Proposition 2.1 there is a smooth function θ satisfying the differential equation

$$\theta' = p\kappa \sin \theta + \tau$$

and the condition

$$\theta(s + T) = \theta(s) + (2n + 1)\pi$$

for some integer n where T is the period of the function α .

Now, if α is planar since $\kappa \neq 0$ it follows immediately that $\tau \equiv 0$. To complete the proof we will show this is impossible. In fact, for $n \geq 0$ there

is a point s_0 such that $\tau(s_0) > 0$ and for $n < 0$ there is a point s_0 such that $\tau(s_0) < 0$. We will prove this proposition for $n = 0$ all other cases are similar. Suppose $\tau(s) \leq 0$ for all s . There is a first point s_1 for which $\theta(s_1) = \pi$. Choose $s_0 > s_1$ so that $0 < \theta(s) \leq \pi$ for $s_0 \leq s \leq s_1$. If $s_0 < s < s_1$ we have

$$\begin{aligned} \int_{s_0}^s f(t) dt &= \int_{s_0}^s \frac{\theta'(t)}{\sin \theta(t)} dt - \int_{s_0}^s \frac{\tau(t)}{\sin \theta(t)} dt \\ &\geq \int_{s_0}^s \frac{\theta'(t)}{\sin \theta(t)} dt \\ &= \ln((\csc \theta(s_0) + \cot \theta(s_0))/(\csc \theta(s) + \cot \theta(s))). \end{aligned}$$

Thus,

$$\ln(\csc \theta(s) + \cot \theta(s)) \geq \ln(\csc \theta(s_0) + \cot \theta(s_0)) - \int_{s_0}^s f(t) dt.$$

However,

$$\lim_{s \rightarrow s_1} \ln(\csc \theta(s) + \cot \theta(s)) = -\infty$$

and we have a contradiction. □

The next theorem shows that any space curve which does not lie in a plane is the axis of an embedding of the Möbius strip.

Theorem 2.5. *Let α be a closed smooth (resp. analytic) curve with nonzero curvature and no self intersections. If α does not lie in a plane then α is the axis of a smooth (resp. analytic) ruled developable Möbius strip.*

Proof. Assume as usual that $\alpha : R \rightarrow R^3$ is parametrized by arc length and is periodic of period T . Also, let τ denote the torsion of α . Then, in view of Proposition 2.1 we must construct a smooth (resp. analytic) function θ satisfying $\theta(s + T) = \theta(s) + (2n + 1)\pi$ for some integer n and such that the function

$$f(s) = \frac{\theta' - \tau}{\sin \theta}$$

is smooth (resp. analytic). In order to construct θ we need the following lemma.

Lemma 2.6. *Suppose $\beta > \gamma$ and $\beta > 0$. Then, there is an even trigonometric polynomial $P(t)$ such that*

$$(1) \int_{-\pi}^{\pi} P(t) dt = 0, (2) P(0) = \gamma \text{ and } (3) P(t) < \beta \text{ for } -\pi \leq t \leq \pi.$$

If $\beta < \gamma$ and $\beta < 0$, then there is an even trigonometric polynomial $P(t)$ such that

$$(1) \int_{-\pi}^{\pi} P(t) dt = 0, (2) P(0) = \gamma \text{ and } (3) P(t) > \beta \text{ for } -\pi \leq t \leq \pi.$$

Proof. We prove the first statement. The proof of the second statement is similar. Let C denote the space of real continuous functions $g : [-\pi, \pi] \rightarrow \mathbb{R}$ such that $g(-\pi) = g(\pi)$ with the supremum norm. Define $H : C \rightarrow \mathbb{R}^2$ by

$$H(g) = \left(\int_{-\pi}^{\pi} g(t) dt, g(0) \right).$$

For the open set $A = \{g \in C \mid g(t) < \beta \text{ for } -\pi \leq t \leq \pi\}$ the image $H(A) = \{(s, y) \mid x < 2\beta\pi, y < \beta\}$ is an open cone. Let $B = A \cap T$ where T is the dense space of trigonometric polynomials. Then $H(A \cap T)$ is dense in $H(A)$ and also convex. Hence $H(A \cap T) = H(A)$ since $H(A)$ is open. Thus, there is a trigonometric polynomial $Q \in A \cap T$ such that $H(Q) = (0, \gamma)$. Finally, set $P(t) = 1/2(Q(t) + Q(-t))$ and observe P is even and satisfies (1), (2), and (3). \square

Since α is not planar its torsion τ does not vanish identically. We use this fact to construct θ . In fact we show if $\tau(s_0) > 0$ (resp. $\tau(s_0) < 0$) for some s_0 , then there exists a function θ of the form

$$\theta(s) = \frac{\pi}{T}(s - s_0) + \sum_{n=1}^N a_n \cos(2n\pi s/T)$$

$$\text{(resp. } \theta(s) = -\frac{\pi}{T}(s - s_0) + \sum_{n=1}^N a_n \cos(2n\pi s/T)\text{)}$$

and a continuous function f satisfying

$$f \sin \theta = \theta' - \tau.$$

Such an f satisfies $f(s+T) = -f(s)$ and, in addition, if τ is C^∞ (resp. real analytic) then f is C^∞ (resp. real analytic).

We will prove this proposition for the case $\tau(s_0) > 0$; the other case is similar. Let $\tau(s_0) = \delta$. By the lemma there is an even trigonometric polynomial P satisfying (1) $P(0) = \frac{\pi}{T} - \delta$ (2) $P(s) < \max\left(\frac{\pi}{2T}, \frac{\pi}{T} - \frac{\delta}{2}\right)$ for $s \in R$ and (3) $\int_{-\pi}^{\pi} P(s) ds = 0$, i.e., P has no constant term.

Suppose

$$P(s) = \sum_{n=1}^N \alpha_n \cos ns$$

and define

$$\theta(s) = \frac{\pi(s-s_0)}{T} - \sum_{n=1}^N \frac{T}{2n\pi} \alpha_n \sin(2n\pi(s-s_0)/T).$$

Then,

$$\theta'(s) = \frac{\pi}{T} - \sum_{n=1}^N \alpha_n \cos(2n\pi(s-s_0)/T).$$

Since

$$\theta'(s) \geq \frac{\pi}{T} - \max\left(\frac{\pi}{2T}, \frac{\pi}{T} - \frac{\delta}{2}\right) = \min\left(\frac{\pi}{2T}, \frac{\delta}{2}\right) > 0$$

θ is an increasing function and $\theta(s_0 + mT) = m\pi$ for each integer m . Thus, we have $\sin \theta(s) = 0$ if and only if $s = s_0 + mT = s_m$. At these points

$$\theta'(s_m) = \frac{\pi}{T} - \sum_{n=1}^N \alpha_n = \frac{\pi}{T} - P(0) = \delta = \tau(s_m).$$

Now

$$\begin{aligned} \lim_{s \rightarrow s_m} \frac{\theta'(s) - \tau(s)}{\sin \theta(s)} &= \lim_{s \rightarrow s_m} \frac{\theta''(s) - \tau'(s)}{\theta'(s) \cos \theta(s)} \\ &= (-1)^m \frac{\theta''(s_0) - \tau'(s_0)}{\theta'(s_0)} \end{aligned}$$

by L'Hopital's rule. Hence, if we define

$$f(s) = \begin{cases} \frac{\theta'(s) - \tau(s)}{\sin \theta(s)}, & s \neq s_m \\ (-1)^m \frac{\theta''(s_0) - \tau'(s_0)}{\theta'(s_0)}, & s = s_m \end{cases}$$

then f is continuous and smooth if τ is smooth. \square

In our proof of the existence of a flat Möbius strip in \mathbb{R}^3 we considered in Theorem 2.2 an embedding where the axis of the ruled surface was everywhere orthogonal to the ruling. This assumption is quite restrictive on the global structure of the geometry of the embedding. As one example of this restriction we have

Theorem 2.7. *Let α be a closed smooth curve with nonvanishing curvature. Then α is an axis of a ruled developable Möbius strip with ruling everywhere orthogonal to α if and only if the total torsion of α is an odd multiple of π .*

Proof. As usual if α is an axis of a ruled developable Möbius strip we consider a parametrization of the surface in the form

$$F(s, u) = \alpha(s) + u\omega(s)$$

where $\omega(s) = pt + qn + rb$. If κ is the curvature and τ the torsion of α , then as before there will be a solution θ of the differential equation

$$\theta' = p\kappa \sin \theta + \tau$$

which satisfies $\theta(s + T) = \theta(s) + (2n + 1)\pi$ for an integer n where T is the period of α . But, since α is orthogonal to the ruling $p \equiv 0$ and θ is actually a solution of

$$\theta' = \tau.$$

Thus, the total torsion is

$$\int_0^T \tau(s) ds = (2n + 1)\pi.$$

On the other hand if the total torsion is an odd multiple of π the proof of Theorem 2.2 shows there is a ruled developable Möbius strip of the required form. \square

3 The Total Torsion of Space Curves

This section is independent from the first part of the paper; it was inspired by the calculation for the total torsion of the space curve $\alpha(\theta) =$

$(a \cos \theta, a \sin \theta, v(\theta))$ lying on the surface of a circular cylinder. We develop an apparently new formula for the total torsion of a closed space curve.

Let $\alpha = \alpha(s)$, $0 \leq s \leq L$ denote an analytic space curve parametrized by arc length. We do not as yet suppose α is closed. However, we do assume throughout that the curvature of α never vanishes. Then the tangent curve $t = \alpha'$ describes an analytic curve on the surface of the unit sphere and t' is never zero. We select a point P on the sphere so that $t(s)$ never visits either P or its antipode; then select coordinates so that P is the north pole, i.e., $P = (0, 0, 1)$. With respect to these coordinates we write

$$\begin{aligned}\alpha(s) &= (f(s), g(s), h(s)) \\ t(s) &= (u(s), v(s), w(s)).\end{aligned}$$

Define the unit vector $\eta = (0, 0, 1)$ and $N = t \times \eta = (v, -u, 0)$. Then (t, η, N) is a moving frame along α which is *not* orthonormal. However, we have

$$t \times \eta = N, \quad t \times N = wt - \eta, \quad \eta \times N = t - w\eta$$

and the corresponding "Frenet" equations

$$\begin{aligned}t' &= -\beta wt + \beta\eta + cN \\ \eta' &= 0 \\ N' &= -ct + cw\eta - \beta wN\end{aligned}$$

where

$$\begin{aligned}c &= \frac{vu' - uv'}{u^2 + v^2} \\ \beta &= \frac{w'}{1 - w^2}.\end{aligned}$$

Here, $w^2 \neq 1$ for all s by the hypothesis on P . Thus, both fractions have nonvanishing denominators.

Proposition 3.1. *For the curve α the curvature κ and the torsion τ are given by*

$$\begin{aligned}\kappa &= ((\beta^2 + c^2)(1 - w^2))^{1/2} \\ \tau &= \frac{\beta'c - c'\beta}{\beta^2 + c^2} + cw\end{aligned}$$

where, since $\kappa > 0$, $\beta^2 + c^2$ never vanishes.

Proof. It is helpful to compute $|t| = |\eta| = 1$, $\langle t, N \rangle = 0$, $\langle t, \eta \rangle = w$, $\langle \eta, N \rangle = 0$ and $|N|^2 = 1 - w^2$. Next, express the usual Frenet frame in terms of (t, η, N) to obtain

$$\begin{aligned} t &= t \\ \kappa n &= -\beta w t + \beta \eta + c N \\ \kappa b &= c w t - c \eta + \beta N. \end{aligned}$$

Then κ^2 is just the length of $-\beta w t + \beta \eta + c N$. To compute τ , differentiate κb and then solve for $\tau = \langle b', n \rangle$. \square

The tangent curve $t = (u, v, w)$ in spherical coordinates is given by

$$\begin{aligned} u &= \sin \phi \cos \theta \\ v &= \sin \phi \sin \theta \\ w &= \cos \phi \end{aligned}$$

where $\phi = \phi(s)$, $\theta = \theta(s)$ for $0 \leq s \leq L$ and $0 < \phi(s) < \pi$ for all s . Moreover, we compute

$$\begin{aligned} c &= -\theta' \\ \beta &= -(\csc \phi) \phi'. \end{aligned}$$

Now suppose $t(s)$ parametrizes a closed curve Γ on the sphere, i.e., $t(L) = t(0)$; this happens in particular if α is closed. We identify the sphere (minus the antipode of P) with the plane by stereographic projection onto the tangent plane at P so that $(2 \tan(\phi/2), \theta)$ are polar coordinates in the plane. Using this identification we define the *rotation index* $\text{Ind } \Gamma$ and the *winding number* $n(\Gamma, Q)$ of Γ with respect to any point Q on the sphere. In particular we define $n(\Gamma, Q) = 0$ when Q is the antipode of P . As defined both the rotation index and the winding number depend on the choice of pole and the orientation of the sphere. However, the differential form $(2n(\Gamma, Q) - \text{Ind } \Gamma) dA$ defined on $\Omega = S^2 \setminus \Gamma$ where dA denotes the (oriented) area element does not depend on either the choice of pole or the orientation of the sphere. This is easy to see after checking that the integer valued function $2n(\Gamma, Q) - \text{Ind } \Gamma$ does not depend on the choice of pole. For $Q \in \Gamma$ we define $2n(\Gamma, Q) - \text{Ind } \Gamma = 0$.

Theorem 3.2. *Under the hypotheses described above the total torsion of α is given by*

$$\int_0^L \tau(s) ds = \frac{1}{2} \int_{S^2} 2n(\Gamma, Q) - \text{Ind } \Gamma dA.$$

Proof. We first compute

$$I_1 = \int_0^L \frac{\beta'c - c'\beta}{\beta^2 + c^2} ds.$$

To do this let Γ_1 be the plane curve with polar coordinates $r = \phi(s)$, $\theta = \theta(s)$. Since Γ_1 is just a radial rescaling of the stereographic projection of Γ we have

$$\begin{aligned} \text{Ind } \Gamma_1 &= \text{Ind } \Gamma \\ n(\Gamma_1, 0) &= n(\Gamma, P). \end{aligned}$$

Now

$$\text{Ind } \Gamma_1 - n(\Gamma_1, 0) = n(\Gamma_2, 0)$$

where Γ_2 is the plane curve with Cartesian coordinates $(\phi', \phi\theta')$. In effect, if ψ is the angle between the radius vector and the tangent vector and if σ is the angle the tangent vector makes with the horizontal axis then $\psi = \sigma - \theta$. But, since $0 < \phi(s) < \pi$ for all s we have

$$n(\Gamma_2, 0) = n(\Gamma_3, 0)$$

where Γ_3 is the plane curve with Cartesian coordinates $((\csc \phi)\phi', \theta')$ i.e. the curve $(-\beta, -c)$. Thus,

$$I_1 = 2\pi(n(\Gamma, P) - \text{Ind } \Gamma).$$

Next we compute

$$I_2 = \int_0^L wc ds = - \int_0^L w\theta' ds$$

which we recognize as a line integral so that

$$I_2 = - \int_{\Gamma} w d\theta = - \int_{\Gamma} \cos \phi d\theta.$$

We claim

$$I_2 = \int_{S^2} n(\Gamma, Q) dA - 2\pi n(\Gamma, P).$$

To show this formula is valid for any piecewise analytic closed curve Γ it is only necessary to establish it for a simple closed curve; the general case follows by induction on the number of components of Ω . So, suppose Γ is a simple closed curve. Then, Ω has two components R_1 and R_2 . If $n(\Gamma, P) = 0$ let R_1 be the component which corresponds to the bounded component of the plane curve Γ_1 given in polar coordinates by (ϕ, θ) . We apply Green's Theorem to obtain

$$-\int_{\Gamma} \cos \phi d\theta = n_1 \iint_{R_1} \sin \phi d\phi d\theta$$

where $n_1 = \pm 1$ is the constant value of $n(\Gamma, Q)$ for $Q \in R_1$. It follows that

$$I_2 = \int_{S^2} n(\Gamma, Q) dA.$$

If $n(\Gamma, P) \neq 0$ we apply Green's Theorem by modifying Γ in the usual way to be the boundary curve of a slit annular region with inner boundary a small circle $\Gamma(r)$ of radius r around P . Then, we obtain

$$\lim_{r \rightarrow 0} \left(n_1 \int_{\Gamma(r)} \cos \phi d\theta \right) - \int_{\Gamma} \cos \phi d\theta = \int_{S^2} n(\Gamma, Q) dA.$$

Thus, in either case,

$$I_2 = \int_{S^2} n(\Gamma, Q) dA - 2\pi n(\Gamma, P).$$

Finally,

$$\begin{aligned} I_1 + I_2 &= \int_{S^2} n(\Gamma, Q) dA - 2\pi \text{Ind } \Gamma \\ &= \frac{1}{2} \int_{S^2} 2n(\Gamma, Q) - \text{Ind } \Gamma dA. \end{aligned}$$

□

To offer some applications of Theorem 3.2 we first suppose Γ , the tangent curve of the closed space curve α , is a simple closed curve on the sphere which is positively oriented with respect to the north pole P . Then, Γ separates the sphere into two components R_1 and R_2 . Assume $P \in R_1$ so that $\text{Ind } \Gamma = 1$, $n(\Gamma, Q) = 1$ for $Q \in R_1$ and $n(\Gamma, Q) = 0$ for $Q \in R_2$. We obtain

$$\int_0^L \tau ds = \text{Area } R_1 - 2\pi = \frac{1}{2}(\text{Area } R_1 - \text{Area } R_2)$$

and, therefore,

$$-2\pi < \int_0^L \tau ds < 2\pi.$$

Recall also the classical result [4, p. 170] which states: the total torsion of a closed curve on the surface of the sphere is zero. For any such curve α we have

$$\text{Ind } \Gamma = \frac{1}{2\pi} \int_{S^2} n(\Gamma, Q) dA$$

and, if Γ happens to be a simple closed curve on the sphere then it partitions the sphere into two components with equal areas.

If, instead of the tangent curve Γ , we consider the original space curve α and suppose α is *cylindrical*, i.e., for $\alpha(s) = (f(s), g(s), h(s))$ the curvature of the plane projection $\alpha(s) = (f(s), g(s))$ does not change sign and $t \times \eta = (g', -f', 0)$ never vanishes, then the tangent curve Γ of α always turns in the same direction about P . In particular, $\text{Ind } \Gamma = n(\Gamma, P)$. Moreover, when $n(\Gamma, Q)$ is nonzero it has the same sign as $n(\Gamma, P)$. Now assume $n(\Gamma, P) > 0$. We have

$$\frac{1}{2} \int_{S^2} \text{Ind } \Gamma dA = 2\pi \text{Ind } \Gamma = 2\pi n(\Gamma, P)$$

so

$$\int_0^L \tau ds = \int_{S^2} n(\Gamma, Q) dA - 2\pi n(\Gamma, P).$$

But,

$$0 < \int_{S^2} n(\Gamma, Q) dA < \int_{S^2} n(\Gamma, P) dA = 4\pi n(\Gamma, P)$$

since $n(\Gamma, Q) \leq n(\Gamma, P)$ for all Q and $n(\Gamma, Q) = 0$ on a component of $S^2 \setminus \Gamma$. Hence,

$$-2\pi |\text{Ind } \Gamma| < \int_0^L \tau ds < 2\pi |\text{Ind } \Gamma|.$$

When $n(\Gamma, P) < 0$ a similar argument shows the same inequality holds. Of course, $\text{Ind } \Gamma$ may be replaced in this inequality by either $n(\Gamma, P)$ or $\text{Ind } \gamma$. We have proved the following fact.

Corollary 3.3. *If α is a cylindrical curve and if the curvature of its plane projection γ does not change sign then the absolute value of its total torsion is bounded by $2\pi|\text{Ind } \gamma|$.*

Finally, we mention some problems which arise naturally from our considerations. (1) Find an analytic closed space curve with no self intersections, nonvanishing curvature and torsion everywhere positive. (2) Find necessary and sufficient conditions for T -periodic functions κ and τ to be the curvature and torsion of an (analytic) closed space curve (without self intersections). This problem seems to have been first formulated by Fenchel [2]. In this regard recall that the theorems of Fenchel and Fary-Milnor [1] that the total curvature of a closed space curve is greater than or equal to 2π and greater than or equal to 4π if it is knotted provide necessary conditions for κ to be the curvature of a closed space curve. (3) Formulate, using continuum mechanics, physical models for the shape of an elastic strip which is twisted into a Möbius strip with the ends welded together and find the possible configurations which the model predicts [5], [6] and [7]. For some mathematical background see Truesdell's survey [7]. (4) Exhibit explicit "simple" examples of the embeddings shown to exist by Theorem 2.2 and Theorem 2.5.

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